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Weakly right po-Noetherian ordered semigroups

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Abstract. In this paper, we present a new way of defining the property of being WRP-Noetherian by making use of principal right poideals. Additionally, we provide a characterization of WRP-Noetherian ordered semigroups through their S-posets. Furthermore, we investigate how the property of being WRP-Noetherian behaves under some semigroup-theoretic constructions, like sub ordered semigroups, and quotients. Specifically, we establish necessary and sufficient conditions for the direct product of two ordered semigroups to be WRP-Noetherian.

1 Introduction and Preliminaries

In the study of certain types of algebraic structures known as (universal) algebras, a finiteness condition is a requirement that is satisfied by all finite members of the class. The concept was first introduced and developed by Noether and Artin in the early 1900s. Since then, finiteness conditions have played an important role in understanding the structure and behavior of rings, groups, semigroups, and many other types of algebras. The property

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of being Noetherian, as one of the most important finiteness conditions, is of fundamental importance in abstract algebra. The concepts of Noetherian and Artinian rings and modules have been extensively researched by several authors and have been useful in the development of the structure theory of rings (For example, see [1, 17]). Noetherian rings are critical in ring theory, playing a significant role in major results such as Krull's intersection theorem and Hilbert's basis theorem.

In the case of right Noetherian semigroups, the research was initiated by Hotzel in [8] and further developed by Kozhukhov in [11]. A monoid S is called weakly right Noetherian if every right ideal is finitely generated. It is considered right Noetherian if every right congruence is finitely generated. These types of semigroups have been extensively studied and have gained a lot of attention in research; see for instance [4, 7, 12]. In the case of a monoid, this condition has been highly used in the theory of acts over monoids. For instance, if a monoid M is right Noetherian, then every right S-act that is finitely generated is also finitely presented [13]. Additionally, it has been proven in [11] that if S is right Noetherian, then every subgroup of S is right Noetherian as well. The concept of weakly right po-Noetherian (or briefly, WRP-Noetherian) ordered semigroups was first introduced in [9] (named Notherian ordered semigroups) and then studied in [15] and [16]. It is proved in [9], that being WRP-Noetherian for an ordered semigroup is equivalent to the fact that the right poideals of the ordered semigroup satisfy the so-called ascending chain condition, moreover it is equivalent to the fact that an ordered semigroup satisfies the maximum condition for right poideals. In addition, they proved that if an ordered semigroup S is WRP-Noetherian and I is its poideal then the Rees quotient ordered semigroup S/I is also WRP-Noetherian. Also, if S/I and I are WRP-Noetherian, then so is S.

The present paper is devoted to the study of WRP-Noetherian ordered semigroups. We present an alternative way of defining the WRP-Noetherian property using principal right poideals. Additionally, we provide a characterization of WRP-Noetherian ordered monoids based on their S-posets. Furthermore, we investigate how the WRP-Noetherian property behaves under various semigroup-theoretic constructions, such as sub ordered semigroups, and quotients. In particular, we establish necessary and sufficient conditions for the direct product of two ordered semigroups to be WRP-

Noetherian.

In the following, we briefly recall some basic definitions about posemigroups and S-posets needed in the sequel. For more information see [3], [5], [15], and [16]. Recall that a monoid (semigroup) S is said to be an *ordered monoid* (*ordered semigroup*) (briefly, *pomonoid* (*posemigroup*)) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A *left poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid left ideal ($SI \subseteq I$) and a poset ideal (that is, a down closed subset of S: $a \leq b, b \in I$ imply $a \in I$).

For every subset X of a pomonoid S,

$$<X> = \downarrow(XS^1) = \{t \in S : \exists x \in X, \exists s \in S^1, t \leq xs\}$$

is the smallest right poideal of S which contains X and is called the right poideal of S generated by X. If X is finite, $<X>$ is called a finitely generated right poideal, and if X is singleton, $<X>$ is called a principal right poideal of S. One can easily proves that $\downarrow(\bigcup_{x\in X} xS^1) = \bigcup_{x\in X} \downarrow(xS^1)$. A posemigroup S with no proper (right) poideals is called (right) simple.

If S is a posemigroup, an *order congruence* θ on S is an equivalence relation on S that is compatible with the binary operation on S, and has the further property that S/θ can be equipped with a partial order so that S/θ is a posemigroup and the natural map $S \to S/\theta$ is a posemigroup homomorphism.

Recall from [2] that if θ is any binary relation on S, we write $s \leq_\theta s'$ if a so-called θ-chain

$$s \leq s_1\theta s'_1 \leq s_2\theta s'_2 \leq \cdots \theta s'_m \leq s'$$

from s to s' for some $s_1, s'_1, ..., s_n, s'_n \in S$, exists in S. Then a congruence θ on S is an order congruence if and only if for every $s, s' \in S$, $s\theta s'$ whenever $s \leq_\theta s' \leq_\theta s$.

Recall from [10] that for each ordered set $(X, \leq_X)$, we can construct a free ordered semigroup (or briefly, free posemigroup) over $(X, \leq_X)$. In fact, one can consider the set

$$X^+ := \{x_1x_2\dots x_n|\ n \in \mathbb{N} \text{ and } x_i \in X, i = 1, 2, \dots, n\}$$

with the operation (concatenation)

$$(x_1x_2\ldots x_n).(y_1y_2\ldots y_m) := x_1x_2\ldots x_ny_1y_2\ldots y_m,$$

and the order (componentwise)

$$x_1x_2\ldots x_n \preceq y_1y_2\ldots y_m \Leftrightarrow n = m \text{ and } x_i \leq_X y_i, \text{ for all } i = 1, 2, \ldots, n.$$

Then the pair $(X^+, ., \preceq)$ is a free posemigroup over $(X, \leq_X)$. Let $(X^+, ., \preceq)$ be the free posemigroup with the basis $(X, \leq_X)$. Then $X^* = X^+\dot{\cup}\{\emptyset\}$ is called the free pomonoid over $(X, \leq_X)$ or the free pomonoid with the basis $(X, \leq_X)$ where concatenation with the empty word $\emptyset$ leaves everything unchanged.

Recall from [14] that Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ on a posemigroup are given as follows: For two elements $a, b \in S$, $a\mathcal{L}b$ if they generate the same principal left poideal, that is $\downarrow(S^1a) = \downarrow(S^1b)$. Similarly, $a\mathcal{R}b$ if they generate the same principal right poideal, that is $\downarrow(aS^1) = \downarrow(bS^1)$. Also, $a\mathcal{J}b$ if they generate the same principal poideal, that is $\downarrow(S^1aS^1) = \downarrow(S^1bS^1)$. It is obvious that, Green's relation $\mathcal{L}$ is a right order congruence on S and $\mathcal{R}$ is a left order congruence on S.

Green's relation $\mathcal{R}$ defines a preorder $\leq_{\mathcal{R}}$ on S, given by

$$a \leq_{\mathcal{R}} b \Leftrightarrow \downarrow(aS^1) \subseteq \downarrow(bS^1).$$

The preorder $\leq_{\mathcal{R}}$ induces a partial order on the set of $\mathcal{R}$-classes of S given by

$$R_a \leq R_b \text{ if and only if } a \leq_{\mathcal{R}} b.$$

We recall the following definition which has been introduced in [9] as Noetherian ordered groupoid (semigroup).

Definition 1.1. A posemigroup S is said to be *WRP-Noetherian* if every right poideal of S is finitely generated.

Recall that a posemigroup S satisfies the ascending chain condition for poideals if, for any sequence of poideals $I_1, I_2, \ldots, I_i, \ldots$ of S such that

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_i \subseteq \ldots$$

there exists an element $n \in \mathbb{N}$ of natural numbers such that $I_m = I_n$ for each $m \in \mathbb{N}, m \geq n$. Also, it is said that, a posemigroup S satisfies the maximum condition for poideals if each nonempty set of poideals of S, partially ordered by inclusion, has a maximal element. That is, for each nonempty set $\mathcal{L}$ of poideals of S, there is an element $M \in \mathcal{L}$ such that there is no element $N \in \mathcal{L}$ with $M \subset N$. Equivalently, if $N \in \mathcal{L}$ and $N \subseteq M$, then $N = M$.

The following theorem provides the equivalent conditions for being WRP-Noetherian for a posemigroup S in terms of the ascending chain condition and maximal condition on right poideals.

Theorem 1.2. ([9], Theorem 2.5) *The following are equivalent for a posemigroup S:*

(i) *S is WRP-Noetherian.*

(ii) *S satisfies the ascending chain condition for right poideals.*

(iii) *S satisfies the maximum condition for right poideals.*

2 Weakly right po-Noetherian posemigroups

We begin this section by providing a characterization for WRP-Noetherian posemigroups in terms of their principal right poideals, as well as their $\mathcal{R}$-class structure.

Theorem 2.1. *The following are equivalent for a posemigroup S:*

(i) *S is WRP-Noetherian;*

(ii) *S has no infinite set of pairwise incomparable principal right poideals and satisfies the ascending chain condition on principal right poideals.*

(iii) *the poset of $\mathcal{R}$-classes of S contains no infinite strictly ascending chain or infinite antichain (An antichain of a poset is a subset consisting of pairwise incomparable elements).*

Proof. (i) $\Rightarrow$ (ii) Clearly, by Theorem 1.2, S fulfills the ascending chain condition on principal right poideals.

We are going to prove that S does not contain an infinite set of pairwise incomparable principal right poideals. Let's assume the opposite for the sake of contradiction.

Suppose that there exists an infinite set $\{\downarrow(a_iS^1) : i \in \mathbb{N}\}$ of pairwise incomparable principal right poideals of S. For each $n \in \mathbb{N}$, let I_n be a right poideal of S generated by $a_1, \ldots, a_n$ (i.e. $I_n = \downarrow(\{a_1, \ldots, a_n\}S^1)$. Now, if for some $m \leq n, I_m = I_n$ then for some $i \leq m, a_n \in \downarrow(a_i S^1)$. This implies that $i = m = n$, since $\downarrow(a_i S^1)$ and $\downarrow(a_n S^1)$ are incomparable. Therefore, we get an infinite strictly ascending chain of right poideals of S,

$$I_1 \subset I_2 \subset \ldots.$$

This contradicts Theorem 1.2.

(ii) $\Rightarrow$ (i) Let S satisfy the ascending chain condition on principal right poideals but it is not WRP-Noetherian. We will show that there exists an infinite set of pairwise incomparable principal right poideals of S.

Since S is not WRP-Noetherian, there exists an infinite strictly ascending chain of right poideals of S,

$$I_1 \subset I_2 \subset \ldots,$$

by Theorem 1.2. Choose $a_1 \in I_1$ and for $k \geq 2, a_k \in I_k - I_{k-1}$. Then since $\downarrow(a_j S^1) \subseteq I_j$ and $a_k \in I_k - I_j, j < k$, the principal right poideal $\downarrow(a_k S^1)$ is not contained in any principal right poideal $\downarrow(a_j S^1)$.

The set $\{\downarrow(a_i S^1) : i \in \mathbb{N}\}$ contains a maximal element, namely $\downarrow(a_{r_1} S^1)$. Therefore, $\downarrow(a_{r_1} S^1)$ is not contained in any $\downarrow(a_j S^1), j \neq r_1$. If it were, then there would exist an infinite strictly ascending chain of principal right poideals of S, which contradicts the hypothesis.

Now consider the infinite set $\{\downarrow(a_i S^1) : i \geq r_1 + 1\}$. This set contains a maximal element, denoted by $\downarrow(a_{r_2} S^1)$. It follows that $\downarrow(a_{r_2} S^1)$ is not contained in $\downarrow(a_j S^1)$ for any $j > r_1$, where $j \neq r_2$. This means that, $\downarrow(a_{r_2} S^1)$ is not contained in $\downarrow(a_j S^1)$ for any j that belongs to the set of natural numbers, except for $j = r_2$. This is because $\downarrow(a_{r_2} S^1)$ is not contained in any $\downarrow(a_j S^1)$ where $j < k$.

We can continue this process infinitely, and obtain an infinite set $\{\downarrow(a_{r_i} S^1) : i \in \mathbb{N}\}$ of pairwise incomparable principal right poideals of S.

(iii) $\Leftrightarrow$ (i) It is easily seen that the poset of $\mathcal{R}$-classes of S is isomorphic to the poset of principal right poideals of S, via the order isomorphism given by $R_a = \{b \in S : \downarrow(aS^1) = \downarrow(bS^1)\} \mapsto \downarrow(aS^1)$. □

Corollary 2.2. *Any posemigroup with a finite number of $\mathcal{R}$-classes is WRP-Noetherian. In particular, all finite posemigroups and all right simple posemigroups (which include pogroups) are WRP-Noetherian.*

By the following example we show that satisfying the ascending chain condition on principal right poideals for a posemigroup S is not sufficient for being WRP-Noetherian.

Example 2.3. Suppose that X is a non-empty poset. Then the free posemigroup F_X on X satisfies the ascending chain condition on principal right poideals. But, F_X is WRP-Noetherian if and only if $|X| = 1$. For, let $y, z \in F_X$. Then $\downarrow(yF_X^1) \subset \downarrow(zF_X^1)$ if and only if z is a proper prefix of y which means that the number of elements of X appearing in y is greater than that of z. Therefore, there does not exist an infinite strictly ascending chain of principal right poideals of F_X.

If X is a trivial poset, then F_X contains no infinite set of pairwise incomparable principal right poideals, so it is WRP-Noetherian by Theorem 2.1. Now, suppose that X has at least two elements, and choose the elements $x_1, x_2 \in X$ with $x_1 \neq x_2$. If $i \neq j$, the element $x_1^i x_2$ is not a prefix of $x_1^j x_2$ and vice versa. So one gets an infinite set $\{\downarrow((x_1^i x_2)F_X^1) : i \in \mathbb{N}\}$ of pairwise incomparable principal right poideals and hence, F_X is not WRP-Noetherian by Theorem 2.1.

Note that the free posemigroup F_X is a sub posemigroup of the free pogroup on X, which is WRP-Noetherian, while F_X for $|X| > 1$ is not WRP-Noetherian. So the property of being WRP-Noetherian is not closed under sub posemigroups.

In the following result we show that, similar to the case for monoids and monoid acts, a posemigroup S is WRP-Noetherian if and only if every finitely generated right S-poset is WRP-Noetherian. This means that it satisfies the ascending chain condition on its down closed sub S-posets. Recall that a (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order.

Proposition 2.4. *For each posemigroup S the following statements are equivalent:*

(i) *S is WRP-Noetherian.*

(ii) *Every finitely generated right S-poset is WRP-Noetherian.*

Proof. (i) $\Rightarrow$ (ii) Suppose that A is a finitely generated right S-poset such that $A = \langle X \rangle = XS^1$ and X is a finite, and B is a down closed sub S-poset of A. For each $x \in X$, define a set $I_x = \{s \in S : xs \in B\}$. For each $x \in X$, if $I_x \neq \emptyset$ then it is a right poideal of S since B is a down closed sub S-poset of A. Since S is WRP-Noetherian, for each $x \in X$ with $I_x \neq \emptyset$ there exists a finite set $U_x \subseteq I_x$ such that $I_x = \langle U_x \rangle$. It is proved that B is generated by the set $U = \bigcup_{x \in X, I_x \neq \emptyset} xU_x$. Suppose $b \in B$. Then $b \in A$ and so $b = xs, x \in X, s \in S$. Hence $s \in I_x$, and so for some $u_x \in U_x$ and $s' \in S, s \leq u_x s'$. Therefore, $b = xs \leq (xu_x)s' \in \downarrow(US^1)$, as desired.

(ii) $\Rightarrow$ (i) Since the right poideals of S are down closed sub S-posets of the right S-poset S (which is cyclic) the result holds. □

The following result shows that a finite union of WRP-Noetherian posemigroups is WRP-Noetherian.

Proposition 2.5. *Let a posemigroup S be a union of its sub posemigroups $S_1, \ldots, S_n$ which are WRP-Noetherian. Then S is WRP-Noetherian.*

Proof. Let I be a right poideal of S. For $1 \leq i \leq n$, let $I_i = I \cap S_i$ which clearly are right poideals of S_i. Then each I_i is generated by some finite set X_i since S_i is WRP-Noetherian. Now, we prove that I is generated by the finite set $X = \bigcup_{i=1}^{n} X_i$. Let $s \in I$. Then for some $1 \leq i \leq n, s \in I \cap S_i = I_i = \downarrow(X_i S_i^1)$, as desired. □

Now, the relationship between a WRP-Noetherian posemigroup and its quotients is considered.

Proposition 2.6. *If S is a WRP-Noetherian posemigroup and θ is an order congruence on S, then S/θ is also WRP-Noetherian.*

Proof. Suppose that $\bar{I}$ is a right poideal of S/θ. Define the set $I = \{s \in S : [s]_\theta \in \bar{I}\}$. It is easily seen that I is a right poideal of S. Then I is generated by a finite set X since S is WRP-Noetherian. Now, we prove that $\bar{I}$ is generated by the finite set $\bar{X} = \{[x]_\theta : x \in X\}$. Let $[s]_\theta \in \bar{I}, s \in I$. Then for some $x \in X$ and $t \in S^1, s \leq xt$. Then $[s]_\theta \leq [x]_\theta [t]_\theta \in \downarrow(\bar{X}(S/\theta)^1)$, as required. □

Note that the converse of the above Proposition does not hold in general. For example, the free posemigroup $F_X, |X| \geq 2$ is not WRP-Noetherian by Example 2.3, while there exist quotients of F_X which are WRP-Noetherian. However, if θ is an order congruence on S contained in the Green's relation $\mathcal{R}$, the converse holds, too.

Proposition 2.7. *Suppose that S is a posemigroup and θ is an order congruence on S contained in the Green's relation $\mathcal{R}$. Then the poset of $\mathcal{R}$-classes of S is isomorphic to the poset of $\mathcal{R}$-classes of S/θ. Particularly, S is WRP-Noetherian if and only if S/θ is WRP-Noetherian.*

Proof. Suppose that $\sum_S$ is the poset of principal right poideals of S, and $\sum_{S/\theta}$ is the poset of principal right poideals of S/θ. Using the fact that the poset of principal right poideals of a posemigroup is isomorphic to the poset of its $\mathcal{R}$-classes, it is enough to prove that $\sum_S$ and $\sum_{S/\theta}$ are isomorphic. Then by Theorem 2.1, S is WRP-Noetherian if and only if S/θ is WRP-Noetherian.

Consider a mapping $\alpha : \sum_S \to \sum_{S/\theta}$, given by

$$\downarrow(sS^1) \mapsto \downarrow([s]_\theta(S/\theta)^1).$$

It is obvious that α is surjective, and it remains to show that it is an order embedding (i.e. $\downarrow(sS^1) \subseteq \downarrow(s'S^1)$ if and only if $\downarrow([s]_\theta(S/\theta)^1) \subseteq \downarrow([s']_\theta(S/\theta)^1)$. It is clear that if $\downarrow(sS^1) \subseteq \downarrow(s'S^1)$ then $\downarrow([s]_\theta(S/\theta)^1) \subseteq \downarrow([s']_\theta(S/\theta)^1)$. Conversely, let $\downarrow([s]_\theta(S/\theta)^1) \subseteq \downarrow([s']_\theta(S/\theta)^1)$. Then $[s]_\theta \in \downarrow([s']_\theta(S/\theta)^1)$ and so for some $[t]_\theta \in (S/\theta)^1, [s]_\theta \leq [s']_\theta[t]_\theta = [s't]_\theta$. Hence $s \leq_\theta s't$, and so for some $s_1, s_1', ..., s_n, s_n' \in S, n \in \mathbb{N}$, $s \leq s_1\theta s_1' \leq ... \leq s_n\theta s_n' \leq s'$. Then we get

$$s \leq s_1 \leq s_1't_1 \leq s_2t_1 \leq \ldots \leq s_n't_n \ldots t_1 \leq s'tt_n \ldots t_1 \in s'S^1$$

for some $t_1, \ldots, t_n \in S^1$ since $\theta \subseteq \mathcal{R}$, which means that $s \in \downarrow(s'S^1)$, as desired. □

As we have discussed before, sub posemigroups of WRP-Noetherian posemigroups are not necessarily WRP-Noetherian. This section aims to study when the property of being WRP-Noetherian can be transferred between a posemigroup S and its sub posemigroup T.

Lemma 2.8. *Suppose that T is a sub posemigroup of a posemigroup S such that $S - T$ is contained in a finite union of $\mathcal{R}$-classes. Then S is WRP-Noetherian if T is WRP-Noetherian.*

Proof. Consider a right poideal I of S. Then $I \cap T$ is a right poideal of T and so there exists a finite set $X \subseteq I \cap T$ with $I \cap T = \downarrow(XT^1)$ since T

is WRP-Noetherian by the assumption. Let $S - T \subseteq R_1 \cup \ldots \cup R_n$ where $R_1, \ldots, R_n$ are $\mathcal{R}$-classes. Fix $r_i \in R_i$ for each $1 \le i \le n$. We prove that $I = {\downarrow}(YS^1)$ where $Y = X \cup \{r_i : R_i \subseteq I, 1 \le i \le n\}$ is a finite set. Let $a \in I$. If $a \in I \cap T$, then $a \in {\downarrow}(XT^1)$. If $a \in S - T$, then $a \in I \cap R_i$ for some $1 \le i \le n$. Hence ${\downarrow}(aS^1) = {\downarrow}(r_iS^1)$ which means that $a \le r_is$ for some $s \in S^1$. Hence, in both cases, we get $a \in {\downarrow}(YS^1)$. □

Let's assume that S is a posemigroup and T is its sub posemigroup. We can define the T-relative Green's relation $\mathcal{R}^T$ on S as follows:

Two elements $a, b \in S$ are related by $\mathcal{R}^T$, denoted as $a\mathcal{R}^Tb$, if and only if ${\downarrow}(aT^1) = {\downarrow}(bT^1)$.

Similarly, we can define the relation $\mathcal{L}^T$ on S, and $\mathcal{H}^T$ as the intersection of $\mathcal{R}^T$ and $\mathcal{L}^T$. We can also define $\mathcal{D}^T$ and $\mathcal{J}^T$ in a similar way.

All of these relations are equivalence relations on S, and if T is a down closed sub posemigroup of S, each class belongs entirely to either T or $S - T$.

Proposition 2.9. *Suppose that S is a posemigroup and T is its sub posemigroup such that its complement $S-T$ contains only finitely many $\mathcal{R}^T$-classes. Then each $\mathcal{R}^S$-class of S is a union of a finite number of $\mathcal{R}^T$-classes.*

Proof. We can consider the case where $S - T \neq \emptyset$, because if $S = T$, trivially the result holds. Suppose R^S is an $\mathcal{R}^S$-class of S. Now, assuming for contradiction, that R^S is a union of an infinite number of $\mathcal{R}^T$-classes, we can deduce that $R^S \cap T$ must contain infinitely many $\mathcal{R}^T$-classes, considering that $S - T$ contains only a finite number of $\mathcal{R}^T$-classes. We claim that for any $t_1 \in R^S \cap T$, there exists an $x \in T$ such that $t_1x \in R^S \cap T$ while for all y in T, $t_1 \nleq t_1xy$. According to the definition, we have $R^S \subseteq {\downarrow}(t_1S^1)$. Let u and v be elements in $S - T$, where uR^Tv. This implies that $t_1uR^Tt_1v$ because R^T is a left congruence. Since $S-T$ contains only finitely many $\mathcal{R}^T$-classes, it follows that ${\downarrow}t_1(S - T) \cap T$ will only intersect the finite number of the infinitely many $\mathcal{R}^T$-classes contained in $R^S \cap T$. Thus, there are infinitely many $\mathcal{R}^T$-classes in $R^S \cap T$ that can only be obtained from t_1 by right multiplication by elements from T. Specifically, one can find an $\mathcal{R}^T$-class R in $R^S \cap T$ that is different from that of t_1, and an element x in T such that $t_1x \in R$. Since t_1 and t_1x are not in the same $\mathcal{R}^T$-class and x is an element of T, we conclude that there is no element y in T such that $t_1 \le t_1xy$.

By choosing t_1 and x as the above, and assuming $t_2 = t_1x$, obviously we get $\downarrow(t_1T^1) \supset \downarrow(t_2T^1)$. Now, we can generate an infinite chain

$$\downarrow(t_1T^1) \supset \downarrow(t_2T^1) \supset \downarrow(t_3T^1) \supset \dots$$

of such elements since t_1 is arbitrary. We can then choose $x_i \in S-T$ for each i such that $t_i \leq t_{i+1}x_i$, because all t_i's belong to a single $\mathcal{R}^S$-class of S, and for all $j \leq i$, we have $t_i \leq t_{j+1}x_jx_{j-1}...x_i$. This implies that $x_jx_{j-1}...x_i \in S-T$, since $\downarrow(t_iT^1) \supset \downarrow(t_{j+1}T^1)$. As $S-T$ contains only finitely many $\mathcal{R}^T$-classes, there exists $l>j>i$ such that $x_lx_{l-1}...x_j R^T x_lx_{l-1}...x_jx_{j-1}...x_i$. This means that there exists an element $t \in T$ such that $x_lx_{l-1}...x_jx_{j-1}...x_i \leq x_lx_{l-1}...x_jt$. Then $t_i \leq t_{l+1}x_lx_{l-1}...x_jx_{j-1}...x_i \leq t_{l+1}x_lx_{l-1}...x_jt \leq t_jt$. This leads to a contradiction since $\downarrow(t_iT^1) \supset \downarrow(t_jT^1)$, and the proof completes. □

Proposition 2.10. *Suppose that S is a posemigroup such that its poideals are exactly its semigroup ideals. If T is a sub posemigroup of S such that $S-T$ is a finite union of $\mathcal{R}^T$-classes, then S is WRP-Noetherian if and only if T is WRP-Noetherian.*

Proof. Suppose that I is a right poideal of T and J equals the right poideal of S generated by I, $J = IS^1$. Since S is WRP-Noetherian, J can be generated by some finite subset $X \subseteq I$. Suppose $S-T$ contains the $\mathcal{R}^T$-classes $R_1, R_2, \dots, R_n$, and for each $1 \leq i \leq n$, fix $r_i \in R_i$. We prove that $Y = X \cup \{xr_i \in I : x \in X, 1 \leq i \leq n\}$ is a finite generating set for I. Let a be an element of I. Then, a belongs to $J = XS^1$, and hence there exist elements $x \in X$ and $s \in S^1$ such that $a = xs$. In the case where $s \in T^1$, $a \in \downarrow(YT^1)$. Let $s \in S-T$. Then there exists some $i \in \{1, \dots, n\}$ such that $s \in R_i$. Thus, we can find $t, u \in T^1$ such that $s \leq r_it$ and $r_i \leq su$. Therefore, $xr_i \leq xsu = au \in I$ and so $xr_i \in I$ since I is a right poideal of T. Hence $a \leq xs \leq (xr_i)t \in YT^1$ which shows that $a \in \downarrow(YT^1)$, as desired.

For the converse, by Proposition 2.9, each $\mathcal{R}^S$-class is a union of $\mathcal{R}^T$-classes, and hence $S-T$ is contained in a finite union of $\mathcal{R}^S$-classes. Therefore, S is WRP-Noetherian by Lemma 2.8. □

Corollary 2.11. *Suppose S is a posemigroup such that its poideals are exactly its semigroup ideals, and T is its sub posemigroup where $S-T$ is finite. Then S is WRP-Noetherian if and only if T is WRP-Noetherian.*

Corollary 2.12. *Let S be a posemigroup. Then S is WRP-Noetherian if and only if S^1 is WRP-Noetherian if and only if S^0 is WRP- Noetherian.*

Suppose that S is a posemigroup and T is its sub posemigroup. It is easily seen that

$$\leq_{\mathcal{R}_T} \subseteq \leq_{\mathcal{R}_S} \cap (T \times T).$$

We say that T preserves $\mathcal{R}$ (in S), or is $\mathcal{R}$-preserving, if

$$\leq_{\mathcal{R}_T} = \leq_{\mathcal{R}_S} \cap (T \times T).$$

In the following result, we prove that the property of being WRP-Noetherian is preserved by $\mathcal{R}$–preserving sub posemigroups.

Proposition 2.13. *Suppose S is a posemigroup and T is a sub posemigroup of S which is $\mathcal{R}$-preserving. If S is WRP-Noetherian, then T is also WRP-Noetherian.*

Proof. Assume that I is a right poideal of T, and $\bar{I} = \downarrow(IS^1)$. As S is WRP-Noetherian, there exists a finite subset $X \subseteq I$ such that $\bar{I} = \downarrow(XS^1)$. Suppose a is an element of I, then $a \in \downarrow(xS^1)$ for some $x \in X$, implying $a \leq_{\mathcal{R}_S} x$. According to the assumption, $a \leq_{\mathcal{R}_T} x$, which means $a \in \downarrow(xT^1)$. Therefore, $I = \downarrow(XT^1)$, which proves that I is finitely generated. □

Definition 2.14. We call an element s of a posemigroup S *poregular* if there exists an element $x \in S$ such that $s \leq sxs$. A posemigroup S is called *poregular* if all of its elements are poregular. An element $x \in S$ such that $s \leq sxs$ is called a *pseudoinverse* of s. An element $s' \in S$ such that $s \leq ss's$ and $s' \leq s'ss'$ is called a *po-inverse* of s. Let s be a poregular element of S, so there exists an element $x \in S$ such that $s \leq sxs$. Put $s' = xsx$. It can be easily seen that $s \leq ss's$ and $s' \leq s'ss'$. Therefore, every poregular element of a posemigroup S has a po-inverse element.

Corollary 2.15. *Suppose S is a WRP-Noetherian posemigroup with the property that its right poideals are exactly its semigroup right ideals, and T is its poregular sub posemigroup. Then T is also WRP-Noetherian.*

Proof. Suppose T is a poregular sub posemigroup of a posemigroup S, and t_1 and t_2 belong to T with $t_1 \leq_{\mathcal{R}_S} t_2$. Then there exists an element $s \in S^1$ such that $t_1 = t_2 s$. If we take any inverse of t_2 and call it t_2', then we have

$t_1 = t_2 s \le t_2 t_2' t_2 s = t_2(t_2' t_1)$, which belongs to $\downarrow(t_2 T^1)$. This implies that $t_1 \le_{\mathcal{R}_T} t_2$, meaning that T preserves $\mathcal{R}$. Thus, according to Proposition 2.13, we get the result. □

Corollary 2.16. *Suppose S is a posemigroup and I is a right poideal of S with the property that for every $i \in I, i \in \downarrow(iI)$. If S is WRP-Noetherian, then I is also WRP-Noetherian.*

Proof. It will be proved that I preserves $\mathcal{R}$ in S. To demonstrate this, we only need to show that $\le_{\mathcal{R}_S} \cap (I \times I) \subseteq \le_{\mathcal{R}_I}$. Let's assume that (i, i') belongs to $I \times I$ and $i \le_{\mathcal{R}_S} i'$. In this case, $i \in \downarrow(i'S^1)$. As I is a right poideal with the mentioned property, one gets

$$i \in \downarrow(i'S^1) \subseteq \downarrow((i'I)S^1) = \downarrow(i'(IS^1)) \subseteq \downarrow(i'I).$$

Therefore, $i \le_{\mathcal{R}_I} i'$, as desired. □

A sub posemigroup T of a posemigroup S is called *right unitary* in S if for any t in T and s in S, if ts is also in T, then s must also be in T.

Corollary 2.17. *Assume S is a posemigroup and T is a right unitary sub posemigroup of S. If S is WRP-Noetherian, then T is also WRP-Noetherian.*

Proof. It is easily seen that a right unitary sub posemigroup is $\mathcal{R}$-preserving and so by Proposition 2.13 we get the result. □

Corollary 2.18. *Suppose S is a WRP-Noetherian posemigroup with a sub posemigroup T such that $S - T$ is a left poideal of S. Then T is also WRP-Noetherian.*

Proof. If the complement of a sub posemigroup is a left poideal, it can be deduced that the sub posemigroup is right unitary. Thus one gets the desired result. □

Corollary 2.19. *Suppose S is a posemigroup and T is its sub posemigroup. If $S - T$ is a WRP-Noetherian poideal of S, then S is WRP-Noetherian if and only if T is WRP-Noetherian.*

Proof. For the direct implication, we can refer to Corollary 2.18. Now, for the converse, we can set I as $S-T$. As we know, T is WRP-Noetherian, so $S/I \cong T \cup \{0\}$ is also WRP-Noetherian, according to Corollary 2.12. With this information, we can apply Theorem 2.18 of [9], to conclude that S is WRP-Noetherian. □

3 WRP-Noetherian poregular posemigroups

In this section, we will study WRP-Noetherian poregular posemigroups. We give a necessary and sufficient condition for a poregular posemigroup to be WRP-Noetherian.

Let $E = \{e \in S : e \leq e^2\}$ and E^n be the set of all products of n elements of E. Further, let $<E>$ be the sub posemigroup of S generated by E if $E \neq \emptyset$; then $<E> = \bigcup_{i \in \mathbb{N}} E^i$.

For any element $s \in S$, we define $V(s)$ as the set of all po-inverses of s. For $X \subseteq S$, we define $V(X)$ to be the set of all po-inverses of elements of X, denoted by $V(X) = \bigcup_{x \in X} V(x)$.

For a positive integer m, we define $V^m(X)$ recursively. Specifically, we define $V^1(X) = V(X)$, and for $m \geq 1, V^{m+1}(X) = V(V^m(X))$.

Example 3.1. 1. If S is a right zero posemigroup then it is a WRP-Noetherian poregular posemigroup, since its only right poideal is itself which is finitely generated.

2. Let $S = (\mathbb{N}, max, \leq)$ where $\leq$ is the usual order on $\mathbb{N}$. Then the only right poideal of S is itself which is principal. Hence S is a WRP-Noetherian poregular pomonoid.

Theorem 3.2. *Let S be a poregular posemigroup. If S is WRP-Noetherian posemigroup then for every subset $U \subseteq E$, there exists a finite set $X \subseteq U$ with the property that for each $u \in U$ there exists $x \in X$ and $s \in S$ such that $u \leq xs$.*

The converse is also true if, for every subset $U \subseteq E$, there exists a finite set $X \subseteq U$ with the property that for each $u \in U$ there exists $x \in X$ such that $u \leq xu$.

Proof. Assume that S is a WRP-Noetherian poregular posemigroup. Let us consider a set U that is a subset of E, and a right poideal $I = \downarrow(US^1)$ of S. Since S is WRP-Noetherian, we can find a finite subset X from U

such that $I = {\downarrow}(XS^1)$. For every $u \in U$, since $u \in I$, there exist $x \in X$ and $s \in S^1$ such that $u \leq xs$, as desired.

Now, assume that for every subset $U \subseteq E$, there exists a finite set $X \subseteq U$ with the property that for each $u \in U$ there exists $x \in X$ such that $u \leq xu$ and I is a right poideal of S. By the assumption, for the set $U = I \cap E$ there exists a finite set $X \subseteq U$ that satisfies the mentioned property. We claim that $I = {\downarrow}(XS^1)$. Let a be an element of I. We can find an inverse of a, say a', such that $a \leq aa'a$. Since $aa' \in U$, we can find an $x \in X$ such that $aa' \leq x(aa')$. Consequently, $a \leq aa'a \leq xaa'a$, which implies that $a \in {\downarrow}(XS^1)$. □

Corollary 3.3. *Let S be a poregular posemigroup such that its right poideals are exactly its semigroup ideals. Then S is WRP-Noetherian if and only if for every subset $U \subseteq E$, there exists a finite set $X \subseteq U$ with the property that for each $u \in U$ there exists $x \in X$ such that $u \leq xu$.*

In the following, we prove that, for the set E of idempotent elements of a poregular posemigroup, $V(E^n) = {\downarrow}E^{n+1}$ which generalizes a result of Fitz-Gerald ([6]) for ordered semigroups. As a corollary, we conclude that the sub posemigroup generated by the idempotents of a poregular posemigroup is itself poregular.

Lemma 3.4. *Let s be an element of a posemigroup S. If $s \in {\downarrow}(sE^n s)$ then $s \in {\downarrow}E^{n+1}$.*

Proof. Let $s \leq sxs, x \in E^n$. Hence $x = e_1 \dots e_n$, where for each $1 \leq i \leq n$, $e_i \in E$. For each $1 \leq i \leq n$, define $u_i = e_1 \dots e_i$ and $v_i = e_i \dots e_n$. Then $u_i v_i \geq x$. For $n \geq 2$ and $2 \leq j \leq n$, define $f_j = v_j s u_{j-1}$. Hence

$$f_j^2 = v_j s u_{j-1} v_j s u_{j-1} = v_j sxs u_{j-1} \geq v_j s u_{j-1} = f_j$$

which means that $f_j \in E$. Then

$$s \leq sxs \leq s(xs)^n \leq s(u_n v_n s)(u_{n-1} v_{n-1} s) \dots (u_1 v_1 s) =$$
$$(su_n)(v_n s u_{n-1}) \dots (v_2 s u_1)(v_1 s) = (sx) f_n \dots f_2 (xs) \in E^{n+1},$$

which implies $s \in {\downarrow}(E^{n+1})$. □

Corollary 3.5. $V(E^n) \subseteq {\downarrow}E^{n+1}$.

Lemma 3.6. *Suppose S is a poregular posemigroup. Then ${\downarrow}E^{n+1} \subseteq V(E^n)$.*

Proof. Assume that $s \leq e_1 \dots e_{n+1}$, where $e_i \in E, 1 \leq i \leq n+1$. Then there exists some $s' \in S$ such that $s \leq ss's$ and $s' \leq s'ss'$, since S is poregular. For each $1 \leq i \leq n+1$, set $u_i = e_1 \dots e_i$ and $v_i = e_i \dots e_n$. Then $u_i v_i \geq s$. Further, for each $1 \leq j \leq n$, define $f_j = v_{j+1}s'u_j$. Hence

$$f_j^2 = v_{j+1}s'u_j v_{j+1}s'u_j = v_{j+1}s'ss'u_j \geq v_{j+1}s'u_j = f_j.$$

Set $z = f_n \dots f_1$. Hence

$$\begin{aligned} s &\leq ss's \leq s(s's)^n \\ &\leq se_{n+1}s'u_n v_n s'u_{n-1}v_{n-1} \dots u_1 v_l \\ &= s(v_{n+1}s'u_n)(v_n s'u_{n-1} \dots (v_2 s'u_1)u_1 \\ &= sf_n \dots f_1 s \\ &= szs, \end{aligned}$$

and

$$\begin{aligned} zsz &= f_n \dots f_1 s f_n \dots f_1 \\ &= f_n \dots f_2 v_2 s' u_1 s v_{n+1} s' u_n \dots v_2 s' u_1 \\ &= f_n \dots f_2 (v_2 s')(u_1 s v_{n+1})(s' u_n v_n) \dots (s' u_2 v_2)(s' u_1) \\ &\geq f_n \dots f_2 (v_2 s')(e_1 s e_{n+1})(s's)^{n-1}(s'u_1) \\ &\geq f_n \dots f_2 (v_2 s' s)(s' s s' u_1) \\ &\geq f_n \dots f_2 (v_2 s' u_1) \\ &= f_n \dots f_2 f_1 \\ &= z. \end{aligned}$$

Therefore, $s \in V(z) \subseteq V(E^n)$, as desired. □

By Lemmas 3.4 and 3.6 one gets the following theorem.

Theorem 3.7. *Suppose S is a poregular posemigroup. Then $V(E^n) = {\downarrow}E^{n+1}$.*

Corollary 3.8. *Suppose S is a poregular posemigroup. Then the sub posemigroup $< \downarrow E >$ of S is poregular.*

As a corollary of Theorem 3.2 and Corollary 3.8, one gets the following.

Corollary 3.9. *Let S be a poregular posemigroup such that for every subset $U \subseteq E$, there exists a finite set $X \subseteq U$ with the property that for each $u \in U$ there exists $x \in X$ such that $u \leq xu$. Then the sub posemigroup $T = <{\downarrow}E>$ of S is WRP-Noetherian.*

4 WRP-Noetherian posemigroups and direct products

In this section, we study the behavior of WRP-Noetherian posemigroups with respect to direct products.

Lemma 4.1. *Assume that S and T are two posemigroups where S is infinite. If $S \times T$ is WRP-Noetherian, then for each $t \in T, t \in {\downarrow}(tT)$.*

Proof. Assuming t is an element of T, we consider the right poideal of $S \times T$ generated by the set $\{(s,t) : s \in S\}$, denoted by I. As $S \times T$ is WRP-Noetherian, there exists a finite set $X \subseteq S$ such that I is generated by the set $\{(x,t) : x \in X\}$. We choose an element $s \in S \setminus X$. Thus, for some $x \in X$ and $w \in (S \times T)^1$, we have $(s,t) \leq (x,t)w$. Since $s \neq x$, we can conclude that $w \in S \times T$, so $w = (u,v)$ for some $u \in S$ and $v \in T$. Consequently, we have $t \leq tv \in {\downarrow}(tT)$, as desired. □

Proposition 4.2. *Assume S and T are posemigroups such that for each $s \in S$ and $t \in T$, $s \in {\downarrow}(sS)$ and $t \in {\downarrow}(tT)$. If S and T are WRP-Noetherian, then their direct product, $S \times T$, is also WRP-Noetherian.*

Proof. Consider I, a right poideal of $S \times T$. For any $s \in S$, we define a set $I_{s,T}$ as ${\downarrow}\{t \in T : (s,t) \in I\}$, and claim that $I_{s,T}$ is a right poideal of T. For, let t belong to $I_{s,T}$ and u belong to T. By the assumption, there exists an element $s' \in S$ such that $s \leq ss'$. Since I is a right poideal of $S \times T$, we can say that $(s,tu) \leq (s,t)(s',u)$ belongs to I, which implies that tu also belongs to $I_{s,T}$. Furthermore, if $t' \in T$ and $t' \leq t$ then $(s,t') \leq (s,t)$ which means that $(s,t') \in I$, since I is a right poideal of $S \times T$. Hence t' also belongs to $I_{s,T}$.

In a similar manner, for each $t \in T$, we define a right poideal $I_{t,S}$ as ${\downarrow}\{s \in S : (s,t) \in I\}$. Now we claim that there exists a finite set $X \subseteq S$ such that for each $s \in S$, there exists $x \in X$ such that $s \in {\downarrow}(xS)$ and $I_{s,T} = I_{x,T}$. Let's suppose that there are an infinite number of right poideals in the form of $I_{s,T}$. It's worth noting that for any s and s' in S, we have $I_{s,T} \subseteq I_{ss',T}$.

Since S is WRP-Noetherian, there exists a finite set $X_1 \subseteq S$ such that $S = {\downarrow}(X_1 S^1)$. Clearly, $S = {\downarrow}(X_1 S)$, since for each $s \in S, s \in {\downarrow}(sS)$, which implies that $X_1 \subseteq {\downarrow}(X_1 S)$. Based on our assumption, we know that there exists an $x_1 \in X_1$ such that there are infinitely many $s \in {\downarrow}(x_1 S)$ with $I_{s,T} \neq I_{x_1,T}$. Put $J_1 = S$. Let's consider the set $J_2 = \{s \in {\downarrow}(x_1 S) : I_{s,T} \neq I_{x_1,T}\}$ and let $s \in J_2$ and $s' \in S$. Then $I_{x_1,T} \subset I_{s,T} \subseteq I_{ss',T}$ and hence $ss' \in J_2$. Also, if $s' \leq s$ then $I_{x_1,T} \subset I_{s,T} \subseteq I_{s',T}$ and so $s' \in J_2$, which means that J_2 is a right poideal of S.

Since S is WRP-Noetherian, there exists a finite set $X_2 \subseteq J_2$ such that $J_2 = {\downarrow}(X_2 S)$, and there exists an $x_2 \in X_2$ such that there are infinitely many $s \in {\downarrow}(x_2 S)$ with $I_{s,T} \neq I_{x_2,T}$. We can continue this process to obtain an infinite ascending chain

$$I_{x_1,T} \subset I_{x_2,T} \subset \dots$$

of right poideals of T. However, this contradicts the fact that T is WRP-Noetherian. Thus, there must be a finite set $Z \subseteq S$ such that $I_{s,T} \in \{I_{z,T} : z \in Z\}$ for every $s \in S$.

Let $z \in Z$. Define I'_z as the right poideal of S, which is generated by the set $L_z = \{s \in S : I_{s,T} = I_{z,T}\}$. Since S is WRP-Noetherian, there exists a finite set $X_z \subseteq L_z$, where $I'_z = {\downarrow}(X_z S)$. We can now set $X = \bigcup_{z \in Z} X_z$, the union of X_z for all $z \in Z$. It is evident that X meets the requirement stated in the claim. Similarly, we can prove that there exists a finite set $Y \subseteq T$ such that for each $t \in T$, there exists $y \in Y$ such that $t \in {\downarrow}(yT)$ and $I_{t,S} = I_{y,S}$.

Now, we show that I is generated by the finite set $I \cap (X \times Y)$. Suppose $(s,t) \in I$. Then, $t \in I_{s,T}$ and $s \in I_{t,S}$. Using the above claim, we can find $x \in X$ and $s' \in S$ such that $s \leq xs'$ and $I_{s,T} = I_{x,T}$. Similarly, we can find $y \in Y$ and $t' \in T$ such that $t \leq yt'$ and $I_{t,S} = I_{y,S}$. Since $s \in I_{t,S} = I_{y,S}$, we can infer that $(s,y) \in I$, which implies that $y \in I_{s,T} = I_{x,T}$. Consequently, we can conclude that $(x,y) \in I$. □

Theorem 4.3. *Suppose S and T are two posemigroups, where S is infinite.*

(i) *If T is also infinite, then $S \times T$ is WRP-Noetherian if and only if both S and T are WRP-Noetherian and for each $s \in S, t \in T, s \in {\downarrow}(sS), t \in {\downarrow}(tT)$.*

(ii) *If T is finite, then $S \times T$ is WRP-Noetherian if and only if S is WRP-Noetherian and for each $t \in T, t \in \downarrow(tT)$.*

Proof. If $S \times T$, is WRP-Noetherian, then both S and T are also WRP-Noetherian since they are homomorphic images of $S \times T$ according to Proposition 2.6. Additionally, by Lemma 4.1, for each $t \in T, t \in \downarrow(tT)$. If T is infinite, then also for each $s \in S, s \in \downarrow(sS)$, by Lemma 4.1. Now, consider the case that T is finite and S is WRP-Noetherian while for each $t \in T, t \in \downarrow(tT)$. Then S^1 is also WRP-Noetherian by Corollary 2.12. By Proposition 4.2, it follows that $S^1 \times T$ is WRP-Noetherian. Since $(S^1 \times T) - (S \times T)$ is finite, it can be deduced from Corollary 2.12 that $S \times T$ is WRP-Noetherian. □

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References

[1] Anderson, F. and Fuller, K., "Rings and Categories of Modules", Springer-Verlag, 1992.

[2] Bulman-Fleming, S., Gutermuth, D., Gilmour, A., and Kilp, M., *Flatness properties of S-posets*, Comm. Alg. 34(4) (2006), 1291-1317.

[3] Bulman-Fleming, S. and Mahmoudi, M., *The category of S-posets*, Semigroup Forum 71(3) (2005), 443-461.

[4] Dandan, Y., Gould, V., Hartmann, M., Rusuk, N., and Zenab, R.E., *Coherency and construction for monoids*, Quart. J. Math. 71 (2020), 1461-1488.

[5] Fakhruddin, S.M., *On the category of S-posets*, Acta Sci. Math. (Szeged) 52 (1988), 85-92.

[6] Fitz-Gerald, D.G., *On inverses of products of idempotents in regular semigroups*, J. Austral. Math. Soc. 13(3) (1972), 335-337.

[7] Gould, V., *Coherent monoids*, J. Australian Math. Soc. 53 (1992), 166-182.

[8] Hotzel, E., *On semigroups with maximal conditions*, Semigroup Forum 11 (1975), 337-362.

[9] Kehayopulu, N. and Tsingelis, M., *Noetherian and Artinian ordered groupoids-semigroups*, Int. J. Math. Math. Sci. 2005(13) (2005), 2041-2051.

[10] Kehayopulu, N. and Tsingelis, M., *Free ordered semigroups*, Sci. Math. Jpn. Online (2010), 401-409.

[11] Kozhukhov, I., *On semigroups with minimal or maximal condition on left congruences*, Semigroup Forum 21 (1980), 337-350.

[12] Miller, C., *Semigroups whose right ideals are finitely generated*, Proc. Edinb. Math. Soc. 64 (2021), 848-883.

[13] Normak, P., *On Noetherian and finitely presented acts* (in Russian), Tartu Ul. Toimetised 431 (1977), 37-46.

[14] Sadhya, S. and Hansda, K., *Generalized Greens relations and GV-ordered semigroups*, Quasigroups Related Systems 30 (2022), 161-168.

[15] Shahbaz, L. and Mahmoudi, M., *Various kinds of regular injectivity for S-posets*, Bull. Iranian Math. Soc. 40(1) (2014), 243-261.

[16] Shahbaz, L. and Mahmoudi, M., *Injectivity of S-posets with respect to down closed regular monomorphisms*, Semigroup Forum 91 (2015), 584-600.

[17] Wisbauer, R., "Foundations of Module and Ring Theory", Gordon and Breach Science Publishers, Reading, 1991.

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