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Operads of higher transformations for globular sets and for higher magmas

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I dedicate this work to André Joyal.

Abstract. In this article we discuss examples of fractal ω-operads. Thus we show that there is an ω-operadic approach to explain existence of the globular set of globular sets[1], the reflexive globular set of reflexive globular sets, the ω-magma of ω-magmas, and also the reflexive ω-magma of reflexive ω-magmas. Thus, even though the existence of the globular set of globular sets is intuitively evident, many other higher structures which *fractality* are less evident, could be described with the same technology, using fractal ω-operads. We have in mind the non-trivial question of the existence of the weak ω-category of the weak ω-categories in the globular setting, which is described in [9] with the same technology up to a contractibility hypothesis.

Introduction

This article is the second in a series of three articles (see [8, 9]). Here we give some relevant examples of ω-operads having the *fractal property* in the sense of [8], exhibited by relevant higher structures where contractibility in the sense of Batanin's article [2] is not involved. The natural direction we propose allows us to

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[1] Globular sets are also called ω-graphs by the French School.

consider four examples of higher structures: globular sets, reflexive globular sets, ω-magmas, and reflexive ω-magmas.

We use two coglobular objects in the category $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ of pointed $\mathbb{T}$-graphs over constant globular sets to build these examples: the object $C^\bullet$ and a subobject $G^\bullet$ of $C^\bullet$, both described in the last section of [8]. Thanks to the object $G^\bullet$, we freely generate the coglobular object of ω-operads for globular sets and the coglobular object of ω-operads for reflexive globular sets. Thanks to the object $C^\bullet$, we freely generate the coglobular object of ω-operads for ω-magmas and the coglobular object of ω-operads for reflexive ω-magmas. It is then easy to show that the ω-operad B^0_G of globular sets, the ω-operad $B^0_{G_u}$ of reflexive globular sets, the ω-operad B^0_M of ω-magmas, the ω-operad $B^0_{M_u}$ of reflexive ω-magmas, all have the fractal property. Using the same technology related to the *standard action in* $\mathbb{T}\text{-}\mathbb{C}AT_1$ (see [8]), we deduce the existence of a globular set of globular sets, a reflexive globular set of reflexive globular sets, a ω-magma of ω-magmas, and a reflexive ω-magma of reflexive ω-magmas.

We suspect that the ω-operad B^0_C of Batanin (see [2, 5, 13]), which algebras are his definition of weak higher categories, has the fractal property. Thus the weak ω-category of weak ω-categories should have a similar description as those of this article, up to the contractibility of a specific ω-operad. This important fact of higher category theory is described in [9].

1 Preamble

We are going to describe four examples of fractal ω-operads in the sense of [8]. Not only is the ω-operad of globular sets actually fractal, but so are the ω-operad of reflexive globular sets, the ω-operad of ω-magmas, and the ω-operad of reflexive ω-magmas.

Remark 1.1. It is important to note that our technology also applies in low dimensions. For example, as an exercise, the reader can check easily that graphs form a graph, or reflexive graphs form a reflexive graph, by interpreting in the language of this article. Indeed we can also show easily that the 1-operad of graphs, and the 1-operad of reflexive graphs are also fractals. Also the 2-operad of 2-graphs is fractal as well, where we consider a finite 3-truncated coglobular objects build with the 2-operad of 2-graphs, the 2-operad of morphisms of 2-graphs, and the 2-operad of transformations of 2-graphs. However this 2-operad of 2-graphs can be also used to show that graphs do form a 2-graphs. But in that case this construction doesn't show the fractality structure[2] of such 2-operad of 2-graphs. Many such low dimensions facts (such that the existence of the category of categories, the

[2]In the sense of [8].

strict 2-category of strict 2-categories, the bicategory of bicategories[3], etc.) can be described easily to show more the pertinence of the technology developed in this article.

Remark 1.2. In [17] Dimitri Tamarkin has shown that *DG*-categories form a *DG*-category. We do not know if such proof is similar to the technology developed in our article. If it is the case, such result might be still true for a higher version of *DG*-categories[4]. But we do not want to give more details of such subject, because we are not enough expert about *DG*-categories.

2 The coglobular object of graphical ω-operads

The category ω-$\mathbb{G}$r of globular sets has trivial *higher transformations.* First we are going to describe these higher transformations as presheaves on appropriate small categories $\mathbb{G}_n$, and then see that they form a globular set that we call *the globular set of globular sets.* It is the combinatorial description of these small categories $\mathbb{G}_n$ which allows a straightforward proof of Proposition 2.2. This proposition basically says that these higher transformations are algebras for adapted 2-coloured ω-operads.

Consider the classical globe category $\mathbb{G}_0$

$$\bar{0} \underset{t_0^1}{\overset{s_0^1}{\rightrightarrows}} \bar{1} \underset{t_1^2}{\overset{s_2^1}{\rightrightarrows}} \bar{2} \cdots\cdots \to \overline{n-1} \underset{t_{n-1}^n}{\overset{s_{n-1}^n}{\rightrightarrows}} \bar{n} \cdots\cdots$$

subject to the relations[5] on cosources s_n^{n+1} and cotargets t_n^{n+1}. For each each $n \geqslant 1$ we are going to build other small categories $\mathbb{G}_n$ resulting in a coglobular object in $\mathbb{C}AT$

$$\mathbb{G}_0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} \mathbb{G}_1 \underset{\kappa_1^2}{\overset{\delta_2^1}{\rightrightarrows}} \mathbb{G}_2 \cdots\cdots \to \mathbb{G}_{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} \mathbb{G}_n \cdots\cdots$$

where $\mathbb{G}_0$ is the globe category. The category $\mathbb{G}_n$ is called the *n-globe category.*

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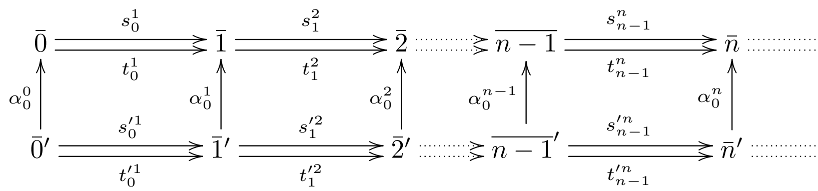
[3] For these everyday cases, we use adapted truncations of operads build in [9].

[4] If such higher version makes sense.

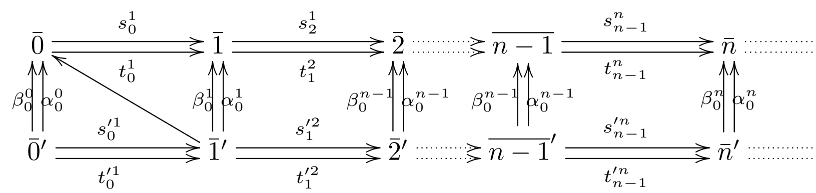
[5] which are describe in [8]

The 1-globe category $\mathbb{G}_1$ is presented by



subject to $\alpha_0^{n+1} \circ s_n'^{n+1} = s_n^{n+1} \circ \alpha_0^n$, $\alpha_0^{n+1} \circ t_n'^{n+1} = t_n^{n+1} \circ \alpha_0^n$.

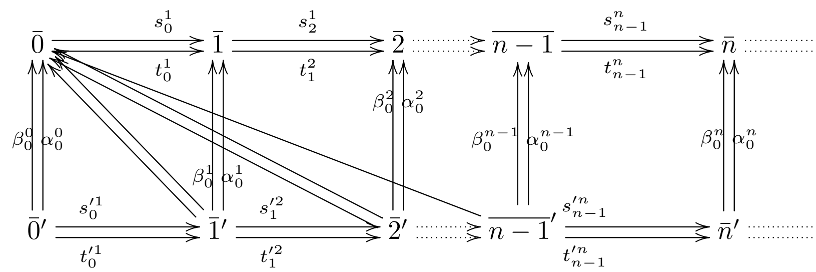
The 2-globe category $\mathbb{G}_2$ is presented by



where in particular we have an arrow $\xi_1 : \bar{1}' \longrightarrow \bar{0}$. Arrows s_n^{n+1}, t_n^{n+1}, α_0^n, β_0^n, and ξ_1 satisfy the following relations

- $\alpha_0^{n+1} \circ s_n'^{n+1} = s_n^{n+1} \circ \alpha_0^n$, $\alpha_0^{n+1} \circ t_n'^{n+1} = t_n^{n+1} \circ \alpha_0^n$,
- $\beta_0^{n+1} \circ s_n'^{n+1} = s_n^{n+1} \circ \beta_0^n$, $\beta_0^{n+1} \circ t_n'^{n+1} = t_n^{n+1} \circ \beta_0^n$,
- $\xi_1 \circ s_0'^1 = \alpha_0^0$ and $\xi_1 \circ t_0'^1 = \beta_0^0$.

More generally the n-globe category $\mathbb{G}_n$ is given by the category



where in particular we have an arrow $\xi_{n-1}:\overline{n-1}'\longrightarrow\bar{0}$, and also for each $1\leqslant p\leqslant n-2$, we have arrows $\bar{p}'\overset{\alpha_p}{\underset{\beta_p}{\rightrightarrows}}\bar{0}$. The arrows s^{n+1}_n, t^{n+1}_n, α^n_0, β^n_0, α_p, β_p, and ξ_{n-1} satisfy the following relations

- $\alpha^{n+1}_0\circ s'^{n+1}_n=s^{n+1}_n\circ\alpha^n_0$, $\alpha^{n+1}_0\circ t'^{n+1}_n=t^{n+1}_n\circ\alpha^n_0$,
- $\beta^{n+1}_0\circ s'^{n+1}_n=s^{n+1}_n\circ\beta^n_0$, $\beta^{n+1}_0\circ t'^{n+1}_n=t^{n+1}_n\circ\beta^n_0$.
- $\alpha_p\circ s'^p_{p-1}=\beta_p\circ s'^p_{p-1}=\alpha_{p-1}$ and $\alpha_p\circ t'^p_{p-1}=\beta_p\circ t'^p_{p-1}=\alpha_{p-1}$, and we put $\alpha_0:=\alpha^0_0$ and $\beta_0:=\beta^0_0$,
- $\xi_{n-1}\circ s'^{n-1}_{n-2}=\alpha_{n-2}$ and $\xi_{n-1}\circ t'^{n-1}_{n-2}=\beta_{n-2}$.

The cosources and cotargets functors $\mathbb{G}_0\overset{\delta^1_0}{\underset{\kappa^1_0}{\rightrightarrows}}\mathbb{G}_1$ are such that δ^1_0 sends $\mathbb{G}_0$ to $\mathbb{G}_0$, and κ^1_0 sends $\mathbb{G}_0$ to $\mathbb{G}'_0$. The cosources and cotargets functors $\mathbb{G}_1\overset{\delta^2_1}{\underset{\kappa^2_1}{\rightrightarrows}}\mathbb{G}_2$ send $\mathbb{G}_0$ to $\mathbb{G}_0$, and $\mathbb{G}'_0$ to $\mathbb{G}'_0$. Also δ^2_1 sends the symbols α^n_0 to the symbols α^n_0, and κ^2_1 sends the symbols α^n_0 to the symbols β^n_0.

Now we consider the case $n\geqslant 3$. The cosource and cotarget functors

$$\mathbb{G}_{n-1}\overset{\delta^n_{n-1}}{\underset{\kappa^n_{n-1}}{\rightrightarrows}}\mathbb{G}_n$$

are constructed as follows. First we remove the cell ξ_{n-1} and the cell β_{n-2} from $\mathbb{G}_n$, and we obtain the category $\mathbb{G}^-_{n-1}$. Clearly we have an isomorphism of categories $\mathbb{G}^-_{n-1}\simeq\mathbb{G}_{n-1}$ (which sends α_{n-2} to ξ_{n-2}), and also the embedding $\mathbb{G}^-_{n-1}\overset{\delta'^{n-1}_n}{\hookrightarrow}\mathbb{G}_n$. The composition of this embedding with the last isomorphism gives $\mathbb{G}_{n-1}\overset{\delta^n_{n-1}}{\longrightarrow}\mathbb{G}_n$.

The cotarget functor κ^n_{n-1} is built similarly: First we remove the cell ξ_{n-1} and the cell α_{n-2} from $\mathbb{G}_n$, and we obtain the category $\mathbb{G}^+_{n-1}$. Clearly we have an isomorphism of categories $\mathbb{G}^+_{n-1}\simeq\mathbb{G}_{n-1}$ (which sends β_{n-2} to ξ_{n-2}), and also the embedding $\mathbb{G}^+_{n-1}\overset{\kappa'^n_{n-1}}{\hookrightarrow}\mathbb{G}_n$. The composite of this embedding with the last isomorphism gives $\mathbb{G}_{n-1} \xrightarrow{\kappa^n_{n-1}} \mathbb{G}_n$. It is easy to see that these functors δ^n_{n-1} and κ^n_{n-1} verify the cosource/cotarget conditions as for the globe category $\mathbb{G}_0$ above.

We denote the category of sets by $\mathbb{S}et$ and the category of large sets by $\mathbb{S}ET$.

When we apply the contravariant functor $[-;\mathbb{S}et](0)$ to the coglobular object in $\mathbb{C}AT$

$$\mathbb{G}^{op}_0 \underset{\kappa^1_0}{\overset{\delta^1_0}{\rightrightarrows}} \mathbb{G}^{op}_1 \underset{\kappa^2_1}{\overset{\delta^1_2}{\rightrightarrows}} \mathbb{G}^{op}_2 \cdots\cdots \mathbb{G}^{op}_{n-1} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} \mathbb{G}^{op}_n \cdots\cdots$$

it is easy to see that we obtain the globular set of globular sets[6]

$$\cdots\cdots [\mathbb{G}^{op}_n;\mathbb{S}et](0) \underset{\beta^n_{n-1}}{\overset{\sigma^n_{n-1}}{\rightrightarrows}} [\mathbb{G}^{op}_{n-1};\mathbb{S}et](0) \cdots\cdots [\mathbb{G}^{op}_1;\mathbb{S}et](0) \underset{\beta^1_0}{\overset{\sigma^1_0}{\rightrightarrows}} [\mathbb{G}^{op}_0;\mathbb{S}et](0)$$

Remark 2.1. If instead we apply to it the contravariant functor $[-;\mathbb{S}et]$ we obtain the globular category of globular sets, which is useful in [9], for example to describe the globular category of the strict ω-categories.

An object of the category of presheaves $[\mathbb{G}^{op}_n;\mathbb{S}et]$ is called an (n,ω)-graph[7]. For instance, if $n \geqslant 3$, then the source functor σ^n_{n-1} is described as follows. Take an (n,ω)-graph $X : \mathbb{G}^{op}_n \longrightarrow \mathbb{S}et$ with underlying $(n-1)$-transformation

$$X(\xi_{n-1}) : X(\bar{0}) \longrightarrow X(\overline{n-1}') ,$$

and then

$$\sigma^n_{n-1}(X)(\xi_{n-2}) : X(\bar{0}) \longrightarrow X(\overline{n-2}')$$

is the underlying $(n-2)$-transformation of $\sigma^n_{n-1}(X)$ defined by:

$$\sigma^n_{n-1}(X)(\xi_{n-2}) = X(s'^n_{n-1}) \circ X(\xi_{n-1}).$$

Similarly for the target functors β^n_{n-1}.

[6] For each $n \in \mathbb{N}$, $[\mathbb{G}^{op}_n;\mathbb{S}et](0)$ means the set of objects of the presheaf category $[\mathbb{G}^{op}_n;\mathbb{S}et]$.

[7] Do not confuse (n,ω)-graphs with the (∞,n)-graphs that we defined in [7]. They are completely different objects. In [7], (∞,n)-graphs are a kind of globular set which plays a central role in defining an algebraic approach of (∞,n)-categories.

Now we are going to give an operadic approach to the globular set of globular sets by using the technology of Section 2 of [8]. Let us denote by $G^{\bullet}$ the coglobular object in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$

$$G^0 \overset{\delta_0^1}{\underset{\kappa_0^1}{\rightrightarrows}} G^1 \overset{\delta_2^1}{\underset{\kappa_1^2}{\rightrightarrows}} G^2 \cdots\cdots \rightrightarrows G^{n-1} \overset{\delta_{n-1}^n}{\underset{\kappa_{n-1}^n}{\rightrightarrows}} G^n \cdots\cdots$$

built just by removing all cells "μ_p^n" and "ν_p^n" from the object $C^{\bullet}$ in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ (described in Section 4 of [8]):

$$C^0 \overset{\delta_0^1}{\underset{\kappa_0^1}{\rightrightarrows}} C^1 \overset{\delta_2^1}{\underset{\kappa_1^2}{\rightrightarrows}} C^2 \cdots\cdots \rightrightarrows C^{n-1} \overset{\delta_{n-1}^n}{\underset{\kappa_{n-1}^n}{\rightrightarrows}} C^n \cdots\cdots$$

If we apply to it the free functor $M : \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \longrightarrow \mathbb{T}\text{-}\mathbb{C}AT_c$ (see Section 2 of [8]) we obtain a coglobular object[8] in $\mathbb{T}\text{-}\mathbb{C}AT_c$

$$B_G^0 \overset{\delta_0^1}{\underset{\kappa_0^1}{\rightrightarrows}} B_G^1 \overset{\delta_2^1}{\underset{\kappa_1^2}{\rightrightarrows}} B_G^2 \cdots\cdots \rightrightarrows B_G^{n-1} \overset{\delta_{n-1}^n}{\underset{\kappa_{n-1}^n}{\rightrightarrows}} B_G^n \cdots\cdots$$

which produces the globular object

$$\cdots\cdots \rightrightarrows \underline{B}_G^n\text{-}\mathbb{A}lg \overset{\sigma_{n-1}^n}{\underset{\beta_{n-1}^n}{\rightrightarrows}} \underline{B}_G^{n-1}\text{-}\mathbb{A}lg \cdots\cdots \rightrightarrows \underline{B}_G^1\text{-}\mathbb{A}lg \overset{\sigma_0^1}{\underset{\beta_0^1}{\rightrightarrows}} \underline{B}_G^0\text{-}\mathbb{A}lg$$

in $\mathbb{C}AT$. We then have the following easy proposition.

Proposition 2.2. *The category $\underline{B}_G^0$-$\mathbb{A}lg$ is the category $[\mathbb{G}_0^{op};\mathbb{S}et]$ of globular sets, $\underline{B}_G^1$-$\mathbb{A}lg$ is the category $[\mathbb{G}_1^{op};\mathbb{S}et]$ of $(1,\omega)$-graphs, and for each integer $n \geqslant 2$, $\underline{B}_G^n$-$\mathbb{A}lg$ is the category $[\mathbb{G}_n^{op};\mathbb{S}et]$ of (n,ω)-graphs.*

Let us denote this coglobular object in $\mathbb{T}\text{-}\mathbb{C}AT_c$ by $B_G^{\bullet}$. Its standard action is given by the following diagram in $\mathbb{T}\text{-}\mathbb{C}AT_1$:

$$Coend(B_G^{\bullet}) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A_G^{op}) \xrightarrow{Coend(Ob(.))} End(A_{0,G})$$

It is the standard action of higher transformations specific to the basic globular set structure. The monochromatic ω-operad $Coend(B_G^{\bullet})$ of coendomorphisms plays a

[8]Here B_G^0 is the initial ω-operad.

central role for globular sets, and we call it the *white operad.* It is straightforward that B^0_G has the fractal property because it is initial in the category $\mathbb{T}$-$\mathbb{C}AT_1$ of ω-operads and thus has a unique morphism

$$B^0_G \xrightarrow{!_G} Coend(B^{\bullet}_G)$$

of ω-operads. If we compose it with the standard action of the globular sets

$$Coend(B^{\bullet}_G) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}_G) \xrightarrow{Coend(Ob(.))} End(A_{0,G})$$

we obtain a morphism of ω-operads

$$B^0_G \xrightarrow{\mathfrak{G}} End(A_{0,G})$$

which expresses an action of the ω-operad B^0_G of globular sets on the globular object $B^{\bullet}_G$-$\mathbb{A}lg(0)$ in $\mathbb{S}ET$ of (n,ω)-graphs $(n \in \mathbb{N})$. This gives a globular set[9] structure on (n,ω)-graphs $(n \in \mathbb{N})$.

3 The functor of contractible units

We denote the category of the reflexive globular sets by ω-$\mathbb{G}$rr (see [16]). We have the adjunction

$$\omega\text{-}\mathbb{G}\mathrm{rr} \underset{R}{\overset{U}{\rightleftarrows}} \mathbb{G}\mathrm{r} \quad (U \dashv R)$$

and the generated monad of reflexive globular sets is denoted by $(\mathbb{R},\eta,\mu)$. Objects of ω-$\mathbb{G}$rr are usually written as $(G,(1^p_n)_{0\leqslant p<n}))$, where the operations $(1^p_n)_{0\leqslant p<n}$ is a chosen reflexive structure on the globular set G. Also we consider the monad $\mathbb{T}$ on the globular sets whose algebras are strict ω-categories. In this paragraph we will build the *functor of the contractible units* for pointed $\mathbb{T}$-graphs (see [13]). It plays the same role for the pointed $\mathbb{T}$-graphs as the previous functor R plays for the globular sets. First we must define *pointed* $\mathbb{T}$*-graphs with contractible units* which are, for the pointed $\mathbb{T}$-graphs, what reflexive globular sets are for globular sets. Throughout this paper we will work with pointed $\mathbb{T}$-graphs and with $\mathbb{T}$-categories over constant globular sets (see [5]), and a subscript "c" on categories will mean that we work with constant globular sets. For instance the category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ of pointed $\mathbb{T}$-graphs is adorned with a "c" to indicate that objects of this category are the pointed $\mathbb{T}$-graphs over constant globular sets.

Now consider an object $(C,d,c;u)$ of the category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$. Here u denotes a chosen point of (C,d,c); that is, the data of a morphism

[9] Some mathematicians might prefer to say *large globular set* or *globular class.*

$$(G, \eta_G, id_G) \xrightarrow{(u, 1_G)} (C, d, c)$$

in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_c$ where G designates the underlying constant globular set of arities of (C, d, c) (or in other words, designates the set of colours of (C, d, c)).

Remark 3.1. A p-cell of G is denoted by $g(p)$ and this notation has the following meaning. The symbol g indicates the "colour", and the symbol p reminds us that we must see $g(p)$ as a p-cell of G because, while G is just a set, we are thinking of it as a constant globular set.

In order to define pointed $\mathbb{T}$-graphs with contractible units we are going first to define an intermediate structure on $\mathbb{T}$-graphs. Consider a $\mathbb{T}$-graph (C, d, c) and, for each $n \in \mathbb{N}$, we denote the set of n-cells of the $\mathbb{T}$-graph (C, d, c) by $C(n)$. Consider also the reflexive globular set $(\mathbb{T}(G), (1^p_n)_{0 \leqslant p < n}))$ such that the operations 1^p_n are freely generated by the monad $\mathbb{T}$. We say that the $\mathbb{T}$-graph (C, d, c) is equipped with a reflexive structure if its underlying globular set C is equipped with a reflexive structure in the usual sense and d is a morphism of reflexive globular sets. Note that G is also equipped with a trivial reflexivity structure $(G, (1^p_n)_{0 \leqslant p < n}))$ such that the operations 1^p_n are defined by $1^p_n(g(p)) = g(n)$, forcing c to be a morphism of reflexive globular sets as well. We denote reflexive $\mathbb{T}$-graphs, where the operations 1^p_n are those of C, by $(C, d, c; (1^p_n)_{0 \leqslant p < n})$. A morphism between two reflexive $\mathbb{T}$-graphs is just a morphism of $\mathbb{T}$-graphs which preserves reflexivity, and the category of reflexive $\mathbb{T}$-graphs over constant globular sets is denoted by $\mathbb{T}$-$\mathbb{G}\mathrm{rr}_c$.

A pointed $\mathbb{T}$-graph $(C, d, c; p)$ over a constant globular set G has contractible units if it is equipped with a monomorphism $\mathbb{R}(G) \xrightarrow{v} C$ such that u factorise as

$$\begin{array}{ccc} & \mathbb{R}(G) & \\ {\scriptstyle \eta(G)} \nearrow & & \searrow \\ G & \xrightarrow[u]{} & C \end{array}$$

and such that the induced $\mathbb{T}$-graph $\mathbb{T}(G) \xleftarrow{d} \mathbb{R}(G) \xrightarrow{c} G$ is reflexive. That is, the restriction of d to $\mathbb{R}(G)$ is a morphism of reflexive globular sets. We denote pointed $\mathbb{T}$-graphs with contractible units by $(C, d, c; p, v, (1^p_n)_{0 \leqslant p < n})$. A morphism

$$(C, d, c; p, v, (1^p_n)_{0 \leqslant p < n}) \xrightarrow{(f,h)} (C', d', c'; p', v', (1^p_n)_{0 \leqslant p < n})$$

of pointed $\mathbb{T}$-graphs with contractible units is given by a morphism of pointed $\mathbb{T}$-graphs (see [13])

$$(C, d, c; p) \xrightarrow{(f,h)} (C', d', c'; p')$$

such that $fv = v'\mathbb{R}(h)$, and $(\mathbb{R}(G), d, c) \xrightarrow{(f,h)} (\mathbb{R}(G'), d', c')$ is a morphism of reflexive $\mathbb{T}$-graphs. Thus morphisms between two $\mathbb{T}$-graphs equipped with contractible units preserve this *structure of contractibility on the units.* The category of pointed $\mathbb{T}$-graphs with contractible units is denoted by $U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$. It is easy to see that $U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ is locally presentable, because it is based on the locally presentable category $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$, and equipped with a *structure of contractibility on the units*, whose operations 1^p_m on the units and their axioms, show easily that $U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ is also projectively sketchable[10].

Also we can easily prove that the forgetful functor

$$U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \xrightarrow{U'} \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is a right adjoint by using basic techniques coming from logic as in [5]. Thus we can apply Proposition 5.5.6 of [3] which shows the monad $\mathbb{T}_U$ induced by this adjunction has rank. Also U' is monadic by the Beck theorem on monadicity. We write R' for the left adjoint of U':

$$U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{R'}{\overset{U'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \ .$$

Furthermore, we have the general fact (which can be found in [10, 11]).

Proposition 3.2 (G.M. Kelly)**.** *Let K be a locally finitely presentable category, and $Mnd_f(K)$ the category of finitary monads on K and strict morphisms of monads. Then $Mnd_f(K)$ is itself locally finitely presentable. If T and S are object of $Mnd_f(K)$, then the coproduct $T\coprod S$ is algebraic, which means that $K^T \underset{K}{\times} K^S$ is equal to $K^{T\coprod S}$ and the diagonal of the pullback square*

$$\begin{array}{ccc} K^T \underset{K}{\times} K^S & \xrightarrow{p_1} & K^S \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle U} \\ K^T & \xrightarrow[V]{} & K \end{array}$$

is the forgetful functor $K^{T\coprod S} \longrightarrow K$. Furthermore the projections $K^T \underset{K}{\times} K^S \longrightarrow K^T$ and $K^T \underset{K}{\times} K^S \longrightarrow K^S$ are monadic.

[10]Good references for sketch theory are [1, 3, 12, 14].

Remark 3.3. According to Steve Lack, this result can be easily generalised to monads having ranks in the context of locally presentable category.

With the functor V defined in Section 2 of [8], we have the following diagram :

$$\begin{array}{ccc} U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} \mathbb{T}\text{-}\mathbb{C}AT_c & \xrightarrow{p_1} & \mathbb{T}\text{-}\mathbb{C}AT_c \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle V} \\ U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} & \xrightarrow[U']{} & \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \end{array}$$

Applying the above proposition, we see that

$$\mathbb{U}\mathbb{T}\text{-}\mathbb{C}AT_c := U\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} \mathbb{T}\text{-}\mathbb{C}AT_c$$

is a locally presentable category[11], and also that the forgetful functor

$$\mathbb{U}\mathbb{T}\text{-}\mathbb{C}AT_c \xrightarrow{O} \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic. Denote by F the left adjoint to O.

In Sections 4 and 6, we apply this functor F to the coglobular objects $G^\bullet$ and $C^\bullet$ of $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$, to obtain respectively, the coglobular object of higher operads for reflexive globular sets, and the coglobular object of higher operads for reflexive ω-magmas.

4 The coglobular objects of the reflexive graphical ω-operads

By taking the globe category $\mathbb{G}_0$ (see Section 2) as basis, we build the reflexive globe category $\mathbb{G}_{0,\mathrm{r}}$ as follow. For each $n \in \mathbb{N}$ we add into $\mathbb{G}_0$ the formal morphism $\overline{n+1} \xrightarrow{1^n_{n+1}} \bar{n}$ such that $1^n_{n+1} \circ s^{n+1}_n = 1^n_{n+1} \circ t^{n+1}_n = 1_{\bar{n}}$. For each $0 \leqslant p < n$ we denote $1^p_n := 1^p_{p+1} \circ 1^{p+1}_{p+2} \circ ... \circ 1^{n-1}_n$.

For each each $n \geqslant 1$ we are going to build similar categories $\mathbb{G}_{n,\mathrm{r}}$ in order to obtain a coglobular object

[11] $\mathbb{U}\mathbb{T}\text{-}\mathbb{C}AT_c$ is the category of coloured ω-operads with chosen contractible units where in particular morphisms of this category preserve contractibility of the units.

$$\xymatrix{\mathbb{G}_{0,\mathrm{r}} \ar@<1ex>[r]^{\delta^1_0} \ar@<-1ex>[r]_{\kappa^1_0} & \mathbb{G}_{1,\mathrm{r}} \ar@/_2pc/[l]_{i^0_1} \ar@<1ex>[r]^{\delta^1_2} \ar@<-1ex>[r]_{\kappa^2_1} & \mathbb{G}_{2,\mathrm{r}} \ar@/_2pc/[l]_{i^1_2} \ar@{..>}[r] & \mathbb{G}_{n-1,\mathrm{r}} \ar@<1ex>[r]^{\delta^n_{n-1}} \ar@<-1ex>[r]_{\kappa^n_{n-1}} & \mathbb{G}_{n,\mathrm{r}} \ar@/_2pc/[l]_{i^{n-1}_n} \ar@{..}[r] & }$$

in $\mathbb{C}AT$, equipped with coreflexivity functors i^n_{n+1}. Each category $\mathbb{G}_{n,\mathrm{r}}$ is called the reflexive n-globe category. It is built as were the categories $\mathbb{G}_n$ $(n \geqslant 1)$ where now we just replace $\mathbb{G}_0$ and $\mathbb{G}'_0$ by $\mathbb{G}_{0,r}$ and $\mathbb{G}'_{0,r}$.

For each $n \geqslant 1$, the cosource and the cotarget functors

$$\mathbb{G}_{n-1,\mathrm{r}} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} \mathbb{G}_{n,\mathrm{r}}$$

are defined as were those for the

$$\mathbb{G}_{n-1} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} \mathbb{G}_{n}$$

of Section 2, where in addition δ^1_0 sends, for all $p \geqslant 0$, the reflexivity morphism 1^p_{p+1} to the reflexivity morphism 1^p_{p+1}, and κ^0_1 sends the reflexivity morphism 1^p_{p+1} to the reflexivity morphism $1'^p_{p+1}$. Also, if $n \geqslant 2$, δ^n_{n+1} and κ^n_{n+1} send, for all $p \geqslant 0$, the reflexivity morphism 1^p_{p+1} to the reflexivity morphism 1^p_{p+1}, and the reflexivity morphism $1'^p_{p+1}$ to the reflexivity morphism $1'^p_{p+1}$. These functors δ^n_{n-1} and κ^n_{n-1} do indeed satisfy the cosource and cotarget conditions.

For each $n \geqslant 1$, the coreflexivity functor

$$\mathbb{G}_{n,\mathrm{r}} \xrightarrow{i^{n-1}_n} \mathbb{G}_{n-1,\mathrm{r}}$$

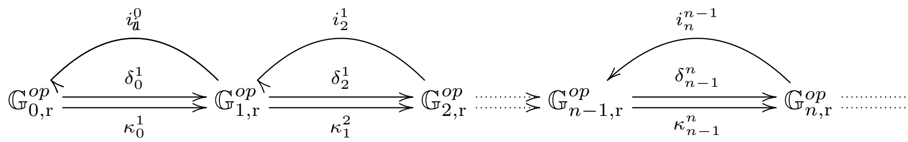
is built as follows: the coreflexivity functor ι^0_1 sends, for all $q \geqslant 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q}'$ to $\bar{q}$, the cosource morphisms s^{q+1}_q and s'^{q+1}_q to s^{q+1}_q, the cotarget morphisms t^{q+1}_q and t'^{q+1}_q to t^{q+1}_q, and the functor morphisms α^q_0 to $1_{\bar{q}}$. Also the coreflexivity functor i^1_2 sends, for all $q \geqslant 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q}'$ to $\bar{q}'$, the cosource morphism s^{q+1}_q to the cosource morphism s^{q+1}_q, the cosource morphism s'^{q+1}_q to the cosource morphism s'^{q+1}_q, the cotarget morphism t^{q+1}_q to the cotarget morphism t^{q+1}_q, the cotarget morphism t'^{q+1}_q to the cotarget morphism t'^{q+1}_q, the functor morphisms α^q_0 to the functor morphisms α^q_0, the functor morphisms β^q_0 to the functor morphisms β^q_0, and the natural transformation morphism ξ_1 to $\alpha^0_0 \circ 1'^0_1$.

Also, for each $n \geqslant 3$, the coreflexivity functor i^{n-1}_n sends, for all $q \geqslant 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q}'$ to $\bar{q}'$, the cosource morphism s^{q+1}_q to the cosource morphism s_q^{q+1}, the cosource morphism $s_q'^{q+1}$ to the cosource morphism $s_q'^{q+1}$, the cotarget morphism t_q^{q+1} to the cotarget morphism t_q^{q+1}, the cotarget morphism $t_q'^{q+1}$ to the cotarget morphism $t_q'^{q+1}$, the functor morphisms α_0^q to the functor morphisms α_0^q, the functor morphisms β_0^q to the functor morphisms β_0^q. Also, if $0 \leqslant p \leqslant n-3$, it sends the p-transformation α_p to the p-transformation α_p, the p-transformation β_p to the p-transformation β_p[12], the $(n-2)$-transformation α_{n-2} and β_{n-2} to the $(n-2)$-transformation ξ_{n-2}, and finally the $(n-1)$-transformation ξ_{n-1} to $\xi_{n-2} \circ 1_{n-1}'^{n-2}$.

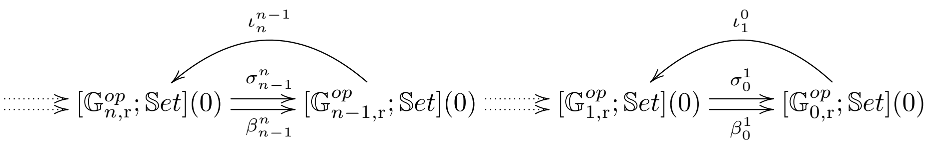
With this construction it is not difficult to show that functors i_n^{n-1} $(n \geqslant 1)$ satisfy the coreflexivity identities

$$i_n^{n-1} \circ \delta_{n-1}^n = 1_{\mathbb{G}_{n-1}^{\mathrm{r}}} = i_n^{n-1} \circ \kappa_{n-1}^n .$$

When we apply the contravariant functor $[-;\mathbb{S}et](0)$ to the coglobular object in $\mathbb{C}AT$



we obtain the reflexive globular set of reflexive globular sets:



An object of the category of presheaves $[\mathbb{G}_n^{op};\mathbb{S}et]$ is called a reflexive (n,ω)-graph.

For instance, if $n \geqslant 3$, the reflexivity functor ι_n^{n-1} can be described as follows. If $X : \mathbb{G}_{n-1}^{op} \longrightarrow \mathbb{S}et$ is an $(n-1,\omega)$-graph and

$$X(\xi_{n-2}) : X(\bar{0}) \longrightarrow X(\overline{n-2}')$$

is its underlying $(n-2)$-transformation, then

$$\iota_n^{n-1}(X)(\xi_{n-1}) : X(\bar{0}) \longrightarrow X(\overline{n-1}')$$

[12]By convention we put $\alpha_0 = \alpha_0^0$ and $\beta_0 = \beta_0^0$. In fact this convention is natural because from our point of view of n-transformations, 1-transformations are the usual natural transformations, and a 0-transformation should be seen as the underlying function F_0 acting on the 0-cells of a functor F.

is the $(n-1)$-transformation defined by $\iota_n^{n-1}(X)(\xi_{n-1}) = X(1'^{n-2}_{n-1}) \circ X(\xi_{n-2})$.

Now we are going to give an operadic approach of the reflexive globular set of reflexive globular sets by using the technology of Section 2 of [8]. Consider the coglobular object $G^{\bullet}$ in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ as in 2. If we apply the free functor $F : \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \longrightarrow \mathbb{UT}\text{-}\mathbb{C}AT_c$ (see Section 3) to it we obtain a coglobular object in $\mathbb{T}\text{-}\mathbb{C}AT_c$:

$$B^0_{G_u} \underset{\kappa^1_0}{\overset{\delta^1_0}{\rightrightarrows}} B^1_{G_u} \underset{\kappa^2_1}{\overset{\delta^1_2}{\rightrightarrows}} B^2_{G_u} \cdots\cdots > B^{n-1}_{G_u} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} B^n_{G_u} \cdots\cdots$$

It is important to notice that the ω-operad $B^0_{G_u}$ is initial in $\mathbb{UT}\text{-}\mathbb{C}AT_1$. Also this coglobular object $B^{\bullet}_{G_u}$ produces the following globular object in $\mathbb{C}AT$

$$\cdots\cdots > \underline{B}^n_{G_u} - \mathbb{A}lg \underset{\beta^n_{n-1}}{\overset{\sigma^n_{n-1}}{\rightrightarrows}} \underline{B}^{n-1}_{G_u} - \mathbb{A}lg \cdots\cdots > \underline{B}^1_{G_u} - \mathbb{A}lg \underset{\beta^1_0}{\overset{\sigma^1_0}{\rightrightarrows}} \underline{B}^0_{G_u} - \mathbb{A}lg$$

and we have:

Proposition 4.1. *The category $\underline{B}^0_{G_u}$-$\mathbb{A}lg$ is the category $[\mathbb{G}^{op}_{0,r};\mathbb{S}et]$ of reflexive globular sets, $\underline{B}^1_{G_u}$-$\mathbb{A}lg$ is the category $[\mathbb{G}^{op}_{1,r};\mathbb{S}et]$ of reflexive $(1,\omega)$-graphs, and for each integer $n \geqslant 2$, $\underline{B}^n_{G_u}$-$\mathbb{A}lg$ is the category $[\mathbb{G}^{op}_{n,r};\mathbb{S}et]$ of reflexive (n,ω)-graphs.*

We denote this coglobular object in $\mathbb{T}\text{-}\mathbb{C}AT_c$ by $B^{\bullet}_{G_u}$. According to the results of Section 2 of [8], we obtain the diagram

$$Coend(B^{\bullet}_{G_u}) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}_{G_u}) \xrightarrow{Coend(Ob(.))} Coend(A^{op}_{0,G_u})$$

in $\mathbb{T}\text{-}\mathbb{C}AT_1$ that we call the standard action of the reflexive globular sets. It is a specific standard action. The monochromatic ω-operad $Coend(B^{\bullet}_{G_u})$ of coendomorphisms plays a central role for reflexive globular sets, and we call it the *blue operad*. Also we have the following result:

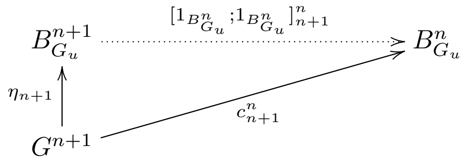
Proposition 4.2. *$B^0_{G_u}$ has the fractal property.*

Proof. The units of the ω-operad $Coend(B^{\bullet}_{G_u})$ are given by the identity morphisms $B^n_{G_u} \xrightarrow{1_{B^n_{G_u}}} B^n_{G_u}$. We are going to exhibit a morphism of ω-operads

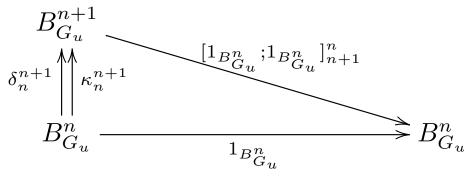
$$B^{n+1}_{G_u} \xrightarrow{[1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1}} B^n_{G_u}$$

which is the contractibility of the unit $1_{B^n_{G_u}}$ with itself.

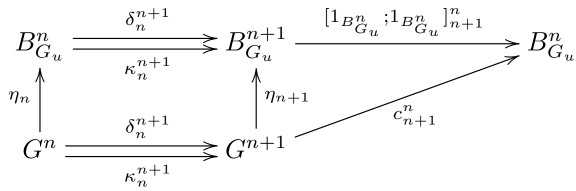
First consider the morphism of $G^{n+1} \xrightarrow{c^n_{n+1}} B^n_{G_u}$ of $\mathbb{T}$-$\mathbb{G}\mathrm{r}_c$, which sends u_m to u_m, v_m to v_m, α^m_0 to α^m_0, β^m_0 to β^m_0, α_p to α_p, β_p to β_p, α_n to ξ_n, β_n to ξ_n, and ξ_{n+1} to $\gamma([u_n;u_n]^n_{n+1};1_{\xi_n})$. This map c^n_{n+1} equips $B^n_{G_u}$ with an operation system of the type G^{n+1}. Now $B^n_{G_u}$ has contractible units, so by the universality of the map η_{n+1}, we obtain a unique morphism $[1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1}$ of ω-operads.



This $(n+1)$-cell $[1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1}$ has as arity the degenerate tree $1^n_{n+1}(1(n))$. Now we just need to prove that the following diagram commutes serially, which will show that the source and target of $[1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1}$ are the unit $1_{B^n_{G_u}}$:



But we have the following diagram which, on the left side commutes serially, and on the right side commutes:



The morphism c^n_{n+1} is a morphism of $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$, as are the morphisms δ^{n+1}_n and κ^{n+1}_n on the bottom of this diagram. Their combinatorial descriptions show easily that we have the equalities $c^n_{n+1} \circ \delta^{n+1}_n = c^n_{n+1} \circ \kappa^{n+1}_n = \eta_n$. So we have the equalities $[1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1} \circ \delta^{n+1}_n = [1_{B^n_{G_u}};1_{B^n_{G_u}}]^n_{n+1} \circ \kappa^{n+1}_n = 1_{B^n_{G_u}}$. This shows that the ω-operad $Coend(B^{\bullet}_{G_u})$ has contractible units, and thus we have a unique morphism

$$B^0_{G_u} \xrightarrow{!_{G_u}} Coend(B^{\bullet}_{G_u})$$

of ω-operads which expresses the fractality of $B^0_{G_u}$. $\square$

If we compose the morphism $!_{G_u}$ with the standard action of the reflexive globular sets

$$Coend(B^{\bullet}_{G_u}) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}_{G_u}) \xrightarrow{Coend(Ob(.))} Coend(A_{0,G_u})op$$

we obtain a morphism of ω-operads

$$B^0_{G_u} \xrightarrow{\mathfrak{G}_u} End(A_{0,G_u})$$

which expresses an action of the ω-operad $B^0_{G_u}$ of reflexive globular sets on the globular object $B^{\bullet}_{G_u}$-$\mathbb{A}lg(0)$ in $\mathbb{S}ET$ of the reflexive (n,ω)-graphs ($n \in \mathbb{N}$). This gives a reflexive globular set structure on reflexive (n,ω)-graphs ($n \in \mathbb{N}$).

5 The coglobular objects of magmatic ω-operads

Consider now the case $P = M$ (*magmatic*). That is, we deal with the category $\mathbb{T}$-$\mathbb{C}AT_c$ of ω-operads. We apply the free functor

$$\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \xrightarrow{M} \mathbb{T}\text{-}\mathbb{C}AT_c$$

to the coglobular object $C^{\bullet}$ in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ of the higher transformations (see Section 3 of [8]) and we obtain a coglobular object $B^{\bullet}_M$ of ω-operads in $\mathbb{T}$-$\mathbb{C}AT_c$

$$B^0_M \underset{\kappa^1_0}{\overset{\delta^1_0}{\rightrightarrows}} B^1_M \underset{\kappa^2_1}{\overset{\delta^1_2}{\rightrightarrows}} B^2_M \cdots\cdots \rightrightarrows B^{n-1}_M \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} B^n_M \cdots\cdots$$

If we write $\mathbb{B}\mathbb{T}$-$\mathbb{C}AT_1$ for the category of ω-operads equipped with a chosen composition system in Batanin's sense, whose morphisms are those which preserve these composition systems, then it is important to note that the ω-operad B^0_M is initial in $\mathbb{B}\mathbb{T}$-$\mathbb{C}AT_1$.

Also, this coglobular object $B^{\bullet}_M$ produces the following globular object in $\mathbb{C}AT$

$$\cdots\cdots \rightrightarrows B^n_M\text{-}\mathbb{A}lg \underset{\beta^n_{n-1}}{\overset{\sigma^n_{n-1}}{\rightrightarrows}} B^{n-1}_M\text{-}\mathbb{A}lg \cdots\cdots \rightrightarrows B^1_M\text{-}\mathbb{A}lg \underset{\beta^1_0}{\overset{\sigma^1_0}{\rightrightarrows}} B^0_M\text{-}\mathbb{A}lg$$

In particular, B^0_M is the ω-operad for ω-magmas, and, for all $n > 0$, algebras for B^n_M are what we call (n,ω)-magmas.

The standard action associated to $B^{\bullet}_M$ is given by the following diagram in $\mathbb{T}$-$\mathbb{C}AT_1$:

$$Coend(B^{\bullet}_M) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}_M) \xrightarrow{Coend(Ob(.))} End(A_{0,M})$$

We call this the standard action of ω-magmas, which is a specific standard action of higher transformations. The monochromatic ω-operad $Coend(B^{\bullet}_M)$ of coendomorphisms plays a central role for ω-magmas. We call it the *yellow operad*. Also we have the following result:

Proposition 5.1. *B^0_M has the fractal property.*

Proof. In 7 we built a composition system for $Coend(B^{\bullet}_M)$. Therefore, just using the universality of B^0_M in $\mathbb{BT}$-$\mathbb{C}AT_1$ we get the result. □

If we compose the morphism $!_M$

$$B^0_M \xrightarrow{!_M} Coend(B^{\bullet}_M)$$

with the standard action associated to $B^{\bullet}_M$, we obtain a morphism of ω-operads

$$B^0_M \xrightarrow{\mathfrak{I}d} End(A_{0,M})$$

which expresses an action of the ω-operad B^0_M of ω-magmas on the globular object $B^{\bullet}_M$-$\mathbb{A}lg(0)$ in SET of (n,ω)-magmas ($n \in \mathbb{N}$), and thus gives an ω-magma structure on (n,ω)-magmas ($n \in \mathbb{N}$).

6 The coglobular object of reflexive magmatic ω-operads

Consider the case $P = M_u$ (*magmatic with contractible units*). That is, we deal with the category $\mathbb{UT}$-$\mathbb{C}AT_c$ of ω-operads with chosen contractible units (see Section 3). We apply the free functor (see Section 3)

$$\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \xrightarrow{F} \mathbb{UT}\text{-}\mathbb{C}AT_c$$

to the coglobular object $C^{\bullet}$ of higher transformations in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ and we obtain a coglobular object $B^{\bullet}_{M_u}$ of ω-operads in $\mathbb{T}$-$\mathbb{C}AT_c$

$$B^0_{M_u} \underset{\kappa^1_0}{\overset{\delta^1_0}{\rightrightarrows}} B^1_{M_u} \underset{\kappa^2_1}{\overset{\delta^1_2}{\rightrightarrows}} B^2_{M_u} \cdots\cdots \rightrightarrows B^{n-1}_{M_u} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} B^n_{M_u} \cdots\cdots$$

If we write $\mathbb{UBT}$-$\mathbb{C}AT_1$ for the category of ω-operads equipped with a composition system in Batanin's sense and which have chosen contractible units, where the morphisms are those which preserve these composition systems and contractibility of the units, then it is important to note that the ω-operad $B^0_{M_u}$ is initial in $\mathbb{UBT}$-$\mathbb{C}AT_1$.

Also this coglobular object $B^\bullet_{M_u}$ produces the following globular object in $\mathbb{C}AT$

$$\cdots\gtrdot B^n_{M_u}\text{-}\mathbb{A}lg \underset{\beta^n_{n-1}}{\overset{\sigma^n_{n-1}}{\rightrightarrows}} B^{n-1}_{M_u}\text{-}\mathbb{A}lg \cdots\gtrdot B^1_{M_u}\text{-}\mathbb{A}lg \underset{\beta^1_0}{\overset{\sigma^1_0}{\rightrightarrows}} B^0_{M_u}\text{-}\mathbb{A}lg$$

In particular $B^0_{M_u}$ is the ω-operad for reflexive ω-magmas (see [7]). The standard action associated to $B^\bullet_{M_u}$ is given by the following diagram in $\mathbb{T}$-$\mathbb{C}AT_1$:

$$Coend(B^\bullet_{M_u}) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}_{M_u}) \xrightarrow{Coend(Ob(.))} End(A_{0,M_u})$$

It is a specific standard action of higher transformations. The monochromatic ω-operad $Coend(B^\bullet_{M_u})$ of coendomorphisms plays a central role for reflexive ω-magmas. We call it the *green operad*. Also we have the following result:

Proposition 6.1. $B^0_{M_u}$ *has the fractal property.*

Proof. In 7 we built a composition system for $Coend(B^\bullet_{M_u})$, and contractibility of its units is proved as in Section 4. Thus we just use the universality of $B^0_{M_u}$ in $\mathbb{UBT}$-$\mathbb{C}AT_1$ to conclude. □

If we compose the morphism $!_{M_u}$

$$B^0_{M_u} \xrightarrow{!_{M_u}} Coend(B^\bullet_{M_u})$$

with the standard action associated to $B^\bullet_{M_u}$, we obtain a morphism of ω-operads

$$B^0_{M_u} \xrightarrow{\mathfrak{C}_u} End(A_{0,M_u})$$

which expresses an action of the ω-operad $B^0_{M_u}$ of reflexive ω-magmas on the globular object $B^\bullet_{M_u}$-$\mathbb{A}lg(0)$ in SET of reflexive (n,ω)-magmas ($n \in \mathbb{N}$), and thus gives a reflexive ω-magma structure on reflexive (n,ω)-magmas ($n \in \mathbb{N}$).

Remark 6.2. In [9] we use the same combinatorics for the C^n but with different monads of arities: In fact we will use monads $\mathbb{T}^n$ (for all $n \geq 2$) on $\mathbb{G}lob^2$ of the strict n-transformations instead, where $\mathbb{G}lob^2$ is the product of the category of globular sets with itself in $\mathbb{C}AT$. If we denote $(1,1)$ the terminal object in $\mathbb{G}lob^2$, then all free strict higher transformations $\mathbb{T}^n(1,1)$ will play the role of arities domain for these C^n[13]. It give us a similar coglobular object $C^\bullet$ for the higher transformations which fits completely the control of coherences cells for its generated higher operads for the strict and the weak higher transformations. We could have proposed this different coglobular object for building this article, but the author believe that higher structures involved here are more simple and also more easily described by using the simpler coglobular object $C^\bullet$ with arities domain $\mathbb{T}(1)+\mathbb{T}(1)$, that we use all along this article.

We suspect that the ω-operad B^0_C of Batanin which algebras are his definition of weak ω-categories is fractal (see [6] and [9]). We also suspect that the ω-operad $B^0_{S_u}$[14] which algebras are the strict ω-categories is fractal as well (see [9]): Suprisingly, we see in [9] that these two questions of fractality, for the strict case and for the weak case, share in fact the same level of difficulty.

7 Composition systems

In this section we describe a composition system in Batanin's sense for the *yellow operad* $Coend(B^\bullet_M)$ described in Section 5, and for the *green operad* $Coend(B^\bullet_{M_u})$ described in Section 6. $B^\bullet_P$, or $B^\bullet$ for short, denotes the coglobular object $B^\bullet_M$ (see Section 5) or the coglobular object $B^\bullet_{M_u}$ (see Section 6) in $\mathbb{T}$-$\mathbb{C}AT_c$. Also, we denote by $B^n \underset{B^p}{\sqcup} B^n$ the 3-coloured ω-operad in $\mathbb{T}$-$\mathbb{C}AT_c$ which is obtained by the pushout of

$$\begin{array}{ccc} B^p & \xrightarrow{\kappa^n_p} & B^n \\ {\scriptstyle\delta^n_p}\downarrow & & \\ B^n & & \end{array}$$

in $\mathbb{T}$-$\mathbb{C}AT_c$, where $\delta^p_n=\delta^n_{n-1}...\delta^p_{p+1}$ and $\kappa^p_n=\kappa^n_{n-1}...\kappa^p_{p+1}$. For integers $0\leqslant p<n$ we are going to define a morphism

$$C^n \xrightarrow{\quad\mu^n_p\quad} B^n \underset{B^p}{\sqcup} B^n$$

[13]Instead of the other useful arities domain $\mathbb{T}(1)+\mathbb{T}(1)$ of this article. These arities domain $\mathbb{T}^n(1,1)$ follow the spirit of constructions of weak higher transformations as in [4], where coherences must be controlled by strict higher transformations.

[14]In [6] and [9], we show that the ω-operad of the strict ω-categories can be presented in a completely similar way as the Batanin's operad B^0_C : It is the initial object in the category of the ω-operads which are *strictly contractible* and equipped with a composition system. This also explain the similar notation for this operad with those of Batanin.

in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ which, depending on the universality property required, gives us a unique morphism

$$B^n \xrightarrow{\mu_p^n} B^n \underset{B^p}{\sqcup} B^n$$

in $\mathbb{T}$-$\mathbb{C}AT_c$, that we still call μ_p^n because there is no risk of confusion. The universal map $C^n \xrightarrow{\eta^n} B^n$ gives us such morphism μ_p^n. The key point in defining these morphisms μ_p^n is first to describe the different compositions $\circ_p^n$ for the strict higher transformations. If $0 < p < n$, we know that for two strict n-transformations σ and τ, we have

$$(\sigma \circ_p^n \tau)(a) := \sigma(a) \circ_{p-1}^{n-1} \tau(a)$$

whose operadic interpretation is given by the cell $\gamma(\mu_{p-1}^{n-1}; \sigma *_{p-1}^{n-1} \tau)$. Then the morphism

$$C^n \xrightarrow{\mu_p^n} B^n \underset{B^p}{\sqcup} B^n$$

in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ sends the principal cell τ of C^n to the $(n-1)$-cell $\gamma(\mu_{p-1}^{n-1}; \sigma *_{p-1}^{n-1} \tau)$ of $B^n \underset{B^p}{\sqcup} B^n$, sends for each $i \in \mathbb{N}$ the i-cell F_i of C^n to the i-cell F_i of $B^n \underset{B^p}{\sqcup} B^n$, and sends the i-cell G_i of C^n to the i-cell H_i of $B^n \underset{B^p}{\sqcup} B^n$. This morphism of $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ is boundary preserving in an evident sense.

For $p = 0$ it is a bit more complex. We are in the situation of the pushout diagram

$$\begin{array}{ccc} B^0 & \xrightarrow{\kappa_n^0} & B^n \\ {\scriptstyle \delta_n^0}\downarrow & & \downarrow{\scriptstyle i_1} \\ B^n & \xrightarrow[i_2]{} & B^n \underset{B^0}{\sqcup} B^n \end{array}.$$

First we describe the composition $\circ_0^n$ for the strict case in order to find the cells that we need in our ω-operad. Consider the following diagram in ω-$\mathbb{C}at$, the strict ω-category of strict ω-categories:

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \Downarrow\tau\ \mathcal{D} \underset{K}{\overset{H}{\rightrightarrows}} \Downarrow\tau\ \mathcal{E}$$

Here $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are 0-cells (that is, strict ω-categories), F, G, H and K are 1-cells (that is, strict ω-functors) and τ and σ are n-cells (that is, strict n-transformations). This picture describes τ and σ with 2-cells, but the reader must see them as n-cells. Also, τ and σ are such that: $s^n_0(\sigma) = \mathcal{C}$, $t^n_0(\sigma) = s^n_0(\tau) = \mathcal{D}$, and $t^n_0(\tau) = \mathcal{E}$. If $a \in \mathcal{C}(0)$, then $F^0 \xrightarrow{\tau(a)} G^0$ is an $(n-1)$-cell of $\mathcal{D}$ and it induces the following commutative square of $(n-1)$-cells in $\mathcal{E}$:

$$\begin{array}{ccc} H^0(F^0(a)) & \xrightarrow{H^{n-1}(\tau(a))} & H^0(G^0(a)) \\ {\scriptstyle \sigma(F^0)}\downarrow & & \downarrow{\scriptstyle \sigma(G^0)} \\ K^0(F^0(a)) & \xrightarrow[K^{n-1}(\tau(a))]{} & K^0(G^0(a)) \end{array}$$

This gives

$$\begin{aligned}(\sigma \circ^n_0 \tau)(a) &= \sigma(G_0(a)) \circ^{n-1}_0 H_{n-1}(\tau(a)) \\ &= K_{n-1}(\tau(a)) \circ^{n-1}_0 \sigma(F_0(a))\end{aligned}$$

which gives the two principal $(n-1)$-cells of $B^n \underset{B^0}{\sqcup} B^n$ that we need:

$$\gamma^{n-1}(\mu^{n-1}_0; \gamma(\sigma; G^0) *^{n-1}_0 \gamma(H^{n-1}; \tau))$$
and
$$\gamma^{n-1}(\mu^{n-1}_0; \gamma(K^{n-1}; \tau) *^{n-1}_0 \gamma(\sigma; F^0)).$$

Then we have two choices of

$$C^n \xrightarrow{\mu^n_o} B^n \underset{B^p}{\sqcup} B^n$$

which send the principal cell τ of C^n to $\gamma^{n-1}(\mu^{n-1}_0; \gamma(\sigma; G^0) *^{n-1}_0 \gamma(H^{n-1}; \tau))$ or to $\gamma^{n-1}(\mu^{n-1}_0; \gamma(K^{n-1}; \tau) *^{n-1}_0 \gamma(\sigma; F^0))$, and for both cases which send, for each $i \in \mathbb{N}$, the i-cell F^i of C^n to the i-cell $\gamma(F_i; H_i)$ of $B^n \underset{B^0}{\sqcup} B^n$, and the i-cell G^i of C^n to the i-cell $\gamma(G_i; K_i)$ of $B^n \underset{B^0}{\sqcup} B^n$. These morphisms of $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ are boundary preserving in an evident sense.

Thanks to the universal property of η^n, we obtain the following unique[15] morphisms of ω-operads μ^n_p and μ^n_0 (the dotted arrows).

[15] We say unique for μ^n_0 because we choose one presentation of μ^n_0 among the two choices as above which are possible for μ^n_0. However it is important to note that under the hypothesis of contractibility of [9], these two choices will connect with a coherence cell or will equalise, depending on the contractibility involved.

$$
\begin{array}{ccc}
B^n & \overset{\mu_p^n}{\dashrightarrow} & B^n \coprod_{B^p} B^n \\
{\scriptstyle \eta^n}\uparrow & \nearrow{\scriptstyle \mu_p^n} & \\
C^n & &
\end{array}
\qquad
\begin{array}{ccc}
B^n & \overset{\mu_0^n}{\dashrightarrow} & B^n \coprod_{B^0} B^n \\
{\scriptstyle \eta^n}\uparrow & \nearrow{\scriptstyle \mu_0^n} & \\
C^n & &
\end{array}
$$

With the identity morphisms of operads $B^n \xrightarrow{1_{B^n}} B^n$

$$
\begin{array}{ccc}
C^0 & \xrightarrow{c} & Coend(B^\bullet) \\
\mu_p^n & \longmapsto & \mu_p^n \\
u_n & \longmapsto & 1_{B^n}
\end{array}
$$

Thus, anticipating Sections 5 and 6, we have the following conclusion:

Proposition 7.1. *The ω-operads of coendomorphisms $Coend(B^\bullet_M)$ and $Coend(B^\bullet_{M_u})$, both have a composition system.*

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References

[1] J. Adámek and J. Rosický, "Locally Presentable and Accessible Categories", Cambridge University Press, 1994.

[2] M. Batanin, *Monoidal globular categories as a natural environment for the theory of weak-n-categories*, Adv. Math. 136 (1998), 39–103.

[3] F. Borceux, "Handbook of Categorical Algebra, Vol. 2", Cambridge University Press, 1994.

[4] K. Kachour, *Définition algébrique des cellules non-strictes*, Cah. Topol. Géom. Différ. Catég. 1 (2008), 1–68.

[5] C. Kachour, *Operadic definition of the non-strict cells*, Cah. Topol. Géom. Différ. Catég. 4 (2011), 1–48.

[6] C. Kachour, "Aspects of Globular Higher Category Theory", Ph.D. Thesis, Macquarie University, 2013.

[7] C. Kachour, *Algebraic definition of weak (∞, n)-categories*, To appear in Theory Appl. Categ. (2015).

[8] C. Kachour, *ω-Operads of coendomorphisms and fractal ω-operads for higher structures*, Categ. General Alg. Structures Appl. 3(1) (2015), 65–88.

[9] C. Kachour, *Steps toward the weak ω-category of the weak ω-categories in the globular setting*, To appear in Categ. General Alg. Structures Appl. 4(1) (2015).

[10] G.M. Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Aust. Math. Soc. 22 (1980), 1–83.

[11] S. Lack, *On the monadicity of finitary monads*, J. Pure Appl. Algebra 140 (1999), 65–73.

[12] L. Coppey and Ch. Lair, *Leçons de théorie des esquisses*, Université Paris VII, (1985).

[13] T. Leinster, "Higher Operads, Higher Categories", London Math. Soc. Lect. Note Series, Cambridge University Press 298, 2004.

[14] M. Makkai and R. Paré, "Accessible Categories: The Foundations of Categorical Model Theory", American Mathematical Society, 1989.

[15] G. Maltsiniotis, *Infini groupoïdes non strictes, d'après Grothendieck*, http://www.math.jussieu.fr/ maltsin/ps/infgrart.pdf (2007).

[16] J. Penon, *Approche polygraphique des ∞-catégories non-strictes*, Cah. Topol. Géom. Différ. Catég. 1 (1999), 31–80.

[17] D. Tamarkin, *What do DG categories form?*, Compos. Math. 143(5) (2007), 1335–1358.

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