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Clustering in Celebrating Professor Themba A. Dube (A TAD Celebration II)

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Commemorating Themba Dube on his 65th birthday

Abstract. This paper is the second in the series celebrating the mathematical works of Professor Themba Dube. In this sequel, we give prominence to Dube's pivotal contributions on pointfree convergence at the unstructured frame level, in the category of locales, and on his noteworthy conceptions on extensions and frame quotients. We distill and draw attention to particular studies of Dube on filters and his novel characterizations of certain conservative pointfree properties by filter and ultrafilter convergence, notably normality, almost realcompactness, and pseudocompactness. We also feature Dube's joint work on convergence and clustering of filters in **Loc** and coconvergence and coclustering of ideals in the category **Frm**.

Keywords: Frame, locale, Katětov extension, Fomin extension, βL, normal, pseudocompact, almost realcompact, Čech-complete, quotient, filter, ultrafilter, clustering, convergence, coconvergence, coclustering.
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1 Introduction

This sequel paper is a continuation of Naidoo [39](2024) in which we presented pertinent studies of Dube on structured frames that of nearness-, metric-, and uniform-frames, and relevant subcategories that he inserted into pointfree topology and manoeuvred within these structured frames. We also steered through Dube's categorical manifestations with certain types of mappings in frames, functorial commutativity, the Stone-Čech compactification (βL), the regular Lindelöf coreflection or Lindelöfication (λL), the realcompact coreflection or Realcompactification (υL), the paracompact coreflection or Paracompactification (πL) and the Booleanization (ßL).

I recall that in one of our conversations, Dube acknowledged that he has an affinity for ideals and it is more his preference. This is self-evident in many of his papers. Nevertheless, I did remark that he is certainly dually filtered. This paper is devoted to giving prominence to Dube's pioneering work on convergence in the category of frames and locales. It is an aggregation of Dube's work, particular filtered since his doctoral thesis until his retirement in 2022. We mainly proposition Dube's unprecedented work specifically on filters in the categories **Frm** and **Loc**. We continue our survey into Dube's pointfree influence and engagements threading through three main pillars that concatenates Dube's prodigious scholarly contributions in pointfree topology during the period 1992 - 2022. We highlight some of Dube's pertinent independent and collaborative work on convergence in frames and locales, and on extensions and quotients of frames. The relevant background material on the category of frames that are referred to in this paper may be accessed from [39]. In the next Section §2 we provide the necessary further data that we require on frames and locales to make the paper self-contained We also formalize some of the symbols and notations that we use.

Section §3 spans over 5 areas of the diverse workings of Dube that all have a common thread in filter characterizations of the conservative properties examined in each independently by Dube and with his coauthors. Section §3 chronicles and recounts Dube's interests in convergence and clustering of filters in frames in which he characterizes normality, almost realcompactness, pseudocompactness (with Matutu) and Čech-completeness (with Mugochi and Naidoo) by the convergence and clustering of certain types of filters that he creates independently and jointly with his coauthors.

The study of convergence in the category of frames has been expansive since Banaschewski and Pultr [23](1990) defined *a filter in a frame to be convergent provided it contains a completely prime filter*. However, in the opposite category of locales, in Dube and Ighedo [15](2016) the authors found that the theory developed on convergence in **Frm** and its utilities serviced by filters was inadequate and not suitable for their study involving P-frames and on mainly characterising points of βL in terms of convergence. They required an entity in locales that is more general than the exemplar of a filter in frames. Filters in the lattice of sublocales $\mathcal{S}\ell(L)$ of a locale L whose members are sublocales met their requirements. This is an unchartered realm in **Loc** that Dube jointly ventures into for discovery and posterity. In Section §4 we recapitulate Dube's seminal joint work on convergence in **Loc**, the results of which bear similar resemblance to many classical topological ones as one would undoubtedly expect. We end in Section §5 by communicating celebratory tributes to Dube extended by Joanne Walters-Wayland, Papiya Bhattacharjee and Tega Ighedo.

2 Preliminaries

For a general background into frames and locales we suggest the book by Picado and Pultr [42] which is our primary source for the symbols, notations, nomenclature and pertinent results that are used.

Frm is the category in which the objects are frames and morphisms are frame homomorphisms (maps between frames that preserve finite meets and arbitrary joins). Equivalently, a frame L is a complete Heyting algebra with (unique) *Heyting operation* $\to$ on L given by

$$a \wedge b \leq c \text{ iff } a \leq b \to c,$$

where $b \to c = \bigvee\{x : b \wedge x \leq c\}$ for any $a, b, c \in L$. This emanates since the map $L \xrightarrow{x\wedge(-)} L$ preserves $\bigvee$ and so has a right Galois adjoint $L \xrightarrow{x\to(-)} L$. The operation $\to$ will be used in the subsequent sections in describing *sublocales*. We will be also be interested in the opposite (dual) category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$, the objects of which are called *locales* (which are the same as frames) and morphisms are *localic* (or *continuous*) maps (which are the right adjoints of frame homomorphisms).

We consider $L \in \mathbf{Frm}$ hereunder and in the next two subsections unless otherwise stated. We recall that each element $x \in L$ has a *pseudocomplement* denoted

$$x^* = \bigvee\{y \in L \colon y \wedge x = 0\}.$$

The following are well known properties of the pseudocomplement for elements x, y in L and any $B \subseteq L$:

$$x \leq x^{**}, \;\; x^{***} = x^*, \;\; \left(\bigvee B\right)^* = \bigwedge_{b \in B} b^*, \;\text{ and }\; (x \wedge y)^{**} = x^{**} \wedge y^{**}.$$

Furthermore, $x \in L$ is *dense* if $x^* = 0$. The collection of all dense elements in L is denoted

$$\mathfrak{d}L = \{x \in L \colon x^* = 0\}.$$

$x \in L$ is called a *regular* element if $x = x^{**}$. The *Booleanization* of L is the set of all regular elements of L that we denote by

$$ßL = \{x \in L \colon x = x^{**}\}.$$

x is *complemented* if $x \vee x^* = 1$.

We will require some properties of a frame (locale) L that we briefly provide in the following Remark.

Remark 2.1.

(1) **RegFrm** is the category of *regular* frames and their homomorphisms. $L \in \mathbf{RegFrm}$ if for each $x \in L$,

$$x = \bigvee\{y \in L \colon y \prec x\}$$

where $y \prec x$ iff $y^* \vee x = 1$.

(2) **CRegFrm** is the category of *completely regular* frames and their homomorphisms. Here $L \in \mathbf{CRegFrm}$ if for each $x \in L$, $x = \bigvee\{y \in L \colon y \prec x\}$ in which $x = \bigvee\{y \in L \colon y \prec\!\prec x\}$ where $y \prec\!\prec x$ iff there is a scale $\{s_q \in L \colon q \in \mathbf{Q} \cap [0,1]\}$ such that $y = s_0, x = s_1$ and $s_q \prec s_p$ whenever $q < p$).

(3) L is *normal* frame if whenever $x \vee y = 1$ there is $s, t \in L$ such that $s \prec x$, $t \prec y$ and $s \wedge t = 0$.

(4) L is *Hausdorff* if $a, b \in L$, $1 \neq a \leq b$ implies that $\exists\, c \in L$ such that $c^* \nleq a$ and $c \nleq b$ (Johnstone and Shu-Hao [33](1988)).

(5) $\mathrm{Pt}(L)$ is the set of all *points* or *prime elements* of L. $p \in L$ is called a *point* of L if $p \neq 1$ and whenever $x, y \in L$ with $x \wedge y \leq p$ we have $x \leq p$ or $y \leq p$.

(6) L is *spatial* if $L \simeq \mathfrak{O}X$ for some topological space X where $\mathfrak{O}X$ is the frame of open sets of X.

The points of a frame provide internal equivalent formulations of spatiality.

Lemma 2.2. *For a frame L the following are equivalent:*

(1) *L is spatial.*

(2) *For each $x \in L$, $x = \bigwedge\{p \in \mathrm{Pt}(L) \colon x \leq p\}$.*

(3) *If $y \nleq x$ in L then there is $p \in \mathrm{Pt}(L)$ such that $x \leq p$ and $y \nleq p$.*

In the above lemma, the equivalence of spatiality and (2) is given in Picado and Pultr [42, Chapter II, Proposition 5.3](2012) whilst the equivalence of (3) appears in Johnstone [32, Chapter II, §1.3 & §1.5](1982). For regular frames, Dube in [9](2011) provides the following equivalent expression of spatiality that embraces points.

Lemma 2.3. [9, Lemma 3.1] *A regular frame L is spatial iff for each $1 \neq x \in L$ there is $p \in \mathrm{Pt}(L)$ such that $x \leq p$.*

2.1 Covering properties of a frame We will use the notation $\subseteq_{<\omega}$ (resp. $\subseteq_\omega$) for finite (resp. countable) subsets. The frame $\wp(X)$ is the powerset of X. A *cover* of a frame L is any subset whose join is the top and $\mathrm{Cov}\, L = \{C \subseteq L \colon \bigvee C = 1\}$ is the collection of all covers of L whilst $\mathrm{Cov}_\omega\, L = \{C \subseteq_\omega L \colon \bigvee C = 1\}$. We also define the set $\mathrm{Cov}_q\, L = \{C \subseteq L \colon \bigvee C \in \mathfrak{d}L\}$ and members of $\mathrm{Cov}_q\, L$ are termed *quasi-covers*. For any $F \subseteq L$ we let

$$\alpha(F) = \bigvee_{x \in F} x^*.$$

$x \in L$ is a *compact* element if $x \leq \bigvee S$ for any $S \subseteq L$ then $x \leq \bigvee T$ for some $T \subseteq_{<\omega} S$. $\mathfrak{k}L$ denotes the collection of all compact elements of L. If $1 \in \mathfrak{k}L$ then the frame L is a *compact* frame (every cover of L has a finite subcover). Banaschewski and Mulvey [22](1980) define $x \in L$ to be *cocompact* provided that the frame $\uparrow x = \{y \in L\colon x \leq y\}$ is compact. We denote the collection of all cocompact elements of L by

$$\mathfrak{co}\mathfrak{k}L = \{x \in L\colon\ \uparrow x \text{ is compact}\}.$$

We also mention that L is

1. *Lindelöf* if for each $A \in \operatorname{Cov} L$ there is $\operatorname{Cov}_\omega L \ni B \subseteq A$ (each cover of L has a countable subcover),
2. *almost compact* if for each $A \in \operatorname{Cov} L$ there is $\operatorname{Cov}_q L \ni B \subseteq_{<\omega} A$ (each cover of L has a finite subset whose join is dense), and
3. *locally almost compact* if $\alpha(F_{\mathfrak{a}}) = 1$ where

$$F_{\mathfrak{a}} = \{x \in L\colon\ \uparrow x^{**} \text{ is almost compact}\}.$$

We will refer to a frame that is not almost compact as a *nalco-frame*. We define an element x in a frame L to be *co-almost compact* if $\uparrow x$ is an almost compact frame and denote the collection of all co-almost compact elements by

$$\mathfrak{coa}\mathfrak{k}L = \{x \in L\colon\ \uparrow x \text{ is almost compact}\}.$$

In Paseka and Šmarda [40](1988) the authors provide an investigation into locally almost compact frames and they provide an extensive treatment of almost compact frames in Paseka and Šmarda [41](1992). It is well known that for regular frames almost compactness and compactness coincide (see Hong [29](1992)). We note that if L is almost compact and $x \in ßL$, then $x \in \mathfrak{coa}\mathfrak{k}L$ [40, Proposition 2.3](1988). Furthermore, if L is Hausdorff then L is compact iff $L = \mathfrak{coa}\mathfrak{k}L$ [40, Proposition 2.9](1988). The co-almost compact elements are again captured in Proposition 3.17 of our paper in our synopsis celebrating Dube's work.

2.2 The cozero part of a frame Let L be a bounded distributive lattice in which for each $S \subseteq_\omega L$, $\bigvee S \in L$ and binary meet distribute over these joins (L satisfies the frame distributive law for countable subsets), then L is called a *σ-frame*. σ**Frm** is the category of σ-frames and their morphisms. σ-frame homomorphisms are maps between σ-frames that preserves all finite meets and countable joints (including the top and bottom elements). In general, in a σ-frame the pseudocomplement of an element does not exist. With the minor modification (to incorporate countable joins) $L \in \sigma\mathbf{Frm}$ is a *regular* if for each $x \in L$, $x = \bigvee Y$ for some $Y \subseteq_\omega \{y \in L : y \prec x\}$ where $y \prec x$ if there is $s \in L$ such that $y \wedge s = 0$ and $s \vee x = 1$. Normality for σ-frames is defined in the same way as that for frames and it is well known that *every regular σ-frame is normal* (see Gilmour [27](1984)).

For $L \in \mathbf{Frm}$, $L \ni c$ is a *cozero element* if $c = h(\mathbb{R}\backslash\{0\})$ for some frame homomorphism $h : \mathfrak{O}\mathbb{R} \to L$. The *cozero part* of a frame L is the set of all cozero elements of L denoted as $\mathrm{Coz}[L]$. For $L \in \mathbf{CRegFrm}$, $\mathrm{Coz}[L]$ join-generates L and $\mathrm{Coz}[L]$ is a regular σ-frame. Equivalent descriptions of a cozero element that are frequently favoured is given in Banaschewski and Gilmour [18, Proposition 1](1996):

Proposition 2.4. *In any frame L and $c \in L$ the following are equivalent:*

(1) $c \in \mathrm{Coz}[L]$.

(2) $c = \bigvee\{x_n \in L : x_n \prec\prec c \;\forall n \in \mathbb{N}\}$.

(3) $c = \bigvee\{x_n \in L : x_n \prec\prec x_{n+1} \;\forall n \in \mathbb{N}\}$.

For more details on σ-frames and $\mathrm{Coz}[L]$, see Reynolds [44](1979), Gilmour [27](1984), Madden and Vermeer [36](1986), Walters [46, 47](1990, 1991) and Banaschewski and Gilmour [18, 19](1996, 2001).

Ball and Walters-Wayland [17](2002) briefly introduced *P-frames*, the pointfree analogue of P-spaces (those spaces in which each cozero set is closed).

> *A frame L is a P-frame if* $\mathrm{Coz}[L]$ *is complemented, that is if* $x \in \mathrm{Coz}[L]$ *then* $x^* \in \mathrm{Coz}[L]$.

Since a topological space X is a P-space iff $\mathfrak{O}X$ is a P-frame, this property is conservative. Notably from Ball and Walters-Wayland [17, Proposition 8.4.8],

> $L \in \mathbf{Frm}$ *is a P-frame iff λL is a P-frame iff υL is a P-frame.*

Dube has made further inroads on this concept and provides a more detailed account of P-frames in Dube [8](2009).

2.3 Filters and convergence in frames For a frame L, $\emptyset \neq F \subseteq L$ is a *filter* in L if $0 \notin F$ and F is an upset ($x \in F$ and $x \leq y$ implies $y \in F$) that is closed under finite meet. F is a *proper* filter if $F \neq L$. We will denote the collection of all filters in L by $\mathfrak{F}(L)$ and consider only proper filters. By dualizing, we have the concept of an *ideal* in a frame. The set of all ideals in L is denoted by $\mathfrak{I}(L)$ and ordered by inclusion $(\mathfrak{I}(L), \subseteq)$ is a compact frame with $\bigcap = \bigwedge$ and for any collection of ideals $\{I_j : j \in J\}$, its join in $\mathfrak{I}(L)$ is the ideal

$$\bigvee_{j\in J} I_j = \left\{ \bigvee F \colon F \subseteq_{<\omega} \bigcup_{j\in J} I_j \right\}$$

given in Picado and Pultr [42, Proposition 4.1.1](2012). For $a \in L$,

$$\uparrow a = \{b \in L \colon a \leq b\} \in \mathfrak{F}(L)$$

is the *principal filter generated by a*. For any $H \subseteq L$, $\langle H \rangle$ denotes the filter generated by H.

Remark 2.5. Given any $a, b \in L$, $S \subseteq L$ and $F \in \mathfrak{F}(L)$, F is classified as:

(1) *prime* if $a \vee b \in F$ then $a \in F$ or $b \in F$. $\mathfrak{F}_{\mathfrak{p}}(L)$ denotes the collection of all prime filters in L.

(2) *disjoint-prime* if $a \vee a^* \in F$ then either $a \in F$ or $a^* \in F$.

(3) *completely prime* if $\bigvee S \in F$ then $F \cap S \neq \emptyset$. The collection of all completely prime filters is denoted $\mathfrak{F}_{\mathfrak{cp}}(L)$. If $p \in \mathrm{Pt}(L)$ then $F_p = \{x \in L \colon x \not\leq p\} \in \mathfrak{F}_{\mathfrak{cp}}(L)$. In contrast, if $F \in \mathfrak{F}_{\mathfrak{cp}}(L)$, then $p_F = \bigvee(L \backslash F) \in \mathrm{Pt}(L)$. Moreover, $p_{F_p} = p$ and $F_{p_F} = F$.

(4) an *ultrafilter* if $G \in \mathfrak{F}(L)$ and $F \subseteq G$ then $F = G$. $\mathfrak{F}_{\mathfrak{ult}}(L)$ is the collection of all ultrafilters in L.

(5) a *σ-filter* if $\bigwedge S \in F$ whenever $S \subseteq_{\omega} F$. The collection of all σ-filters in L is denoted $\mathfrak{F}_{\sigma}(L)$.

(6) *regular* if for each $y \in F$ $\exists$ $x \in F$ such that $x \prec y$. $\mathfrak{F}_{\mathfrak{reg}}(L)$ denotes the set of regular filters of L.

(7) *completely regular* if for each $y \in F \; \exists \; x \in F$ such that $x \prec\!\prec y$. $\mathfrak{F}_{\mathfrak{creg}}(L)$ denotes the set of completely regular filters of L.

(8) *saturated* if $\mathfrak{d}L \subseteq F$.

We note that a filter base F is called regular (respectively, completely regular) if F satisfies (6) (respectively, (7)). In the above, Remark (2) and (8) emanate from Carlson and Porter [24](2009) and is given in pointfree form in Dube [7](2015).

Dualizing the above notions we characterize the corresponding types of ideals as prime-, maximal- and σ-ideal. The corresponding respective collections of these types of ideals are respectively denoted $\mathfrak{I}_{\mathfrak{p}}(L), \mathfrak{I}_{\mathfrak{max}}(L)$ and $\mathfrak{I}_{\sigma}(L)$. The set of all minimal prime ideals of L will be denoted $\mathfrak{I}_{\mathfrak{p}_{\min}}(L)$. The classical separation lemma of Stone for distributive lattices (see, for instance, Johnstone [32](1982)) is a fundamental existential result for prime ideals and we will refer to it in the subsequent part of the paper.

Lemma 2.6 (Stone's Separation Lemma)**.** *Let F be a filter and I be an ideal in a distributive lattice L, and suppose that $F \cap I = \emptyset$. Then there is a prime ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$.*

The original notion of *convergence* in pointfree topology is ascribed to Banaschewski and Pultr in [23](1990). Their study primarily involved providing an alternate description of the completion of a uniform frame via its Samuel compactification (the compact regular coreflection of a uniform frame). Apart from this description of the completion, they considered the classical Cauchy completeness of a uniform space in pointfree form. In so doing, a prerequisite was the notion of a convergent filter in a frame. Given $F \in \mathfrak{F}(L)$, they defined F to be convergent if there is $G \in \mathfrak{F}_{\mathfrak{cp}}(L)$ such that $G \subseteq F$ (*a filter F is convergent if it contains a completely prime filter*).

Hong [30](1995) presented a first concise investigation into pointfree convergence and introduced a novel cover approach to convergence and clustering of filters in frames derived from the conservative property of the convergence of open filters in spaces. Given $F \in \mathfrak{F}(L)$,

1. F *converges* or is *convergent* in L provided that $F \cap A \neq \emptyset$ for each $A \in \operatorname{Cov} L$.

2. F *clusters* or is *clustered* in L provided that $\sec F \cap A \neq \emptyset$ for each $A \in \operatorname{Cov} L$ where $\sec F = \{x \in L : \text{ for any } a \in F,\ a \wedge x \neq 0\}$.

We introduce the following notations for the convergence and clustering of a filter in a frame L. Given $F \in \mathfrak{F}(L)$,

1. $F \rightarrow L$ denotes that F converges in L. We note that $F \rightarrow L$ iff $\forall$ $A \in \operatorname{Cov} L$, $a \in F$ for some $a \in A$. If $G \in \mathfrak{F}(L)$ and $G \supseteq F \rightarrow L$ then $G \rightarrow L$.
2. $F \multimap L$ denotes that F clusters in L. Hong [30, Proposition 1.3](1995) provides an equivalent formulation of a clustered filter:

$$F \multimap L \text{ iff } \alpha(F) \neq 1.$$

Furthermore, if $F \rightarrow L$ then $F \multimap L$. If $G \in \mathfrak{F}(L)$ and $G \subseteq F \multimap L$ then $G \multimap L$.
3. F is called *free* if $F \not\multimap L$, that is $\alpha(F) = 1$ (see Strauss and Zhang [48](1999)). The collection of all free filters (resp. prime, resp. ultrafilters) in L is denoted $\mathfrak{F}^{F}(L)$ (resp. $\mathfrak{F}^{F}_{\mathfrak{p}}(L)$, resp. $\mathfrak{F}^{F}_{\mathfrak{ult}}(L)$).

Free filters in a frame L are termed *α-filters* by Paseka and Šmarda in their papers [40, 41](1988, 1992). They observed that

$$\textit{for any } G \in \mathfrak{F}^{F}(L),\ \bigwedge G = 0$$

and they provided an elegant characterization of almost compact frames using such filters:

A frame L is almost compact iff $\mathfrak{F}^{F}_{\mathfrak{ult}}(L) = \emptyset$.

They also showed that

A nalco-frame L is locally almost compact iff $F_{\mathfrak{a}} \in \mathfrak{F}^{F}(L)$.

We note the following concerning filters, ultrafilters and some covering properties characterized by them.

Remark 2.7. (1) If $F \in \mathfrak{F}_{\mathfrak{cp}}(L)$ then $F \rightarrow L$. For regular frames, convergence is equivalently formulated using completely prime filters. If L is a regular frame and $F \in \mathfrak{F}(L)$ then $F \textcircled{$\to$} L$ iff F contains a completely prime filter (Banaschewski and Hong [21]). Hong's notion of convergence of filters is weaker than that of the Banaschewski-Pultr definition. Nevertheless, both descriptions coincide for regular frames.

(2) If $F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ then $F = \sec F$ and consequently $\mathfrak{d}L \subseteq F$ (F contains all dense elements). We also have that every filter is contained in an ultrafilter (by the usual application of Zorn's Lemma). Furthermore, if $F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ then $F \textcircled{$\multimap$} L$ iff $F \textcircled{$\to$} L$.

(3) For regular frames, compactness and Lindelöfness have characterizations by the convergence and clustering of filters. For a regular frame L, we have by Hong [30, Corollary 1.5](1995) that

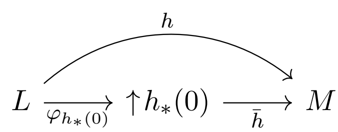
$$\begin{aligned} L \text{ is compact} \quad &\text{iff} \quad F \textcircled{$\multimap$} L \text{ for every } F \in \mathfrak{F}(L) \\ &\text{iff} \quad F \textcircled{$\to$} L \text{ for every } F \in \mathfrak{F}_{\mathfrak{ult}}(L), \end{aligned}$$

and by Naidoo [37, Theorem 4.6](2007) that

$$L \text{ is Lindelöf iff } F \textcircled{$\multimap$} L \text{ for every } F \in \mathfrak{F}_{\sigma}(L).$$

We now briefly discuss the notion of frame extensions that we will require in the next section. We call an onto frame homomorphism $h \colon L \to M$ a *quotient*. For any frame homomorphism $h \colon L \to M$ its *right adjoint* is denoted $h_* \colon M \to L$ and $h \colon L \to M$ is *dense* iff $h_*(0_M) = 0_L$.

In a frame L and $a \in L$, the map $\varphi_a \colon L \to {\uparrow} a$ defined by $\varphi_a(x) = a \vee x$ is a quotient called the *closed quotient* at a. Every frame homomorphism $h \colon L \to M$ has a *dense-onto* factorization via the closed quotient at $h_*(0)$.



$h = \bar{h} \circ \varphi_{h_*(0)}$ where $\bar{h}$ is the dense frame homomorphism mapping as h.

An *extension* of a frame L is any pair (M, h) where M is any frame and $h \colon M \to L$ is a dense quotient which we call the *extension map*.

Remark 2.8. Let (M, h) be an extension of a frame L. Then (M, h) is called:

(1) *strict* if $h_*(L)$ generates M,

(2) *spatial over* L if whenever $h(a) = h(b)$ and $a \not\leq b$, then $\exists\ p \in \mathrm{Pt}(M)$ such that $b \leq p$ and $a \not\leq p$,

(3) *perfect* if $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)\ \forall\ a \in L$, Equivalently, if $h_*(a \vee b) = h_*(a) \vee h_*(b)\ \forall\ a, b \in L$ with $a \wedge b = 0$.

In the above Remark, (1) and (2) can be found in Banaschewski and Hong [20](1999) and (3) emanates from Baboolal [16](2011). See also Dube and Mugochi [13](2015). Hong in [30](1995) defines a *simple extension* of a frame L determined by a set of filters using a subframe of the product $L \times \wp(\mathfrak{X})$ where $\mathfrak{X} \subseteq \mathfrak{F}(L)$ as follows. For each $a \in L$, let

$$\mathfrak{X}_a = \{F \in \mathfrak{X} \colon a \in F\}$$

and

$$s_{\mathfrak{X}}L = \{(a, \mathfrak{S}) \in L \times \wp(\mathfrak{X}) \colon \mathfrak{S} \subseteq \mathfrak{X}_a\}.$$

Then $s_{\mathfrak{X}}L$ is a subframe of $L \times \wp(\mathfrak{X})$. Furthermore, the map $s \colon s_{\mathfrak{X}}L \to L$ given by $s(a, \mathfrak{S}) = a$ is an open, dense quotient with right adjoint $s_* \colon L \to s_{\mathfrak{X}}L$ given by $s_*(a) = (a, \mathfrak{X}_a)$. The pair $(s_{\mathfrak{X}}L, s)$ is called the *simple extension* of L with respect to $\mathfrak{X}$ and s is the *simple extension map*. The subframe of $s_{\mathfrak{X}}L$ generated by $s_*(L)$ is denoted by $t_{\mathfrak{X}}L$ and $s\ |_{t_{\mathfrak{X}}L} = t \colon t_{\mathfrak{X}}L \to L$ is a dense quotient. Then $(t_{\mathfrak{X}}L, t)$ is the *strict extension* of L determined by $\mathfrak{X}$.

2.4 Sublocales Let $L \in \mathbf{Loc}$. A subset $S \subseteq L$ is a sublocale of L if

(S1) $\bigwedge A \in S$ for all $A \subseteq S$, and

(S2) for all $x \in L$ and $s \in S$, $x \to s \in S$.

By (S1) $1_L = \bigwedge \emptyset \in S$ for any $S \in \mathcal{S}\ell(L)$ so that every sublocale is nonempty. The lattice of all sublocales of L, denoted by $\mathcal{S}\ell(L)$, ordered by inclusion is complete with meet and join given by

$$\bigwedge_{j \in J} S_j = \bigcap_{j \in J} S_j \quad \text{and} \quad \bigvee S_j = \Big\{\bigwedge A : A \subseteq \bigcup_{j \in J} S_j\Big\},$$

for any family $\{S_j : j \in J\}$ of sublocales of L. The *least* (*smallest*) sublocale of L is $0_{\mathcal{S}\ell(L)} = \{1\} = \mathsf{O}$ and is contained in any other sublocale. The *greatest* sublocale is, of course, $1_{\mathcal{S}\ell(L)} = L$.

For $S \in \mathcal{S}\ell(L)$, its *supplement* is the sublocale defined as

$$\sup(S) = \bigcap\{R \in \mathcal{S}\ell(L) \colon R \vee S = L\} = \bigvee\{T \in \mathcal{S}\ell(L) \colon T \cap S = \mathsf{O}\}.$$

S is *complemented* if there is $T \in \mathcal{S}\ell(L)$ such that $S \cap T = \mathsf{O}$ and $T \vee S = L$. Not all sublocales have complements. If $S \in \mathcal{S}\ell(L)$ is complemented, then $\sup(S)$ is the complement of S in $\mathcal{S}\ell(L)$. We will denote the lattice of all complemented sublocales of L by

$$\mathcal{S}\ell_c(L) = \{S \in \mathcal{S}\ell(L) \colon S \text{ is complemented}\}.$$

We refer the reader to Isbell [31](1991) and Plewe [43](2000) for further details on the lattice $\mathcal{S}\ell_c(L)$.

$\mathcal{S}\ell(L)$ is a *co-frame* meaning that for any $S \in \mathcal{S}\ell(L)$ and any family $\{T_j : j \in J\}$ of sublocales we have that

$$S \vee \bigcap_{j \in J} T_j = \bigcap_{j \in} (S \vee T_j).$$

For $x \in L$, we have the *open* and *closed* sublocale induced by x, given respectively by

$$\mathfrak{o}(x) = \{x \to a : a \in L\} = \{a : x \to a = a\}$$

and

$$\mathfrak{c}(x) = \{a \in L : x \leq a\} = \uparrow x.$$

The following are some well-known identities and properties of open and closed sublocales.

Lemma 2.9. *For $L \in \mathbf{Loc}$, $x, y \in L$ and $\{x_j : j \in J\} \subseteq L$ we have*

(OSC1) $\mathfrak{o}(0_L) = \mathfrak{c}(1_L) = \mathsf{O}$ *and* $\mathfrak{o}(1_L) = \mathfrak{c}(0_L) = L$,

(OSC2) $\mathfrak{o}(x)$ *and* $\mathfrak{c}(x)$ *are complements of each other in* $\mathcal{S}\ell(L)$,

(OSC3) $x \leq y$ *iff* $\mathfrak{c}(x) \supseteq \mathfrak{c}(y)$ *iff* $\mathfrak{o}(x) \subseteq \mathfrak{o}(y)$,

(OSC4) $\mathfrak{o}(x) \cap \mathfrak{o}(y) = \mathfrak{o}(x \wedge y)$ *and* $\mathfrak{c}(x) \vee \mathfrak{c}(y) = \mathfrak{c}(x \wedge y)$,

(OSC5) $\bigcap_{j \in J} \mathfrak{c}(x_j) = \mathfrak{c}\left(\bigvee_{j \in J} x_j\right)$ *and* $\bigvee_{j \in J} \mathfrak{o}(x_j) = \mathfrak{o}\left(\bigvee_{j \in J} x_j\right)$, *and*

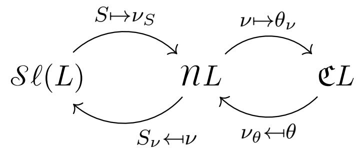
(OSC6) $\mathfrak{c}(x) \subseteq \mathfrak{o}(y)$ *iff* $x \vee y = 1_L$.

Sublocales are equivalently represented by congruences and nuclei as we briefly illustrate (see Picado and Pultr [42, Chapter III §5](2012) or Johnstone [32, Chapter II §2](1982) for the finer details).

We recall that for any frame L, a *congruence* on L is an equivalence relation $\theta \subseteq L \times L$ that is a subframe of $L \times L$. The lattice of congruences, $\mathfrak{C}L$, of a frame L partially ordered by inclusion is complete with intersection for the meet. Moreover, $\mathfrak{C}L$ is a frame with top $\nabla = 1_{\mathfrak{C}L} = L \times L$ and bottom $\Delta = 0_{\mathfrak{C}L} = \{(x, x) : x \in L\}$. Each $x \in L$ is associated with the least (or smallest) congruence containing $(0_L, x)$, namely $\nabla_x = \{(a, b) \in L \times L : a \vee x = b \vee x\}$ (the *closed* congruence), and with the least congruence containing $(x, 1_L)$, namely $\Delta_x = \{(a, b) \in L \times L : a \wedge x = y \wedge x\}$ (the *open* congruence). Furthermore, $\nabla_x \vee \Delta_x = \nabla$ and $\nabla_x \cap \Delta_x = \Delta$ so that ∇_x and Δ_x are complements of each other in $\mathfrak{C}L$.

A *nucleus* on a frame L is a monotone map $\nu : L \to L$ satisfying the following properties for each $x, y \in L$:
(N1) $x \leq \nu(x)$,
(N2) $\nu(\nu(x) = \nu(x)$, and
(N3) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$.

$\mathcal{N}L$ denotes the collection of all nuclei on a frame L that is partially ordered by $\nu, \kappa \in \mathcal{N}L$, $\nu \leq \kappa$ iff $\nu(x) \leq \kappa(x)$ for each $x \in L$. $\mathcal{N}L$ is a frame with meet defined for $N = \{\nu_j : j \in J\} \subseteq \mathcal{N}L$ by $\bigwedge N : L \to L$ given by $\bigwedge N(x) = \bigwedge_{j \in J} \nu_j(x)$. Given any nucleus ν on L, the associated congruence on L is defined by $\theta_\nu = \{(x, y) \in L \times L : \nu(x) = \nu(y)\}$. For any congruence $\theta \in \mathfrak{C}L$, we define $\nu_\theta : L \to L$ by $\nu_\theta(x) = \bigvee\{y : (y, x) \in \theta\}$ which gives the associated nucleus on L. On the other hand, given any sublocale $S \in \mathcal{S}\ell(L)$, $\nu_S : L \to L$ given by $\nu_S(x) = \bigwedge\{s \in S : x \leq s\}$ defines the associated nucleus on L. For any nucleus $\nu : L \to L$, the associated sublocale is $S_\nu = \nu(L)$. Sublocales, congruences and nuclei are thus in one-to-one correspondences with each other:



The open and closed sublocales ($\mathfrak{o}(x)$ and $\mathfrak{c}(x)$) correspond to the open and closed congruences (respectively, Δ_x and ∇_x). As frames, we have the isomorphisms

$$\mathcal{S}\ell(L)^{\mathrm{op}} \simeq \mathcal{N}L \simeq \mathfrak{C}L.$$

Particularly, meets and join in $\mathcal{S}\ell(L)$ correspond to respectively join and meets in $\mathfrak{C}L$ (and $\mathcal{N}L$).

For $S \in \mathcal{S}\ell(L)$, the *closure* of S in L, denoted by $\overline{S}$, is the least closed sublocale containing S, computed as

$$\overline{S} = \bigcap_{x\in L}\{\mathfrak{c}(x) : S \subseteq \mathfrak{c}(x)\} = \mathfrak{c}\Big(\bigwedge S\Big) = \uparrow\Big(\bigwedge S\Big).$$

The following properties of closures can then be realised for $S \in \mathcal{S}\ell(L)$:
(Cl1) $S \subseteq \overline{S}$.
(Cl2) $\overline{\overline{S}} = \overline{S}$.
(Cl3) $\overline{S \vee T} = \overline{S} \vee \overline{T}$.
(Cl4) $\overline{\mathfrak{o}(x)} = \mathfrak{c}(x \to 0_L) = \mathfrak{c}(x^*) = \uparrow(x^*)$ for each $x \in L$.

A sublocale S is *dense* in L iff $\overline{S} = L$ iff $0_L \in S$. It is well established that *every locale L has a smallest dense sublocale* $ß_L$ (see Johnstone [32, Chapter II, §2.3, Lemma, pg.50](1982) or Picado and Pultr [42, Chapter III, §7, Proposition, pg. 40](2012)) where $ß_L = \{x \to 0 \colon x \in L\}$.

3 Clustering in frames

In this section, Dube's mathematical works especially on the clustering of filters in frames is elucidated. In §3.1 we feature Dube's earliest conceptions on pointfree clustering emanating from his doctoral thesis [1](1992) followed by his insertion and coverage of specialized filters that he brings to the fore in [2](2002). Dube's compass points next towards weaker forms of realcompactness in [4](2006) that forms the basis in his subsequent encounter with convergence that we elaborate in §3.2. We also acknowledge Dube's joint work with Matutu in [5](2007) with particular interest on the interaction of pseudocompactness with filters.

In §3.3 we muse on the Katětov extension $(\kappa L, \kappa_L)$ of a frame L and confront Dube's unfolding of the Fomin extension $(\sigma L, \sigma_L)$ of L and its coincidence with the Stone-Čech compactification βL for a completely regular frame L. Penulimately, in §3.4 we touch upon the concept of Čech-completeness developed jointly by Dube together with Mugochi and the author in [11](2014) and revisit Dube's collaborative work with Mugochi in [13](2015) on remote points. We also narrate Dube's prerequisite work on frame quotients in [3](2005) that is fundamental in grounding his joint work in [13](2015). We end with §3.5 that gives a brief account of a stronger version of clustering in pointfree form that Dube introduces with the author in addressing a pointfree version of a folklore result on convergence in spaces.

3.1 Filters and their balance Dube's initial imprint on convergence in pointfree topology was rather subtle, first appearing in his doctoral thesis [1](1992). Dube invents the notion of a *near* subset which inadvertently cloaks Hongs [30](1995) notion of convergence of filters in frames:

$$A \subseteq L \textit{ is } \text{near} \textit{ if } \forall\ B \in \operatorname{Cov} L\ \exists\ b \in B \textit{ such that } b \wedge a \neq 0\ \forall\ a \in A.$$

Equivalently,

$$A \subseteq L \textit{ is near iff } A \cap \sec B \neq \emptyset\ \forall\ B \in \operatorname{Cov} L.$$

Dube then shows, in his doctoral thesis [1](1992), that

$$A \subseteq L \textit{ is near iff } \{a^* \colon a \in A\} \notin \operatorname{Cov} L,$$

that is

$$A \textit{ is near iff } \alpha(A) \neq 1.$$

For $F \in \mathfrak{F}(L)$, this transcribes to $F \mathbin{\bigcirc\kern-0.75em{\multimap}} L$ iff F is near. Dube's original concept of a near subset actually delivers the pointfree notion of Hong's clustering of filters.

Dube's first impactful contribution to the theory of convergence in frames appears in Dube [2](2002) in which he characterizes normality in frames via the use of certain distinguished filters that he defines as the balanced and closed-generated ones. Moreover, Dube provides some of the essential classical theory on filters of point set topology in pointfree form that we highlight below. Stone's Separation Lemma is used to show (1) below.

Proposition 3.1. *Let $L \in \mathbf{Frm}$ and $F \in \mathfrak{F}(L)$.*
(1) *If $0 \neq x \in F$ then $\exists\, G \in \mathfrak{F}_{\mathfrak{p}}(L)$ such that $x \notin G$ and $F \subseteq G$.*
(2) $F = \bigcap\{P \in \mathfrak{F}_{\mathfrak{p}}(L) \colon F \subseteq P\}$.
(3) *$F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ iff $\forall\, x \in L$ either $x \in F$ or $x^* \in F$.*
(4) *$F \in \mathfrak{F}_{\mathfrak{p}}(L)$ iff F is maximal with respect to missing some $M \subseteq \mathfrak{d}L$ where M is closed under finite joins.*
(5) *$F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ iff F is maximal with respect to containing $\mathfrak{d}L$.*

Dube then defines his distinguished filters and ideals as follows.

Remark 3.2. Let $F \in \mathfrak{F}(L)$.

(a) $b(F) = \bigcap\{U \in \mathfrak{F}_{\mathfrak{ult}}(L) \colon F \subseteq U\} \in \mathfrak{F}(L)$ and $b(F)$ is called the *balance* of F. If $F = b(F)$ then F is said to be *balanced*. We denote the collection of all balanced filters in a frame L by $\mathfrak{F}_{\mathfrak{bal}}(L)$.

(b) If $I \in \mathfrak{I}(L)$ then $\gamma(I) = \{x \in L \colon \exists\, y \in I, x \vee y = 1\} \in \mathfrak{F}(L)$. If there is $J \in \mathfrak{I}(L)$ such that $F = \gamma(J)$ then F is called *closed-generated*.

(c) If $G \in \mathfrak{F}(L)$, then $\delta(G) = \{x \in L \colon \exists\, y \in G, x \wedge y = 0\} \in \mathfrak{I}(L)$. Given any $J \in \mathfrak{I}(L)$, if there is $F \in \mathfrak{F}(L)$ such that $J = \delta(F)$ then J is called *open-generated*.

(d) If $F = \gamma(\delta(F))$ then F is called *stably closed-generated*.

Using the above original and innovative pointfree concepts Dube discerns the following neat characterizations of balanced filters, ultrafilters and maximal ideals with elegant and succinct proofs.

Proposition 3.3. *Let $L \in \mathbf{Frm}$, $F \in \mathfrak{F}(L)$ and $J \in \mathfrak{I}(L)$. Then*

(1) *$F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ iff $F \in \mathfrak{F}_{\mathfrak{p}}(L) \cap \mathfrak{F}_{\mathfrak{bal}}(L)$ iff $L \backslash F \in \mathfrak{I}_{\mathfrak{p}_{\min}}(L)$ iff $\delta(F) = L \backslash F$.*

(2) *$F \in \mathfrak{F}_{\mathfrak{bal}}(L)$ iff $\mathfrak{d}L \subseteq F$.*

(3) $b(F) = \{x \in L \colon x^{**} \in F\} = \langle F \cup \mathfrak{d}L \rangle = \sec^2 F$.

(4) $\sec F = \bigcup\{U \in \mathfrak{F}_{\mathfrak{ult}}(L) \colon U \supseteq F\}$.

(5) *$J \in \mathfrak{I}_{\mathfrak{max}}(L)$ iff $\gamma(J) = L \backslash J$.*

Dube also presents a characterization of when $F \in \mathfrak{F}_{\mathfrak{reg}}(L)$ and this is precisely when F is stably closed-generated. Furthermore, he imparts the following:

For $L \in \mathbf{RegFrm}$, $\mathfrak{F}^{\mathsf{F}}_{\mathfrak{ult}}(L) \subseteq \mathfrak{F}_{\mathfrak{reg}}(L)$ *iff* $\mathfrak{F}^{\mathsf{F}}_{\mathfrak{p}}(L) \subseteq \mathfrak{F}_{\mathfrak{ult}}(L)$.

His main objective of the paper in characterizing normality in frames is disseminated using his prototype distinguished filters.

Proposition 3.4. [2, Proposition 9] *A frame is normal iff its closed-generated filters are precisely the stably closed-generated ones.*

3.2 Almost realcompactness and pseudocompactness Dube's subsequent exposé on convergence in pointfree topology is to be found in Dube [4](2006) in his creation of weaker forms of realcompactness. In this section we highlight Dube's conception of almost realcompactness and filter characterizations with pseudocompactness.

Remark 3.5. Let $L \in \mathbf{Frm}$ and S be any sublattice of L. Then $J \in \mathfrak{I}(S)$ is called

(a) *σ-proper* if $\bigvee_L T \neq 1$ for each $T \subseteq_\omega J$, and

(b) *completely proper* if $\bigvee_L J \neq 1$.

L is a *realcompact* frame if for any $J \in \mathfrak{I}_{\mathfrak{max}}(\mathrm{Coz}[L])$ that is σ-proper is completely proper (Banaschewski and Gilmour [19](2001)).

In Dube [4](2006) a weaker form of the Banaschewski and Gilmour notion of realcompactness is developed by Dube. By replacing the sublattice $\mathrm{Coz}[L]$ by ßL, the Booleanization of L, Dube defines a frame L to be *almost realcompact* if for any $J \in \mathfrak{I}_{\mathfrak{max}}(ßL)$ that is σ-proper is completely proper. Every realcompact frame is almost realcompact and if L is Boolean, then $\mathrm{Coz}[L] = ßL$ so that realcompactness and almost realcompactness coincide for Boolean frames. Dube defines $F \in \mathfrak{F}(L)$ to be *σ-fixed* if $\alpha(S) \neq 1$ for any $S \subseteq_\omega L$ and shows that almost realcompactness is characterized by the convergence and clustering of certain designated filters.

Proposition 3.6. [4, Proposition 3.4] *For* $L \in \mathbf{Frm}$ *the following are equivalent:*

(1) L *is almost realcompact.*

(2) $F \mathrel{\text{\textcircled{$\multimap$}}} L \ \forall\ F \in \mathfrak{F}(L)$ *where* $F \cap A \neq \emptyset \ \forall\ A \in \mathrm{Cov}_\omega L$.

(3) $F \mathrel{\text{\textcircled{$\rightarrow$}}} L \ \forall\ F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ *where* F *is σ-fixed.*

For a regular frame L, L is almost realcompact iff $F \textcircled{$\rightarrow$} L \; \forall \; F \in \mathfrak{F}(L)$ where $F \cap A \neq \emptyset \; \forall \; A \in \operatorname{Cov}_\omega L$.

Coincidently, a similar type result of the above proposition is achieved by Hong [30](1995) in characterizing almost compactness by filter convergence.

Proposition 3.7. [30, Corollary 1.4] *For $L \in \mathbf{Frm}$ the following are equivalent:*

(1) *L is almost compact.*

(2) $F \textcircled{$\multimap$} L \; \forall \; F \in \mathfrak{F}(L)$.

(3) $F \textcircled{$\rightarrow$} L \; \forall \; F \in \mathfrak{F}_{\mathfrak{ult}}(L)$.

For regular frames, almost compactness and compactness are equivalent so that the above proposition holds for regular frames with almost compact being replaced by compact (see Remark 2.7(3)). Lindelöf and almost compact frames are almost realcompact. Almost realcompactness is shown to be a conservative property in Dube [4, Corollary 3.9](2006).

Dube's next intervention in an internal characterization of a conservative property using convergence in frames appears in Dube and Matutu [5](2007). Here, pseudocompactness is given a filter characterization by the clustering of a distinguished filter base.

Proposition 3.8. [5, Proposition 4.1] *For any frame L, L is pseudocompact iff every countable completely regular filter base $F \textcircled{$\multimap$} L$.*

3.3 The Fomin extension and the Stone-Čech compactfication

Dube takes us on a frame-theoretic excursion of the Katětov and Fomin extensions in Dube [6](2007). We mainly unfold the Fomin extension that Dube establishes and purveys in [6]. The preceding paper is dedicated by Dube to the memory of the late Professor Sergio de Ornelas Salbany (1941-2005) who was an eminent topologist and colleague in the Department of Mathematical Sciences at the University of South Africa. Incidentally, in September 2012 Dube chaired the organising committee of an international conference at the University of South Africa on Topology, Algebra and Category Theory (TACT2012) that was in honour of Professor Sergio Salbany. Dube was one of the guest editors of the proceedings of the conference (see Dube, Naidoo and Brümmer [12]).

We recall that a topological space is called *H-closed* (*Hausdorff-closed*) if it is closed in any Hausdorff space in which it is embedded. An *extension* of a topological space X is a space Y in which X is a dense subspace of Y. In Katětov [34](1940) the author showed that every Hausdorff space X possesses an H-closed extension κX which has since been coined the *Katětov extension of* X. Katětov continued with his deliberations on κX and presented other analogous descriptions of κX in Katětov [35](1947). A general method of constructing extensions of topological spaces is prepensed in Fomin [26](1943). In the latter, Fomin also constructs another H-closed extension σX of a Hausdorff space X (benamed the *Fomin extension* of X). The Katětov and Fomin extensions of a Hausdorff space X are further investigated in Flachsmeyer [25](1966) and the author shows that $\kappa X = \sigma X$ iff $\kappa X \setminus X$ is finite.

The pointfree analogue, the Katětov H-closed extension of a frame, has been constructed using the collection of all free ultrafilters in a frame and have been studied by Paseka and Šmarda [41](1992), and independently by Hong [29](1992). In §2.2 we described the simple extension of a frame $(s_{\mathfrak{X}}L, s)$ determined by a set of filters $\mathfrak{X}$ given by Hong [30](1995) and the associated strict extension $(t_{\mathfrak{X}}L, t)$. For a $L \in \mathbf{Frm}$ and $a \in L$, we let

$$\mathfrak{F}^F_{\mathsf{ult}_a}(L) = \{G \in \mathfrak{F}^F_{\mathsf{ult}}(L) \colon a \in G\}$$

and consider the product frame $L \times \wp(\mathfrak{F}^F_{\mathsf{ult}_a}(L))$. Let

$$\kappa L = \{(a, \mathfrak{X}) \colon \mathfrak{X} \subseteq \mathfrak{F}^F_{\mathsf{ult}_a}(L)\}.$$

Then κL is a subframe of the product $L \times \wp(\mathfrak{F}^F_{\mathsf{ult}_a}(L))$ and the map $\kappa_L \colon \kappa L \to L$ defined by $\kappa_L(a, \mathfrak{X}) = a$ is a dense quotient. The simple extension $(\kappa L, \kappa_L)$ is called the *Katětov extension* of L. The right adjoint of the Katětov extension map is given by $(\kappa_L)_* \colon L \to \kappa L$ where $(\kappa_L)_*(a) = (a, \mathfrak{F}^F_{\mathsf{ult}_a}(L))$ for each $a \in L$.

Dube foregrounds the *Fomin extension* that also conscripts ultrafilters and convergence in frames. He defines the *Fomin extension* $(\sigma L, \sigma)$ of a frame L to be the associated strict extension of the Katětov extension $(\kappa L, \kappa_L)$ determined by $\mathfrak{F}^F_{\mathsf{ult}}(L)$. σL is the subframe of κL generated by $(\kappa_L)_*(L)$ and $\sigma_L \colon \sigma L \to L$ is the *Fomin* (*strict*) *extension map* given by the restriction of κ_L to σL. Dube mainly sets out to determine when σL

and βL are isomorphic for $L \in \mathbf{CRegFrm}$ and achieves this by various filter characterizations. He first shows the following discerning result on the Fomin extension for any $F \in \mathbf{Frm}$,

$$\sigma L \in \mathbf{RegFrm} \text{ iff } L \in \mathbf{RegFrm} \text{ and } \mathfrak{F}^{\digamma}_{\mathfrak{ult}}(L) \subseteq \mathfrak{F}_{\mathfrak{reg}}(L).$$

Dube then distinguishes certain types of covers in a frame. We denote

$$\mathrm{Cov}_p\, L = \{C \in \mathrm{Cov}\, L \colon \exists\, \mathrm{Cov}_q\, L \ni B \subseteq_{<\omega} C\}.$$

Dube calls any $C \in \mathrm{Cov}_p\, L$ a *p-cover* and presents the following equivalent conditions for regular frames.

Lemma 3.9. [6, Lemma 4.2] *For $L \in \mathbf{RegFrm}$ the following are equivalent:*

(1) $\mathfrak{F}^{\digamma}_{\mathfrak{ult}}(L) \subseteq \mathfrak{F}_{\mathfrak{reg}}(L)$.

(2) $\mathfrak{d}L \subseteq \mathfrak{cok}L$.

(3) *Every $C \in \mathrm{Cov}_p\, L$ has a finite subcover.*

Dube then establishes his main result on the Fomin extension and the Stone-Čech compactification of a completely regular frame.

Proposition 3.10. [6, Proposition 4.3] *For $L \in \mathbf{CRegFrm}$ the following are equivalent:*

(1) $\sigma L \simeq \beta L$.

(2) $\sigma L \in \mathbf{RegFrm}$.

(3) $\mathfrak{F}^{\digamma}_{\mathfrak{ult}}(L) \subseteq \mathfrak{F}_{\mathfrak{reg}}(L)$.

(4) $\mathfrak{F}^{\digamma}_{\mathfrak{p}}(L) \subseteq \mathfrak{F}_{\mathfrak{ult}}(L)$.

(5) $\mathfrak{d}L \subseteq \mathfrak{cok}L$.

(6) *Every $C \in \mathrm{Cov}_p\, L$ has a finite subcover.*

3.4 Čech-complete frames and remote points revisted Continuing with the quest in providing filter characterizations of conservative properties, we turn our attention to the joint work by Dube, Mugochi and Naidoo [11](2014) which we highlighted in Naidoo [39](2024) particularly for nearness frames. This collaborative paper would be the next formidable encounter with convergence by Dube jointly with Mugochi and the author in which they introduce a new type of filter and give an ultrafilter characterization of the conservative property of *Čech-completeness* for frames. We illuminate this next with some new notation.

Remark 3.11. Let $L \in \mathbf{Frm}$ and $\mathscr{C} \subseteq \operatorname{Cov} L$ be *any* collection of covers of L. Then:

(a) $F \in \mathfrak{F}(L)$ is called *$\mathscr{C}$-Cauchy* provided that $F \mathrel{\bigcirc\mkern-15mu\to}_{\mathscr{C}} L$ (F converges in L localised/relative/with respect to $\mathscr{C}$), meaning $F \cap C \neq \emptyset$ for every cover $C \in \mathscr{C}$. Notably, if $\mathfrak{F}(L) \ni G \supseteq F \mathrel{\bigcirc\mkern-15mu\to}_{\mathscr{C}} L$ then $G \mathrel{\bigcirc\mkern-15mu\to}_{\mathscr{C}} L$.

(b) The frame L is called *Čech-complete* (resp. *strongly Čech-complete*) if there is $\mathscr{C} \subseteq_{\omega} \operatorname{Cov} L$ such that for every $\mathfrak{F}(L) \ni F \mathrel{\bigcirc\mkern-15mu\to}_{\mathscr{C}} L$, $F \mathrel{\bigcirc\mkern-15mu\multimap} L$ (resp. $F \mathrel{\bigcirc\mkern-15mu\to} L$).

Strong Čech-completeness implies Čech-completeness. Furthermore, Čech-completeness is conservative for regular spaces and also has a characterization in terms of ultrafilter convergence.

Proposition 3.12. [11, Proposition 3.2] *A frame L is Čech-complete iff there is $\mathscr{C} \subseteq_{\omega} \operatorname{Cov} L$ such that for every $\mathfrak{F}_{\mathsf{ult}}(L) \ni F \mathrel{\bigcirc\mkern-15mu\to}_{\mathscr{C}} L$, $F \mathrel{\bigcirc\mkern-15mu\to} L$.*

Dube and his coauthors then indulge in the preservation and reflection of Čech-completeness and strong Čech-completeness under suitable morphisms between frames and other attributes of these two pointfree properties which the reader may further engage in [11].

Pursuant to Dube, Mugochi and Naidoo [6](2014), Dube's immediate succession with an amplified preoccupation with convergence and extensions in pointfree topology is a joint work with Mugochi on localic remote points in Dube and Mugochi [13](2015). The proclivity by the authors is on establishing criteria on when extensions have remote points. This predisposition appeals primarily to filters. The joint article evolved from Dube's original work in Dube [3, 7](2005, 2009). These earlier two papers also engage diversely with clustering of filters and conservative properties that lays the foundation for [13].

We next provide some of the data on onto frame homomorphisms and definitions that we require from Dube [3, 7](2005, 2009).

Remark 3.13. Let $h\colon L \to M$ be a quotient. Then h is:

(1) *bounded* (*bdd* for brevity) if $\forall\ C \in \operatorname{Cov} L\ \exists\ B \subseteq_{<\omega} C$ such that $h(B) \in \operatorname{Cov} M$,

(2) *almost bounded* (*a-bdd* for brevity) if $\forall\ C \in \operatorname{Cov} L\ \exists\ B \subseteq_{<\omega} C$ such that $h((\bigvee B)^*) = 0$,

(3) an *H-quotient* if $\forall\ C \subseteq L$ such that $h(C) \in \operatorname{Cov} L\ \exists\ B \subseteq_{<\omega} C$ such that $h((\bigvee B)^*) = 0$,

(4) *extension-closed* if $\forall\ C \in \operatorname{Cov} M\ \exists\ B \in \operatorname{Cov} L$ such that $h(B) = C$ iff $h_*(C) \in \operatorname{Cov} L\ \forall\ C \in \operatorname{Cov} M$,

(5) *nowhere dense* if $\forall\ 0 \neq x \in L\ \exists\ 0 \neq y \leq x$ such that $h(y) = 0$.

Nowhere dense quotients are characterized by the denseness of the image of the bottom under their right adjoints. From Dube [7, Lemma 3.2](2009) we have that

a quotient $h\colon L \to M$ *is nowhere dense iff* $h_*(0) \in \mathfrak{d}L$.

We next give some of the pertinent filter characterizations of the above concepts that are shown creatively by Dube.

Proposition 3.14. *Let* $h\colon L \to M$ *be a quotient.*

(1) *If* $L \in$ **RegFrm**, *then* h *is bdd iff* $\forall\ F \in \mathfrak{F}(M)$, $\mathfrak{F}(L) \ni h^{-1}(F) \ominus L$ *iff* h *is a-bdd.*

(2) h *is a-bdd iff* $\forall\ F \in \mathfrak{F}(L)$ *such that* $h(x) \neq 0\ \forall\ x \in F$, $F \ominus L$.

(3) *If* h *is dense then* h *is an* H*-quotient iff* $\forall\ F \in \mathfrak{F}(L)$, $h(F) \ominus M$.

The following results concern the closed quotients $\varphi_a\colon L \to {\uparrow} a$ at a for $a \in L$ where L is any frame.

Proposition 3.15. *Let* $a \in L$. *Then* φ_a *is*

(1) *a-bdd iff* $a \in F\ \forall\ F \in \mathfrak{F}^{F}(L)$.

(2) *nowhere dense iff* $a \in \mathfrak{d}L$.

(3) *a-bdd iff* ${\uparrow} a$ *is almost compact whenever* $a \in ßL$.

(4) *an H-quotient iff φ_a is bdd iff φ_a is a-bdd iff $\uparrow a$ is compact iff $\uparrow a$ is almost compact whenever $L \in$ **RegFrm**.*

By the results of Proposition 3.3, Dube shows how his above concepts of frame quotients can be neatly applied to characterize free prime filters as ultrafilters in a frame L.

Proposition 3.16. [3, Proposition 5.17] *$\mathfrak{F}^{F}_{\mathfrak{p}}(L) \subseteq \mathfrak{F}_{\mathfrak{ult}}(L)$ iff each closed nowhere dense quotient in L is a-bdd.*

Concerning the locally almost compact frames of Paseka and Šmarda [40](1988), Dube shows the following result for such frames.

Proposition 3.17. *If L is a nalco-frame that is locally almost compact then $\langle \mathfrak{coak} L\rangle = \bigcap\{F \subseteq L \colon F \in \mathfrak{F}^{F}(L)\} \in \mathfrak{F}^{F}(L)$.*

The pointfree form of the property of almost H-closedness is delivered by Dube:

A frame L is almost H-closed if $|\mathfrak{F}^{F}_{\mathfrak{ult}}(L)| \leq 1$.

H-closedness for frames is then characterized by almost compactness.

L is H-closed iff $\forall\ a, b \in L$ with $a \wedge b = 0$, at least one of the frames $\uparrow a^*$ or $\uparrow b^*$ is almost compact.

Dube then provides the characterization he coveted amongst nalco-frames.

Proposition 3.18. [3, Proposition 5.18] *For a nalco-frame L the following are equivalent:*

(1) *L is almost H-closed and every closed nowhere dense quotient of L is almost compact.*

(2) *L is almost H-closed and every closed nowhere dense quotient of L is a quotient of an almost compact quotient of L.*

(3) $|\mathfrak{F}^{F}_{\mathfrak{p}}(L)| = 1$.

(4) $|\mathfrak{F}^{F}(L)| = 1$

We now peruse the conservative notion of the remoteness of points of a frame extension of a given frame that is established in Dube and Mugochi [13](2015) and tweak some notation.

Remark 3.19. Let (M, h) be an extension of a frame L and $p \in \mathrm{Pt}(M)$. Then:

(1) *p is remote from L* if for every nowhere dense quotient $\eta\colon L \to N$ we have $h_*(\eta_*(0)) \not\leq p$. We denote the property *p is remote from L* by the notation $p \,\textcircled{\scriptsize$\ltimes$}_{\mathrm{Pt}(M)}\, L$ (for brevity $p \,\textcircled{\scriptsize$\ltimes$}\, L$). The collection of all points of M that are remote from L is designated by

$$\mathrm{Pt}(M \ltimes L) = \{p \in \mathrm{Pt}(M)\colon p \,\textcircled{\scriptsize$\ltimes$}\, L\}.$$

(2) Let $U^p = \{a \in L\colon h_*(a) \not\leq p\}$. Then $U^p \in \mathfrak{F}(L)$.

(3) Let $I_p = \{a \in L\colon h_*(a) \leq p\}$ so that $I_p = L \backslash U^p$. Then $I^p \in \mathfrak{J}(L)$.

(4) $p \,\textcircled{\scriptsize$\ltimes$}\, L$ iff $\forall\ d \in \mathfrak{d}L$, $h_*(d) \not\leq p$ iff U^p is saturated (see Dube and Mugochi [13, Proposition 3.2]).

Returning to the Katětov extension $(\kappa L, \kappa_L)$ of a frame L, where

$$\kappa L = \{(a, \mathfrak{X})\colon \mathfrak{X} \subseteq \mathfrak{F}^{F}_{\mathfrak{ult}_a}(L)\}$$

and $\kappa_L\colon \kappa L \to L$ given by $\kappa_L(a, \mathfrak{X}) = a$, we have that

$$\mathrm{Pt}(\kappa L) = \{(1, \mathfrak{X} \backslash \{F\})\colon F \in \mathfrak{X}\} \cup \{(p, \mathfrak{X})\colon p \in \mathrm{Pt}(L)\}$$

from Paseka and Šmarda [41, Proposition 3.9(2)](1992).

For each $F \in \mathfrak{X}$, $F \in \mathfrak{F}_{\mathfrak{ult}}(L)$ so that by Proposition 3.3 we have that $\mathfrak{d}L \subseteq F$. Thus for any $d \in \mathfrak{d}L$ and $F \in \mathfrak{X}$ we have that

$$(\kappa_L)_*(d) = (d, \mathfrak{F}^{F}_{\mathfrak{ult}_d}(L)) = (d, \mathfrak{X}) \not\leq (1, \mathfrak{X} \backslash \{F\}).$$

Thus, by Remark 3.19(4) above we have that

$$\mathrm{Pt}(\kappa L \ltimes L) = \{(1, \mathfrak{X} \backslash \{F\}\colon F \in \mathfrak{X}\}$$

The above description of the remote points of the Katětov extension is a delightful example given by Dube and Mugochi in [13] that articulates the cohesion of Dube's contributions and finesse in pointfree topology.

Dube and Mugochi then proceed to analyse remote points in perfect extensions and in so doing they introduce dual balanced filters, the balanced ideals. In a frame L,

$$J \in \mathfrak{I}(L) \text{ is balanced if for any } x \in L,\ x^{**} \in J \text{ whenever } x \in J.$$

We denote $\mathfrak{I}_{\mathfrak{bal}}(L)$ the collection of all balanced ideals of L. Given any perfect extension (M, h) of a frame L the elements of $\mathrm{Pt}(M \ltimes L)$ characterized by ultrafilters and ideals are distilled from [13, Proposition 5.1 & 5.2] and given below. The filter $F_p = \{x \in M : x \leq p\}$ is discussed in Remark 2.5(3).

Proposition 3.20. *If (M, h) is a perfect extension of a frame L and $p \in \mathrm{Pt}(M)$ then the following are equivalent:*

(1) $p \circledast L$.

(2) $U^p \in \mathfrak{F}_{\mathfrak{ult}}(L)$.

(3) $I_p \in \mathfrak{I}_{\mathfrak{p}_{\min}}(L)$.

(4) $I_p \in \mathfrak{I}_{\mathfrak{bal}}(L)$.

(5) $h(F_p) \in \mathfrak{F}_{\mathfrak{ult}}(L)$.

Perquisites of the above result are then shared by Dube and his coauthor:

1. If (M, h) is a perfect extension of a frame L then $\mathrm{Pt}(M \ltimes L) \neq \emptyset$ iff $\exists$ $F \in \mathfrak{F}_{\mathfrak{cp}}(M)$ such that $h(F) \in \mathfrak{F}_{\mathfrak{ult}}(L)$.
2. $\mathrm{Pt}(M \ltimes L) \simeq \{F \in \mathfrak{F}_{\mathfrak{cp}}(M) : h(F) \in \mathfrak{F}_{\mathfrak{ult}}(L)\}$.

A fitting conclusion to Dube's joint study on remote points is given in the determination of when an extension (M, h) of a frame L that is spatial over L is perfect. The latter invites disjoint-prime filters and trace filters as solvents that is passed over for the reader's satiety.

3.5 Strong clustering We annexe a quick look into pointfree clustering that is pursued by Dube and the author in [14](2015). It is folklore that *a filter in a topological space clusters iff it is contained in a convergent filter.* However, this result does not hold in the category of frames as illustrated in Dube and Naidoo [14, Example 6.1](2015).

Dube, together with the author, examines this primordial result innate to clustered filters in spaces in pointfree form in [14]. They introduce a stronger notion of clustering of filters in frames to ameliorate this limitation in the covering interpretation of clustering:

> *A filter F in a frame L is strongly clustered or strongly clusters in L if there is $p \in \mathrm{Pt}(L)$ such that $\alpha(F) \leq p$.*

Remark 3.21. For a frame L and $F \in \mathfrak{F}(L)$ we note the following.
(1) $F \odot L$ denotes that F strongly clusters in L.
(2) If $F \odot L$ then $F \circledcirc L$.
(3) If $\mathfrak{F}(L) \ni G \subseteq F \odot L$ then $G \odot L$.
(4) If $L \in$ **RegFrm** and $F \ominus L$ then $F \odot L$.

The idea of strong clustering in frames developed jointly by Dube provides an adequate curation of the result in spaces in pointfree form.

Proposition 3.22. [14, Proposition 6.4] *Let $L \in$ **RegFrm** and consider any $F \in \mathfrak{F}(L)$. Then $F \odot L$ iff $\exists\ G \in \mathfrak{F}(L)$ such that $F \subseteq G \ominus L$.*

Furthermore, the two conceptualizations of clustering coincide for regular frames under spatiality.

Proposition 3.23. [14, Proposition 6.5] *Let $L \in$ **RegFrm**. For each $\mathfrak{F}(L) \ni F \circledcirc L$, $F \odot L$ iff L is spatial.*

4 Convergence in Locales

In this section, $L \in$ **Loc** (unless stated otherwise) and we refer to §2.4 for the relevant abridgement of sublocales. We refer to Dube and Ighedo [15](2016) for a comprehensive treatment of our précis in this section. We will retain our notations for the specific class of filters on a locale (see §2.3) and also the notations for the convergence and clustering of filters with appropriate modification as required.

4.1 Convergence of filters on a locale Given any sublattice $\mathscr{A}$ of $\mathcal{S}\ell(L)$, we let $\mathfrak{F}(\mathscr{A}) = \{\mathscr{F} : \mathscr{F}$ is a proper filter in $\mathscr{A}\}$ and we note that

$\mathsf{O} \notin \mathcal{F}$ for each $\mathcal{F} \in \mathfrak{F}(\mathcal{A})$. For any $\mathcal{F} \in \mathfrak{F}(\mathcal{A})$, $\mathcal{F}$ is called an *$\mathcal{A}$-filter on L.*

The lattice of all open sublocales of L induced by the elements of L is denoted

$$\mathcal{O}(L) = \{\mathfrak{o}(a) \colon a \in L\}.$$

We symbolize

$$\mathcal{C}_{coz}(L) = \{\mathfrak{c}(a) \colon a \in \mathrm{Coz}[L]\}$$

for the lattice of all closed sublocales induced by cozero elements of L.

If $p \in \mathrm{Pt}(L)$ and $\mathcal{F} \in \mathfrak{F}(\mathcal{A})$ for any sublattice $\mathcal{A}$ of $\mathcal{S}\ell(L)$, then

1. a *neighbourhood* (for brevity *nhood*) of p is any member of $\mathcal{O}(L)$ that contains p and
2. $\mathcal{F}$ *converges to* p, which we write as $\mathcal{F} \ominus p$, if every nhood of p contains a member of $\mathcal{F}$. The point p is a *limit* of $\mathcal{F}$.

For a frame L and $F \in \mathfrak{F}(L)$ we denoted $F \ominus L$ (and merely say F *converges*) for the convergence of F in L in the Banaschewsk and Pultr [23](1990) sense as well as that of Hong [30](1995). For a locale L and any $\mathcal{A}$-filter $\mathcal{F}$ on L, we will write $\mathcal{F} \ominus L$ (and say $\mathcal{F}$ *converges*) to mean that $\exists\ p \in \mathrm{Pt}(L)$ such that $\mathcal{F} \ominus p$. The notion of $\mathcal{F} \ominus L$ for a locale L is shown by Dube and Ighedo to generalize that of the Banschewski and Putr completely prime filter notion in [23](1990) for T_1-locales defined by Rosický and Šmarda [45](1985). Here, for a filter F in a frame L they construct a filter $\mathcal{F}$ on L induced by F of the type (1), (2) and (3) in the Remark below and show that $F \ominus L$ iff $\mathcal{F} \ominus L$ for T_1-locales.

Remark 4.1. Given a specific sublattice $\mathcal{A}$ of sublocales, Dube and Ighedo provide the following nomenclature (with justification) for the corresponding $\mathcal{A}$-filters on the locale L.

(1) $\mathcal{F} \in \mathfrak{F}(\mathcal{S}\ell(L))$ is a *filter on L.*
(2) $\mathcal{F} \in \mathfrak{F}(\mathcal{S}\ell_c(L))$ is a *Cartan filter on L.*
(3) $\mathcal{F} \in \mathfrak{F}(\mathcal{O}(L))$ is an *open filter on L.*
(4) $\mathcal{F} \in \mathfrak{F}(\mathcal{C}_{coz}(L))$ is a *z-filter on L.*

In particular for Hausdorff locales the following is disseminated.

Proposition 4.2. *Let L be a Hausdorff locale and $\mathcal{A}$ be a sublattice of $\mathcal{S}\ell(L)$.*

(1) *If $\mathfrak{F}(\mathcal{A}) \ni \mathcal{F} \ominus p \in \mathrm{Pt}(L)$, then $\bigcap\{\overline{F}\colon F \in \mathcal{F}\} = \{p, 1\}$.*
(2) *Limits are unique in L.*

The converse of (2) remains an open question. Next, for a given sublocale $S \subseteq L$ convergence of an $\mathcal{A}$-filter on S to a point in L is defined:

> *Let $S \in \mathcal{S}\ell(L)$. If $\mathcal{F}$ is a filter (or Cartan filter) on S, then $\mathcal{F} \ominus p \in \mathrm{Pt}(L)$ if $\forall\ a \in L$ with $p \in \mathfrak{o}(a)$ $\exists\ F \in \mathcal{F}$ such that $F \subseteq \mathfrak{o}(a)$.*

If L is a frame and (M, h) is an extension of L, then $h_*(L)$ is a dense sublocale of L. The map $h_*\colon L \to h_*(L)$ is a frame isomorphism with inverse $h_*^{-1}\colon h_*(L) \to L$ mapping as h. With the above expression of convergence of an $\mathcal{A}$-filter on a sublocale converging to a point of L, it is observed that for any frame L

> *if (M, h) is an extension of L, then for any filter or Cartan filter $\mathcal{F}$ on the sublocale $h_*(L)$, $\mathcal{F} \ominus p \in \mathrm{Pt}(L)$ iff $\forall\ a \in M$ with $p \in \mathfrak{o}(a)$, $\mathfrak{o}_{h_*(L)}(h_*h(a)) \in \mathcal{F}$.*

This then motivates a dual notion of convergence of ideals that we watershed in the next section that Dube and Ighedo define and investigate.

4.2 Coconvergence of ideals in a frame

Let $L \in \mathbf{Frm}$ and (M, h) be an extension of L with $\mathrm{Pt}(M) \neq \emptyset$. Let $J \in \mathfrak{I}(L)$ (or $J \in \mathfrak{I}(\mathrm{Coz}[L])$) be proper. Conconvergence of the ideal J is then defined:

> *J coconverges to $p \in \mathrm{Pt}(M)$ if, for every $m \in M$ with $m \vee p = 1$, there is $u \in J$ such that $h(m) \vee u = 1$.*

We introduce the notation $J \circledcirc p$ to denote that the ideal J coconverges to p. Now if $\mathcal{F}$ is filter on the sublocale $h_*(L)$, then

$$J_{\mathcal{F}} = \{a \in L\colon \mathfrak{c}_{h_*(L)}(h_*(a)) \in \mathcal{F}\}$$

is a proper ideal in L, and if J is an ideal in L then

$$\mathcal{F}_J = \{S \in \mathcal{S}\ell(h_*(L))\colon S \supseteq \mathfrak{c}_{h_*(L)}(h_*(a)) \text{ for some } a \in J\}$$

is a filter on $h_*(L)$. Furthermore, $J_{\mathcal{F}_J} = J$ and $\mathcal{F}_{J_{\mathcal{F}}} \subseteq \mathcal{F}$. Considering the poset $(\mathfrak{F}(h_*(L)), \subseteq)$ and the frame of ideals of L the authors exhibit an adjunction

$$\mathfrak{I}(L) \underset{J_{\mathcal{F}} \leftarrow\!\shortmid \mathcal{F}}{\overset{J \mapsto \mathcal{F}_J}{\rightleftarrows}} \mathfrak{F}(h_*(L))$$

and show the following for specific extensions:

1. If (M, h) is a T_1-extension of L, $J \in \mathfrak{I}(L)$ and $p \in \mathrm{Pt}(M)$, then $J \textcircled{$\looparrowright$} p$ iff $\mathcal{F}_J \textcircled{$\rightarrow$} p$.
2. If (M, h) is a regular-extension of L, $\mathcal{F} \in \mathfrak{F}(h_*(L))$ and $p \in \mathrm{Pt}(M)$, then $J_{\mathcal{F}} \textcircled{$\looparrowright$} p$ iff $\mathcal{F} \textcircled{$\rightarrow$} p$.
3. If (M, h) is a compact-Hausdorff-extension of L and J is an ideal of L then $J \textcircled{$\looparrowright$} p \in \mathrm{Pt}(M)$ iff $\bigvee\limits_{u \in J} h_*(u) = p$.
4. If $L \in \mathbf{CRegFrm}$, $I \in \mathrm{Pt}(\beta L)$ and J is a proper ideal of L, then $J \textcircled{$\looparrowright$} I$ iff $J \supseteq I$.

Dube together with his coauthor then apply the above theory that they have developed in the category **CRegFrm** and successfully characterize the points of βL in terms of filter characterizations and the notion of a *sharp point* of βL that Dube introduced jointly in Dube and Matlabyana [10, Definition 4.2](2013):

$I \in \mathrm{Pt}(\beta L)$ is sharp if for any $c \in \mathrm{Coz}[L]$, $(\beta_L)_*(c) \subseteq I$ implies $c \in I$.

Recall that $\beta_L \colon \beta L \to L$ is the map given by join and the right adjoint is the map $(\beta_L)_* \colon L \to \beta L$ given by $(\beta_L)_*(a) = \{x \in L \colon x \prec\!\prec a\}$ for each $a \in L$. Their main result follows.

Proposition 4.3. [15, Proposition 3.2] *For $I \in \mathrm{Pt}(\beta L)$ the following are equivalent:*

(1) *I is a sharp point.*

(2) *$\mathbf{A}^I \subseteq \mathcal{F}$ for every prime filter $\mathcal{F}$ on $(\beta_L)_*(L)$ such that $\mathcal{F} \textcircled{$\rightarrow$} I$.*

(3) $\mathbf{A}^I \subseteq \mathcal{F}$ *for every ultrafilter* $\mathcal{F}$ *on* $(\beta_L)_*(L)$ *such that* $\mathcal{F} \ominus I$.

(4) $\mathbf{A}^I \subseteq \mathcal{F}$ *for every Cartan ultrafilter* $\mathcal{F}$ *on* $(\beta_L)_*(L)$ *such that* $\mathcal{F} \ominus I$.

In the above Proposition, for $I \in \mathrm{Pt}(\beta L)$

$$\mathbf{A}^I = \{\mathfrak{c}_{(\beta_L)_*(L)}((\beta_L)_*(c))\colon c \in \mathbf{U}^I\}.$$

and

$$\mathbf{U}^I = \{c \in \mathrm{Coz}[L]\colon (\beta_L)_*(c) \subseteq I\}.$$

4.3 Clustering of filters on a locale In this section we advocate the machinery developed by Dube and Ighedo in which they proffer for locales similar filter characterizations of compactness that Hong [30](1995) establishes using his covering approach to convergence and clustering of filters in frames. To this end they introduce the concept of *clustering of a filter on a locale* as follows: Let $L \in \mathbf{Loc}$, $S \in \mathcal{S}\ell(L)$ and $\mathcal{F}$ be a filter or Cartan filter on S. Then

> $\mathcal{F}$ *clusters at* $p \in \mathrm{Pt}(L)$ *if every nhood of* p *meets every member of* $\mathcal{F}$*, that is for every* $a \in L$ *with* $p \in \mathfrak{o}(a)$, $\mathfrak{o}(a) \cap F \neq \mathsf{O}$ *for every* $F \in \mathcal{F}$.

We retain the notation and write $\mathcal{F} \circleddash p$ whenever $\mathcal{F}$ clusters at p for a filter $\mathcal{F}$ on a locale L and $p \in \mathrm{Pt}(L)$. It is then observed that

1. $\mathcal{F} \circleddash p$ iff $p \in \bigcap_{F \in \mathcal{F}} \overline{F}$ and
2. if $\mathcal{F} \ominus p$ then $\mathcal{F} \circleddash p$.

Compact locales are then characterized by the convergence and clustering of filters on the locale reminiscent of those of Hong [30](1995) for frames.

Proposition 4.4. [15, Proposition 4.2] *For any locale* L *the following are equivalent:*

(1) L *is compact.*

(2) *Every filter on* L *clusters.*

(3) *Every Cartan filter on* L *clusters.*

(4) *Every prime filter on L clusters.*

(5) *Every ultrafilter on L clusters.*

Dube and Ighedo also introduce *coclustering* of ideals in frames in parallel to conconvergence:

> *Let (M, h) be an extension of L and $J \in \mathfrak{I}(L)$. J coclusters at a point $p \in \mathrm{Pt}(M)$ if $\bigvee_{u \in J} h_*(u) = p$.*

They conclude their study on providing equivalent conditions for a locale to be compact in terms of coconvergence and coclustering of ideals:

> $L \in$ **Loc** *is compact iff every ideal in L coclusters iff every prime ideal in L coconverges iff every maximal ideal in L coconverges.*

5 Celebratory tributes

The following personal tributes are conveyed to Themba Dube in celebration of his lifetime mathematical achievements on the occasion of his $\geq 65^{\text{th}}$ birthday and retirement.

5.1 Dr. Joanne Walters-Wayland (Jo)

Jo Walters-Wayland is a member of the Center of Excellence in Computation, Algebra and Topology (CECAT) at Chapman University, USA.

A magnetic personality, a booming voice and a dignified presence - not attributes one usually associates with an accomplished mathematician, and yet those are not even all that are immediate when meeting Professor Themba Dube.

I had the privilege to meet Themba many decades ago when we were both grad. students - he attended a conference at the University of Cape Town - I was drawn to the attributes mentioned above and we shared an obsession and love for frame theory; none of which has changed all these years later.

Unfortunately, we had less opportunity to collaborate than I would have liked as my home ended up being in California. However I am very proud of the papers we did co-author and I always enjoyed his company (and that of his students) as the numerous international conferences we both attended.

Themba has always been an inspiration to me, and to this day, whenever I am asked, or wonder about, an idea related to my area of frame theory, I always check as to whether Themba Dube or Bernhard Banaschewski have already written about it - and very often they have!

Ngiyabonga Themba!

Jo

5.2 Prof Papiya Bhattacharjee

Papiya Bhattacharjee is currently the Assistant Chair of the Department of Mathematics and Statistics at the Florida Atlantic University, Charles E. Schmidt College of Science, Boca Raton, and a former Associate Professor in the School of Science (Mathematics) at the Pennsylvania State University, Erie, The Behrend College, USA.

I met Professor Themba Dube for the first time at the University of Florida, in the United States, while we were attending the conference on Ordered Algebraic Structures (OAL), in March 2012. Professor Dube travelled to the USA with Professor Naidoo to attend the conference. During my days as a graduate student, studying the theory of frames, I became familiar with Professor Dube's work through his papers. In 2012, when I first met him in person, I was star-struck! I soon found out that it was extremely easy to have conversations with Professor Dube and Professor Naidoo; hence began our mathematical collaborations. Our research work continued in the following years, leading to two published articles in frame theory with Professor Dube.

Soon after, I was invited to attend the TACT 2012 Conference at University of South Africa in Pretoria, in September of 2012. In the years 2013 and 2016, I hosted and organized the OAL13 and OAL16 conferences at Penn State Behrend, in the United States. It was my great pleasure that Professor Dube accepted and attended both conferences and presented his beautiful mathematical work. In 2014, I was both pleasantly surprised and honored when I received an invitation from the Topology Research Group at UNISA to visit and continue my collaborative work with Professor Dube.

I must admit that the back and forth visitation and collaboration with Professor Dube and Professor Naidoo, between the United States and South Africa, has created fond memories that I will cherish forever.

I admire Professor Themba Dube and have extreme respect for him. I enjoy discussing mathematics with him greatly and I strongly hope that we will continue our mathematical collaborations in the future. I wish Professor Dube all the best in his future endeavors.

5.3 Prof Oghenetega Ighedo (Tega)

Tega Ighedo is currently an Associate Professor at the Center of Excellence in Computation, Algebra and Topology (CECAT) at Chapman University, USA. She was the first female doctoral student of Themba Dube, and graduated as the first African black woman to receive her PhD in pure mathematics in the 140 year history of the University of South Africa. Tega was also a former Associate Professor in the Department of Mathematical Sciences at Unisa.

Good day everyone. I also want to join the list of great friends and colleagues paying tribute to Professor Themba Dube.

Deciding to pursue a PhD degree at the University of South Africa and having Themba Dube as my advisor is one of the best decisions I have made so far in my academic career. He is an excellent teacher, an outstanding advisor and a superb mentor. He is fully committed to pushing his students up to the level he wants them to be.

He is one person I know that no place is too sacred for discussing mathematics, always having a pen lurking around somewhere on him. Paper napkins at restaurants become writing pads, and university shuttles between the Pretoria campus and the Florida campus on Unisa become lecture halls for discussing the research topics one is working on.

An incident that I will never forget occurred on one of our trips to a conference. Professor Themba Dube, Dr. Jissy Nsonde-Nsayi (He was still a graduate student then, a brilliant student under the supervision of Themba Dube) and I were travelling together for a conference. We arrived at the O.R. Tambo International Airport, Johannesburg, South Africa early enough and checked in our luggage. We were waiting to board, and I was sitting next to Professor Themba Dube. I recall him looking around and asking where Jissy was. I looked and beheld Jissy sitting very far away. We called him to come over. He came over and we requested him to sit with us. He bluntly said no, he would not sit with us. We asked why? He said: "Because Professor Dube is going to ask me about the work he gave me to do, which of course I am busy with and will show him when we get to our destination and will not discuss this here". He left us and went to sit by himself. I looked at Professor Dube and we burst into laughter.

His culture of hard work and passion to share his knowledge became a huge benefit to a lot of us. I would go with him from place to place, country to country in the African continent to recruit students for graduate studies in pointfree topology. He is a great pure mathematician with a huge sense of humour. He is also an excellent presenter of research findings, making everyone listening to him feel as if they are experts in what he is presenting; a quality which is rare in presenting findings in some areas of pure mathematics. We are grateful for his huge contribution to the area of pointfree topology.

I wish him all the good things that come with retirement, and I hope he will let us keep tapping into his huge wealth of knowledge from time to time.

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References

[1] Dube, T., "Structures in Frames", Ph.D. Thesis, University of Durban-Westville, 1992.

[2] Dube, T., *Balanced and closed-generated filters in frames*, Quaest. Math. 25 (2002), 73-81.

[3] Dube, T., *Bounded quotients of frames*, Quaest. Math. 28 (2005), 55-72.

[4] Dube, T., *An algebraic view of weaker forms of realcompactness*, Algebra Universalis 55 (2006), 187-202.

[5] Dube, T., and Matutu, P., *Pointfree pseudocompactness revisited*, Topology Appl. 154 (2007), 2056-2062.

[6] Dube, T., *Katetov revisited: a frame-theoretic excursion*, Quaest. Math. 30 (2007), 365-380.

[7] Dube, T., *Remote points and the like in pointfree topology*, Acta Math. Hungar. 123(3) (2009), 203-222.

[8] Dube, T., *Concerning P-frames, essential P-frames, and strongly zero-dimensional frames*, Algebra Universalis 61(1) (2009), 115-138.

[9] Dube, T., *A broader view of the almost Lindelof property*, Algebra Universalis 65 (2011), 263-276.

[10] Dube, T. and Matlabyana, M., *Cozero complemented frames*, Topology Appl. 16 (2013), 1345-1352.

[11] Dube, T., Mugochi, M.M., and Naidoo, I., *Čech-completeness in pointfree topology*, Quaest. Math. 37(1) (2014), 49-65.

[12] Dube, T., Naidoo, I., and Brümmer, G.C.L.B., Guest Editors, In: Topology Appl. (*Special Issue: TACT2012*), 168 (2014).

[13] Dube, T. and Mugochi, M.M., *Localic remote points revisited*, Filomat 29(1) (2015), 111-120.

[14] Dube, T. and Naidoo, I., *More on uniform paracompactness in pointfree topology*, Math. Slovaca 65 (2015), 273-288.

[15] Dube, T. and Ighedo, O., *Characterising points which make P-frames*, Topology Appl. 200 (2016), 146-159.

[16] Baboolal, D., *Perfect compactifications of frames*, Czechoslovak Math. J. 61(3) (2011), 845-861.

[17] Ball, R.N. and Walters-Wayland, J., "C- and C^*-quotients in pointfree topology", Dissertationes Math. 412(412) (2002), 1-62.

[18] Banaschewski, B. and Gilmour, C., *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. 37(3) (1990), 577-587.

[19] Banaschewski, B. and Gilmour, C., *Realcompactness and the cozero part of a frame*, Appl. Categ. Structures 9 (2001), 395-417.

[20] Banaschewski, B. and Hong, S.S., *Filters and strict extensions of frames*, Kyungpook Math. J. 39(1) (1999), 215-230.

[21] Banaschewski, B. and Hong, S.S., *Extension by continuity in pointfree topology*, Appl. Categ. Structures 8 (2000), 475-486.

[22] Banaschewski, B. and Mulvey, C., *Stone-Čech compactification of locales I*, Houston J. Math. 6(3) (1980), 301-312.

[23] Banaschewski, B. and Pultr, A., *Samuel compactification and completion of uniform frames*, Math. Proc. Cambridge Philos. Soc. 108 (1990), 63-78.

[24] Carlson, N.A. and Porter, J.R., *Open ultrafilters and maximal points*, Topology Appl. 156 (2009), 2317-2325.

[25] Flachsmeyer, J., *Zur theorie derH-abgeschlossen Erweiterungen*, Math. Z. 94 (1966), 349-381.

[26] Fomin, S., *Extensions of topological spaces*, Ann. Math. 44(3) (1943), 471-480.

[27] Gilmour, C.R.A., *Realcompact spaces and regular σ-frames*, Math. Proc. Cambridge Philos. Soc. 96 (1984), 73-79.

[28] Grätzer, G., "Lattice Theory: Foundation", Birkhäuser, Springer-Basel AG, 2011.

[29] Hong, S.S., *Simple extension of frames*, Math. Res. 67 (1992), 156-159.

[30] Hong, S.S., *Convergence in frames*, Kyungpook Math. J. 35 (1995), 85-93.

[31] Isbell, J., *First steps in descriptive theory of locales*, Trans. Amer. Math. Soc. 327(1) (1991), 353-371.

[32] Johnstone, P.T., "Stone Spaces", Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, 1982.

[33] Johnstone, P. and Shu-Hao, S., *Weak products and Hausdorff locales*, In: Borceux, F. (Eds) Categorical Algebra and its Applications, Lect. Notes Math. 1348 (1988), 173-193.

[34] Katětov, M., *Über H-abgeschlossene und bikompakte Raüme*, Časopis pro Pěstování Matematiky a Fysiky 69(2) (1940), 36-49.

[35] Katětov, M., *H-closed extensions of topological spaces*, Časopis pro Pěstování Matematiky a Fysiky 72(1) (1947), 17-32.

[36] Madden, J. and Vermeer, H., *Lindelöf locales and realcompactness*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 472-480.

[37] Naidoo, I., *Strong Cauchy completeness in uniform frames*, Acta Math. Hungar. 116(3) (2007), 273-284.

[38] Naidoo, I., *An Interview with Themba Andrew Dube* (*A TAD Interview*), Categ. Gen. Algebr. Struct. Appl. 20(1) (2024), 5-32.

[39] Naidoo, I., *Celebrating Professor Themba A. Dube* (*A TAD Celebration I*), Categ. Gen. Algebr. Struct. Appl. 20(1) (2024), 33-103.

[40] Paseka, J. and Šmarda, B., *On some notions related to compactness for locales*, Acta Univ. Caroli. Math. Phys. 29(2) (1988), 51-65.

[41] Paseka, J. and Šmarda, B., *T_2-frames and almost compact frames*, Czechoslovak Math. J. 42(3) (1992), 385-402.

[42] Picado, J. and Pultr, A., "Frames and Locales: Topology without Points", Frontiers in Mathematics, Birkhäuser/Springer Basel AG, 2012.

[43] Plewe, T., *Quotient Maps of Locales*, Appl. Categ. Structures 8 (2000), 17-44.

[44] Reynolds, G., *Alexandroff Algebras and complete regularity*, Proc. Amer. Math. Soc.76 (1979), 322-326.

[45] Rosický, J. and Šmarda, B., *T_1-locales*, Math. Proc. Cambridge Philos. Soc. 98(1) (1985), 81-86.

[46] Walters, J.L., "Uniform Sigma Frames and the Cozero Part of Uniform Frames", Masters Dissertation, University of Cape Town, 1990.

[47] Walters, J.L., *Compactifications and uniformities on σ-frames*, Comment. Math. Univ. Carolin. 32(1) (1991), 189-198.

[48] Zhang, F.L. and Strauss, D., *Katětov-extension of a Frame*, In: *Festschrift*, Proceedings, Symposium on Categorical Topology, Banaschewski, B., Gilmour, G.R.A., and Herrlich, H. (Eds), University of Cape Town (1999), 261-271.

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