Categories and
General Algebraic Structures
with Applications
Volume 3, Number 1, July 2015, 65-88

WWW.CGASA.ir

ω-Operads of coendomorphisms and fractal ω-operads for higher structures

Camell Kachour

I dedicate this work to Michael Batanin.

Abstract. In this article we introduce the notion of *Fractal ω-operad* emerging from a natural ω-operad associated to any coglobular object in the category of higher operads in Batanin's sense, which in fact is a coendomorphism ω-operads. We have in mind coglobular object of higher operads which algebras are kind of higher transformations. It follows that this natural ω-operad acts on the globular object associated to these higher transformations. To construct the natural ω-operad we introduce some general technology and give meaning to saying an ω-operad possesses the *fractal property*. If an ω-operad B^0_P has this property then one can define a globular object of all higher B^0_P-transformations and show that the globular object has a B^0_P-algebra structure.

1 Introduction

This article is the first in a series of three articles (see [14, 15]). Here we present a notion of *fractal property* which may be possessed by an ω-operad in the sense of Batanin [2]. The property already exists in the simple case of the globular set of globular sets (see [14]). Behind the technology of *fractal ω-operads* of this article,

Keywords: Higher categories; higher operads; weak higher transformations.
Mathematics Subject Classification [2010]: 03B15, 03C85, 18A05, 18C20, 18D05, 18D50, 18G55, 55U35, 55U40.
Received: 4 May 2015, Accepted: 9 July 2015
ISSN Print: 2345-5853 Online: 2345-5861

we had in mind the desire to find an elegant way to prove and describe a globular approach of the weak ω-category of weak ω-categories. Actually the article [15] shows that the weak ω-category of weak ω-categories has a similar description up to the contractibility of specific higher operads. These main ideas were exposed for the first time in September 2010, in the Australian Category Seminar at Macquarie University [11, 16].

The main result of this article can be summarized as follows. Let $\mathbb{T}$-$\mathbb{C}AT_c$ be the category of $\mathbb{T}$-categories over constant globular sets; in other words, it is the category of coloured ω-operads over constant globular sets (see Section 2.3 and Section 3). We start with the basic data of a coglobular object $B^{\bullet}_P$ of ω-operads in $\mathbb{T}$-$\mathbb{C}AT_c$ equipped with a structure P. We say that the first ω-operad B^0_P (the "0-step") of this coglobular object $B^{\bullet}_P$ has the *fractal property*, provided there is a morphism of ω-operads between B^0_P and the corresponding ω-operad of coendomorphisms $Coend(B^{\bullet}_P)$ associated to $B^{\bullet}_P$. If this is the case, then all algebras for all ω-operads B^n_P ($n \in \mathbb{N}$) organise into a single algebra of B^0_P.

The Batanin theory of higher operads which we describe in the first section takes a large place of this article, and the new concept of *fractal ω-operad* we develop in Section 3, takes few pages compared to the first section where nothing are new in it. However this first section is needed to fully understand the new contributions of the Section 3, which is the core of our article. Applications of this concept of fractal ω-operad are given in the articles [14, 15].

Section 2 summarises Batanin's theory of higher operads (see [2]) with the goal of extracting Corollary 2.12, which is the basis for our article, and this corollary is just a consequence of Proposition 7.2 in [2]. The material surrounding Corollary 2.12 was described in [2]: globular categories, globular functors, monoidal globular categories (called MG-categories), monoidal globular functors (called MG-functors), augmented monoidal globular categories (called AMG-categories), globular objects of a globular category, and so on. However we expose these concepts using a more modern approach, essentially following Weber [23]. Then we explain in detail the two most important MG-categories for Batanin's theory of ω-operads: the MG-category $\mathbb{T}$ree of trees and the MG-category $\mathbb{S}pan$ of spans in Set (see 2.2), also described in modern terms by [3, 4, 22, 24]. In Section 2.3 we briefly describe $\mathbb{T}$-categories, where $\mathbb{T}$ is the monad for strict ω-categories on globular sets. $\mathbb{T}$-categories are important for this article because we take the view that an ω-operad in the sense of Batanin is a $\mathbb{T}$-category over the terminal globular set; see Section 2.4.

In Section 3, we state the main result of the article. Using Corollary 2.12 mentioned above, to each coglobular object $W^{\bullet}$ in $\mathbb{T}$-$\mathbb{C}AT_c$, we associate its *standard action in* $\mathbb{T}$-$\mathbb{C}AT_1$ which, roughly speaking, is a diagram in $\mathbb{T}$-$\mathbb{C}AT_1$ made of two morphisms of ω-operads. In particular each coglobular object $W^{\bullet}$ shows us two important ω-operads: the ω-operad W^0 (the "0-step" of the coglobular object $W^{\bullet}$), and the associated ω-operad

$$Coend(W) := (HOM(W^n, W^t))_{n\in\mathbb{N}, t\in\mathbb{T}ree}$$

of its coendomorphisms. The ω-operad W^0 is *fractal* provided there exists a morphism of ω-operads from it to $Coend(W)$.

The last section describes an important coglobular object in the category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ (see 2.3) which contains all combinatorics we need to build many kind of higher transformations as described in [14]. This coglobular object is denoted $C^\bullet$ and contains all kind of basic operations for higher transformations. It generates canonical standard actions which are useful in [14] for describing different relevant fractal ω-operads. The logical nature of this coglobular object $C^\bullet$ shows quickly that the corresponding free generated operads have finite ranks. In the article [15] we describe a very similar coglobular object which is more specifically adapted for building higher operads for strict and weak higher transformations (see also the last remark in the article [14]).

2 Batanin's theory of ω-Operads

Thoughout this paper, if $\mathbb{C}$ is a category then $\mathbb{C}(0)$ is the class of its objects (but we often omit "0" when there is no confusion) and $\mathbb{C}(1)$ is the class of its morphisms. The symbol := means "by definition is". Also Set denotes the category of sets, and SET denotes the category of large sets (for instance the proper class of ordinals is an object of SET, but not in Set). Similarly $\mathbb{C}at$ denotes the 2-category of small categories, and $\mathbb{C}AT$ denotes the 2-category of categories.

The theory of higher operads was developed for the first time by Michael Batanin in his seminal article [2]. What is more, he produced a theory of higher operads in the general context of his monoidal globular categories.

In this section, we summarise the general approach of the theory of higher operads of Michael Batanin, because it is in this general approach that the important Corollary 2.12 was formulated. This corollary is the key result to develop the main technology of this article. It is a result about the existence of the ω-operad of coendomorphisms which, as we will see, plays an important role for many kinds of *higher structure*. A *higher structure* for us means a structure based on globular sets. For instance, ω-magmas are a basic example of such higher structure, but we will consider also reflexive ω-magmas as an other kind of higher structure, and also other more complex higher structures such as the weak ω-categories.

2.1 MG-categories and AMG-categories

A lot of material which surrounds Corollary 2.12 is described in [2]: globular categories, globular functors, monoidal globular categories (called MG-categories), monoidal globular functors (called MG-functors), augmented monoidal globular categories (called AMG-categories), globular objects of a globular category, and so on.

Definition 2.1. The globe category $\mathbb{G}$ is defined as follows. For each $n \in \mathbb{N}$, objects of $\mathbb{G}$ are formal objects $\bar{n}$. Morphisms of $\mathbb{G}$ are generated by the (formal) cosource and cotarget $\bar{n} \underset{t_n^{n+1}}{\overset{s_n^{n+1}}{\rightrightarrows}} \overline{n+1}$ such that we have the relations $s_n^{n+1} s_{n-1}^n = s_n^{n+1} t_{n-1}^n$ and $t_n^{n+1} t_{n-1}^n = t_n^{n+1} s_{n-1}^n$. For each $0 \leqslant p < n$, we put $s_p^n := s_{n-1}^n \circ s_{n-2}^{n-1} \circ ... \circ s_p^{p+1}$ and $t_p^n := t_{n-1}^n \circ t_{n-2}^{n-1} \circ ... \circ t_p^{p+1}$

Definition 2.2. Starting with the globe category $\mathbb{G}$ above, we build the reflexive globe category $\mathbb{G}_\mathrm{r}$ as follow. For each $n \in \mathbb{N}$ we add in $\mathbb{G}$ the formal morphism $\overline{n+1} \xrightarrow{1_{n+1}^n} \bar{n}$ such that $1_{n+1}^n \circ s_n^{n+1} = 1_{n+1}^n \circ t_n^{n+1} = 1_{\bar{n}}$. For each $0 \leqslant p < n$, we put $1_n^p := 1_{p+1}^p \circ 1_{p+2}^{p+1} \circ ... \circ 1_n^{n-1}$

The category of globular sets is the category of presheaves ω-$\mathbb{G}\mathrm{r} := [\mathbb{G}^{op}; Set]$ (see for example [2]), the category of large globular sets is the category of presheaves ω-$\mathbb{G}\mathrm{R} := [\mathbb{G}^{op}; SET]$, and the 2-category of globular categories is the 2-category of prestacks $\mathbb{G}\mathbb{C}AT := [\mathbb{G}^{op}; \mathbb{C}AT]$.

Definition 2.3. Consider the terminal globular category 1 and a globular category $\mathcal{C}$. A globular object $(W, \mathcal{C})$ in $\mathcal{C}$ is a morphism $1 \xrightarrow{W} \mathcal{C}$ in $\mathbb{G}\mathbb{C}AT$.

Let us put ω-$\mathbb{G}\mathrm{rr} := [\mathbb{G}_\mathrm{r}^{op}; Set]$, the category of the reflexive globular sets (see [21]). We have the adjunction

$$U \dashv R : \omega\text{-}\mathbb{G}\mathrm{rr} \longrightarrow \omega\text{-}\mathbb{G}\mathrm{r}$$

and we write $(\mathbb{R}, \eta, \mu)$ for the generated monad whose algebras are reflexive globular sets. Objects of ω-$\mathbb{G}\mathrm{rr}$ are usually denoted by $(G, (1_n^p)_{0 \leqslant p < n}))$, where the operations $(1_n^p)_{0 \leqslant p < n}$ form a chosen reflexive structure on the globular set G.

Let us denote by ω-$\mathbb{C}at$ the category of strict ω-categories. The forgetful functor $\omega\text{-}\mathbb{C}at \xrightarrow{U} \omega\text{-}\mathbb{G}\mathrm{r}$, which associates to any strict ω-category C its underlying globular set $U(C)$, is monadic. The corresponding adjunction generates a cartesian monad $\mathbb{T}$ which is the monad of strict ω-categories on globular sets.

Consider $\mathbb{C}AT_{Pull}$, the 2-category of categories with pullbacks, with morphisms functors which preserve these pullbacks, and with 2-cells natural transformations between these functors. The functor $\mathbb{C}at(-)$ which associates to any object C in $\mathbb{C}AT_{Pull}$ the 2-category $\mathbb{C}at(C)$ of internal categories in it, is a 2-functor

$$\mathbb{C}AT_{Pull} \xrightarrow{\mathbb{C}at(-)} 2\text{-}\mathbb{C}AT$$

where 2-$\mathbb{C}AT$ denotes the 2-category of 2-categories. Thus for the case of the monad $\mathbb{T}$ on ω-$\mathbb{G}$r we can associate the 2-monad $\mathcal{T} = \mathbb{C}at(\mathbb{T})$ on $\mathbb{GC}AT$.

Definition 2.4 ([23])**.** An MG-category is a normal pseudo $\mathcal{T}$-algebra for the 2-monad $\mathcal{T}$ on $\mathbb{GC}AT$, MG-functors are strong $\mathcal{T}$-morphisms, and MG-natural transformations are algebra 2-cells of $\mathcal{T}$. These data form the 2-category $\mathbb{MGC}AT$ of MG-categories.

There is a coherence result in [2] that any MG-category is equivalent to a strict MG-category (a strict MG-category is just an internal strict ω-category in $\mathbb{C}AT$). Because of this coherence theorem we will not mention explicitly the coherence isomorphisms in the MG-categories which can be found in [2]. Also, the 2-category $\mathbb{MGC}AT$ has a cartesian monoidal structure, which allows us to make the following definition.

Definition 2.5 ([23])**.** An AMG-category is a pseudo monoid in $\mathbb{MGC}AT$. An AMG-functor is an MG-functor $f : A \longrightarrow A'$ equipped with a strong monoidal structure. An AMG-natural transformation $\phi : f \Longrightarrow f'$ is an MG-natural transformation such that ϕ is a monoidal 2-cell. These data form the 2-category $\mathbb{AMGC}AT$ of the AMG-categories.

2.2 Main examples of monoidal globular categories

Globular categories can be defined also as internal categories in ω-$\mathbb{G}$r because of the canonical isomorphism $\mathbb{C}at(\omega\text{-}\mathbb{G}\mathrm{r}) \simeq [\mathbb{G}^{op}; \mathbb{C}at]$. We will use this presentation to define the strict MG-category of n-trees as a discrete internal category

$$\begin{tikzcd}
\mathbb{T}(1) & \mathbb{T}(1) \arrow[l, shift right=1.5] \arrow[l, shift left=1.5] \arrow[r, shift left=3] \arrow[r, shift right=3] \arrow[l, shift left=0] & \mathbb{T}(1) \arrow[l, shift right=1.5] \arrow[l, shift left=1.5] \arrow[l, shift left=4.5] \arrow[l, shift right=4.5]
\end{tikzcd}$$

This MG-category $\mathbb{T}$ree has a canonical globular object given by the unit of $\mathbb{T}$: $1 \longrightarrow \mathbb{T}\mathrm{ree}$, $1(n) \longmapsto 1(n)$, where 1 denotes the terminal globular category, and $1(n)$[1] denotes the n-linear tree. It is shown in [3, 4] that it has the following universal property: if $\mathcal{C}$ is an MG-category and $(\mathcal{C}, W)$ is a globular object in it, then there is a unique, up to isomorphism, MG-functor $W(-)$ which makes commutative the following triangle.

$$\begin{tikzcd}
\mathbb{T}\mathrm{ree} \arrow[dr, dotted, "W(-)"] & \\
1 \arrow[u] \arrow[r, "W"'] & \mathcal{C}
\end{tikzcd}$$

[1] which is denoted by U_n in [2]

Let us set up some notation. Tensors of the monoidal globular category of n-trees are denoted by symbols $\star^n_p$.

$$\star^n_p : \mathbb{T}\mathrm{ree}_n \underset{\mathbb{T}\mathrm{ree}_p}{\times} \mathbb{T}\mathrm{ree}_n \longrightarrow \mathbb{T}\mathrm{ree}_n$$

Also an n-tree t is degenerate if it is of the form $t = Z^k_n(t')$ where t' is a k-tree such that $0 \leqslant k < n$. In [2] the author used the letter "Z" to express the reflexivity of an MG-category, but we prefer use the notation "1" to express these reflexivities for the specific case of n-trees, to emphasis that a degenerate tree $t = 1^k_n(t')$ is also an n-cell in the strict ω-category $\mathbb{T}(1)$. For example, for the n-linear tree $1(n)$, the $(n+1)$-tree $t = 1^n_{n+1}(1(n))$ of $\mathbb{T}(1)$ is degenerate.

Each n-tree t has a unique decomposition

$$1^{k_1}_{i_1}(1(k_1)) \star^{sup(i_1,i_2)}_{i'_1} 1^{k_2}_{i_2}(1(k_2)) \star^{sup(i_2,i_3)}_{i'_2} \ldots \star^{sup(i_{m-1},i_m)}_{i'_{m-1}} 1^{k_m}_{i_m}(1(k_m))$$

where for each $1 \leqslant j \leqslant m-1$, we have $i'_j < k_{j+1} \leqslant i_{j+1}$ and $i'_j < k_j \leqslant i_j$, and if $k_j = i_j$ by convention we put $1^{k_j}_{i_j}(1(k_j)) = 1(k_j)$. From this unique decomposition, the n-tree t can be represented by the matrix of numbers

$$\begin{pmatrix} i_1 & & i_2 & . & . & . & i_{m-1} & & i_m \\ & i'_1 & . & . & . & . & . & i'_{m-1} & \end{pmatrix}$$

which we call the *Grothendieck notation* for the n-tree t (see [1, 9, 20]). Many authors have given their own approach to n-trees (see for instance [2, 5, 8, 10, 19, 22]), and all these approaches are equivalent.

The second class of important examples of MG-category is given by the *Span and Cospan construction.* For each $n \in \mathbb{N}$, consider the following formal partially ordered set $Oct(n)$.

$$\begin{tikzcd}
 & \bullet_n \arrow[dl] \arrow[dr] & \\
\bullet^+_{n-1} \arrow[d] \arrow[drr] & & \bullet^-_{n-1} \arrow[d] \arrow[dll] \\
\bullet^+_{n-2} \arrow[d, dotted] & & \bullet^-_{n-2} \arrow[d, dotted] \\
\bullet^+_1 \arrow[d] \arrow[drr] & & \bullet^-_1 \arrow[d] \arrow[dll] \\
\bullet^+_0 & & \bullet^-_0
\end{tikzcd}$$

Let $Oct^+(n-1)$ be the poset obtained from $Oct(n)$ by removing $\bullet^-_{n-1}$ and $\bullet_n$. Similarly, let $Oct^-(n-1)$ be the poset obtained from $Oct(n)$ by removing $\bullet^+_{n-1}$ and $\bullet_n$.

$$\begin{tikzcd}
 & \bullet^+_{n-1} \arrow[dl] \arrow[dr] & \\
\bullet^+_{n-2} \arrow[d] \arrow[drr] & & \bullet^-_{n-2} \arrow[d] \arrow[dll] \\
\bullet^+_{n-3} \arrow[d, dotted] & & \bullet^-_{n-3} \arrow[d, dotted] \\
\bullet^+_1 \arrow[d] \arrow[drr] & & \bullet^-_1 \arrow[d] \arrow[dll] \\
\bullet^+_0 & & A^-_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet^-_{n-1} \arrow[dl] \arrow[dr] & \\
\bullet^+_{n-2} \arrow[d] \arrow[drr] & & \bullet^-_{n-2} \arrow[d] \arrow[dll] \\
\bullet^+_{n-3} \arrow[d, dotted] & & \bullet^-_{n-3} \arrow[d, dotted] \\
\bullet^+_1 \arrow[d] \arrow[drr] & & \bullet^-_1 \arrow[d] \arrow[dll] \\
\bullet^+_0 & & \bullet^-_0
\end{tikzcd}$$

We obtain the following diagram in $\mathbb{C}at$

$$\begin{tikzcd}
 & Oct^+(n-1) \arrow[dr, "i_n^+"] & \\
Oct(n-1) \arrow[ur, "e_n^+"] \arrow[dr, "e_n^-"'] & & Oct(n) \\
 & Oct^-(n-1) \arrow[ur, "i_n^-"'] &
\end{tikzcd}$$

such that functors i_n^+, i_n^- are just canonical inclusions, and the functors e_n^+ and e_n^- are obvious isomorphisms. Put $s_{n-1}^n = i_n^+ \circ e_n^+$ and $t_{n-1}^n = i_n^- \circ e_n^-$. The family of functors $Oct(n-1) \underset{t_{n-1}^n}{\overset{s_{n-1}^n}{\rightrightarrows}} Oct(n)$ $(n \geqslant 1)$, defines an object of $\mathbb{GC}AT$.

Furthermore, for any category $C \in \mathbb{C}AT$, the category $Span_n(C) := [Oct(n); C]$ of functors into C is called *the category of n-spans in C*. The previous functors s_{n-1}^n and t_{n-1}^n induce a family of functors

$$Span_n(C) \underset{t_{n-1}^n}{\overset{s_{n-1}^n}{\rightrightarrows}} Span_{n-1}(C)$$

(that we still denote by s_{n-1}^n and t_{n-1}^n because there is no risk of confusion), which defines an object of $\mathbb{GC}AT$. Dually, for any category $C \in \mathbb{C}AT$, the category $Cospan_n(C) := [Oct(n)^{op}; C]$ of presheaves is called *the category of n-cospans in C*. The functors s_{n-1}^n and t_{n-1}^n between the $Oct(n)$, also induce a family of functors

$$Cospan_n(C) \underset{t_{n-1}^n}{\overset{s_{n-1}^n}{\rightrightarrows}} Cospan_{n-1}(C) \quad ,$$

which is still an object of $\mathbb{GC}AT$. These two constructions are functorial and define the Span and Cospan constructions

$$\mathbb{C}AT \underset{Cospan}{\overset{Span}{\rightrightarrows}} \mathbb{GC}AT \ .$$

The case of a category C with pullbacks is more interesting for the span construction because the corresponding globular category $Span(C)$ is canonically equipped with an MG-structure. We have a dual result for categories with pushouts and their cospans (see Example 7 of Section 3 in [2]). For example consider a category C with pushouts and the two 2-cospans x and y in C (x is the diagram on the left).

$$\begin{tikzcd}
 & A_2 & & & B_2 & \\
A_1^+ \arrow[ur] & & A_1^- \arrow[ul] & B_1^+ \arrow[ur] & & A_1^+ \arrow[ul] \\
A_0^+ \arrow[u] \arrow[urr] & & A_0^- \arrow[u] \arrow[ull] & A_0^+ \arrow[u] \arrow[urr] & & A_0^- \arrow[u] \arrow[ull]
\end{tikzcd}$$

The 1-cospans $s_1^2(x)$ and $t_1^2(y)$ are equal to the following 1-cospan

$$\begin{tikzcd}
 & A_1^+ & \\
A_0^+ \arrow[ur] & & A_0^- \arrow[ul]
\end{tikzcd}$$

and $x \otimes_1^2 y$ is given by the following 2-cospan in C.

$$\begin{tikzcd}
 & A_2 \underset{A_1^+}{\sqcup} B_2 & \\
A_1^+ \arrow[ur] & & B_1^+ \arrow[ul] \\
A_0^+ \arrow[u] \arrow[urr] & & A_0^- \arrow[u] \arrow[ull]
\end{tikzcd}$$

Consider the category $\mathbb{C}AT_{Push}$ of categories with pushouts and with morphisms functors which preserve these pushouts. Dually consider the underlying category of the 2-category $\mathbb{C}AT_{Pull}$ introduced in Section 2.1. We have the following diagram

$$\begin{tikzcd}
\mathbb{C}AT_{Push} \arrow[d, "(.)^{op}"'] \arrow[dr, "Cospan"] & \\
 & \mathbb{GMC}AT \\
\mathbb{C}AT_{Pull} \arrow[ur, "Span"'] &
\end{tikzcd}$$

where $(.)^{op}$ is the basic isomorphism of categories coming from duality, and where the functors *Cospan* and *Span* are defined on morphisms to yield MG-functors since the morphisms are functors preserving pushouts and pullbacks, respectively.

Remark 2.6. If $\mathbb{C}AT^*_{Pull}$ denotes the category of categories with pullbacks and initial objects, and morphisms functors which preserve pullbacks and initial objects, and $\mathbb{C}AT^*_{Push}$ denotes the category of categories with pushouts and initial objects, and morphisms functors which preserve pushouts and initial objects, then we have the following constructions

$$\begin{tikzcd}
\mathbb{C}AT^*_{Push} \arrow[dd, "(.)^{op}"'] \arrow[dr, "Cospan"] & \\
& \mathbb{AMGC}AT \\
\mathbb{C}AT^*_{Pull} \arrow[ur, "Span"'] &
\end{tikzcd}$$

Now consider a category C with pushouts and a globular object (C, W) in $Cospan(C)$, which is also a coglobular object in C. Thanks to the universality of the map $1 \longrightarrow \mathbb{T}\mathrm{ree}$ above there exists a unique map

$$W(-) : \mathbb{T}\mathrm{ree} \longrightarrow Cospan(C)\ .$$

This map $W(-)$ sends each n-tree t to an n-coglobular object in C:

$$W(t) = (W^0 \underset{\kappa^1_0}{\overset{\delta^1_0}{\rightrightarrows}} W^{\partial^{n-1}t} \underset{\kappa^2_1}{\overset{\delta^1_2}{\rightrightarrows}} W^{\partial^{n-2}t} \cdots\cdots\rightrightarrows W^{\partial t} \underset{\kappa^n_{n-1}}{\overset{\delta^n_{n-1}}{\rightrightarrows}} W^t),$$

where the $\partial^k t$ denotes the truncation of the n-tree t at level k $(1 \leqslant k \leqslant n-1)$. In this n-coglobular object $W(t)$, the term W^t denotes the colimit in C of the diagram

$$\begin{tikzcd}
W^{i_1} & & W^{i_2} \arrow[r, dotted, no head] & W^{i_{m-2}} & & W^{i_{m-1}} \\
& W^{i'_1} \arrow[ul, "\kappa^{i'_1}_{i_1}"] \arrow[ur, "\delta^{i'_1}_{i_2}"'] \arrow[rrr, dotted, no head] & & & W^{i'_{m-1}} \arrow[ul, "\kappa^{i'_{m-1}}_{i_{m-2}}"] \arrow[ur, "\delta^{i'_{m-1}}_{i_{m-1}}"'] &
\end{tikzcd}$$

coming from the Grothendieck presentation of the n-tree t.

Span := *Span*(*Set*) is an important MG-category. Examples of n-spans in *Set* are given by the *HOM construction* defined as follows. For each globular category $\mathcal{C} \in \mathbb{GCAT}$, and each pair of objects $A, B \in \mathcal{C}_n$, we associate the n-span $HOM(A,B)$:

$$\begin{tikzcd}
& HOM(A,B)_n \arrow[dl, "s^n_{n-1}"'] \arrow[dr, "t^n_{n-1}"] & \\
HOM(A,B)_{n-1} \arrow[d, "s^{n-1}_{n-2}"'] \arrow[drr, "t^{n-1}_{n-2}" near start] & & HOM(A,B)_{n-1} \arrow[d, "t^{n-1}_{n-2}"] \arrow[dll, "s^{n-1}_{n-2}" near end] \\
HOM(A,B)_{n-2} \arrow[d, dotted] & & HOM(A,B)_{n-2} \arrow[d, dotted] \\
HOM(A,B)_1 \arrow[d, "s^1_0"'] \arrow[drr, "t^1_0" near start] & & HOM(A,B)_1 \arrow[d, "t^1_0"] \arrow[dll, "s^1_0" near end] \\
HOM(A,B)_0 & & HOM(A,B)_0
\end{tikzcd}$$

in Set, having $HOM(A,B)_n := hom_{\mathcal{C}_n}(A,B)$, and, for all $0 \leqslant k < n$, $HOM(A,B)_k := hom_{\mathcal{C}_k}(s^n_k(A), s^n_k(B))$, where $(s^{k+1}_k)_{0 \leqslant k \leqslant n-1}$ and $(t^{k+1}_k)_{0 \leqslant k \leqslant n-1}$ are given by the source and target functors of the globular category $\mathcal{C}$.

Now consider a category C with pushouts and a globular object (C, W) in $Cospan(C)$. If t is an n-tree we can associate between $W(1(n))$ and $W(t) \in Cospan(C)_n$ the n-span $HOM(W(1(n)), W(t))$, such that elements of the set $HOM(W(1(n)), W(t))_n$ are diagrams of the form

$$
\begin{tikzcd}[row sep=large, column sep=huge]
W^n \arrow[r, "f_n"] & W^t \\
W^{n-1} \arrow[u, shift left, "\delta^n_{n-1}"] \arrow[u, shift right, "\kappa^n_{n-1}"'] \arrow[r, shift left, "f^-_{n-1}"] \arrow[r, shift right, "f^+_{n-1}"'] & W^{\partial t} \arrow[u, shift left, "\delta^t_{\partial t}"] \arrow[u, shift right, "\kappa^t_{\partial t}"'] \\
W^{n-(k-1)} \arrow[u, dotted, shift left] \arrow[u, dotted, shift right] \arrow[r, shift left, "f^-_{n-(k-1)}"] \arrow[r, shift right, "f^+_{n-(k-1)}"'] & W^{\partial^{k-1}t} \arrow[u, dotted, shift left] \arrow[u, dotted, shift right] \\
W^{n-k} \arrow[u, shift left, "\delta^{n-(k-1)}_{n-k}"] \arrow[u, shift right, "\kappa^{n-(k-1)}_{n-k}"'] \arrow[r, shift left, "f^-_{n-k}"] \arrow[r, shift right, "f^+_{n-k}"'] & W^{\partial^k t} \arrow[u, shift left, "\delta^{\partial^{k-1}t}_{\partial^k t}"] \arrow[u, shift right, "\kappa^{\partial^{k-1}t}_{\partial^k t}"'] \\
W^1 \arrow[u, dotted, shift left] \arrow[u, dotted, shift right] \arrow[r, shift left, "f^-_1"] \arrow[r, shift right, "f^+_1"'] & W^{\partial^{n-1}t} \arrow[u, dotted, shift left] \arrow[u, dotted, shift right] \\
W^0 \arrow[u, shift left, "\delta^1_0"] \arrow[u, shift right, "\kappa^1_0"'] \arrow[r, shift left, "f^-_0"] \arrow[r, shift right, "f^+_0"'] & W^0 \arrow[u, shift left, "\delta^{\partial^{n-1}t}_0"] \arrow[u, shift right, "\kappa^{\partial^{n-1}t}_0"']
\end{tikzcd}
$$

which commute serially; that is:

- $f_n\delta^n_{n-1} = \delta^t_{\partial t} f^-_{n-1}$, $f_n\kappa^n_{n-1} = \kappa^t_{\partial t} f^+_{n-1}$
- $\forall 1 \leqslant k \leqslant n-1$, $f^-_{n-(k-1)}\delta^{n-(k-1)}_{n-k} = \delta^{\partial^{k-1}t}_{\partial^k t} f^-_{n-k}$, $f^-_{n-(k-1)}\kappa^{n-(k-1)}_{n-k} = \kappa^{\partial^{k-1}t}_{\partial^k t} f^-_{n-k}$ and $f^+_{n-(k-1)}\delta^{n-(k-1)}_{n-k} = \delta^{\partial^{k-1}t}_{\partial^k t} f^+_{n-k}$, $f^+_{n-(k-1)}\kappa^{n-(k-1)}_{n-k} = \kappa^{\partial^{k-1}t}_{\partial^k t} f^+_{n-k}$.

See also Paragraph 9.2 in [19].

Remark 2.7. Spans in sets can be seen in a conceptual way: In [22], Ross Street has shown that objects of *Span* are internal sets in the petit topos ω-$\mathbb{G}$r of globular sets, and in [24] Mark Weber has shown that *Span* is a discrete opfibration classifier in the 2-topos $\mathbb{G}\mathbb{C}AT$ of globular categories.

We can summarise many constructions of this section with the following diagram in 2-$\mathbb{C}AT$

$$\begin{tikzcd}
\mathbb{C}AT_{push} \arrow[rr, "j"] \arrow[dr, "Cospan"] \arrow[dd, "(-)^{op}"'] & & \mathbb{C}AT \arrow[dr, "Cospan"] \arrow[dd, "(-)^{op}"'] & \\
& \mathbb{M}\mathbb{G}\mathbb{C}AT \arrow[rr, "i"] & & \mathbb{G}\mathbb{C}AT \\
\mathbb{C}AT_{pull} \arrow[rr, "k"] \arrow[ur, "Span"'] & & \mathbb{C}AT \arrow[ur, "Span"'] &
\end{tikzcd}$$

Recall that in this section we denote by 1 the terminal globular category, and according to Definition 2.1 we denote the globe category by $\mathbb{G}$.

Lemma 2.8. *We have the following identifications*

- $(1 \downarrow i)$ *is the comma category of the globular objects* $1 \xrightarrow{W} \mathcal{C}$ *such that* $\mathcal{C} \in \mathbb{M}\mathbb{G}\mathbb{C}AT$*,*

- $(1 \downarrow i \circ Cospan)$ *is the comma category of the globular objects* $1 \xrightarrow{W} Cospan(C)$ *such that* $C \in \mathbb{C}AT_{push}$*,*

- $(\mathbb{G} \downarrow j)$ *is the comma category of the globular objects* $\mathbb{G} \xrightarrow{W} C$ *in* C *such that* $C \in \mathbb{C}AT_{push}$*,*

- *We have the following isomorphisms of categories*

$$(1 \downarrow i \circ Cospan) \xrightarrow{\sim} (\mathbb{G} \downarrow j) \qquad (1 \downarrow i \circ Span) \xrightarrow{\sim} (\mathbb{G}^{op} \downarrow k) \ .$$

2.3 Digression on $\mathbb{T}$-categories

Let us recall the approach to ω-operads by Tom Leinster using $\mathbb{T}$-categories[2] (see his book [19]). We recall the notions of $\mathbb{T}$-graph and $\mathbb{T}$-category defined in [13, 19].

[2] For an arbitrary cartesian monad $\mathbb{M}$ on a category with pullbacks the notion of $\mathbb{M}$-category was first suggested by Albert Burroni in 1971; see [7].

Consider the bicategory Span($\mathbb{T}$) as defined in Leinster's book [19]. A $\mathbb{T}$-graph (C, d, c) is a diagram of ω-$\mathbb{G}$r such as

$$\mathbb{T}(G) \xleftarrow{d} C \xrightarrow{c} G$$

$\mathbb{T}$-graphs are endomorphisms of Span($\mathbb{T}$) and they form a category $\mathbb{T}$-$\mathbb{G}$r.

If we fix $G \in \omega$-$\mathbb{G}$r(0), the endomorphisms on G in Span($\mathbb{T}$) form a subcategory of $\mathbb{T}$-$\mathbb{G}$r which is denoted $\mathbb{T}$-$\mathbb{G}\mathrm{r}_G$. The category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_G$ is monoidal with tensor given by:

$$(C, d, c) \otimes (C', d', c') := (\mathbb{T}(C) \times_{\mathbb{T}(G)} C', \mu(G)\mathbb{T}(d)\pi_0, c\pi_1),$$

and with unit given by $I(G) = (G, \eta(G), 1_G)$. The object $I(G)$ is also an identity morphism of Span($\mathbb{T}$). The globular set G is called the globular set of globular arities, or the globular set of arities for short.

Remark 2.9. A p-cell of G is denoted by $g(p)$ and this notation has the following meaning. The symbol g indicates the "colour", and the symbol p records that $g(p)$ is a p-cell of G. This is useful because G has to be seen as an globular set even though it is just a set.

A $\mathbb{T}$-graph (C, d, c) equipped with a morphism $I(G) \xrightarrow{p} (C, d, c)$ is called a *pointed* $\mathbb{T}$-*graph*. This means that one has a 2-cell $I(G) \xrightarrow{p} (C, d, c)$ of Span($\mathbb{T}$) such that $dp = \eta(G)$ and $cp = 1_G$. A pointed $\mathbb{T}$-graph is denoted by $(C, d, c; p)$. We define in a natural way the category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_p$ of pointed $\mathbb{T}$-graphs, and also the category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,G}$ of G-pointed $\mathbb{T}$-graphs; their morphisms keep pointing in an obvious direction.

A constant globular set is an globular set G such that, for all $n, m \in \mathbb{N}$, we have $G(n) = G(m)$, and such that the source and target maps are identities. We write ω-$\mathbb{G}\mathrm{r}_c$ for the corresponding category of constant globular sets. We write $\mathbb{T}$-$\mathbb{G}\mathrm{r}_c$ for the subcategory of $\mathbb{T}$-$\mathbb{G}$r consisting of $\mathbb{T}$-graphs with underlying globular sets of globular arity which are constant globular sets. We write $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$ for the subcategory of $\mathbb{T}$-$\mathbb{G}\mathrm{r}_p$ consisting of pointed $\mathbb{T}$-graphs with underlying globular sets of globular arity which are constant globular sets. Also, for a given G in ω-$\mathbb{G}\mathrm{r}_c$, we write $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c,G}$ for the fiber subcategory in $\mathbb{T}$-$\mathbb{G}\mathrm{r}_{p,c}$.

Definition 2.10. Consider a $\mathbb{T}$-graph (C, d, c). If $k \geqslant 1$, two k-cells x, y of C are parallel if $s^k_{k-1}(x) = s^k_{k-1}(y)$ and if $t^k_{k-1}(x) = t^k_{k-1}(y)$. In that case we write $x \| y$.

A $\mathbb{T}$-category is a monad in the bicategory Span($\mathbb{T}$) or, equally, a monoid in the monoidal category $\mathbb{T}$-$\mathbb{G}\mathrm{r}_G$ (for a specific G). The category of $\mathbb{T}$-categories will be denoted by $\mathbb{T}$-$\mathbb{C}AT$, and that of $\mathbb{T}$-categories over the same globular set of globular arities G will be denoted by $\mathbb{T}$-$\mathbb{C}AT_G$. Specifically, a $\mathbb{T}$-category $(B, d, c; \gamma, p) \in$

$\mathbb{T}$-$\mathbb{C}AT$ is given by the morphism $(B,d,c)\otimes(B,d,c)\xrightarrow{\gamma}(B,d,c)$ of operadic composition and the operadic unit $I(G)\xrightarrow{p}(B,d,c)$, satisfying axioms of associativity and unity that we can find in Leinster's book [19]. Note that $(B,d,c;\gamma,p)$ has $(B,d,c;p)$ as natural underlying pointed $\mathbb{T}$-graph. Algebras for a $\mathbb{T}$-category are just algebras for its underlying monad.

2.4 Endomorphism and coendomorphism ω-operads

Let $\mathcal{C}\in\mathbb{GC}AT$. Recall from [2] that the category of collections ω-$\mathbb{C}oll(\mathcal{C})$ in $\mathcal{C}$ has as objects globular functors $\mathbb{T}ree\xrightarrow{A}\mathcal{C}$ and as morphisms, globular natural transformations between such globular functors. It is straightforward to see that this defines a strict 2-functor $\mathbb{C}oll:=Hom_{\mathbb{GC}AT}(\mathbb{T}ree,-)$:

$$\mathbb{GC}AT\xrightarrow{\mathbb{C}oll}\mathbb{C}AT$$

Theorem 6.1 of [2] gives criteria for finding many categories of collections with monoidal structure. Colimits commuting with the monoidal structure of an AMG-category are defined in Definition 5.3 of [2].

Theorem 2.11. *If $\mathcal{C}$ is an AMG-category such that colimits in $\mathcal{C}$ commute with its monoidal structure, then $\mathbb{C}oll(\mathcal{C})$ has a natural monoidal structure.*

For our purpose the main example of such an AMG-category as in this theorem is *Span*. The monoidal category $\mathbb{C}oll(Span)$ is equivalent to the monoidal category $\mathbb{T}$-$\mathbb{G}r_1$ of $\mathbb{T}$-graphs over the terminal globular set 1 (see 2.3 and [19]). The category of monoids in $\mathbb{T}$-$\mathbb{G}r_1$ is denoted $\mathbb{T}$-$\mathbb{C}AT_1$, and objects of this category are thus ω-operads of Batanin in *Span*. So in this article we see the ω-operad K of Batanin[3] as a specific $\mathbb{T}$-category in $\mathbb{T}$-$\mathbb{C}AT_1$.

Now we are ready to express the main result of this section, which in fact is just a corollary of Proposition 7.2 of [2].

Corollary 2.12. *For each object (C,W) in $(\mathbb{G}\downarrow j)$ we can associate an ω-operad $Coend(W)$ of coendomorphisms, given by the collection*

$$Coend(W):=(HOM(W^n,W^t))_{n\in\mathbb{N},t\in\mathbb{T}ree}\ .$$

Also for each morphism

$$(C,W)\xrightarrow{f}(C',W')$$

[3] In the article [13, 15] we have prefered to denote it B^0_C to emphasise that it is the first step of a sequence of higher operads: The higher operads B^n_C ($n\geqslant 1$) of the weak higher transformations. The letter "B" indicates "Batanin", and the subscript C means contractible.

in $(\mathbb{G} \downarrow j)$ *we can associate a morphism*

$$Coend(W) \xrightarrow{Coend(f)} Coend(W')$$

of ω*-operads. Furthermore this construction is functorial; it defines a functor*

$$(\mathbb{G} \downarrow j) \xrightarrow{Coend} \mathbb{T}\text{-}\mathbb{C}AT_1 \ .$$

Also, for each object (C, W) *in* $(\mathbb{G}^{op} \downarrow k)$*, we can associate the* ω*-operad* $End(W)$ *of endomorphisms, given by the collection*

$$End(W) := (HOM(W^t, W^n))_{n \in \mathbb{N}, t \in \mathbb{T}ree}.$$

Also, for each morphism

$$(C, W) \xrightarrow{f} (C', W')$$

in $(\mathbb{G}^{op} \downarrow k)$*, we can associate a morphism*

$$End(W) \xrightarrow{End(f)} End(W')$$

of ω*-operads. Furthermore this construction is functorial; it defines a functor*

$$(\mathbb{G}^{op} \downarrow k) \xrightarrow{End} \mathbb{T}\text{-}\mathbb{C}AT_1 \ .$$

Proposition 2.13. *If* $W \in (\mathbb{G}^{op} \downarrow k)$ *then* $End(W) \xrightarrow{\sim} Coend(W^{op})$ *in* $\mathbb{T}\text{-}\mathbb{C}AT_1$*.*

Definition 2.14. If $B \in \mathbb{T}\text{-}\mathbb{C}AT_1$ then an algebra for B in the sense of Batanin is given by a morphism

$$B \longrightarrow End(W)$$

in $\mathbb{T}\text{-}\mathbb{C}AT_1$, where $W : \mathbb{G}^{op} \longrightarrow Set$ is an object of ω-$\mathbb{G}$r.

Proposition 2.15 ([19])**.** *If* $B \in \mathbb{T}\text{-}\mathbb{C}AT_1$*, then algebras for* B *in the sense of Batanin and algebras for* B *in the sense of Leinster (see Section 2.3) coincide.*

3 Standard actions associated to a coglobular object in $\mathbb{T}\text{-}\mathbb{C}AT_c$

A $\mathbb{T}$-category over any globular set can be seen as a coloured ω-operad (see [13, 19]), and the category $\mathbb{T}\text{-}\mathbb{C}AT$ of coloured ω-operads is locally presentable, thus it is a category with pushouts. However it is in the context of the locally presentable category $\mathbb{T}\text{-}\mathbb{C}AT_c$ of $\mathbb{T}$-categories over constant globular sets (see the Section $\mathbb{T}$-*graphs with contractible units* of the article [14] and the article [13]) that we are going to build *the standard actions associated to a coglobular object in* $\mathbb{T}\text{-}\mathbb{C}AT_c$. This concept is an application of the previous section to the category $\mathbb{T}\text{-}\mathbb{C}AT_c$.

Definition 3.1. A coglobular ω-operadic object $(\mathbb{T}\text{-}\mathbb{C}AT_c, W)$ in $\mathbb{T}\text{-}\mathbb{C}AT_c$ is called *algebraic* if it is additionally equipped with an ω-operadic morphism

$$W(0) \xrightarrow{\quad w \quad} Coend(W)\ .$$

Definition 3.2. An ω-operad A is *fractal* if there exists an algebraic coglobular ω-operadic object of the form $(\mathbb{T}\text{-}\mathbb{C}AT_c, W)$ in $\mathbb{T}\text{-}\mathbb{C}AT_c$ with $W(0) = A$.

Remark 3.3. At this stage it is important to notice that we can generalise these definitions easily: Any coglobular object of higher operads in a category where pushouts are well defined, leads to such notion of algebraic coglobular ω-operadic object and fractal higher operads (see [15]). We shall not require such generality in the present article and in the article [14].

Consider the following diagram in $\mathbb{C}AT_{Push}$

$$\mathbb{T}\text{-}\mathbb{C}AT_c \xrightarrow{\quad \mathbb{A}lg(.) \quad} \mathbb{C}AT^{op} \xrightarrow{\quad Ob(.) \quad} SET^{op}$$

For each coglobular object $(\mathbb{T}\text{-}\mathbb{C}AT_c, W)$ in $\mathbb{T}\text{-}\mathbb{C}AT_c$, we have the following diagram in $(\mathbb{G} \downarrow j)$.

$$\begin{tikzcd}
 & \mathbb{G} \arrow[dl, "W"'] \arrow[d, "A^{op}"] \arrow[dr, "A_0^{op}"] & \\
\mathbb{T}\text{-}\mathbb{C}AT_c \arrow[r, "\mathbb{A}lg(.)"'] & \mathbb{C}AT^{op} \arrow[r, "Ob(.)"'] & SET^{op}
\end{tikzcd}$$

If we apply the functor *Coend* of Corollary 2.12 to this diagram, and if we use Proposition 2.13, we obtain the following definition

Definition 3.4. The standard action in $\mathbb{T}\text{-}\mathbb{C}AT_1$ associated to the coglobular object $(\mathbb{T}\text{-}\mathbb{C}AT_c, W) \in (\mathbb{G} \downarrow j)$ in $\mathbb{T}\text{-}\mathbb{C}AT_c$ is defined by the following diagram in $\mathbb{T}\text{-}\mathbb{C}AT_1$.

$$Coend(W) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A^{op}) \xrightarrow{Coend(Ob(.))} End(A_0)$$

Now we are ready to explain the philosophy of the standard action associated to a coglobular object in $\mathbb{T}\text{-}\mathbb{C}AT_c$. The category $\mathbb{T}\text{-}\mathbb{C}AT_c$ is locally finitely presentable and the forgetful functor

$$\mathbb{T}\text{-}\mathbb{C}AT_c \xrightarrow{\quad V \quad} \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic (see [19]), thus according to Proposition 5.5.6 of [6], V has rank. Let us call its left adjoint M and denote by $\mathbb{T}_M$ the finitary monad generated by the adjunction.

Now consider a category $P\mathbb{T}\text{-}\mathbb{C}AT_c$ of ω-operads, equipped with a structure that we call "P", whereby the category is locally finitely presentable and equipped with a monadic forgetful functor

$$U_P : P\mathbb{T}\text{-}\mathbb{C}AT_c \longrightarrow \mathbb{T}\text{-}\mathbb{C}AT_c \ .$$

Various concrete choices for P will be considered in the article [14], when in particular we shall consider some specific standard actions (see Section 4) called *standard action for higher transformations* because it is built with the coglobular object of higher transformations $C^\bullet$ in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$. The main problem for our philosophy is to be able to build relevant examples of algebraic coglobular ω-operadic objects (see Definition 3.1) in this context of the combinatorics for higher transformations. More concretely, we want to build a morphism of ω-operads between the monochromatic ω-operad B^0_P (the "0-step" of the coglobular object $B^\bullet_P$) and the monochromatic ω-operad $Coend(B^\bullet_P)$ (built with the whole coglobular object $B^\bullet_P$). If such a morphism exists, which means that B^0_P is fractal (see Definition 3.2), then we have a morphism of operads

$$B^0_P \longrightarrow End(A_{0,P})$$

which shows that B^0_P-algebras and all their higher transformations form a B^0_P-algebra. This motivates us to use the word *fractal* for such ω-operads.

We denote by F_P the left adjoint to U_P.

$$\begin{tikzcd} P\mathbb{T}\text{-}\mathbb{C}AT_c \arrow[r, shift left, "U_P"] \arrow[r, phantom, "\top"] & \mathbb{T}\text{-}\mathbb{C}AT_c \arrow[l, shift left, "F_P"] \arrow[r, shift left, "V"] \arrow[r, phantom, "\top"] & \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \arrow[l, shift left, "M"] \end{tikzcd}$$

Thus we are in a situation where $V \circ U_P$ is monadic and the induced monad $\mathbb{T}_P$ on $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$ has rank. Also we get the functor

$$P := F_P \circ M : \mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c} \longrightarrow P\mathbb{T}\text{-}\mathbb{C}AT_c$$

which assigns the free PT-category on each pointed $\mathbb{T}$-graph.

4 Standard actions for higher transformations.

We finish this article by describing an important class of *actions* which is used in the article [14] to describe several interesting fractal ω-operads.

Consider the following coglobular object $C^\bullet$ in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$, that we call the coglobular object for the higher transformations in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$, because it includes precisely the combinatorics we need for such higher transformations[4] (see [13]):

[4] In [15] we use a slightly different coglobular object $C^\bullet$ which is used to generate operads for strict and weak higher transformations, but with the same globular combinatorics C^n of this section, for each integer $n \in \mathbb{N}$.

$$C^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} C^1 \underset{\kappa_1^2}{\overset{\delta_2^1}{\rightrightarrows}} C^2 \cdots\cdots\rightrightarrows C^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} C^n \cdots\cdots$$

Let us recall what is involved in this coglobular object. Pointings p of each collection involved in this specific coglobular object are denoted with the symbol λ. The term C^0 is Batanin's system of compositions; that is, there is the collection $\mathbb{T}(1) \xleftarrow{d^0} C^0 \xrightarrow{c^0} 1$ where C^0 precisely contains the symbols $\mu_p^m \in C^0(m)(0 \leq p < m)$ for the compositions of ω-categories, plus the operadic unary symbols $u_m \in C^0(m)$. More specifically:

$\forall m \in \mathbb{N}$, C^0 contains an m-cell u_m such that: $s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}$ (if $m \geq 1$); $d^0(u_m) = 1(m)(= \eta(1 \cup 2)(1(m)))$, $c^0(u_m) = 1(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C^0 contains an m-cell μ_p^m such that: If $p = m-1$, $s_{m-1}^m(\mu_{m-1}^m) = t_{m-1}^m(\mu_{m-1}^m) = u_{m-1}$. If $0 \leq p < m-1$, $s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}$. Also $d^0(\mu_p^m) = 1(m) \star_p^m 1(m)$, and inevitably $c^0(\mu_p^m) = 1(m)$.

Furthermore C^0 contains a 1-cell μ_0^1 such that $s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0$, $d^0(\mu_0^1) = 1(1) \star_0^1 1(1)$, also inevitably $c^0(\mu_0^1) = 1(1)$.

The system of composition C^0 has a standard pointing λ^0 which is defined by: $\forall m \in \mathbb{N}$, $\lambda^0(1(m)) = u_m$.

Firstly we will define a collection (C, d, c) which will be useful to build the collections of n-transformations ($n \in \mathbb{N}^*$). C contains two copies of the symbols of C^0, each having a distinct colour: symbols formed with the letters μ and u are those of colour 1, and those formed with the letters ν and v are those of colour 2. Let us be more precise:

$\forall m \in \mathbb{N}$, C contains an m-cell u_m such that: $s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}$ (if $m \geq 1$) and $d(u_m) = 1(m)$, $c(u_m) = 1(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C contains an m-cell μ_p^m such that: If $p = m-1$, $s_{m-1}^m(\mu_{m-1}^m) = t_{m-1}^m(\mu_{m-1}^m) = u_{m-1}$. If $0 \leq p < m-1$, $s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}$. Also $d(\mu_p^m) = 1(m) \star_p^m 1(m)$, $c(\mu_p^m) = 1(m)$.

Furthemore C contains a 1-cell μ_0^1 such that $s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0$ and $d(\mu_0^1) = 1(1) \star_0^1 1(1)$, $c(\mu_0^1) = 1(1)$.

Besides, $\forall m \in \mathbb{N}$, C contains an m-cell v_m such that: $s_{m-1}^m(v_m) = t_{m-1}^m(v_m) = v_{m-1}$ (if $m \geq 1$) and $d(v_m) = 2(m)$, $c(v_m) = 2(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C contains an m-cell ν_p^m such that: If $p = m-1$, $s_{m-1}^m(\nu_{m-1}^m) = t_{m-1}^m(\nu_{m-1}^m) = v_{m-1}$. If $0 \leq p < m-1$, $s_{m-1}^m(\nu_p^m) = t_{m-1}^m(\nu_p^m) = \nu_p^{m-1}$. Also $d(\nu_p^m) = 2(m) \star_p^m 2(m)$, $c(\nu_p^m) = 2(m)$.

Furthemore C contains a 1-cell ν_0^1 such that $s_0^1(\nu_0^1) = t_0^1(\nu_0^1) = v_0$ and $d(\nu_0^1) = 2(1) \star_0^1 2(1)$, $c(\nu_0^1) = 2(1)$.

C^1 is the system of operations of ω-functors. It is built by starting with C and adding to it a single symbol for functor (for each cell level): $\forall m \in \mathbb{N}$ the F^m m-cell is added, which is such that: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$. Also $d^1(F^m) = 1(m)$ and $c^1(F^m) = 2(m)$.

C^2 is the system of operations of natural ω-transformations. C^2 is built on C by adding to it two symbols of functor (for each cell level) and a symbol of natural transformation. More precisely

$\forall m \in \mathbb{N}$ we add the m-cell F^m such that: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$. Also $d^2(F^m) = 1(m)$ and $c^2(F^m) = 2(m)$.

Then $\forall m \in \mathbb{N}$ we add the m-cell H^m such that: If $m \geq 1$, $s_{m-1}^m(H^m) = t_{m-1}^m(H^m) = H^{m-1}$. Also $d^2(H^m) = 1(m)$ and $c^2(H^m) = 2(m)$.

And finally we add 1-cell τ such that: $s_0^1(\tau) = F^0$ and $t_0^1(\tau) = H^0$. Also $d^2(\tau) = 1_{1(0)}$ and $c^2(\tau) = 2(1)$.

Observe that the 2-coloured collections C^i $(i = 1,2)$ are naturally equipped with a pointing λ^i defined by $\lambda^i(1(m)) = u_m$ and $\lambda^i(2(m)) = v_m$.

In order to define the general theory of n-transformations ($n \in \mathbb{N}^*$), it is necessary to define the systems of operations C^n for the higher n-transformations ($n \geq 3$). This paragraph can be left out in the first reading. Each collection C^n is built on C by adding to it the required cells. They contain four large groups of cells: the symbols of source and target ω-categories, the symbols of operadic units (obtained on the basis of C), the symbols of the ω-functors (sources and targets), and the symbols of the n-transformations (natural ω-transformations, ω-modification, etc). More precisely, on the basis of C:

Symbols for ω-Functors $\forall m \in \mathbb{N}$, C^n contains m-cells α_0^m and β_0^m such that: if $m \geq 1$, then $s_{m-1}^m(\alpha_0^m) = t_{m-1}^m(\alpha_0^m) = \alpha_0^{m-1}$ and $s_{m-1}^m(\beta_0^m) = t_{m-1}^m(\beta_0^m) = \beta_0^{m-1}$. Furthermore $d^n(\alpha_0^m) = d^n(\beta_0^m) = 1(m)$ and $c^n(\alpha_0^m) = c^n(\beta_0^m) = 2(m)$.

Symbols for Higher n-Transformations $\forall p$, with $1 \leq p \leq n-1$, C^n contains p-cells α_p and β_p which are such that: $\forall p$ with $2 \leq p \leq n-1$, $s_{p-1}^p(\alpha_p) = s_{p-1}^p(\beta_p) = \alpha_{p-1}$ and $t_{p-1}^p(\alpha_p) = t_{p-1}^p(\beta_p) = \beta_{p-1}$. If $p = 1$, then $s_0^1(\alpha_1) = s_0^1(\beta_1) = \alpha_0^0$ and $t_0^1(\alpha_1) = t_0^1(\beta_1) = \beta_0^0$. Moreover, $\forall p$ with $1 \leq p \leq n-1$, $d^n(\alpha_p) = d^n(\beta_p) = 1_p^0(1(0))$ and $c^n(\alpha_p) = c^n(\beta_p) = 2(p)$. Finally C^n

contains an n-cell ξ_n such that $s^n_{n-1}(\xi_n) = \alpha_{n-1}$, $b^n_{n-1}(\xi_n) = \beta_{n-1}$ and $d^n(\xi_n) = 1^0_n(1(0))$ and $c^n(\xi_n) = 2(n)$.

We can see that $\forall n \in \mathbb{N}^*$, the 2-coloured collection C^n is naturally equipped with the pointing $1 \cup 2 \xrightarrow{\lambda^n} (C^n, d, c)$ defined as:

$$\forall m \in \mathbb{N}, \lambda^n(1(m)) = u_m \text{ and } \lambda^n(2(m)) = v_m.$$

The set $\{C^n | n \in \mathbb{N}\}$ has a canonical structure of coglobular object. This coglobular object is generated by diagrams

$$\begin{tikzcd} C^n \arrow[r, shift left, "\delta^n_{n+1}"] \arrow[r, shift right, "\kappa^n_{n+1}"'] & C^{n+1} \end{tikzcd}$$

of pointed 2-coloured collections. For $n \geq 2$, these diagrams are defined as follows. First the $(n+1)$-coloured collection contains the same symbols of operations as C^n for the j-cells with $0 \leq j \leq n-1$ or $n+2 \leq j < \omega$. For the n-cells and the $(n+1)$-cells the symbols of operations will change: C^n contains the n-cell ξ_n whereas C^{n+1} contains the n-cells α_n and β_n, in addition, contains the $(n+1)$-cell ξ_{n+1}. If one denotes by $C^n - \xi_n$ the n-coloured collection obtained on the basis of C^n by taking from it the n-cell ξ_n, then δ^n_{n+1} is defined as follows: $\delta^n_{n+1}|_{C^n - \xi_n}$ (i.e the restriction of δ^n_{n+1} to $C^n - \xi_n$) is the canonical injection $C^n - \xi_n \hookrightarrow C^{n+1}$ and $\delta^n_{n+1}(\xi_n) = \alpha_n$. In a similar way κ^n_{n+1} is defined as follows: $\kappa^n_{n+1}|_{C^n - \xi_n} = \delta^n_{n+1}|_{C^n - \xi_n}$ and $\kappa^n_{n+1}(\xi_n) = \beta_n$. Notice that δ^n_{n+1} and κ^n_{n+1} keep pointing; that is, we have for all $n \geq 1$ the equalities $\delta^n_{n+1}\lambda^n = \lambda^{n+1}$ and $\kappa^n_{n+1}\lambda^n = \lambda^{n+1}$.

The morphisms of 2-coloured pointing collections of the diagram

$$\begin{tikzcd} C^0 \arrow[r, shift left, "\delta^0_1"] \arrow[r, shift right, "\kappa^0_1"'] & C^1 \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^1_2"'] & C^2 \arrow[r, shift left, "\delta^2_3"] \arrow[r, shift right, "\kappa^2_3"'] & C^3 \end{tikzcd}$$

have a similar definition:

We have for all integers $0 \leq p < n$ and for all $\forall m \in \mathbb{N}$:

$\delta^0_1(\mu^n_p) = \mu^n_p$; $\delta^0_1(u_m) = u_m$; $\kappa^0_1(\mu^n_p) = \nu^n_p$; $\kappa^0_1(u_m) = v_m$.

Also: $\delta^1_2(\mu^n_p) = \mu^n_p$; $\delta^1_2(u_m) = u_m$; $\delta^1_2(\nu^n_p) = \nu^n_p$; $\delta^1_2(v_m) = v_m$; $\delta^1_2(F^m) = F^m$. And $\kappa^1_2(\mu^n_p) = \mu^n_p$; $\kappa^1_2(u_m) = u_m$; $\kappa^1_2(\nu^n_p) = \nu^n_p$; $\kappa^1_2(v_m) = v_m$; $\kappa^1_2(F^m) = H^m$.

Finally: $\delta^2_3(\mu^n_p) = \mu^n_p$; $\delta^2_3(u_m) = u_m$; $\delta^2_3(\nu^n_p) = \nu^n_p$; $\delta^2_3(v_m) = v_m$; $\delta^2_3(F^m) = \alpha^m_0$; $\delta^2_3(H^m) = \beta^m_0$; $\delta^2_3(\tau) = \alpha_1$. And $\kappa^2_3(\mu^n_p) = \mu^n_p$; $\kappa^2_3(u_m) = u_m$; $\kappa^2_3(\nu^n_p) = \nu^n_p$; $\kappa^2_3(v_m) = v_m$; $\kappa^2_3(F^m) = \alpha^m_0$; $\kappa^2_3(H^m) = \beta^m_0$; $\kappa^2_3(\tau) = \beta_1$.

The pointed 2-coloured collections C^n $(n \in \mathbb{N}^*)$ are the systems of operations of n-transformations.

If we apply the functor P of the Section 3 to this coglobular object we obtain a coglobular object in $P\mathbb{T}\text{-}\mathbb{C}AT_c$

$$B_P^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} B_P^1 \underset{\kappa_1^2}{\overset{\delta_2^1}{\rightrightarrows}} B_P^2 \cdots\rightrightarrows B_P^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} B_P^n \cdots$$

which is also, when we forget its structure "P", a coglobular object $W = B_P^\bullet$ of $\mathbb{T}\text{-}\mathbb{C}AT_c$, and thus we obtain its resulting standard action

$$Coend(B_P^\bullet) \xrightarrow{Coend(\mathbb{A}lg(.))} Coend(A_P^{op}) \xrightarrow{Coend(Ob(.))} End(A_{0,P})$$

where in particular, $Coend(B_P^\bullet)$ is the monochromatic ω-operad of coendomorphisms associated to this coglobular object.

The coglobular object $C^\bullet$ freely generates higher operads of different kind of rather simple higher transformations which are described in the article [14], and is an important step for building their standard actions. In the article [14] we use it with four specific functors P to prove that the ω-operad of globular *sets*, the ω-operad of reflexive globular *sets*, the ω-operad of ω-*magmas* [5], and the ω-operad of *reflexive* ω-*magmas*, are all fractal.

Remark 4.1. In fact ω-operad of globular *sets*, ω-operad of reflexive globular *sets*, and ω-operads of their corresponding higher transformations, use a more basic coglobular object $G^\bullet$ in $\mathbb{T}\text{-}\mathbb{G}\mathrm{r}_{p,c}$: If we remove the symbols μ_p^m and ν_p^m from the coglobular object $C^\bullet$ described just above, we obtain such $G^\bullet$.

Remark 4.2. In the article [15] we use a different, but very similar, coglobular object of higher transformations $C^\bullet$ in order to build strict and weak higher transformations.

Acknowledgement

I am grateful to Michael Batanin and Ross Street for their mathematical support and encouragement. I am grateful to Mark Weber who explained me some technical points that I was not able to understand by myself. I am also grateful to Richard Garner who shared with me his point of view on the algebraic small object argument, and to Clémens Berger, Denis-Charles Cisinski, and Rémy Tuyéras for many discussions about abstract homotopy theory. I am grateful to some of my friends

[5] An ω-magma is a globular set equipped with compositions $\circ_p^m$ like strict ω-categories, but without requiring the axioms of associativity and unities (see [14]).

who have supported me many times with skype: Samir Berrichi, Jean-Pierre Ledru, Nizar Slimani, and Karim Moui; and, other friends who supported me in Australia: Frank Valkenborgh, Chris McMillan, Estelle Helene Borrey, Edwin Nelson, Khalid Nasamo, and Tom Hakkinen. Also, thanks to the referee for his/her comments which improved the paper.

References

[1] D. Ara, *Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catégorique*, http://arxiv.org/pdf/math/0607820v2 (2010).

[2] M. Batanin, *Monoidal globular categories as a natural environment for the theory of weak-n-categories*, Adv. Math. 136 (1998), 39–103.

[3] M. Batanin and R. Street, *The universal property of the multitude of trees*, J. Pure Appl. Algebra 154 (2000), 3–13.

[4] M. Batanin, *The Eckmann-Hilton argument and higher operads*, Adv. Math. 217 (2008), 334–385.

[5] C. Berger, *A cellular nerve for higher categories*, Adv. Math. 169 (2002), 118–175.

[6] F. Borceux, "Handbook of Categorical Algebra, Vol. 2", Cambridge University Press, 1994.

[7] A. Burroni, *T-catégories (catégories dans un triple)*, Cah. Topol. Géom. Differ. Catég. 12 (1971), 215–321.

[8] D.Ch. Cisinski, *Batanin higher groupoids and homotopy types*, Contemp. Math. 143 (2007), 171–186.

[9] A. Grothendieck, "Pursuing Stacks", Typed manuscript, 1983.

[10] A. Joyal, *Disks, duality and Θ-categories*, (1997), Preprint.

[11] C. Kachour, *Toward the operadical definition of the weak omega category of the weak omega categories, Part 3: The Red Operad*, Australian Category Seminar, Macquarie University (2010).

[12] K. Kachour, *Définition algébrique des cellules non-strictes*, Cah. Topol. Géom. Differ. Catég. 1 (2008), 1–68.

[13] C. Kachour, *Operadic definition of the non-strict cells*, Cah. Topol. Géom. Differ. Catég. 4 (2011), 1–48.

[14] C. Kachour, *Operads of higher transformations for globular sets and for higher magmas*, Categ. General Alg. Structures Appl. 3(1) (2015), 89–111.

[15] C. Kachour, *Steps toward the weak ω-category of the weak ω-categories in the globular setting*, To appear in Categ. General Alg. Structures Appl. 4(1) (2015).

[16] C. Kachour, "Aspects of Globular Higher Category Theory", Ph.D. Thesis, Macquarie University, 2013.

[17] G.M. Kelly and A.J. Power, *Adjunctions whose counits are coequalizers and presentations of finitary enriched monads*, J. Pure Appl. Algebra 89 (1993), 163–179.

[18] S. Lack, *On the monadicity of finitary monads*, J. Pure Appl. Algebra 140 (1999), 65–73.

[19] T. Leinster, "Higher Operads, Higher Categories", London Math. Soc. Lect. Note Series, Cambridge University Press 298, 2004.

[20] G. Maltsiniotis, *Grothendieck ∞-groupoids, and still another definition of ∞-categories*, Available online : http://arxiv.org/pdf/1009.2331v1.pdf (2010).

[21] J. Penon, *Approche polygraphique des ∞-catégories non-strictes*, Cah. Topol. Géom. Différ. Catég. 1 (1999), 31–80.

[22] R. Street, *The petit topos of Globular sets*, J. Pure Appl. Algebra 154 (2000), 299–315.

[23] M. Weber, *Operads within monoidal pseudo algebras*, Appl. Categ. Structures 13 (2005), 389–420.

[24] M. Weber, *Yoneda structures from 2-toposes*, Appl. Categ. Structures 15 (2007), 259–323.

Camell Kachour, *Department of Mathematics, Macquarie University, Sydney, Australia.*
Email: camell.kachour@gmail.com