Categories and
General Algebraic Structures
with Applications

In press.

Pure filters and topological spaces on triangle algebras

P. Franclin Demlabi, A.G. Tallee Kakeu*, B.B. Koguep Njionou, and Celestin Lele

Abstract. In this paper, we delve into the lattice of filters of a triangle algebra. Moreover, we establish the prime filter theorem, and investigate the algebraic structure of the set of co-annihilators of a triangle algebra. In addition, we explore the concept of pure filter within the framework of triangle algebras. Furthermore, we describe the topological properties of the prime filter space of a triangle algebra by equipping the lattice of prime filters with the Zariski topology. Thanks to the notion of pure filters in triangle algebras, we also provide a characterization of the open stable sets with respect to the stable topology, a topology that is coarser than the Zariski topology.

1 Introduction

Zadeh's (see [19]) approach to fuzzy set theory is still popular and has gained considerable momentum in recent years. Indeed, Zadeh used the real unit interval $[0;1]$ as a set of truth values, with the intersection and

* Corresponding author
Keywords: Triangle algebra, filter, pure filter, Zariski topology, stable topology.
Mathematics Subject Classification [2010]: 03G05, 03B52, 06A15, 06B23, 06B75, 06D15, 06F30, 08A72.
Received: 20 July 2024, Accepted: 23 September 2024.
ISSN: Print 2345-5853, Online 2345-5861.

union modeled by the minimum and maximum, respectively. Taking into account the potentiel of incomparability among elements in the truth value set, Goguen (see [9]) substituted the real unit interval with a bounded lattice and used triangular norms and co-norms to extend logical conjunction and disjunction [12, 13]. In line with the residuation principle, this led to the algebraic structure known as residuated lattice (see [18]), which is often used as basic structure of truth degrees in fuzzy logic. Given that the truth value of a statement is mostly gradual than strict, that is, its precision is usually unknown and restricted to an interval, Van Gasse et al (see [14, 15, 17]) constructed residuated lattices from triangular lattices, also called interval-valued residuated lattices (IVRLs). Subsequently, they equipped these IVRLs with approximation operators and introduced a third angular point, resulting in the so-called extended interval residuated lattices, whose corresponding logical algebraic structure is represented by triangle algebras [14, 17].

A triangle algebra $\mathcal{L} := (L, \wedge, \vee, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ is a residuated lattice to which two approximation operators (necessity and possibility) and a constant (uncertainty) other than 0 and 1, have been added. As subsets of partially ordered sets satisfying specific properties, filters are crucial, since they are set of provable formulas. Also, they are intimately related to congruence relations, which are essential in the study of quotient sets. In 2010, Van Gasse et. al introduced the notion of filter in triangle algebras, involving the approximation operator ν, making the filters of triangle algebras different from that of other algebraic structures such as residuated lattices and their subclasses [16]. They defined Boolean filters and prime filters in triangle algebras, establishing some relationships between them. Subsequently, Zahiri et al., in a series of papers [20–22] further investigated properties of triangle algebras by exploring other types of filters.

In the present paper, given a triangle algebra $\mathcal{L}$, we examine the lattice of filters of a triangle algebra. Since tools from topology are used to interpret algebraic varieties, we prove that $\mathrm{Spec}(L)$, the set of all prime filters of $\mathcal{L}$, can be equipped with the Zariski topology τ_L as well as the stable topology, in light of what has been done in the context of bounded distributive lattices [7], and residuated lattices [2].

The paper is organized as follows: firstly, we recall some preliminary notions of triangle algebras in Section 2. In Section 3, we study the algebraic structure of the set of filters of a triangle algebra, as well as the set of co-annihilators of a triangle algebra. Moreover, the prime filter theorem is established. Section 4 investigates the notion of pure filter of a triangle algebra, along with some of its properties. Section 5 is devoted to the spectral topology on triangle algebras. We show that $(\mathrm{Spec}(L), \tau_L)$ is a compact T_0 space, and characterize triangle algebras for which $(\mathrm{Spec}(L), \tau_L)$ is connected. In Section 6, we examine the stable topology S_L, whose stable sets are completely described by pure filters.

2 Preliminaries

In this section, we recall some important notions and results on residuated lattices, and triangle algebras, which will be needed in the sequel.

Definition 2.1. [18] A *residuated lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

(ii) $(L, \odot, 1)$ is a commutative monoid ;

(iii) $x \odot y \leq z$ if and only if $x \leq y \to z$, for any $x, y, z \in L$ (residuation principle).

In this paper, we will use the following notations:

- $\neg x$ for $x \to 0$;
- $x \leftrightarrow y$ for $(x \to y) \wedge (y \to x)$;
- x^n for $\underbrace{x \odot x \odot \cdots \odot x}_{n \quad times}$, with $n \in \mathbb{N}^*$. Conventionally, $x^0 = 1$.

Definition 2.2. [23] Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a residuated lattice. The order of $x \in L$, denoted by $ord(x)$, is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.

Proposition 2.3. [14, 16]
Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a residuated lattice. For every $x, y, z \in L$, we have:

(RL1) $x \odot y \leq x \wedge y$; $x \leq y \to (x \odot y)$, $y \leq x \to y$;

(RL2) $x \odot (x \to y) \leq x \wedge y$; *particularly,* $x \odot \neg x = 0$;

(RL3) $x \vee y \leq (x \to y) \to y$; *particularly,* $x \leq \neg\neg x$;

(RL4) $(x \to y) \odot z \leq x \to (y \odot z)$, $(x \to y) \odot (y \to z) \leq x \to z$, $x \to y \leq (x \odot z) \to (y \odot z)$; $(x \odot y) \to z = x \to (y \to z)$; *particularly,* $\neg(x \odot y) = x \to \neg y$;

(RL5) $x \to (y \wedge z) = (x \to y) \wedge (y \longrightarrow z)$;

(RL6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$; *particularly,* $\neg(x \vee y) = \neg x \wedge \neg y$;

(RL7) $x \to y = sup\{z \in L : x \odot z \leq y\}$; *particularly,* $1 \to x = x$ *and* $x \to x = 1$;

(RL8) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$, *and* $(x \vee y)^{mn} \leq x^m \vee y^n$, *for every* $m, n \geq 1$.

(RL9) $x \odot 0 = 0$, $\neg(0) = 1$, $\neg(1) = 0$; $\neg\neg\neg x = \neg x$; $\neg(x \odot y) = x \to \neg y = y \to \neg x$.

Definition 2.4. [16] Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a residuated lattice. A *filter* of $\mathcal{L}$ is a nonempty subset F of L such that, for every $x, y \in L$:
(F_1) if $x \in F$ and $x \leq y$, then $y \in F$;
(F_2) if $x, y \in F$, then $x \odot y \in F$.

Another way to describe filters in residuated lattices is through the concept of deductive systems. A deductive system of a residuated lattice L is a nonempty subset F of L containing 1 such that for all $x, y \in L$, $x \to y \in F$ and $x \in F$ imply $y \in F$. It is well-known that the notions of filter and deductive system coincide in residuated lattices [4].

Let $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice. We recall from [14, 16, 17] that the *triangularization or triangular lattice* of $\mathcal{L}$ is the bounded lattice $\mathbb{T}(\mathcal{L})$ of the closed intervals of L defined by:

$$\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \vee_{Int(\mathcal{L})}, \wedge_{Int(\mathcal{L})}, [0,0], [1,1])$$

such that $Int(\mathcal{L}) = \{[x_1, x_2] : x_1, x_2 \in L \text{ and } x_1 \leq x_2\}$, and for all $x_1, x_2, y_1, y_2 \in L$,

- $[x_1, x_2] \vee_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$;
- $[x_1, x_2] \wedge_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$;

- $[x_1, x_2] \leq_{Int(\mathcal{L})} [y_1, y_2]$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

The set $D(\mathcal{L}) = \{[x, x] : x \in L\}$ is called *Diagonal* of $\mathbb{T}(\mathcal{L})$.

From [14, 17], an *interval-valued residuated lattice (IVRL)* is a residuated lattice $(\mathrm{Int}(\mathcal{L}), \vee, \wedge, \odot, \rightarrow_{\odot}, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$ such that the diagonal $D(\mathcal{L})$ is closed under $\odot$ and $\rightarrow_{\odot}$, that is, $[x, x] \odot [y, y] \in D(\mathcal{L})$ and $[x, x] \rightarrow_{\odot} [y, y] \in D(\mathcal{L})$, for all x, y in L.

The structure $(Int(\mathcal{L}), \vee, \wedge, \odot, \rightarrow_{\odot}, pr_v, pr_h, [0, 0], [0, 1], [1, 1])$ is called *extended IVRL*, where $u = [0, 1]$ is a constant interval, pr_v and pr_h are respectively called vertical and horizontal projections defined from $Int(\mathcal{L})$ to $Int(\mathcal{L})$ by $pr_v([x_1, x_2]) = [x_1, x_1]$ and $pr_h([x_1, x_2]) = [x_2, x_2]$.

Definition 2.5. [14, 16] A *triangle algebra* is a structure $\mathcal{L} = (L, \vee, \wedge, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on L, u $(0 \neq u \neq 1)$ a constant, all satisfying the following conditions:

$(T.1)$ $\nu x \leq x$; $(T.1')$ $x \leq \mu x$;
$(T.2)$ $\nu x \leq \nu\nu x$; $(T.2')$ $\mu\mu x \leq \mu x$;
$(T.3)$ $\nu(x \wedge y) = \nu x \wedge \nu y$; $(T.3')$ $\mu(x \wedge y) = \mu x \wedge \mu y$;
$(T.4)$ $\nu(x \vee y) = \nu x \vee \nu y$; $(T.4')$ $\mu(x \vee y) = \mu x \vee \mu y$;
$(T.5)$ $\nu u = 0$; $(T.5')$ $\mu u = 1$;
$(T.6)$ $\nu\mu x = \mu x$; $(T.6')$ $\mu\nu x = \nu x$;
$(T.7)$ $\nu(x \rightarrow y) \leq \nu x \rightarrow \nu y$;
$(T.8)$ $(\nu x \leftrightarrow \nu y) \odot (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)$;
$(T.9)$ $\nu x \rightarrow \nu y \leq \nu(\nu x \rightarrow \nu y)$.

Every triangle algebra is isomorphic to an extended IVRL [14].

For terminology and theory of triangle algebra, we refer the reader to [14, 16, 21, 22].

Proposition 2.6. [14, 16] *Let* $\mathcal{L} = (L, \wedge, \vee, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ *be a triangle algebra. Then, for any* $x, y \in L$,

(1) $\nu(x \odot y) = \nu x \odot \nu y$;

(2) $\mu(x \odot y) \leq \mu x \odot \mu y$.

Remark 2.7. [14] Let $\mathcal{L} = (L, \wedge, \vee, \nu, \mu, 0, u, 1)$ be a triangle algebra.

(i) For any $x, y \in L$, if $\nu x = \nu y$ and $\mu x = \mu y$, then $x = y$.

(ii) For all $x \in L$, $\nu\nu x = \nu x$ and $\mu\mu x = \mu x$.

(iii) The operators ν and μ are increasing.

(iv) $\nu 1 = 1$ and $\mu 0 = 0$.

Lemma 2.8. [5] *Let $\mathcal{L} = (L, \wedge, \vee, \nu, \mu, 0, u, 1)$ be a triangle algebra and $x, y \in L$. If $\nu x \vee y = 1$, then $x \odot y = x \wedge y$.*

Let $\mathcal{L} = (L, \vee, \wedge)$ be a lattice. A map $f \colon L \longrightarrow L$ is a *closure operator* on L, if it satisfies the following properties for all $x, y \in L$ [1]:

(i) $f(f(x)) = f(x)$;

(ii) if $x \leq y$, then $f(x) \leq f(y)$;

(iii) $x \leq f(x)$.

For any lattice $\mathcal{L}$ and a map $C \colon L \longrightarrow L$, an element x of L is said to be closed if $C(x) = x$. The poset of all the closed elements of L will be denoted by L_C.

Proposition 2.9. [1] *Let $C \colon L \longrightarrow L$ be a closure operator on a lattice $(L, \wedge, \vee)$. Then, $(L_C, \wedge, \vee')$ is a complete lattice in which $x \vee' y = c(x \vee y)$, for all $x, y \in L_C$.*

Given two triangle algebras $\mathcal{L}_1 = (L_1, \wedge_1, \vee_1, \odot_1, \rightarrow_1, \nu_1, \mu_1, 0, u, 1)$ and $\mathcal{L}_2 = (L_2, \wedge_2, \vee_2, \odot_2, \rightarrow_2, \nu_2, \mu_2, 0, u, 1)$, a map $f : L_1 \longrightarrow L_2$ is a *morphism* of triangle algebras, if for any $x, y \in L_1$ (see [23]),

1. $f(0) = 0$ and $f(1) = 1$;
2. $f(x \wedge_1 y) = f(x) \wedge_2 f(y)$ and $f(x \vee_1 y) = f(x) \vee_2 f(y)$;
3. $f(x \odot_1 y) = f(x) \odot_2 f(y)$ and $f(x \rightarrow_1 y) = f(x) \rightarrow_2 f(y)$;
4. $f(\nu_1 x) = \nu_2(f(x))$ and $f(\mu_1 x) = \mu_2(f(x))$.

In the rest of this paper, unless otherwise indicated, a triangle algebra $(L, \wedge, \vee, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ will be simply denoted by $\mathcal{L}$.

Definition 2.10. [16, 20, 21] Let $\mathcal{L}$ be a triangle algebra. A *filter* of $\mathcal{L}$ is a nonempty subset F of L which satisfies (F_1), (F_2), and (F_3) given by: if $x \in F$, then $\nu x \in F$, for every $x, y \in L$.

We shall notice from [16] that any filter F of $\mathcal{L}$ has the property (F_3') given by: for all $x \in L$, $x \in F$ if and only if $\nu x \in F$.

For any triangle algebra $\mathcal{L}$, an element $x \in L$ is said to be *exact* if $\nu x = x$. The set of exact elements of L is denoted by $E(L)$, and $\varepsilon(L) = (E(L), \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a subalgebra of $\mathcal{L}$. A nonempty subset F of L is called *IVRL-extended filter* if the property (F_3') holds and $F \cap E(L)$ is a filter of the residuated lattice $\varepsilon(L)$ [16]. Moreover, F is a filter of $\mathcal{L}$ if and only if F is an IVRL-extended filter.

$\mathcal{F}(L)$ will stand for the set of filters of a triangle algebra $\mathcal{L}$.

A filter F of $\mathcal{L}$ is said to be proper if $F \neq L$.

Definition 2.11. [16, 23] Let $\mathcal{L}$ be a triangle algebra. A proper filter F of $\mathcal{L}$ is called *prime* if $\nu(x \vee y) \in F$ implies $\nu x \in F$ or $\nu y \in F$, for all $x, y \in L$.

Proposition 2.12. [16, 21] *Let $\mathcal{L}$ be a triangle algebra and F a filter of $\mathcal{L}$. Then, the following assertions are equivalent:*

(1) *F is a prime filter;*

(2) *If F_1 and F_2 are filters of $\mathcal{L}$ such that $F = F_1 \cap F_2$, then $F = F_1$ or $F = F_2$;*

(3) *If F_1 and F_2 are filters of $\mathcal{L}$ such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.*

The set of prime filters of a triangle algebra $\mathcal{L}$ is called the spectrum of $\mathcal{L}$ and will be denoted $\mathrm{Spec}(L)$. Obviously, for a morphism of triangle algebras $f : L_1 \longrightarrow L_2$, if $P \in \mathrm{Spec}(L_2)$, then $f^{-1}(P) \in \mathrm{Spec}(L_1)$.

Recall from [21, 23] that a proper filter F of $\mathcal{L}$ is said to be *maximal* if for any filter G of $\mathcal{L}$, $F \subseteq G$ implies $G = F$ or $G = L$.

We will denote by $Max(L)$ the set of all maximal filters of $\mathcal{L}$. Note that $Max(L) \subseteq \mathrm{Spec}(L)$.

A triangle algebra $\mathcal{L}$ is said to be *local* if it has exactly one maximal filter.

Proposition 2.13. [23] *A triangle algebra $\mathcal{L}$ is local if and only if $ord(\nu x) < \infty$ or $ord(\neg \nu x) < \infty$, for all $x \in L$.*

Definition 2.14. Let $\mathcal{L}$ be a triangle algebra. A prime filter P of $\mathcal{L}$ is said to be *minimal* if for any prime filter Q of $\mathcal{L}$, $Q \subseteq P$ implies $P = Q$.

Throughout this paper, we denote by $Min(L)$ the set of minimal prime filters of $\mathcal{L}$.

For any subset X of a triangle algebra $\mathcal{L}$, the set $\langle X\rangle$ will stand for the smallest filter of $\mathcal{L}$ containing X, called the filter generated by X.

Proposition 2.15. [11] *Let X be a subset of a triangle algebra $\mathcal{L}$. Then,* $\langle X\rangle := \{x \in L \mid \exists(n, x_i) \in \mathbb{N}^* \times X, x \geq \nu x_1 \odot \nu x_2 \odot \cdots \odot \nu x_n, \forall i \in \{1, 2, \cdots, n\}\}$. *In particular,* $\langle a\rangle = \{x \in L \mid x \geq (\nu a)^n, n \in \mathbb{N}^*\}$, *for every* $a \in L$.

From [11], for all $X, Y \subseteq L$, we have:

(i) if $X \subseteq Y$, then $\langle X\rangle \subseteq \langle Y\rangle$;

(ii) $\langle X \cup Y\rangle = \{z \in L \mid z \geq \nu x \odot \nu y, x \in X \text{ and } y \in Y\}$.

Remark 2.16. Let $\mathcal{L}$ be a triangle algebra and $x, y \in L$. Then, $x \leq y$ implies $\langle y\rangle \subseteq \langle x\rangle$.

Definition 2.17. [22] Let $\mathcal{L}$ be a triangle algebra and X a nonempty subset of L. The set $X^\top := \{a \in L \mid \nu a \vee x = 1, \forall x \in X\}$ is called the *co-annihilator* of X. We will denote $\{x\}^\top$ by $x^\top$.

Below are some properties of co-annihilators in triangle algebras.

Proposition 2.18. [22] *Let $\mathcal{L}$ be a triangle algebra. Then, for all nonempty subsets X and Y of L:*

(1) $X^\top$ *is a filter of* L,

(2) $X \subseteq Y$ *implies* $Y^\top \subseteq X^\top$,

(3) $X \subseteq X^{\top\top}$,

(4) $X^\top = X^{\top\top\top}$,

(5) $X^\top = \langle X\rangle^\top$,

(6) $(\bigcup_{i\in I} X_i)^\top = \bigcap_{i\in I} X_i^\top \subseteq (\bigcap_{i\in I} X_i)^\top$, *for all* $X_i \subset L$,

(7) $X^\top = \bigcap_{x\in X} x^\top$,

(8) $\langle X\rangle \cap X^\top = \{1\}$.

Corollary 2.19. [22] *Let $\mathcal{L}$ be a triangle algebra. For any $x, y \in L$, we have:*

(1) $x \leq y$ *implies* $x^\top \subseteq y^\top$,

(2) $x^\top \cap y^\top = (x \odot y)^\top$,

(3) $x^\top = L$ *if and only if* $x = 1$,

(4) $x^\top = (\nu x)^\top$.

3 The lattice of filters of a triangle algebra

In this section, we investigate the lattice structure of all filters of a triangle algebra. Moreover, we state and prove the prime filter theorem in a triangle algebra.

Proposition 3.1. *Let $\mathcal{L}$ be a triangle algebra. Then, $(\mathcal{F}(L), \cap, \bigsqcup, \{1\}, L)$ is a complete distributive lattice, where the operator $\bigsqcup$ is defined by $F \bigsqcup G = \langle F \cup G \rangle$, for any $F, G \in \mathcal{F}(L)$.*

Proof. (1) Consider the application $\beta : F \mapsto \langle F \rangle$, for any $F \in \mathcal{P}(L)$, the power set of L. We show that β is a closure operator on $\mathcal{P}(L)$: let $F \in \mathcal{P}(L)$, then we have $\beta(\beta(F)) = \langle\langle F \rangle\rangle = \langle F \rangle = \beta(F)$. Also, $F \subseteq \langle F \rangle = \beta(F)$. In addition, for any $G \in \mathcal{P}(L)$ such that $F \subseteq G$, we have $\langle F \rangle \subseteq \langle G \rangle$, that is, $\beta(F) \subseteq \beta(G)$. Therefore, β is a closure operator on $\mathcal{P}(L)$. Moreover, $\mathcal{F}(L)$ is the set of closed elements of β, since $\beta(F) = F$, for any $F \in \mathcal{F}(L)$. Hence, applying Proposition 2.9, $\mathcal{F}(L)$ is a complete lattice in which $F \bigsqcup G = \langle F \cup G \rangle$, for any $F, G \in \mathcal{F}(L)$.

(2) Let $F, G, H \in \mathcal{F}(L)$. Since $G \cap H \subseteq G, H$, it follows that $F \bigsqcup (G \cap H) \subseteq (F \bigsqcup G) \cap (F \bigsqcup H)$.

To show the converse, let $a \in (F \bigsqcup G) \cap (F \bigsqcup H)$. Then, $a \in \langle F \cup G \rangle$ and $a \in \langle F \cup H \rangle$, that is, there are $x, t \in F$, $y \in G$ and $z \in H$ such that $a \geq \nu x \odot \nu y$ and $a \geq \nu t \odot \nu z$, that is, $a \vee a \geq (\nu x \odot \nu y) \vee (\nu t \odot \nu z) \geq (\nu x \odot \nu t \odot \nu y) \vee (\nu x \odot \nu t \odot \nu z) \overset{RL8}{=} (\nu x \odot \nu t) \odot (\nu y \vee \nu z) = \nu(x \odot t) \odot \nu(y \vee z)$. But $x \odot t \in F$ and $y \vee z \in G \cap H$, which implies that $\nu(x \odot t) \in F$ and $\nu(y \vee z) \in G \cap H$. Thus, $a \in \langle F \cup (G \cap H) \rangle$, and therefore, $(F \bigsqcup G) \cap (F \bigsqcup H) \subseteq F \bigsqcup (G \cap H)$.

Hence, $(\mathcal{F}(L), \cap, \bigsqcup, \{1\}, L)$ is a complete distributive lattice. □

We recall that given a lattice $(L, \wedge, \vee)$ (with 0 as the smallest element), the pseudocomplement of an element $a \in L$ is the greatest element a^* of L such that $a \wedge a^* = 0$. A lattice is said to be pseudocomplemented if all its elements have a pseudocomplement (see [1]).

Proposition 3.2. *Let $\mathcal{L}$ be a triangle algebra. Then, the lattice $(\mathcal{F}(L), \cap, \bigsqcup, \{1\}, L)$ is pseudocomplemented, and for all $F \in \mathcal{F}(L)$, the pseudocomplement of F is $F^\top$.*

Proof. From Proposition 2.18(8), we have $F \cap F^\top = \{1\}$. Now let $G \in \mathcal{F}(L)$ such that $F \cap G = \{1\}$. We know that for any $a \in G$, $\nu a \in G$. Let $x \in F$, then $x \leq \nu a \vee x$ and $\nu a \leq \nu a \vee x$. This implies that $\nu a \vee x \in F \cap G$. Thus, $\nu a \vee x = 1$ (since $F \cap G = \{1\}$). Therefore, $a \in F^\top$ and hence $G \subseteq F^\top$. □

Lemma 3.3. *Let $\mathcal{L}$ be a triangle algebra, $x, y \in L$. Then, we have:*

(1) $\langle x \rangle \bigsqcup \langle y \rangle = \langle x \odot y \rangle$.

(2) $\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$.

Proof. Let $x, y \in L$. (1)

Since $x \odot y \leq x, y$, we deduce from Remark 2.16 that, $\langle x \rangle, \langle y \rangle \subseteq \langle x \odot y \rangle$. Thus, $\langle x \odot y \rangle$ is a filter of $\mathcal{L}$ containing $\langle x \rangle \cup \langle y \rangle$.

Now let $F \in \mathcal{F}(L)$ such that $\langle x \rangle, \langle y \rangle \subseteq F$. Then, $x, y \in F$, which implies that $\nu x, \nu y \in F$. For any $t \in \langle x \odot y \rangle$, we have $t \geq [\nu(x \odot y)]^n = (\nu x)^n \odot (\nu y)^n \in F$, $n \in \mathbb{N}^*$.Thus, $t \in F$ and it follows that $\langle x \odot y \rangle \subseteq F$. we have just shown that $\langle x \odot y \rangle$ is the smallest filter of $\mathcal{L}$ containing $\langle x \rangle \cup \langle y \rangle$. Hence, $\langle x \rangle \bigsqcup \langle y \rangle = \langle x \odot y \rangle$.

(2) By applying Remark 2.16 on $x, y \leq x \vee y$, we obtain $\langle x \vee y \rangle \subseteq \langle x \rangle, \langle y \rangle$, implying that, $\langle x \vee y \rangle \subseteq \langle x \rangle \cap \langle y \rangle$.

Conversely, let $t \in \langle x \rangle \cap \langle y \rangle$. Then, there are $m, n \in \mathbb{N}^*$ such that $t \geq (\nu x)^n$ and $t \geq (\nu y)^m$, which implies that $t \geq (\nu x)^n \vee (\nu y)^m \overset{(RL8)}{\geq} [\nu(x \vee y)]^{mn}$. Thus, $t \in \langle x \vee y \rangle$, that is, $\langle x \rangle \cap \langle y \rangle \subseteq \langle x \vee y \rangle$. Hence, $\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$. □

Proposition 3.4. *Let $\mathcal{L}$ be a triangle algebra and $F, G \in \mathcal{F}(L)$. Then, the set $\{x \in L \mid \langle \nu x \rangle \cap F \subseteq G\}$ is a filter of $\mathcal{L}$.*

Proof. Let $F, G \in \mathcal{F}(L)$, and let $M = \{x \in L \mid \langle \nu x \rangle \cap F \subseteq G\}$. Clearly, M is not empty, since $1 \in M$.

• Let $x, y \in L$ such that $x \leq y$ and $x \in M$. Then, $\nu x \leq \nu y$ which implies from Remark 2.16 that $\langle \nu y \rangle \subseteq \langle \nu x \rangle$. Thus, $\langle \nu y \rangle \cap F \subseteq \langle \nu x \rangle \cap F \subseteq G$, as $x \in$M. Therefore, $y \in M$.

• If $x, y \in M$, then $\langle \nu x \rangle \cap F \subseteq G$ and $\langle \nu y \rangle \cap F \subseteq G$. Using Lemma 3.3, we have $\langle \nu(x \odot y) \rangle \cap F = \langle \nu x \odot \nu y \rangle \cap F = (\langle \nu x \rangle \bigsqcup \langle \nu y \rangle) \cap F = (\langle \nu x \rangle \cap F) \bigsqcup (\langle \nu y \rangle \cap F) \subseteq G \bigsqcup G = G$. Thus, $x \odot y \in M$.

• Let $x \in L$ such that $x \in M$. Then $\langle \nu \nu x \rangle \cap F = \langle \nu x \rangle \cap F \subseteq G$, that is, $\nu x \in M$.

Hence, M is a filter of $\mathcal{L}$. □

We recall from [8] that a Heyting algebra is a lattice $(L, \wedge, \vee)$ with 0 such that for every $a, b \in L$, there exists an element $a \to b \in L$ (called the pseudocomplement of a with respect to b) such that for every $x \in L$, $a \wedge x \leq b$ if and only if $x \leq a \to b$ (that is, $a \to b = \sup\{x \in L : a \wedge x \leq b\}$).

Theorem 3.5. *Let $\mathcal{L}$ be a triangle algebra.*

Then, $(\mathcal{F}(L), \cap, \bigsqcup, \to, \{1\}, L)$ is a complete Heyting algebra in which $F \to G := \{x \in L \mid \langle \nu x \rangle \cap F \subseteq G\}$, for all $F, G \in \mathcal{F}(L)$.

Proof. It follows from Proposition 3.1 that $(\mathcal{F}(L), \cap, \bigsqcup, \to, \{1\}, L)$ is a complete distributive lattice.

Now let $F, G, H \in \mathcal{F}(L)$. According to Proposition 3.4, the set $\{x \in L \mid \langle \nu x \rangle \cap F \subseteq G\}$ is a filter, that is, $F \to G \in \mathcal{F}(L)$. In addition assume that $H \cap F \subseteq G$. Then, for any $x \in H$, we have $\langle \nu x \rangle \subseteq H$, thus $\langle \nu x \rangle \cap F \subseteq H \cap F \subseteq G$, that is, $x \in F \to G$. Therefore, $H \subseteq F \to G$.

Conversely, assume that $H \subseteq F \to G$ and let $t \in H \cap F$. Then, $t \in H$ and $t \in F$, which implies that $t \in F \to G$, that is, $\langle \nu(t) \rangle \cap F \subseteq G$. Since $t \in \langle t \rangle \subseteq \langle \nu(t) \rangle$ and $t \in F$, we obtain $t \in \langle \nu(t) \rangle \cap F \subseteq G$, thus, $H \cap F \subseteq G$.

Hence, $(\mathcal{F}(L), \cap, \bigsqcup, \to, \{1\}, L)$ is a complete Heyting algebra. □

It is worth noticing that $F^{\top} = F \to \{1\}$, for all $F \in \mathcal{F}(L)$.

For any triangle algebra $\mathcal{L}$, we define $CoAnn(\mathcal{L}) := \{X^{\top} \in \mathcal{F}(L) \mid X \subseteq L\}$.

Theorem 3.6. *Let $\mathcal{L}$ be a triangle algebra.*

Then, $(CoAnn(\mathcal{L}), \cap, \sqcup, ^{\top}, \{1\}, L)$ is a complete Boolean algebra, with $F \sqcup G = (F^{\top} \cap G^{\top})^{\top}$, for any $F, G \in \mathcal{F}(L)$.

Proof. (1) Consider the map $\gamma : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$ defined by $\gamma(X) = X^{\top\top}$, for any $X \subseteq L$.

By Proposition 2.18(4), we have $\gamma(\gamma(X)) = \gamma(X)$. Also, by Proposition 2.18(3), we have $X \subseteq X^{\top\top} = \gamma(X)$. In addition, for any $X, Y \in \mathcal{P}(L)$ such that $X \subseteq Y$, it follows from Proposition 2.18(2) that $X^{\top\top} \subseteq Y^{\top\top}$, that is, $\gamma(X) \subseteq \gamma(Y)$. Hence, γ is a closure operator on $\mathcal{P}(L)$.

Moreover, for any $F \in CoAnn(\mathcal{L})$, there is $X \subseteq L$ such that $F = X^{\top}$. From Proposition 2.18(4), $CoAnn(\mathcal{L})$ is the set of closed elements of γ. Therefore, applying Proposition 2.9 yields that $CoAnn(\mathcal{L})$ is a complete lattice.

(2) Since $CoAnn(\mathcal{L}) \subseteq \mathcal{F}(L)$ which is a distributive lattice, so is $CoAnn(\mathcal{L})$.

(3) Let $F \in CoAnn(\mathcal{L})$, then by Proposition 3.2, $F \cap F^{\top} = \{1\}$. Moreover, applying Proposition 2.18 (8), we have $F \sqcup F^{\top} = (F^{\top} \cap F^{\top\top})^{\top} = 1^{\top} = L$.

Hence, $(CoAnn(\mathcal{L}), \cap, \sqcup, ^{\top}, \{1\}, L)$ is a complete Boolean algebra. □

Theorem 3.7. *(Prime filter theorem) Let F be a filter of a triangle algebra $\mathcal{L}$ and I a $\vee$-closed set such that $F \cap I = \emptyset$. Then, there is a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let F be a filter and I a $\vee$-closed set of a triangle algebra $\mathcal{L}$ such that $F \cap I = \emptyset$. Let $\prod = \{G \in \mathcal{F}(L) \mid F \subseteq G$ and $G \cap I = \emptyset\}$. Obviously, $\prod \neq \emptyset$ as $F \in \prod$. Since $\prod$ satisfies Zorn's lemma criteria, it has a maximal element P. It remains to show that P is prime.

Let $x, y \in L$ such that $\nu(x \vee y) \in P$. Suppose by contrary that $\nu x \notin P$ and $\nu y \notin P$. Then, $P \subsetneq P \bigsqcup \langle \nu x \rangle$ and $P \subsetneq P \bigsqcup \langle \nu y \rangle$. It follows that $P \bigsqcup \langle \nu x \rangle$ and $P \bigsqcup \langle \nu y \rangle$ are filters of $\mathcal{L}$ containing F (since $F \subseteq P \subsetneq P \bigsqcup \langle \nu x \rangle$ and $F \subseteq P \subsetneq P \bigsqcup \langle \nu y \rangle$), and by maximality of P, we obtain $(P \bigsqcup \langle \nu x \rangle) \cap I \neq \emptyset$ and $(P \bigsqcup \langle \nu y \rangle) \cap I \neq \emptyset$. Thus, for any $a \in (P \bigsqcup \langle \nu x \rangle) \cap I$ and $b \in (P \bigsqcup \langle \nu y \rangle) \cap I$, we have $a \vee b \in I$. Since $a, b \leq a \vee b$, we deduce that $a \vee b \in P \bigsqcup \langle \nu x \rangle$ and $a \vee b \in P \bigsqcup \langle \nu y \rangle$ (from the fact that $P \bigsqcup \langle \nu x \rangle$ and $P \bigsqcup \langle \nu y \rangle$ are filters of $\mathcal{L}$). Thus, $a \vee b \in (P \bigsqcup \langle \nu x \rangle) \cap (P \bigsqcup \langle \nu y \rangle) = P \bigsqcup (\langle \nu x \rangle \cap \langle \nu y \rangle) = P \bigsqcup (\langle \nu x \vee \nu y \rangle) = P \bigsqcup (\langle \nu(x \vee y) \rangle)$. But by hypothesis, $\nu(x \vee y) \in P$, which implies that $a \vee b \in P \bigsqcup P = P$. Thus, $a \vee b \in P \cap I$ which contradicts the fact that $P \cap I = \emptyset$. Therefore, P is prime. □

Corollary 3.8. *Let $\mathcal{L}$ be a triangle algebra.*

(1) *Let F be a filter of $\mathcal{L}$ and $x \in L$ such that $x \notin F$. Then, there exists a prime filter P of $\mathcal{L}$ such that $F \subseteq P$ and $x \notin P$.*

(2) *For any $x \in L$ such that $x \neq 1$, there is a prime filter P of $\mathcal{L}$ satisfying $x \notin P$;*

(3) $\bigcap\{P \in \mathcal{F}(L) \mid P \in \mathrm{Spec}(L)\} = \{1\}$.

(4) *For any proper filter F of $\mathcal{L}$, there exists a prime filter P of $\mathcal{L}$ containing F.*

Corollary 3.9. *Let F be a proper filter and I a $\vee$-closed set of a triangle algebra $\mathcal{L}$ such that $F \cap I = \emptyset$. Then, there exists a minimal prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Corollary 3.10. *Let $\mathcal{L}$ be a triangle algebra and F a filter of $\mathcal{L}$. Then, for all $x \in L \setminus F$ with $x \neq 1$, there exists a minimal prime filter P such that $F \subseteq P$ and $x \notin P$.*

In the following section, we introduce the notion of *pure filter*, which will be used to describe the stable sets of the stable topology on triangle algebras.

4 Pure filters of triangle algebras

For any filter F of a triangle algebra $\mathcal{L}$, consider the set

$$\sigma(F) := \{x \in L \mid \exists (y, z) \in x^{\top} \times F \text{ such that } \nu y \odot z = 0\}.$$

Lemma 4.1. *Let $\mathcal{L}$ be a triangle algebra and $F, G \in \mathcal{F}(L)$. Then,*

(1) *$\sigma(F)$ is a filter of $\mathcal{L}$ and $\sigma(F) \subseteq F$;*

(2) *$F \subseteq G$ implies $\sigma(F) \subseteq \sigma(G)$;*

(3) *$\sigma(F \cap G) = \sigma(F) \cap \sigma(F)$;*

(4) *$\sigma(F) \bigsqcup \sigma(F) \subseteq \sigma(F \bigsqcup G)$.*

Proof. Let $F, G \in \mathcal{F}(L)$.

(1) We have $\sigma(F) \neq \emptyset$, since $1 \in \sigma(F)$. Let $x_1, x_2 \in L$ such that $x_1 \in \sigma(F)$ and $x_1 \leq x_2$. Then, there is $(y, z) \in x_1^{\top} \times F$ such that $\nu y \odot z = 0$. But, $x_1 \leq x_2$ implies $x_1^{\top} \subseteq x_2^{\top}$. Thus, $y \in x_2^{\top}$, and therefore $x_2 \in \sigma(F)$.

Now, if $x_1, x_2 \in \sigma(F)$, then there are $y_1 \in x_1^\top$, $y_2 \in x_2^\top$ and $z_1, z_2 \in F$ such that $\nu y_1 \odot z_1 = 0 = \nu y_2 \odot z_2$.

We set $z = z_1 \odot z_2$ and $y = y_1 \vee y_2$. Then, $z \in F$, as F is a filter. Moreover, $\nu y \vee (x_1 \odot x_2) \overset{RL8}{\geq} (\nu y \vee x_1) \odot (\nu y \vee x_2) \geq (\nu y_1 \vee x_1) \odot (\nu y_2 \vee x_2) = 1 \odot 1 = 1$. Thus $y \in (x_1 \odot x_2)^\top$.

In addition, $\nu y \odot z = (\nu y_1 \vee \nu y_2) \odot z \overset{RL8}{=} (\nu y_1 \odot z) \vee (\nu y_2 \odot z) = (\nu y_1 \odot z_1 \odot z_2) \vee (\nu y_2 \odot z_2 \odot z_1) = (0 \odot z_2) \vee (0 \odot z_1) = 0 \vee 0 = 0$. Therefore, $x_1 \odot x_2 \in \sigma(F)$.

Let $x \in L$ such that $x \in \sigma(F)$. Then, there is $(y, z) \in x^\top \times F$ such that $\nu y \odot z = 0$. By Remark 2.7(ii) and Proposition 2.6(1), we have $\nu y \odot \nu z = \nu\nu y \odot \nu z = \nu(\nu y \odot z) = \nu 0 = 0$. But, $(y, z) \in x^\top \times F$ such that $\nu y \odot \nu z = 0$ implies $[y \in (\nu x)^\top$ and $\nu z \in F$ such that $\nu y \odot \nu z = 0]$, that is, $\nu x \in \sigma(F)$. Hence, $\sigma(F)$ is a filter of $\mathcal{L}$.

Moreover, for any $x \in \sigma(F)$, there is $(y, z) \in x^\top \times F$ such that $\nu y \odot z = 0$. This implies that $z = z \odot 1 = z \odot (\nu y \vee x) \overset{RL8}{=} (z \odot \nu y) \vee (z \odot x) = 0 \vee (z \odot x) = z \odot x$. Thus, $z \leq x$, which implies that $x \in F$, that is, $\sigma(F) \subseteq F$.

(2) Let $F \subseteq G$ and $x \in \sigma(F)$. Then, $\exists (y, z) \in x^\top \times F$ such that $\nu y \odot z = 0$. But, $z \in F \subseteq G$, which implies that $x \in \sigma(G)$. Thus, $\sigma(F) \subseteq \sigma(G)$.

(3) Since $F \cap G \subseteq F, G$, applying (2), we obtain $\sigma(F \cap G) \subseteq \sigma(F) \cap \sigma(G)$.

Conversely, let $x \in \sigma(F) \cap \sigma(G)$. Then, $x \in \sigma(F)$ and $x \in \sigma(G)$, which implies that there exist $z_1 \in F$, $z_2 \in F$ and $y_1, y_2 \in x^\top$ such that $\nu y_1 \odot z_1 = 0 = \nu y_2 \odot z_2$. Setting $z = z_1 \vee z_2$ and $y = y_1 \odot y_2$, we have $(y, z) \in x^\top \times F \cap G$. It follows that $\nu y \odot z = \nu y \odot (z_1 \vee z_2) = (\nu y \odot z_1) \vee (\nu y \odot z_2) = (\nu y_2 \odot \nu y_1 \odot z_1) \vee (\nu y_1 \odot \nu y_2 \odot z_2) = (\nu y_2 \odot 0) \vee (\nu y_1 \odot 0) = 0 \vee 0 = 0$. Thus, $x \in \sigma(F \cap G)$, that is, $\sigma(F) \cap \sigma(G) \subseteq \sigma(F \cap G)$.

(4) From $F, G \subseteq F \bigsqcup G$, we have $\sigma(F), \sigma(G) \subseteq \sigma(F \bigsqcup G)$. Thus, $\sigma(F) \bigsqcup \sigma(G) \subseteq \sigma(F \bigsqcup G)$. □

Definition 4.2. Let $\mathcal{L}$ be a triangle algebra. A filter F of $\mathcal{L}$ is called *pure filter* of $\mathcal{L}$ if $\sigma(F) = F$ (that is, $F \subseteq \sigma(F)$).

Example 4.3. Let $L = \{[0,0], [0,a], [0,b], [a,a], [b,b], [0,1], [a,1], [b,1], [1,1]\}$. Consider the Hasse diagram pictured in Figure 1 and the operators $\odot$, $\Rightarrow$ displayed in Table 1:

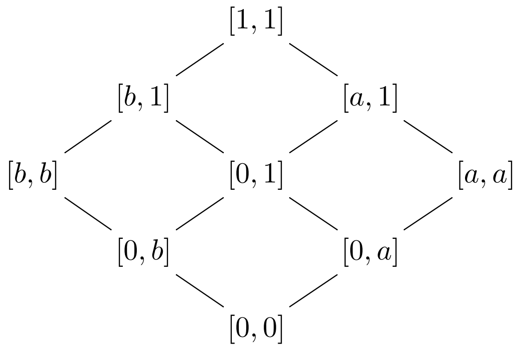

Figure 1: Hasse diagram of $\mathcal{L}$ in Example 4.3

Table 1: Operators $\odot$ and $\Rightarrow$ from Example 4.3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
1	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Defining the operators ν, μ, $\star$ and $\rightarrow$ by:

$$\nu([x_1, x_2]) = [x_1, x_1]$$

$$\mu([x_1, x_2]) = [x_2, x_2]$$

$$[x_1, x_2] \star [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2]$$

$$[x_1, x_2] \rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]\,,$$

and $(L, \wedge, \vee, \star, \rightarrow, \nu, \mu, [0,0]\,, [0,1]\,, [1,1])$ is a triangle algebra [22].

One can easily verify that the filter $F_1 = \{[b,b]\,, [b,1]\,, [1,1]\}$ is a pure filter.

Note that pure filters are not always prime nor maximal, as illustrated by the pure filter $F = \{[1,1]\}$ in Example 4.3, since $\nu([b,1] \vee [a,1]) = [1,1] \in F$, but $\nu[b,1] = [b,b] \notin F$ and $\nu[a,1] = [a,a] \notin F$.

Remark 4.4. For any triangle algebra $\mathcal{L}$, $\{1\}$ and L are trivial pure filters of $\mathcal{L}$.

We present an example of filter of a triangle algebra that is not pure.

Example 4.5. Consider the Hasse diagram depicted in Figure 2, and the operators $\odot$, $\Rightarrow$ given in Table 2. Then, $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice [3].

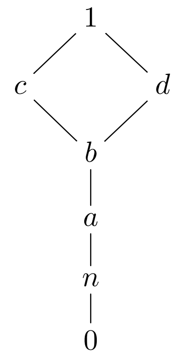

Figure 2: Hasse diagram of $\mathcal{L}$ in Example 4.5

Defining the operators ν, μ, $\star$ and $\rightarrow$ by:

$$\nu([x_1, x_2]) = [x_1, x_1], \quad \mu([x_1, x_2]) = [x_2, x_2],$$

$$[x_1, x_2] \star [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2],$$

$$[x_1, x_2] \rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2],$$

Then $(L, \wedge, \vee, \star, \rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra [11, 14].

The filter $F = \{[c, c], [c, 1], [1, 1]\}$ is not a pure filter of $\mathcal{L}$, since $\sigma(F) = \{[1, 1]\} \neq F$.

We should observe that maximal filters (and hence prime filters) are not always pure filters, as is the case for the filter

$$F = \{[a,a], [a,b], [a,d], [a,c], [a,1], [b,b], [b,c], [b,d], [b,1], [c,c], [c,1], [1,1], [d,d], [d,1], [1,1]\}$$

from Example 4.5 that is maximal, but $\sigma(F) = \{[1, 1]\}$, that is, F is not pure.

Table 2: Operators $\odot$ and $\Rightarrow$ from Example 4.5

$\Rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	1	1	1	1
c	0	n	a	d	1	d	1
d	n	n	a	c	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

Proposition 4.6. *Let $\mathcal{L}$ be a triangle algebra and $F, G \in \mathcal{F}(L)$. If F and G are pure filters of $\mathcal{L}$, so are $F \cap G$ and $F \bigsqcup G$.*

Proof. By using (3) of Lemma 4.1, we have $\sigma(F \cap G) = \sigma(F) \cap \sigma(G) = F \cap G$, that is, $F \cap G$ is a pure filter of $\mathcal{L}$. Also, using (4) of Lemma 4.1, we have $\sigma(F) \bigsqcup \sigma(G) \subseteq \sigma(F \bigsqcup G)$, that is, $F \bigsqcup G \subseteq \sigma(F \bigsqcup G)$. Thus $\sigma(F \bigsqcup G) = F \bigsqcup G$. $\square$

Lemma 4.7. *Let F be a filter of a triangle algebra $\mathcal{L}$ such that $\sigma(F) \neq \{1\}$. Then, there exists $z \in F$ such that $ord(\neg \nu z) = \infty$.*

Proof. Let $x \in \sigma(F)$ such that $x \neq 1$. Then $\exists (y, z) \in x^{\top} \times F$ such that $\nu y \odot z = 0$. This implies that, $\nu y \odot \nu z = \nu \nu y \odot \nu z = \nu(\nu y \odot z) = \nu 0 = 0$. Thus, $\nu z \to \neg \nu y = \neg(\nu y \odot \nu z) = \neg 0 = 1$, that is, $\nu z \leq \neg \nu y$, which implies that $\neg(\neg \nu y) \leq \neg \nu z$, and $(\neg\neg \nu y)^n \leq (\neg \nu z)^n$, for all $n \geq 1$. It suffices to show that $ord(\neg\neg \nu y) = \infty$.

Since $y \in x^{\top}$, we have $\nu y \vee x = 1$, which implies that $\neg\neg\nu y \vee x = 1$ (as $\nu y \leq \neg\neg\nu y$). This implies from $(RL8)$ that $(\neg\neg\nu y)^n \vee x^n = 1$, for all $n \geq 1$. Suppose by contrary that there exists $n \in \mathbb{N}^*$ such that $(\neg\neg\nu y)^n = 0$. This will imply that $x^n = 1$, that is, $x = 1$ (as $x^n \leq x$), which contradicts the assumption $x \neq 1$. Therefore $ord(\neg\neg\nu y) = \infty$, and hence $ord(\neg\nu z) = \infty$. □

Corollary 4.8. *Let $\mathcal{L}$ be a triangle algebra. If $\mathcal{L}$ is local, then $\sigma(F) = \{1\}$, for any proper filter F of $\mathcal{L}$.*

Proof. Assume that $\mathcal{L}$ is local, and suppose by contrary that there is a proper filter F such that $\sigma(F) \neq \{1\}$. According to Lemma 4.7, there exists $z \in F$ such that $ord(\neg\nu z) = \infty$. Since $\mathcal{L}$ is local, we deduce from Proposition 2.13 that , $ord(\nu z) < \infty$, that is, $\exists n \in \mathbb{N}^*$ such that $\nu z^n = 0$. This implies that $0 \in F$ (as $z \in F$), that is, $F = L$, which contradicts the fact that F is proper. □

The subsequent section presents some results on the spectral topology τ_L for a triangle algebra $\mathcal{L}$.

5 The spectral topology for a triangle algebra

Throughout what follows, for a triangle algebra $\mathcal{L} = (L, \wedge, \vee, \nu, \mu, 0, u, 1)$, we set $r(X) := \{P \in \text{Spec }(L) \mid X \nsubseteq P\}$, for all $X \subseteq L$.

Proposition 5.1. *Let $\mathcal{L}$ be a triangle algebra.*

(1) *For all $X, Y \in \mathcal{P}(L)$, $X \subseteq Y$ if and only if $r(X) \subseteq r(Y)$;*

(2) *$r(\{1\}) = \emptyset$ and $r(L) = r(0) = \text{Spec}(L)$;*

(3) *For all $X, Y \in \mathcal{P}(L)$, $r(X) \cap r(Y) = r(X \cap Y)$;*

(4) *If $\{X_i\}_{i\in I} \subseteq \mathcal{P}(L)$, then $\bigcup\limits_{i\in I} r(X_i) = r(\bigcup_{i\in I} X_i)$,*

(5) *For all $X \subseteq L$, $r(\langle X \rangle) = r(X)$.*

Proof. (1) Let X and Y be two subsets of L such that $X \subseteq Y$. For any $P \in r(X)$, we have $X \nsubseteq P$. This implies that $Y \nsubseteq P$, that is, $P \in r(Y)$.

Conversely, if $Y = L$, then clearly $X \subseteq Y$. Now assume that $Y \neq L$ and suppose by contrary that $X \nsubseteq Y$. Then, there exists $x \in X \setminus Y$. According to Corollary 3.8(1), there is a prime filter P such that $Y \subseteq P$ (that is, $P \notin r(Y)$) and $x \notin P$, that is, $x \in X \setminus P$, which means that $X \nsubseteq P$. It follows that $P \notin r(Y)$ and $P \in r(X)$, which contradicts the hypothesis $r(X) \subseteq r(Y)$. (2) Suppose by contrary that $r(\{1\}) \neq \emptyset$ and let $F \in r(\{1\})$. Then, $F \in \text{Spec }(L)$ and $\{1\} \nsubseteq F$, that is, $1 \notin F$, which is not possible since F is a filter. By definition of $r(0)$, it is clear that $r(0) \subseteq \text{Spec }(L)$.

Conversely, for any $F \in \text{Spec }(L)$, F is a proper filter of $\mathcal{L}$, that is, $0 \notin F$, and therefore $\{0\} \nsubseteq F$. Thus $F \in r(0)$ and hence, $r(0) =$Spec (L). Similarly, it follows that $r(L) = \text{Spec}(L)$.

(3) Let $X, Y \in \mathcal{P}(L)$. Since $X \cap Y \subseteq X, Y$, then $r(X \cap Y) \subseteq r(X) \cap r(Y)$, from (1).

Conversely, if $P \in r(X) \cap r(Y)$, then P is a prime filter of $\mathcal{L}$ such that $X \nsubseteq P$ and $Y \nsubseteq P$. Thus, from Proposition 2.12(3), $X \cap Y \nsubseteq P$, that is, $P \in r(X \cap Y)$. Therefore, $r(X) \cap r(Y) \subseteq r(X \cap Y)$.

(4) Let $\{X_i\}_{i \in I} \subseteq L$. Then, for all $P \in \text{Spec }(L)$, we have $P \in r(\bigcup\limits_{i \in I} X_i)$ if and only if $\bigcup_{i \in I} X_i \nsubseteq P$ if and only if $\exists i \in I$ such that $X_i \nsubseteq P$ if and only if $\exists i \in I$ such that $P \in r(X_i)$ if and only if $P \subseteq \bigcup\limits_{i \in I} r(X_i)$. Thus, $\bigcup_{i \in I} r(X_i) = r(\bigcup_{i \in I} X_i)$.

(5) Let $X \subseteq L$ and $P \in \text{Spec }(L)$. Clearly, $X \subseteq P$ if and only if $\langle X \rangle \subseteq P$. Thus, $X \nsubseteq P$ if and only if $\langle X \rangle \nsubseteq P$, that is, $P \in r(X)$ if and only if $P \in r(\langle X \rangle)$. Therefore, $r(X) = r(\langle X \rangle)$. □

If we consider $\tau_L := \{r(X) \mid X \in \mathcal{P}(L)\}$, then we deduce from Proposition 5.1 (2), (3) and (4) that τ_L is a topology on Spec (L), called Zariski topology (or spectral topology).

For illustration, we provide the following example.

Example 5.2. Consider the triangle algebra $\mathcal{L}$ from Example 4.3. Then, $\text{Spec}(L) = \{F_1, F_2\}$, with $F_1 = \{[b, b], [b, 1], [1, 1]\}$ and $F_2 = \{[a, a], [a, 1], [1, 1]\}$. The collections $\tau_{1,L} = \{\emptyset, \{F_1\}, \text{Spec}(L)\}$, $\tau_{2,L} = \{\emptyset, \{F_2\}, \text{Spec}(L)\}$, and $\tau_{3,L} = \{\emptyset, \text{Spec}(L)\}$ are topologies on $\text{Spec}(L)$. But the set $\{\emptyset, \{F_1\}, \{F_2\}, \text{Spec}(L)\}$ is not a spectral topology, since $F_1 \cup F_2 = \{[b, b], [b, 1], \{[a, a], [a, 1], [1, 1]\} \notin \text{Spec}(L)$.

For all $a \in L$, we set $r(a) := \{P \in \text{Spec}(L) \mid a \notin P\}$.

Proposition 5.3. *Let $\mathcal{L}$ be a triangle algebra, $a, b \in L$ and $X \subseteq L$. Then,*

(1) $a \leq b$ *implies* $r(b) \subseteq r(a)$;

(2) $r(a) = r(\langle a \rangle)$;

(3) $r(a) = \emptyset$ *if and only if* $a = 1$;

(4) $r(a) =$*Spec*(L) *if and only if* $\langle a \rangle = L$ *(particularly,* $r(0) =$*Spec*(L)*)*;

(5) $r(a) \cap r(b) = r(\nu(a \vee b))$;

(6) $r(a) \cup r(b) = r(\nu(a \wedge b)) = r(\nu(a \odot b))$;

(7) $r(X) = \bigcup_{x \in X} \{r(x)\}$;

(8) $r(a) = r(b)$ *if and only if* $\langle a \rangle = \langle b \rangle$.

Proof. Let $a, b \in L$ and $X \in \mathcal{P}(L)$.

(1) Assume that $a \leq b$ and $P \in r(b)$. Then, $b \notin P$, implying that $a \notin P$. Thus, $P \in r(a)$ and therefore, $r(b) \subseteq r(a)$;

(2) Let $P \in \mathrm{Spec}(L)$. Since $a \in P$ if and only if $\langle a \rangle \subseteq P$, we have $a \notin P$ if and only if $\langle a \rangle \nsubseteq P$. Thus, $P \in r(a)$ if and only if $P \in r(\langle a \rangle)$. Hence, $r(a) = r(\langle a \rangle)$.

(3) Suppose that $a = 1$. Then, by Proposition 5.1(2), $r(a) = \emptyset$.

Conversely, assume that $r(a) = \emptyset$ and suppose by contrary that $a \neq 1$. Then, according to Corollary 3.8(2), there exists a prime filter P such that $a \notin P$, that is, $P \in r(a)$, contradicting the fact that $r(a) = \emptyset$. Hence $a = 1$.

(4) Consider $r(a) = \mathrm{Spec}(L)$. Assume by contrary that $\langle a \rangle \neq L$. Then, according to Corollary 3.8(4), there exists a prime filter P such that $\langle a \rangle \subseteq P$, that is, $a \in P$. This implies that $P \notin r(a)$, which is a contradiction to $r(a) = \mathrm{Spec}(L)$. Therefore, $\langle a \rangle = L$.

Conversely, if $\langle a \rangle = L$, then by Proposition 5.1 (2) and (5), we have $r(a) = r(\langle a \rangle) = r(L) =\mathrm{Spec}(L)$.

(5) Let $P \in r(a) \cap r(b)$. Then, $a \notin P$ and $b \notin P$, which implies that $\nu a \notin P$ and $\nu b \notin P$. Then, $\nu(a \vee b) \notin P$, since P is prime. Therefore, $P \in r(\nu(a \vee b))$, that is, $r(a) \cap r(b) \subseteq r(\nu(a \vee b))$.

Conversely, if $P \in r(\nu(a \vee b))$, then $\nu(a \vee b) \notin P$. Suppose by contrary that $a \in P$ or $b \in P$. Then, $\nu a \in P$ or $\nu b \in P$. Since $\nu a \leq \nu a \vee \nu b$ and $\nu b \leq \nu a \vee \nu b$, then $\nu(a \vee b) = \nu a \vee \nu b \in P$, which is absurd. Thus, $a \notin P$ and $b \notin P$, that is, $P \in r(a)$ and $P \in r(b)$, and therefore $P \in r(a) \cap r(b)$.

Hence $r(\nu(a \vee b)) = r(a) \cap r(b)$.

(6) If $P \in r(a) \cup r(b)$, then $P \in r(a)$ or $P \in r(b)$, that is, $a \notin P$ or $b \notin P$. Suppose by contrary that $\nu(a \wedge b) \in P$. From $\nu(a \wedge b) = \nu a \wedge \nu b \leq \nu a, \nu b$, we deduce that $\nu a, \nu b \in P$, and therefore $a, b \in P$, which is a contradiction. Thus, $\nu(a \wedge b) \notin P$, that is, $P \in r(\nu(a \wedge b))$.

For the converse, let $P \in r(\nu(a \wedge b))$. Then, $\nu(a \wedge b) \notin P$. Suppose by contrary that $a \in P$ and $b \in P$. This implies that $\nu a \in P$ and $\nu b \in P$, and therefore $\nu(a \odot b) = \nu a \odot \nu b \in P$, implying that $\nu(a \wedge b) = \nu a \wedge \nu b \in P$, which is absurd. Thus, $a \notin P$ or $b \notin P$, that is, $P \in r(a)$ or $P \in r(b)$.

Hence, $r(a) \cup r(b) = r(\nu(a \wedge b))$. Analogously, we show that $r(a) \cup r(b) = r(\nu(a \odot b))$.

(7) Let $P \in \mathrm{Spec}(L)$. Then, we have $P \in r(X)$ if and only if $X \nsubseteq P$ if and only if $\exists x \in X$ such that $x \notin P$ if and only if $\exists x \in F$ such that $P \in r(x)$ if and only if $P \in \bigcup_{x \in X} \{r(x)\}$. Thus, $r(X) = \bigcup_{x \in X} \{r(x)\}$.

(8) Using (2) from this Proposition 5.3 and Proposition 5.1(1), we have $r(a) = r(b)$ if and only if $r(\langle a \rangle) = r(\langle b \rangle)$ if and only if $\langle a \rangle = \langle b \rangle$. □

Let $(L, \wedge, \vee)$ be a complete lattice. An element $x \in L$ is said to be *compact* if for all $X \subseteq L$, $x \leq \vee X$ implies that there exists a finite subset Y of X such that $x \leq \vee Y$ (see [8]).

Proposition 5.4. *Let $\mathcal{L}$ be a triangle algebra.*

(1) *The family $\{r(a) \mid a \in L\}$ is a basis for the topology τ_L on Spec(L);*

(2) *The compact open subsets of Spec(L) are exactly the sets of the form $r(a)$, with $a \in L$.*

Proof. (1) For all $X \subseteq L$, let $r(X)$ be an open subset of $\mathrm{Spec}(L)$. Then, $r(X) = \bigcup_{a \in X} \{r(a)\}$, from Proposition 5.3(7).

(2) We first show that for all $a \in L$, $r(a)$ is a compact element in $\mathrm{Spec}(L)$. Let $\{r(a_i)\}_{i \in I}$ be a non empty family of open subsets of $\mathrm{Spec}(L)$ such that $r(a) \subseteq \bigcup_{i \in I} r(a_i)$. Then, from Proposition 5.3 (6), we have $r(a) \subseteq r(\bigodot_{i \in I} \nu(a_i))$. According to Proposition 5.3(2) and Proposition 5.1(1), we have $\langle a \rangle \subseteq \langle \bigodot_{i \in I} \nu(a_i) \rangle$, that is, there are $n \geq 1$ and $i_1, i_2, \cdots, i_n \in I$ such that $\nu(a) \geq \nu a_{i_1} \odot \nu a_{i_2} \odot \cdots \odot \nu a_{i_n}$. We deduce from Proposition 5.3 (1) and (6)

that $r(a) \subseteq r(\nu(a)) \subseteq r(\nu a_{i_1} \odot \nu a_{i_2} \odot \cdots \odot \nu a_{i_n}) = r(a_{i_1}) \cup r(a_{i_2}) \cup \cdots \cup r(a_{i_n})) = \bigcup_{1\leq j\leq n} r(a_j)$. Thus, $r(a) \subseteq \bigcup_{1\leq j\leq n} r(a_j)$. Hence $r(a)$ is compact.

Let $r(X)$ be a compact open subset of Spec(L). Then, according to Proposition 5.3 (7), we have $r(X) = \bigcup_{a\in X} \{r(a)\}$. But $r(X)$ is compact, that is, there are $n \geq 1$ and $a_1, a_2, \cdots, a_n \in X$ such that $r(X) = \bigcup_{1\leq i\leq n} r(a_i) = r(\bigwedge_{1\leq i\leq n} \nu(a_i))$. Hence the result. □

Theorem 5.5. *Let $\mathcal{L}$ be a triangle algebra. Then,*

(1) Spec(L) *is compact;*

(2) Spec(L) *is a T_0-space.*

Proof. (1) According to Proposition 5.1(2), $r(0)$ =Spec(L). But $r(0)$ is compact. Hence, Spec(L) is compact, by Proposition 5.4(2).

(2) Let $P, Q \in$ Spec(L) such that $P \neq Q$. Then, we have $P \nsubseteq Q$ or $Q \nsubseteq P$. Without loss of generality, suppose that $P \nsubseteq Q$. Then, there exists $a \in P$ such that $a \notin Q$, that is, $P \notin r(a)$ and $Q \in r(a)$. Since $r(a)$ is an open set for the topology τ_L, we conclude that Spec(L) is a T_0-space. □

A topological space (X, τ) is said to be connected if for all disjoint open subsets U and V of X, $X = U \cup V$ implies $U = \emptyset$ or $V = \emptyset$. We will denote by $B(L)$ the set of all complemented elements in $\mathcal{L}$.

Theorem 5.6. *Let $\mathcal{L}$ be a triangle algebra. Then the following assertions are equivalent:*

(1) (Spec$(L), \tau_L$) *is connected;*

(2) $B(L) = \{0, 1\}$.

Proof. (1)⇒(2) Suppose that (Spec$(L), \tau_L$) is connected. Clearly $0, 1 \in B(L)$, that is, $\{0, 1\} \subseteq B(L)$.

Conversely, for any $a \in B(L)$, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. But $r(a) \cup r(b) = r(\nu(a \wedge b)) = r(\nu 0) = r(0)$ =Spec(L) and $r(a) \cap r(b) = r(\nu(a \vee b)) = r(\nu 1) = r(1) = \emptyset$. Thus, $r(a) = \emptyset$ or $r(b) = \emptyset$, since (Spec$(L), \tau_L$) is connected). We obtain from Proposition 5.3(3) that, $a = 1$ or $b = 1$. Since, $b = 1$ if and only if $a = 0$, it follows that $a = 1$ or $a = 0$. Hence, $B(L) = \{0, 1\}$.

(2)⇒(1) Suppose that $B(L) = \{0,1\}$. Assume by contrary that $(\mathrm{Spec}(L), \tau_L)$ is not connected, that is, there are open nonempty disjoint subsets P and Q of $\mathrm{Spec}(L)$, such that $\mathrm{Spec}(L) = P \cup Q$. Since the family $\{r(a) \mid a \in L\}$ is a basis for the topology τ_L on $\mathrm{Spec}(L)$, then there are $X, Y \subseteq L$ such that $P = \bigcup_{x \in X} r(x)$ and $Q = \bigcup_{y \in Y} r(y)$. But according to Theorem 5.5(1), $\mathrm{Spec}(L)$ is compact, that is, there exist $x_1, x_2, \cdots, x_n \in X$ and $y_1, y_2, \cdots, y_m \in Y$ such that $P = \bigcup_{1 \le i \le n} r(x_i) = r(\nu(x_1 \odot x_2 \odot \cdots \odot x_n))$ and $Q = \bigcup_{1 \le i \le m} r(y_i) = r(\nu(y_1 \odot y_2 \odot \cdots \odot y_m))$. Setting $x = \nu(x_1 \odot x_2 \odot \cdots \odot x_n)$ and $y = \nu(y_1 \odot y_2 \odot \cdots \odot y_m)$, we obtain $\mathrm{Spec}(L) = r(x) \cup r(y)$ and $r(x) \cap r(y) = \emptyset$. But, $r(0) =\mathrm{Spec}(L) = r(x) \cup r(y) = r(\nu(x \wedge y))$ and $r(1) = \emptyset = r(x) \cap r(y) = r(\nu(x \vee y))$. Thus, from Proposition 5.3(8), $\langle \nu(x \vee y) \rangle = \langle 1 \rangle$ and $\langle \nu(x \wedge y) \rangle = \langle 0 \rangle$, that is, $\nu(x \vee y) = 1$ and $\langle \nu(x \wedge y) \rangle = L$. Therefore, $\nu x \vee \nu y = 1$ and $\langle \nu x \wedge \nu y \rangle = L$, which implies that there exists $n \ge 1$ such that $(\nu x \wedge \nu y)^n = 0$ and also $1 = (\nu x \vee \nu y)^n \overset{RL8}{\le} \nu x^n \vee \nu y^n$, that is, $1 = \nu x^n \vee \nu y^n$. Applying Lemma 2.8, we have $\nu \nu x^n \wedge \nu y^n = \nu x^n \odot \nu y^n = (\nu x \odot \nu y)^n = (\nu x \wedge \nu y)^n = 0$. Which means that $\nu x^n, \nu y^n \in B(L) = \{0,1\}$ and νx^n is a complement of νy^n. If $\nu x^n = 1$, then $\nu x = 1$ and therefore $x = 1$, that is, $P = r(1) = \emptyset$. Which is absurd as P is nonempty. Similarly, if $\nu x^n = 0$, then $\nu y^n = 1$, that is, $y = 1$ and thus $Q = r(1) = \emptyset$. Which is also absurd as Q is nonempty. Hence, $(\mathrm{Spec}(L), \tau_L)$ is connected. □

In the upcoming section, we will describe the stable open sets with respect to the stable topology, using pure filters.

6 The stable topology for a triangle algebra

Given a triangle algebra $\mathcal{L}$, for any filter F of $\mathcal{L}$, the set $r(F) = \{P \in \mathrm{Spec}(L) \mid F \nsubseteq P\}$ is an open set of $\mathrm{Spec}(L)$, while its complement $v(F) := \mathrm{Spec}(L) \setminus r(F) = \{P \in \mathrm{Spec}(L) \mid F \subseteq P\}$ is a closed set of $\mathrm{Spec}(L)$.

Recall that a subset A of a set X is said to be stable under ascent (respectively descent) in X, if for all $x, y \in X$, $x \le y$ and $x \in A$ implies $y \in A$ (respectively $x \le y$ and $y \in A$ imples $x \in A$). Therefore, A is said to be stable in X if A is simultaneously stable under ascent and descent in X.

It follows that $r(F)$ and $v(F)$ are respectively stable under descent and stable under ascent in $\mathrm{Spec}(L)$. Obviously, the clopen sets of $\mathrm{Spec}(L)$ are stable in Spec(L). In this section, we describe the stable sets relative to the stable topology.

Definition 6.1. Let $\mathcal{L}$ be a triangle algebra. We call *stable topology* on $\mathcal{L}$, denoted S_L, the collection of open subsets stable under ascent of Spec(L), that is, $S_L = \{r(F) \in \tau_L \mid F \in \mathcal{F}(L)$ and $r(F)$ is stable under ascent in Spec$(L)\}$.

Example 6.2. Let Spec$(L) = \{F_1, F_2\}$ be the set of prime filters of the triangle algebra $\mathcal{L}$ from Example 5.2. Then, $S_{1,L} = \{\emptyset, \{F_1\}, Spec(L)\}$, $S_{2,L} = \{\emptyset, \{F_2\}, Spec(L)\}$, and $S_{3,L} = \{\emptyset, Spec(L)\}$ are stable topologies on Spec(L).

The example below highlights the fact that the stable and spectral topologies are not identical.

Example 6.3. Consider the triangle algebra $\mathcal{L}$ from Example 4.5. Then, Spec$(L) = \{F_1, F_2, F_3, F_4\}$, where

$$F_1 = \{[c,c], [c,1], [1,1]\}, F_2 = \{[d,d], [d,1], [1,1]\},$$

$$F_3 = \{[b,b], [b,c], [b,d], [b,1], [c,c], [c,1], [d,d], [d,1], [1,1]\},$$

$$F_4 = \{[a,a], [a,b], [a,d], [a,c], [a,1], [b,b], [b,c], [b,d], [b,1], [c,c], [c,1], [d,d], [d,1], [1,1]\}.$$

The collection $\tau_L = \{\emptyset, \{F_1\}, \{F_3\}, \{F_4\}, \text{Spec}(L)\}$, is a spectral topology on Spec(L) but not a stable topology, since $\{F_1\}$ is not stable under ascent, as $F_1 \subseteq F_3$ but $F_3 \notin \{F_1\}$. However, the collection $S_L = \{\emptyset, \text{Spec}(L)\}$ is a stable topology on Spec(L).

Theorem 6.4. *Let $\mathcal{L}$ be a triangle algebra and F a filter of $\mathcal{L}$. Then, $r(F)$ is stable under ascent in Spec(L) if and only if F is a pure filter of $\mathcal{L}$.*

Proof. ($\Leftarrow$) Suppose that F is a pure filter of $\mathcal{L}$, that is, $\sigma(F) = F$, and let $P, Q \in \text{Spec}(L)$ such that $P \subseteq Q$ and $P \in r(F)$. Then, $F \nsubseteq P$, that is, there exists $x \in F$ (and thus, $\nu(x) \in F$) such that $x \notin P$ (that is, $\nu x \notin P$). Since $\sigma(F) = F$, then $x \in \sigma(F)$ and therefore there exists $(y,z) \in x^\top \times F$ such that $\nu y \odot z = 0$. This implies that $\nu(y \odot z) = \nu y \odot \nu z = \nu\nu y \odot \nu z = \nu(\nu y \odot z) = \nu 0 = 0$. Since $y \in x^\top$, then $\nu y \vee x = 1$, implying that $\nu(y \vee x) \in P$, as

$\nu(y \vee x) = \nu y \vee \nu x = \nu\nu y \vee \nu x = \nu(\nu y \vee x) = \nu 1 = 1 \in P$. But $\nu x \notin P$ and P is a prime filter, thus $\nu y \in P$. Assume by contrary that $Q \notin r(F)$, then $F \subseteq Q$ and therefore $z \in Q$. Since $y, z \in Q$, then $y \odot z \in Q$ meaning that $0 = \nu(y \odot z) \in Q$, that is, $Q = L$, which contradicts the fact that Q is proper. Thus, $r(F)$ is stable under ascent in Spec(L).

($\Rightarrow$) Suppose that $r(F)$ is stable under ascent in Spec(L) and assume by contrary that F is not a pure filter of $\mathcal{L}$. Then, $F \nsubseteq \sigma(F)$, which implies that there exists $x \in F$ such that $x \notin \sigma(F)$, that is, $x \neq 1$. We deduce from Corollary 3.10 that there exists a minimal prime filter P such that $\sigma(F) \subseteq P$ and $x \notin P$. But $F \nsubseteq P$, as P is minimal, that is, $P \in r(F)$. Since $x \notin \sigma(F)$, then for all $(y, z) \in x^\top \times F$, we have $\nu y \odot z \neq 0$, that is, $y \odot z \neq 0$. This implies that $0 \notin x^\top \bigsqcup F$, that is, $x^\top \bigsqcup F$ is a proper filter of $\mathcal{L}$. Then, according to Corollary 3.8(4), there exists a prime filter Q such that $x^\top \bigsqcup F \subseteq Q$. By minimality of P, we have $P \subseteq Q$. Given that $F \subseteq x^\top \bigsqcup F$, it yields that $F \subseteq Q$, that is, $Q \notin r(F)$. We finally obtain $P, Q \in$ Spec(L) such that $P \subseteq Q$ and $P \in r(F)$ but $Q \notin r(F)$, which is absurd since $r(F)$ is stable under ascent in Spec(L). Hence, $\sigma(F) = F$, that is, F is a pure filter of $\mathcal{L}$. □

By duality, we have the following result:

Theorem 6.5. *Let $\mathcal{L}$ be a triangle algebra and F a filter of $\mathcal{L}$. Then, $v(F)$ is closed stable under descent in Spec(L) if and only if F is a pure filter of $\mathcal{L}$.*

Corollary 6.6. *Let $\mathcal{L}$ be a triangle algebra. The map $F \longmapsto r(F)$ is a bijection between the set of pure filters of $\mathcal{L}$ and S_L, as well as $F \longmapsto v(F)$.*

Theorem 6.7. *Let $\mathcal{L}$ be a triangle algebra and F a pure filter of $\mathcal{L}$ such that $F \neq L$. Then, there exists a minimal prime filter P such that $F \subseteq P$.*

Proof. Let F be a pure filter of $\mathcal{L}$ such that $F \neq L$. Suppose by contrary that for any minimal prime filter P, $F \nsubseteq P$. Then, $F \in r(F)$ and since F is a pure filter, $r(F)$ is stable under ascent in Spec(L), from Theorem 6.4. Let $Q \in$ Spec(L), then by Corollary 3.10, there exists a minimal prime filter Q such that $P \subseteq Q$. Since $r(F)$ is stable under ascent in Spec(L) and $P \in r(F)$, then $Q \in r(F)$, that is, Spec$(L) \subseteq r(F)$. Therefore, $r(F) =$ Spec(L). It follows from Proposition 5.3 (4) that $F = L$, which is absurd since $F \neq L$ by assumption. □

Theorem 6.8. *Let $\mathcal{L}$ be a triangle algebra and F a pure filter of $\mathcal{L}$. Let $P_1, P_2 \in Min(L)$ and $P \in Spec(L)$ such that $P_1 \subseteq P$ and $P_2 \subseteq P$. Then, $F \subseteq P_1$ if and only if $F \subseteq P_2$.*

Proof. Let F be a pure filter of a triangle algebra $\mathcal{L}$. Let $P_1, P_2 \in Min(L)$ and $P \in \mathrm{Spec}(L)$ such that $P_1 \subseteq P$ and $P_2 \subseteq P$. Suppose that $F \subseteq P_1$ and assume by contrary that $F \nsubseteq P_2$. Then, $P_2 \in r(F)$. But F is a pure filter of $\mathcal{L}$, which implies from Theorem 6.4 that $r(F)$ is stable under ascent in $\mathrm{Spec}(L)$. Since $P_2 \subseteq P$, then $P \in r(F)$, that is, $F \nsubseteq P$. In addition, $F \subseteq P_1$ and $P_1 \subseteq P$ (by minimality of P_1), therefore $F \subseteq P$ which is absurd (as $F \nsubseteq P$). Thus, $F \subseteq P_2$. □

For any maximal filter M of a triangle algebra $\mathcal{L}$, let $\widehat{M} := \{P \in \mathrm{Spec}(L) \mid P \subseteq M\}$.

Corollary 6.9. *Let $\mathcal{L}$ be a triangle algebra, F a pure filter of $\mathcal{L}$ and M a maximal filter of $\mathcal{L}$. Then, $[F \subseteq P, \forall P \in \widehat{M}]$ or $[F \nsubseteq P, \forall P \in \widehat{M}]$.*

Consequently, for any local triangle algebra $\mathcal{L}$, the topology S_L is trivial.

7 Conclusion

This study aimed to examine the lattice of filters of a triangle algebra and establish the spectral and stable topology on its spectrum. We demonstrated that the set of filters of a triangle algebra forms a complete pseudocomplemented distributive lattice, also known as a Heyting algebra. Additionally, we obtained that the set of co-annihilators of a triangle algebra is a Boolean algebra. Furthermore, we naturally defined the Zariski topology on the set of prime filters of a triangle algebra, showing that it is a compact T_0-space. Moreover, introducing the notion of pure filter, we described the open stable sets of the stable topology on triangle algebras.

In our future work, we will explore the concept of α-filter of triangle algebras, a concept derived from co-annihilators, by examining the set of α-filters of a triangle algebra, as it has been done for residuated lattices and subclasses of residuated lattices [6, 10]. We plan to determine the relations among pure filters, α-filters, and other existing types of filters in triangle algebras. We will also investigate various topological properties of the space of prime α-filters.

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P. Franclin Demlabi *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, Cameroon.*

Email: franklindemlabi345@gmail.com

A.G. Tallee Kakeu *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, Cameroon.*

Email: arianekakeugabie@yahoo.fr

B.B. Koguep Njionou *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, Cameroon.*

Email: blaise.koguep@univ-dschang.org

Celestin Lele *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, Cameroon.*

Email: celestinlele@yahoo.com