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Z-ideals and Z-congruences on semiring $\mathcal{R}^+(L)$

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Abstract. For a frame L, $\mathcal{R}^+(L)$ denotes the nonnegative real valued continuous functions on L. We define the concept of z-ideals in this semiring and give a characterization of its z-ideals in terms of cozero elements of L. Also, we show that there is a one-one correspondence between z-ideals and z-congruences on a ring $\mathcal{R}(L)$ and a semiring $\mathcal{R}^+(L)$. We establish a relationship between z-congruence relation on $\mathcal{R}(L)$ and z-congruence relation on $\mathcal{R}^+(L)$. A new characterization of P-frames is given via z-congruences on $\mathcal{R}^+(L)$. Also, we show that there is a bijection between the minimal prime ideals of $\mathcal{R}(L)$ and coz-ultrafilter on L.

1 Introduction

The notion of semirings was introduced in [25] in 1934. In fact, semirings are algebraic systems that generalize both rings and distributive lattices and have many applications in diverse branches of mathematics and computer

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science. Semirings have two binary operations of addition and multiplication, which are connected by the ring-like distributive laws. However, unlike in rings, subtraction is not allowed in semirings that are not rings. As we know, in the study of ring structure, ideals play an important role; the same is true for semirings. Although the concept of ideal in semirings is different from this concept in rings. The lack of subtraction in semirings shows that many results in rings have no equivalent in semirings. To solve the subtraction problem in semiring, the concept of k-ideal was introduced in [18]. After the introduction of the k-ideal, several studies have been carried out on semirings (see [13, 17, 20, 21]).

The ring $C(X)$ was studied extensively by Gillman and Jerison [15]. The positive cone of this lattice ordered ring is denoted by $C^+(X)$, that is,

$$C^+(X) = \{f \in C(X) : f \geq 0\}.$$

Moreover, $C^+(X)$ is a partially ordered commutative semiring with additive identity 0 and multiplicative identity 1. The semiring $C^+(X)$ emerged as an important area in literature in [2, 3]. Later, Vechtomov et al. [27] studied the semiring $C^+(X)$ extensively. Congruences on semirings were studied by Varankina et al. [26]. They described maximal congruences on semirings $C^+(X)$. In 1993, Acharyya et al. [2] introduced the z-congruence on $C^+(X)$ and showed that there is a bijection between the set of all z-congruences on $C^+(X)$ and the set of all z-filters on X. Mohammadian [22] introduced the concept of positive semirings, and by using the fact that maximal ideals contain an element of a positive semiring, he gave the concept of z-ideals in this kind of semiring and investigated some properties of these ideals.

Since M. Stone worked on Stone duality in the 1930s and showed that topology can be viewed from an algebraic point of view (lattice-theoretic), the pointfree version of $C(X)$ has also been studied. The ring of real valued continuous functions on a frame L, which is the pointfree version of $C(X)$, is the set of all frame homomorphisms $\alpha : \mathcal{L}(\mathbb{R}) \to L$, where $\mathcal{L}(\mathbb{R})$ is the frame of reals, which is isomorphic to the frame of open subsets of $\mathbb{R}$. This ring is denoted by $\mathcal{R}(L)$ (see [4] and [5] for details). In this article, we study the semiring $\mathcal{R}^+(L)$ of all nonnegative real valued continuous functions on L, which is the pointfree version of $C^+(X)$.

This paper is organized as follows. Section 2 presents the basic concepts and preliminaries, which will be used in the next sections. In Section 3, we give a characterization of z-ideals in a semiring $\mathcal{R}^+(L)$ in terms of cozero elements of L, which we shall need throughout. In Section 4, we introduce the concepts of z-congruence in ring $\mathcal{R}(L)$ and semiring $\mathcal{R}^+(L)$. We give a correlation between z-congruences on $\mathcal{R}(L)$ and z-congruences on $\mathcal{R}^+(L)$. Also, we show that there is a one-one correspondence between z-ideals and z-congruences on a ring $\mathcal{R}(L)$ and a semiring $\mathcal{R}^+(L)$. In Section 5, we consider the lattice $\big(z\mathcal{S}\ell(L), \subseteq\big)$, where $z\mathcal{S}\ell(L) := \big\{\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) : \alpha \in \mathcal{R}(L)\big\}$, and consider a z-filter on this lattice. We examine the relationships between z-filters on $\mathcal{S}\ell(L)$ and proper congruences on the ring $\mathcal{R}(L)$ and the semiring $\mathcal{R}^+(L)$. In Section 6, we check equivalence conditions that a frame L is an F-frame and a P-frame. We show that there is a bijection between the minimal prime ideals of $\mathcal{R}(L)$ and coz-ultrafilter on L.

2 Preliminaries

In this section, we give some basic concepts and preliminaries, which will be used in next sections.

2.1 Ring of $\mathcal{R}(L)$ It is well known that a complete lattice L is called a frame if $a \wedge \bigvee X = \bigvee_{x \in X}(a \wedge x)$ for every $(a, X) \in L \times \mathcal{P}(L)$. The frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,
(R3) $(p, q) = \bigvee \big\{(r, s) : p < r < s < q\big\}$,
(R4) $\top = \bigvee \big\{(p, q) : p, q \in \mathbb{Q}\big\}$.

Note that the pairs (p, q) in $\mathcal{L}(\mathbb{R})$ and the open intervals $\langle p, q\rangle = \big\{x \in \mathbb{R} : p < x < q\big\}$ in the frame $\mathfrak{O}\mathbb{R}$ have the same role. Let $\mathcal{R}(L)$ be the set of all frame maps from $\mathcal{L}(\mathbb{R})$ to a completely regular frame L, which is an f-ring. The reader can see [5] for more details of all these facts.

The properties of mapping $\mathrm{coz} : \mathcal{R}(L) \to L$, defined by $\mathrm{coz}(\varphi) = \varphi(-, 0) \vee \varphi(0, -)$, which are often used by us, read as follows:

(1) $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)$.

(2) $\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2)$.

(3) $\alpha \in \mathcal{R}(L)$ is invertible if and only if $\mathrm{coz}(\alpha) = \top$.

(4) $\operatorname{coz}(\alpha) = \bot$ if and only if $\alpha = \mathbf{0}$.

For $A \subseteq \mathcal{R}(L)$, let $\operatorname{Coz}(A) := \{\operatorname{coz}(\alpha) \colon \alpha \in A\}$ and let the cozero part of L, denoted by $\operatorname{Coz}(L)$, be the regular sub-σ-frame consisting of all the cozero elements of L. It is known that L is completely regular if and only if $\operatorname{Coz}(L)$ generates L. For $A \subseteq \operatorname{Coz}(L)$, we write $\operatorname{Coz}^{\leftarrow}(A)$ to designate the family of frame maps $\{\alpha \in \mathcal{R}(L) \colon \operatorname{coz}(\alpha) \in A\}$.

2.2 Sublocale It is well known that a subset S of a frame L is called a sublocale of L if $\bigwedge A \in S$ and $a \longrightarrow s \in S$ for every $(A, a, s) \in \mathcal{P}(S) \times L \times S$. A sublocale is an independent frame, where the meets (and hence the partial order) and the Heyting implication are computed in L. The lattice of all sublocales of a frame L is denoted by $\mathcal{S\ell}(L)$. The finite meet in this lattice is the intersection of them and the join of every subset $\{S_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{S\ell}(L)$ in this lattice is given by

$$\bigvee_{\lambda \in \Lambda} S_\lambda = \Big\{ \bigwedge X \colon X \subseteq \bigcup_{\lambda \in \Lambda} S_\lambda \Big\}.$$

The pair $(\mathcal{S\ell}(L), \subseteq)$ is a coframe, which $\mathsf{O} := \{\top\}$ and L are bottom and top elements of $\mathcal{S\ell}(L)$, respectively. For every $a \in L$, we say $\mathfrak{o}_L(a) := \{a \longrightarrow x \colon (a \longrightarrow x) = x \in L\}$ is an open sublocale of L and $\mathfrak{c}_L(a) := \{x \in L \colon a \leq x\} = {\uparrow} a$ is a closed sublocale of L. Also, the zero and the cozero sublocales corresponding to each $\alpha \in \mathcal{R}(L)$ are, respectively, the sublocales $\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big)$ and $\mathfrak{o}_L\big(\operatorname{coz}(\alpha)\big)$. We denote the set of all zero sublocales of L by $z\mathcal{S\ell}(L)$. Some of the properties of open and closed sublocales, which will be used freely, are as follows:

(1) $\mathfrak{o}_L(\bot) = \mathsf{O} = \mathfrak{c}_L(\top)$ and $\mathfrak{o}_L(\top) = L = \mathfrak{c}_L(\bot)$.

(2) $\mathfrak{o}_L(a \wedge b) = \mathfrak{o}_L(a) \cap \mathfrak{o}_L(b)$ and $\mathfrak{o}_L(\bigvee_i a_i) = \bigvee_i \mathfrak{o}_L(a_i)$.

(3) $\mathfrak{c}_L(a \wedge b) = \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ and $\mathfrak{c}_L(\bigvee_i a_i) = \bigwedge_i \mathfrak{c}_L(a_i)$.

The **closure** of a sublocale S of L, denoted $\overline{S}$ or $\operatorname{cl}_L(S)$, and its **interior**, denoted S° or $\operatorname{int}_L(S)$, are the sublocales

$$\operatorname{cl}_L(S) = \bigcap \{\mathfrak{c}_L(a) \colon S \subseteq \mathfrak{c}_L(a)\} = \mathfrak{c}_L\Big(\bigwedge S\Big),$$

and

$$\operatorname{int}_L(S) = \bigvee \{\mathfrak{o}_L(a) \colon \mathfrak{o}_L(a) \subseteq S\} = \mathfrak{o}_L\Big(\bigwedge (L \setminus S)\Big).$$

2.3 Semiring We recall from [16, 24] that a **semiring** is a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(1) $(S, +)$ is a commutative monoid with identity element 0.

(2) $(S, \cdot)$ is a commutative monoid with identity element 1.

(3) Multiplication distributes over addition.

(4) $0r = 0 = r0$ for all $r \in S$.

An element r of a semiring S is a unit if and only if there exists an element r' of S satisfying $rr' = 1 = r'r$. We denote the set of all units of S by $U(S)$. A semiring S is said to be **positive** if for each $x \in S$, $1 + x \in U(S)$. A nonempty subset I of S is called an **ideal** of S if $a + b \in I$ and $ra \in I$ for all $a, b \in I$ and $r \in S$. An ideal I of S is said to be **proper** if $I \neq S$. Moreover, S and $\{0\}$ are said to be trivial ideals of S. Denote by $\mathcal{ID}(S)$ the family of all ideals of S. For an ideal I of S, the set $\overline{I} = \{x \in S \colon x + a = b \text{ for some } a, b \in I\}$ is called the **subtractive closure** or k**-closure** of I in S. The set $\overline{I}$ is an ideal of S such that $I \subseteq \overline{I}$ and $I = \overline{\overline{I}}$. An ideal I of S is called a **subtractive ideal** or k**-ideal** of S if $\overline{I} = I$. Denote by $\mathcal{KI}(S)$ the family of all k-ideals of S. Also, a proper ideal M of a semiring S is called a **maximal ideal** of S if $M \subseteq I \subseteq S$ for any ideal I of S implies either $I = M$ or $I = S$. We denote the set of all maximal ideals of S by $\mathrm{Max}(S)$.

2.4 Congruence relation on semirings An equivalence relation ρ defined on a semiring S which satisfies the additional condition that if $(a, b) \in \rho$ and $(c, d) \in \rho$ then $(a + c, b + d) \in \rho$ and $(ac, bd) \in \rho$ is called a **congruence** relation. It is easy to see that this definition is equivalent to; a congruence relation ρ on S is an equivalence relation, such that $(a, b) \in \rho$ implies $(a + x, b + x), (ax, bx) \in \rho$ for every $a, b, x \in S$. The family of all congruences on S is denoted by Cong(S). The set Cong(S) with respect to the inclusion generates an algebraic lattice: $\rho \subseteq \tau$ means that $a\rho b$ implies $a\tau b$ for all $a, b \in S$. Congruence ρ on S is called **cancellative** if $(a + x, b + x) \in \rho$ implies $(a, b) \in \rho$ for every $a, b, x \in S$. Also, a cancellative congruence ρ is called **regular congruence** if there exists $(e_1, e_2) \neq (0, 0)$, where e_1 and e_2 are distinct elements in S such that $(a + e_1a, e_2a) \in \rho$ for each $a \in S$.

Consider the ring $\mathcal{R}(L)$. The positive cone of this lattice ordered ring is denoted by $\mathcal{R}^+(L)$, that is,

$$\mathcal{R}^+(L) := \big\{\alpha \in \mathcal{R}(L) \colon \alpha \geq \mathbf{0}\big\}.$$

In fact, $\mathcal{R}^+(L)$ is a partially ordered commutative semiring with additive identity $\mathbf{0}$ and multiplicative identity $\mathbf{1}$. It is easy to see that, for every congruence relation ρ on $\mathcal{R}(L)$,

$$\rho^c := \rho \cap \mathcal{R}^+(L) \times \mathcal{R}^+(L)$$

is a congruence relation on $\mathcal{R}^+(L)$ and for every congruence relation ρ on $\mathcal{R}^+(L)$, the relation

$$\rho^e := \big\{(\alpha, \beta) \colon \alpha, \beta \in \mathcal{R}(L) \text{ and } \alpha - \beta = \gamma - \delta \text{ for some } (\gamma, \delta) \in \rho\big\}$$

is a congruence relation on $\mathcal{R}(L)$.

3 z-ideals in semiring $\mathcal{R}^+(L)$

The concept of z-ideal in positive semirings was introduced in [22]. In this section, we give a characterization of z-ideals in the semiring $\mathcal{R}^+(L)$ in terms of cozero elements of L, which we shall need throughout.

An ideal J of L is said to be **completely regular** if for each $x \in J$, there exists $y \in J$ such that $x \prec\!\!\prec y$. For a completely regular L, the frame of its completely regular ideals is denoted by βL. The join map $\beta L \to L$ is dense onto and referred to as the Stone-Čech compactification of L. We denote its right adjoint by r_L. A straightforward calculation shows that $r_L(a) = \{x \in L \colon x \prec\!\!\prec a\}$ for each $a \in L$. For each $I \in \beta L$, the ideals $\mathbf{M}^I$ and $\mathbf{O}^I$ of $\mathcal{R}(L)$ are defined by $\mathbf{M}^I = \{\varphi \in \mathcal{R}(L) \colon r_L(coz\varphi) \subseteq I\}$ and $\mathbf{O}^I = \{\varphi \in \mathcal{R}(L) \colon r_L(coz\varphi) \prec\!\!\prec I\}$(see [9, 12]).

Clearly, $\mathbf{O}^I \subseteq \mathbf{M}^I$. Since, for any $I \in \beta L$ and $a \in L$, $r_L(a) \prec\!\!\prec I$ if and only if $a \in I$, it follows that $\mathbf{O}^I = \{\varphi \in \mathcal{R}(L) \colon coz(\varphi) \in I\}$. The following is shown in [9]:

1. A subset Q of $\mathcal{R}(L)$ is a maximal ideal iff there is a unique $I \in \Sigma\beta L$ such that $Q = \mathbf{M}^I$, where $\Sigma\beta L$ is the set of all prime elements of βL.

2. If P is an ideal of $\mathcal{R}(L)$, there exists $J \in \beta L$ such that $\mathbf{O}^J \subseteq P \subseteq \mathbf{M}^J$.
3. For any $I \in \Sigma\beta L$, $\mathbf{M}^I$ is the unique maximal ideal containing $\mathbf{O}^I$.

Let $\mathrm{Max}(\mathcal{R}(L))$ be the set of all maximal ideals of a ring $\mathcal{R}(L)$. For $\alpha \in \mathcal{R}(L)$ and $A \subseteq \mathcal{R}(L)$, let $M_\alpha = \bigcap\{M \in Max(\mathcal{R}(L)) \colon \alpha \in M\}$ and $M_A = \bigcap\{M \in Max(\mathcal{R}(L)) \colon M \supseteq A\}$. By [7, Lemma 3.7], for every ideal Q of $\mathcal{R}(L)$,

$$M_Q = \big\{\varphi \in \mathcal{R}(L) \colon r_L\big(\mathrm{coz}(\varphi)\big) \leq \bigvee_{\alpha \in Q} r_L\big(\mathrm{coz}(\alpha)\big)\big\},$$

and for every $\alpha \in \mathcal{R}(L)$,

$$M_\alpha = \big\{\varphi \in \mathcal{R}(L) \colon \mathrm{coz}(\varphi) \leq \mathrm{coz}(\alpha)\big\}.$$

Now, let S be a semiring, let $a \in S$, and let M_a be the intersection of all maximal ideal containing a. If S is a positive semiring, then by [22, Theorem 2],

$$M_a = \big\{x \in S \colon \forall y \in S, a + y \notin U(S) \Rightarrow a + x + y \notin U(S)\big\}.$$

Definition 3.1. An ideal I of semiring S is called a **z-ideal** if for every $a \in I$, $M_a \subseteq I$.

We recall that for every family $\{a_i\}_{i\in I}$ of elements of L, $\bigvee_{i\in I} a_i = \top$ if and only if $\bigcap_{i\in I} \mathfrak{c}_L\big(a_i\big) = \mathsf{O}$. Then a frame L is compact if and only if for every family $\mathcal{F} = \{\mathfrak{c}_L(a_i)\}_{i\in I}$ of closed sublocales of L, $\bigcap \mathcal{F} = \mathsf{O}$ implies there exists a finite subset $\big\{i_1, \ldots, i_n\big\}$ of I such that $\bigcap_{j=1}^{n} \mathfrak{c}_L\big(a_{i_j}\big) = \mathsf{O}$. Also, if L is a compact frame, then there exists a maximal element m of L such that $a \leq m$ for every $a \in L \setminus \big\{\top\big\}$.

In the following proposition, we investigate the relationship between the maximal ideals of the semiring $\mathcal{R}^+(L)$ and the maximal ideals of the ring $\mathcal{R}(L)$. We use the above points to prove this proposition.

Proposition 3.2. *Let M be an ideal of $\mathcal{R}^+(L)$. Then M is a maximal ideal of $\mathcal{R}^+(L)$ if and only if there exists a unique element I of $\Sigma\beta L$ such that $M = \mathbf{M}^I \cap \mathcal{R}^+(L)$.*

Proof. Necessity. Let M be a maximal ideal of $\mathcal{R}^+(L)$. It is evident that for every finite subset A of M,

$$\bigwedge_{\alpha \in A} \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) = \mathfrak{c}_L\left(\mathrm{coz}\left(\sum_{\alpha \in A} \alpha\right)\right) \neq \mathbf{0}.$$

From the compactness of βL, it follows that there exists an element (J, I) in $\beta L \times \Sigma\beta L$ such that $J \in \bigcap_{\alpha\in M} \mathrm{cl}_{\beta L} r_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) \neq \mathsf{O}$ and $J \subseteq I$. Since $M \subseteq \mathbf{M}^J \cap \mathcal{R}^+(L) \subseteq M^I \cap \mathcal{R}^+(L)$, we deduce from the maximality of M that $M = \mathbf{M}^I \cap \mathcal{R}^+(L)$.

Sufficiency. Let $I \in \Sigma\beta L$ with $M = \mathbf{M}^I \cap \mathcal{R}^+(L)$ be given. Suppose $\alpha \in \mathcal{R}^+(L) \setminus M$. Then $\alpha \notin \mathbf{M}^I$, which implies from the maximality of $\mathbf{M}^I$ that there exists an element (β, γ) in $\mathcal{R}(L) \times \mathbf{M}^I$ such that $\alpha\beta + \gamma = \mathbf{1}$, and we obtain $\mathrm{coz}(\alpha^2\beta^2 + \gamma^2) \geq \mathrm{coz}(\alpha\beta + \gamma) = \top$. Thus we have $\alpha^2\beta^2 + \gamma^2 \in (M, \alpha) \cap U\big(\mathcal{R}^+(L)\big)$, which implies that $(M, \alpha) = \mathcal{R}^+(L)$. Therefore, M is a maximal ideal of $\mathcal{R}^+(L)$. □

Let $\alpha \in \mathcal{R}^+(L)$ and let M^+_α be the intersection of all maximal ideal of $\mathcal{R}^+(L)$ containing α. Then

$$M^+_\alpha = \{\beta \in \mathcal{R}^+(L)\colon \forall\gamma \in \mathcal{R}^+(L), \mathrm{coz}(\alpha + \gamma) \neq \top \Rightarrow \mathrm{coz}(\alpha + \beta + \gamma) \neq \top\}.$$

In the following proposition, we give a relation between M_α and M^+_α for every $\alpha \in \mathcal{R}^+(L)$.

Proposition 3.3. *For every $\alpha \in \mathcal{R}^+(L)$, $M^+_\alpha = M_\alpha \cap \mathcal{R}^+(L)$.*

Proof. By Proposition 3.2, we have

$$\begin{aligned} M^+_\alpha &= \bigcap_{\alpha\in M,\ M\in Max(\mathcal{R}^+(L))} M \\ &= \bigcap_{\alpha\in\mathbf{M}^I,\ I\in\Sigma\beta L} \big(\mathbf{M}^I \cap (\mathcal{R}^+(L)\big) \\ &= \Big(\bigcap_{\alpha\in\mathbf{M}^I,\ I\in\Sigma\beta L} \mathbf{M}^I\Big) \cap \mathcal{R}^+(L) \\ &= \Big(\bigcap_{\alpha\in M,\ M\in Max(\mathcal{R}(L))} M\Big) \cap \mathcal{R}^+(L) \\ &= M_\alpha \cap \mathcal{R}^+(L). \end{aligned}$$

□

Now, by the above proposition, we show that the behavior of z-ideals in the semiring $\mathcal{R}^+(L)$ is completely similar to the behavior of z-ideals in the ring $\mathcal{R}(L)$.

Proposition 3.4. *For any ideal Q of $\mathcal{R}^+(L)$, the following conditions are equivalent:*

(1) *Q is a z-ideal.*

(2) *For any $\alpha, \beta \in \mathcal{R}^+(L)$, $\alpha \in Q$ and $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ imply $\beta \in Q$.*

Proof. (1)⇒(2) Suppose $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$, where $\alpha, \beta \in \mathcal{R}^+(L)$ and $\alpha \in Q$. Then, $\beta \in M_\alpha$ and so $\beta \in M_\alpha \cap \mathcal{R}^+(L)$. Hence by Proposition 3.3, $\beta \in M_\alpha^+$. Now, since Q is a z-ideal and $\alpha \in Q$, we have $M_\alpha^+ \subseteq Q$. Therefore $\beta \in Q$.

(2)⇒(1) Let $\alpha \in Q$ and let $\beta \in M_\alpha^+$. Since $\beta \in \mathcal{R}^+(L)$, by Proposition 3.3, we have $\beta \in M_\alpha$ and so $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$. Thus $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\beta)$ and since $\alpha\beta \in Q$, we have $\beta \in Q$ by (2). Then $M_\alpha^+ \subseteq Q$. Therefore Q is a z-ideal. □

4 z-congruences in ring $\mathcal{R}(L)$ and semiring $\mathcal{R}^+(L)$

In this section, we introduce the concept of z-congruence and study the relationship between z-congruence and z-ideals in ring $\mathcal{R}(L)$ and semiring $\mathcal{R}^+(L)$.

Let L be a completely regular frame. We recall from [15] the following concepts:

(1) If $\mathcal{F}$ is a proper filter on $\mathrm{Coz}(L)$, then it is called a **coz-filter** on L.

(2) A prime coz-filter on L is a coz-filter $\mathcal{F}$ such that $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in \mathcal{F}$ implies $\mathrm{coz}(\alpha) \in \mathcal{F}$ or $\mathrm{coz}(\beta) \in \mathcal{F}$.

(3) A coz-filter $\mathcal{G}$ on L is a **coz-ultrafilter** if whenever $\mathcal{G} \subseteq \mathcal{F}$, where $\mathcal{F}$ is a coz-filter, then $\mathcal{G} = \mathcal{F}$.

Definition 4.1. Throughout this paper, we define

(1) $\rho_Q := \{(\alpha, \beta) \colon \alpha, \beta \in \mathcal{R}(L) \text{ and } \alpha - \beta \in Q\}$ for every subset Q of $\mathcal{R}(L)$

(2) $\rho_Q^+ := \{(\alpha, \beta) \colon \alpha, \beta \in \mathcal{R}^+(L) \text{ and } \alpha - \beta \in Q\}$ for every subset Q of $\mathcal{R}^+(L)$.

(3) $Q_\rho := \{\alpha - \beta \colon \alpha, \beta \in \mathcal{R}(L) \text{ and } (\alpha, \beta) \in \rho\}$ for every binary relation ρ on $\mathcal{R}(L)$.

(4) $Q^+_\rho := \{\alpha-\beta\colon \alpha,\beta \in \mathcal{R}^+(L) \text{ and } (\alpha,\beta)\in\rho\}$ for every binary relation ρ on $\mathcal{R}^+(L)$.

Definition 4.2. We call a proper congruence ρ on

(1) $\mathcal{R}(L)$ a z**-congruence** if $\alpha,\beta \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha-\beta) \in \mathrm{coz}(Q_\rho)$ implies $(\alpha,\beta)\in\rho$.

(2) $\mathcal{R}^+(L)$ a z**-congruence** if $\mathrm{coz}(\alpha-\beta) \in \mathrm{coz}(Q^+_\rho)$ implies $(\alpha,\beta)\in\rho$ for every $\alpha,\beta\in\mathcal{R}^+(L)$.

Also, the set of all z-congruences on $\mathcal{R}(L)$ (or on $\mathcal{R}^+(L)$) will be denoted by $\mathrm{zCong}(L)$ (or $\mathrm{zCong}^+(L)$).

Remark 4.3. Let ρ be a z-congruence on $\mathcal{R}^+(L)$, and let $\alpha,\beta,\gamma\in\mathcal{R}^+(L)$ with $(\alpha+\gamma,\beta+\gamma)\in\rho$ be given. Then $\mathrm{coz}(\alpha-\beta)=\mathrm{coz}\big(\alpha+\gamma-(\beta+\gamma)\big)\in \mathrm{coz}(Q^+_\rho)$, which implies by the definition of z-congruence that $(\alpha,\beta)\in\rho$. Therefore, every z-congruence on $\mathcal{R}^+(L)$ is cancellative.

Lemma 4.4. *If ρ is a z-congruence on $\mathcal{R}^+(L)$, then $\rho^{ec}=\rho$.*

Proof. It is clear that $\rho\subseteq\rho^{ec}$. Conversely, let $(\alpha,\beta)\in\rho^{ec}$. Then $\alpha,\beta\in\mathcal{R}^+(L)$ and $(\alpha,\beta)\in\rho^e$, which implies that $\alpha-\beta=\gamma-\delta$ for some $(\gamma,\delta)\in\rho$, and so $\alpha+\delta=\beta+\gamma$. Since ρ is a congruence relation, we have $(\alpha+\gamma+\delta,\beta+\gamma+\delta)\in\rho$. Then, by Remark 4.3, $(\alpha,\beta)\in\rho$ and we see that $\rho^{ec}\subseteq\rho$. Therefore, $\rho^{ec}=\rho$. □

Lemma 4.5. *For every $\alpha,\beta\in\mathcal{R}(L)$, there exist $\gamma,\delta\in\mathcal{R}^+(L)$ such that $\alpha-\beta=\gamma-\delta$.*

Proof. We put $(a,b)=\big((\alpha-\beta)(0,-),(\beta-\alpha)(0,-)\big)$,

$$\begin{cases} A:=\mathfrak{c}_L(a), \\ B:=\mathfrak{c}_L(b), \end{cases} \quad \begin{cases} \gamma_1:=\nu_A(\alpha-\beta), \\ \gamma_2:=\nu_B\mathbf{0}, \end{cases} \quad \text{and} \quad \begin{cases} \delta_1:=\nu_A\mathbf{0}, \\ \delta_2:=\nu_B(\beta-\alpha). \end{cases}$$

It is evident that for every $p,q\in\mathbb{Q}$,

$$\begin{aligned} \delta_2(p,q)\vee a\vee b &= (\beta-\alpha)(p,q)\vee(\beta-\alpha)(0,-)\vee\mathrm{coz}(\alpha-\beta) \\ &= \begin{cases} \top & \text{if } p<0<q \\ \mathrm{coz}(\alpha-\beta) & \text{if } 0\le p \text{ or } q<0 \end{cases} \\ &= \delta_1(p,q)\vee\mathrm{coz}(\alpha-\beta) \\ &= \delta_1(p,q)\vee a\vee b, \end{aligned}$$

and a similar argument shows that

$$\gamma_1(p,q) \vee a \vee b = \gamma_2(p,q) \vee a \vee b.$$

Since $a \wedge b = \bot$, we conclude from [6, Proposition 1.7] that there exists a pair unique elements γ, δ in $\mathcal{R}(L)$ such that

$$\begin{cases} \nu_A\delta(s) = \delta(s) \vee a = \delta_1(s), \\ \nu_B\delta(s) = \delta(s) \vee b = \delta_2(s), \end{cases} \quad \begin{cases} \nu_A\gamma(s) = \gamma(s) \vee a = \gamma_1(s), \\ \nu_B\gamma(s) = \gamma(s) \vee b = \gamma_2(s), \end{cases}$$

for all $s \in \mathcal{L}(\mathbb{R})$. Then for every $p, q \in \mathbb{Q}$,

$$\begin{aligned}
(\gamma-\delta)(p,-) &= \big[(\gamma-\delta)(p,-) \vee a\big] \wedge \big[(\gamma-\delta)(p,-) \vee b\big] \\
&= \left[\bigvee_{t\in\mathbb{Q}} \big(\gamma(t,-) \vee a\big) \wedge \big(\delta(-,t-p) \vee a\big)\right] \wedge \\
&\quad \left[\bigvee_{t\in\mathbb{Q}} \big(\gamma(t,-) \vee b\big) \wedge \big(\delta(-,t-p) \vee b\big)\right]
\end{aligned}$$

$$\begin{aligned}
&= \left[\bigvee_{t\in\mathbb{Q}} \big(\gamma_1(t,-) \wedge \delta_1(-,t-p)\big)\right] \wedge \left[\bigvee_{t\in\mathbb{Q}} \big(\gamma_2(t,-) \wedge \delta_2(-,t-p)\big)\right] \\
&= (\gamma_1-\delta_1)(p,-) \wedge (\gamma_2-\delta_2)(p,-) \\
&= \big(\nu_A(\alpha-\beta) - \nu_A\mathbf{0}\big)\big)(p,-) \wedge \big(\nu_B\mathbf{0} - \nu_B(\beta-\alpha)\big)(p,-) \\
&= \nu_A(\alpha-\beta)(p,-) \wedge \nu_B(\alpha-\beta)\big)(p,-) \\
&= \big((\alpha-\beta)(p,-) \vee a\big) \wedge \big((\alpha-\beta)(p,-) \vee b\big) \\
&= (\alpha-\beta)(p,-) \vee (a \wedge b) \\
&= (\alpha-\beta)(p,-)
\end{aligned}$$

and similarly,

$$(\gamma-\delta)(-,q) = (\alpha-\beta)(-,q).$$

Therefore, $\gamma - \delta = \alpha - \beta$. □

Proposition 4.6. *The following statements are true:*

(1) *If ρ is a z-congruence on $\mathcal{R}(L)$, then ρ^c is a z-congruence on $\mathcal{R}^+(L)$. Moreover,*

$$\{\mathrm{coz}(\alpha-\beta)\colon \alpha,\beta \in \mathcal{R}(L), (\alpha,\beta)\in\rho\} = \{\mathrm{coz}(\alpha-\beta)\colon \alpha,\beta\in\mathcal{R}^+(L), (\alpha,\beta)\in\rho^c\}.$$

(2) *If ρ is a z-congruence on $\mathcal{R}^+(L)$, then ρ^e is a z-congruence on $\mathcal{R}(L)$.*

Proof. (1). Let $\alpha,\beta\in\mathcal{R}^+(L)$ with $\mathrm{coz}(\alpha-\beta)\in\mathrm{coz}(Q^+_{\rho^c})$ be given. Then there exist $\gamma,\delta\in\mathcal{R}^+(L)$ such that $(\gamma,\delta)\in\rho^c$ and $\mathrm{coz}(\alpha-\beta)=\mathrm{coz}(\gamma-\delta)$. Then $(\gamma,\delta)\in\rho$ and $\mathrm{coz}(\alpha-\beta)=\mathrm{coz}(\gamma-\delta)\in\mathrm{coz}(Q_\rho)$. Since ρ is a z-congruence on $\mathcal{R}(L)$, $(\alpha,\beta)\in\rho\cap\big(\mathcal{R}^+(L)\times\mathcal{R}^+(L)\big)=\rho^c$. Therefore, ρ^c is a z-congruence on $\mathcal{R}^+(L)$.

(2). Let $\alpha,\beta\in\mathcal{R}(L)$ with $\mathrm{coz}(\alpha-\beta)\in\mathrm{coz}(Q_{\rho^e})$ be given. Then there exist $(\gamma,\delta)\in\rho^e$ and $(\eta,\mu)\in\rho$ such that $\mathrm{coz}(\alpha-\beta)=\mathrm{coz}(\gamma-\delta)$ and $\gamma-\delta=\eta-\mu$. On the other hand, by Lemma 4.5, there exist $h,k\in\mathcal{R}^+(L)$ such that $\alpha-\beta=h-k$. Hence

$$\mathrm{coz}(h-k)=\mathrm{coz}(\alpha-\beta)=\mathrm{coz}(\gamma-\delta)=\mathrm{coz}(\eta-\mu)\in\mathrm{coz}(Q^+_\rho)(\text{or }\ \mathrm{coz}(Q_{\rho^e})).$$

Since ρ is a z-congruence and $h,k\in\mathcal{R}^+(L)$, we have $(h,k)\in\rho$. Then $(\alpha,\beta)\in\rho^e$. Therefore, ρ^e is a z-congruence on $\mathcal{R}(L)$. □

Proposition 4.7. *The following statements are true:*

(1) *If ρ is a proper congruence relation on $\mathcal{R}(L)$, then Q_ρ is a proper ideal of $\mathcal{R}(L)$ and $\rho=\rho_{Q_\rho}$. In particular, if ρ is a z-congruence relation on $\mathcal{R}(L)$, then Q_ρ is a z-ideal of $\mathcal{R}(L)$.*

(2) *If ρ is a proper congruence relation on $\mathcal{R}^+(L)$, then Q^+_ρ is a proper ideal of $\mathcal{R}^+(L)$ and $\rho=\rho^+_{Q^+_\rho}$. In particular, if ρ is a z-congruence relation on $\mathcal{R}^+(L)$, then Q^+_ρ is a z-ideal of $\mathcal{R}^+(L)$.*

(3) *If Q is a proper ideal of $\mathcal{R}(L)$, then ρ_Q is a proper congruence on $\mathcal{R}(L)$ and $Q=Q_{\rho_Q}$. In particular, if Q is a z-ideal of $\mathcal{R}(L)$, then ρ_Q is a z-congruence on $\mathcal{R}(L)$.*

(4) *If Q is a proper ideal of $\mathcal{R}^+(L)$, then k^+_Q is a proper congruence on $\mathcal{R}^+(L)$ and if Q is a k-ideal of $\mathcal{R}^+(L)$, then $Q=Q^+_{k^+_Q}$. In particular, if a k-ideal Q of $\mathcal{R}^+(L)$ is a z-ideal of $\mathcal{R}^+(L)$, then k^+_Q is a z-congruence on $\mathcal{R}^+(L)$.*

Proof. (1). Let $\eta, \mu \in Q_\rho$ and $\sigma \in \mathcal{R}(L)$ be given. Then there are $\alpha, \beta, \gamma, \delta \in \mathcal{R}(L)$ with $(\alpha, \beta), (\gamma, \delta) \in \rho$ such that $\eta = \alpha - \beta$ and $\mu = \gamma - \delta$. Thus $(\alpha + \gamma, \beta + \delta), (\alpha\sigma, \beta\sigma) \in \rho$ and so, $\eta + \mu, \eta\sigma \in Q_\rho$. If $\mathbf{1} \in Q_\rho$, then $\mathbf{1} = \alpha - \beta$ for some $(\alpha, \beta) \in \rho$, which implies that $(\mathbf{1}, \mathbf{0}) \in \rho$. Consequently, ρ is not proper, and this is a contradiction. Therefore, Q_ρ is a proper ideal of $\mathcal{R}(L)$.

Let $(\alpha, \beta) \in Q_\rho \times \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. Then there exists an element $(\gamma, \delta) \in \rho$ such that $\alpha = \gamma - \delta$, which implies that $\mathrm{coz}(\beta - \mathbf{0}) = \mathrm{coz}(\alpha) = \mathrm{coz}(\gamma - \delta)$, and so $(\beta, \mathbf{0}) \in \rho$. Thus $\beta = \beta - \mathbf{0} \in Q_\rho$. Then Q_ρ is a z-ideal of $\mathcal{R}(L)$.

(2). The proof is similar to the proof of part (1).

(3). Since $\alpha - \alpha = \mathbf{0} \in Q$, we have $(\alpha, \alpha) \in \rho_Q$ for every $\alpha \in \mathcal{R}(L)$. If $\alpha, \beta, \gamma \in \mathcal{R}(L)$ with $(\alpha, \beta), (\beta, \gamma) \in \rho_Q$, then $\alpha - \gamma = \alpha - \beta + \beta - \gamma \in Q$, which implies that $(\alpha, \gamma) \in \rho_Q$. Hence, ρ_Q is an equivalence relation on $\mathcal{R}(L)$. Let $\alpha, \beta, \gamma, \delta \in \mathcal{R}(L)$ with $(\alpha, \beta), (\gamma, \delta) \in \rho_Q$ be given. Then $\alpha - \beta, \gamma - \delta \in Q$, which implies that $(\alpha + \gamma) - (\beta + \delta) \in Q$ and

$$\alpha\gamma - \beta\delta = \alpha\gamma - \beta\gamma + \beta\gamma - \beta\delta = (\alpha - \beta)\gamma + (\gamma - \delta)\beta \in Q.$$

Hence, $(\alpha + \gamma, \beta + \delta), (\alpha\gamma, \beta\delta) \in \rho_Q$. Therefore, ρ_Q is a congruence relation on $\mathcal{R}(L)$. It is evident that $Q = Q_{\rho_Q}$.

If Q is a z-ideal of $\mathcal{R}(L)$ and $\alpha, \beta \in \mathcal{R}(L)$ with

$$\mathrm{coz}(\alpha - \beta) \in \mathrm{coz}(Q_{\rho_Q}) = \mathrm{coz}(Q),$$

then $\alpha - \beta \in Q$, which implies that $(\alpha, \beta) \in \rho_Q$. Hence, ρ_Q is a z-congruence on $\mathcal{R}(L)$.

(4). From $\mathbf{0} \in Q$ and $\alpha + \mathbf{0} = \alpha + \mathbf{0}$, $(\alpha, \alpha) \in k_Q^+$ for every $\alpha \in \mathcal{R}^+(L)$. If $\alpha, \beta, \gamma \in \mathcal{R}^+(L)$ with $(\alpha, \beta), (\beta, \gamma) \in k_Q^+$, then there exist $f, g, h, k \in Q$ such that $\alpha + f = \beta + g$ and $\beta + h = \gamma + k$. Therefore

$$\alpha + f + h = \beta + g + h = \gamma + k + h.$$

Since Q is an ideal of $\mathcal{R}^+(L)$, we conclude that $(\alpha, \gamma) \in k_Q^+$. Hence, k_Q^+ is an equivalence relation on $\mathcal{R}^+(L)$. Let $\alpha, \beta, \gamma, \delta \in \mathcal{R}^+(L)$ with $(\alpha, \beta), (\gamma, \delta) \in k_Q^+$ be given. Then there exist $f, g, h, k \in Q$ such that $\alpha + f = \beta + g$ and $\gamma + h = \delta + k$, which implies that $\alpha + \gamma + f + h = \beta + \delta + g + k$ and

$$\begin{aligned}(\alpha + f)(\gamma + h) = (\beta + g)(\delta + k) \Rightarrow \alpha\gamma + \alpha h + \gamma f + fh \\ = \beta\delta + \beta k + \delta g + gk.\end{aligned}$$

Hence, $(\alpha+\gamma,\beta+\delta),(\alpha\gamma,\beta\delta)\in k_Q^+$. Therefore, k_Q^+ is a congruence relation on $\mathcal{R}^+(L)$.

Now, we show $Q=Q^+_{k_Q^+}$. Let $\alpha\in Q$. Since $\mathbf{0}\in Q$ and $\alpha+\mathbf{0}=\mathbf{0}+\alpha$, so $(\alpha,0)\in k_Q^+$, which implies $\alpha=\alpha-\mathbf{0}\in Q^+_{k_Q^+}$. Conversely, let $\alpha\in Q^+_{k_Q^+}$. Then there exist $f,g\in\mathcal{R}^+(L)$ such that $\alpha=f-g$ and $(f,g)\in k_Q^+$. By the definition of k_Q^+, there exist $\gamma,\delta\in Q$ such that $f+\gamma=g+\delta$. Hence

$$\alpha+\gamma=f+\gamma-g=g+\delta-g=\delta\in Q.$$

Since Q is a k-ideal and $\gamma,\alpha+\gamma\in Q$, we have $\alpha\in Q$. Hence $Q^+_{k_Q^+}\subseteq Q$.

If Q is a z-ideal of $\mathcal{R}^+(L)$ and $\alpha,\beta\in\mathcal{R}^+(L)$ with

$$\operatorname{coz}(\alpha-\beta)\in\operatorname{coz}(Q^+_{k_Q^+})=\operatorname{coz}(Q),$$

then $\alpha-\beta\in Q$, which implies that $(\alpha,\beta)\in k_Q^+$. Hence, k_Q^+ is a z-congruence on $\mathcal{R}^+(L)$. □

We recall from [17] that a proper k-ideal I of a semiring S is called **k-maximal** if it is not properly contained in another proper k-ideal. Hence, by [17, Proposition 3.3], an ideal I of a semiring S is k-maximal if and only if it is a k-ideal and a maximal ideal of S.

Lemma 4.8. *Let Q be a k-ideal of $\mathcal{R}^+(L)$ and let ρ be a cancellable congruence on $\mathcal{R}^+(L)$. Then $k^+_{Q^+_\rho}=\rho$*

Proof. Let $\alpha,\beta\in\mathcal{R}^+(L)$ with $(\alpha,\beta)\in\rho$ be given. Then, there exists $f\in Q^+_\rho$ such that $\alpha-\beta=f$. Hence, $\alpha+0=\beta+f$, which means $(\alpha,\beta)\in k^+_{Q^+_\rho}$. Then, $\rho\subseteq k^+_{Q^+_\rho}$. Conversely, let $\alpha,\beta\in\mathcal{R}^+(L)$ with $(\alpha,\beta)\in k^+_{Q^+_\rho}$ be given. Then there exist $\gamma,\delta\in Q^+_\rho$ such that $\alpha+\gamma=\beta+\delta$, which implies from the definition of Q^+_ρ that $\gamma=f-g$ and $\delta=h-k$ for some $(f,g),(h,k)\in\rho$. Since ρ is a congruence relation, $(g+h,f+k)\in\rho$ and so $(g+h)-(f+k)\in Q^+_\rho$, which implies that $\alpha-\beta\in Q^+_\rho$. Hence, there exist $\alpha',\beta'\in\mathcal{R}^+(L)$ such that $(\alpha',\beta')\in\rho$ and $\alpha-\beta=\alpha'-\beta'$. Then $\alpha'+\beta=\alpha+\beta'$. On the other hand, $(\alpha'+\beta,\beta'+\beta)\in\rho$ and so $(\alpha+\beta',\beta'+\beta)\in\rho$. Since, ρ is a cancellative relation on $\mathcal{R}^+(L)$, we conclude $(\alpha,\beta)\in\rho$. Thus, $k^+_{Q^+_\rho}\subseteq\rho$. □

Proposition 4.9. *The following statements are true:*

(1) *If ρ is a maximal congruence on $\mathcal{R}(L)$, then ρ is a z-congruence on $\mathcal{R}(L)$ and Q_ρ is a maximal ideal of $\mathcal{R}(L)$.*

(2) *If ρ is a maximal congruence on $\mathcal{R}^+(L)$, then ρ is a z-congruence on $\mathcal{R}^+(L)$ and Q_ρ^+ is a maximal ideal of $\mathcal{R}^+(L)$.*

(3) *If M is a maximal ideal of $\mathcal{R}(L)$, then ρ_M is a maximal congruence on $\mathcal{R}(L)$.*

(4) *If M is a k-maximal ideal of $\mathcal{R}^+(L)$, then k_M^+ is a maximal congruence on the class of cancellative congruence on $\mathcal{R}^+(L)$.*

Proof. (1). Let J be a proper ideal of $\mathcal{R}(L)$ such that $Q_\rho \subseteq J$. Then $\rho = \rho_{Q_\rho} \subseteq \rho_J$, and by the maximality of congruences on $\mathcal{R}(L)$, we have $\rho = \rho_J$, which implies $Q_\rho = Q_{\rho_J} = J$. Hence, Q_ρ is a maximal ideal of $\mathcal{R}(L)$. Also, by part (3) of Proposition 4.7, $\rho = \rho_{Q_\rho}$ is a z-congruence on $\mathcal{R}(L)$, since Q_ρ is a z-ideal of $\mathcal{R}(L)$.

(2). The proof is similar to that of part (1).

(3). Let M be a maximal ideal of $\mathcal{R}(L)$. Since every maximal ideal of $\mathcal{R}(L)$ is a z-ideal, we conclude from Proposition 4.7 that ρ_M is a z-congruence on $\mathcal{R}(L)$. Now, let ρ be a congruence on $\mathcal{R}(L)$ and let $\rho_M \subseteq \rho$. Then $M = M_{\rho_M} \subseteq M_\rho$, and by the maximality of ideals of $\mathcal{R}(L)$ we have $M = M_\rho$, which implies $\rho_M = \rho_{M_\rho} = \rho$. Hence, ρ_M is a maximal congruence on $\mathcal{R}(L)$.

(4). Let ρ be a cancellative congruence relation on $\mathcal{R}^+(L)$ and let $k_M^+ \subseteq \rho$. Then $M = M^+_{k_M^+} \subseteq M_\rho^+$, and by the maximality of ideals of $\mathcal{R}^+(L)$ we have $M = M_\rho^+$, which implies from Lemma 4.8 that $k_M^+ = k^+_{M_\rho^+} = \rho$. Hence, k_M^+ is a maximal cancellative congruence on $\mathcal{R}^+(L)$. □

5 z-filter on the ring $\mathcal{R}(L)$ and semiring $\mathcal{R}^+(L)$

It is evident that $\big(z\mathcal{S}\ell(L), \subseteq\big)$ is a lattice. In this section, we examine the relationships between z-filters on $\mathcal{S}\ell(L)$ and proper congruences on the ring $\mathcal{R}(L)$ and the semiring $\mathcal{R}^+(L)$.

Definition 5.1. A proper filter of $z\mathcal{S}\ell(L)$ is called a **z-filter** on $\mathcal{S}\ell(L)$. Therefore, if $\mathcal{F}$ is a z-filter on $\mathcal{S}\ell(L)$, then

(1) $\mathsf{O} \notin \mathcal{F} \subseteq z\mathcal{S}\ell(L)$,

(2) for every $a, b \in \mathcal{F}$, $a \wedge b \in \mathcal{F}$, and

(3) if $b \in \mathcal{F}$, $a \in z\mathcal{S}\ell(L)$, and $b \leq a$, then $a \in \mathcal{F}$.

Also, the set of all z-filter on $\mathcal{S}\ell(L)$ will be denoted by zFil$\mathcal{S}\ell(L)$.

Let $\alpha, \beta \in \mathcal{R}(L)$. We put $E(\alpha, \beta) := \mathfrak{c}_L\big(\operatorname{coz}(\alpha - \beta)\big)$, and $E(\rho) := \big\{E(\alpha, \beta) \colon (\alpha, \beta) \in \rho\big\}$ for every binary relation ρ on $\mathcal{R}(L)$. Also, for every subset Q of $\mathcal{R}(L)$, let

$$\mathfrak{c}_L\operatorname{coz}(Q) := \big\{\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \colon \alpha \in Q\big\}.$$

Proposition 5.2. *Let L be a frame. Then, the following statements are true:*

(1) *If ρ be a proper congruence relation on $\mathcal{R}(L)$, then $E(\rho)$ is a z-filter on $\mathcal{S}\ell(L)$.*

(2) *For any ideal Q of $\mathcal{R}(L)$, $\mathfrak{c}_L\operatorname{coz}[Q] = E(\rho_Q)$.*

(3) *If $\mathcal{F}$ is a z-filter on $\mathcal{S}\ell(L)$, then*

$$E^{-1}(\mathcal{F}) := \big\{(\alpha, \beta) \in \mathcal{R}(L) \times \mathcal{R}(L) \colon \mathfrak{c}_L\big(\operatorname{coz}(\alpha - \beta)\big) \in \mathcal{F}\big\}$$

is a proper congruence on $\mathcal{R}(L)$.

Proof. (1). If $\mathsf{O} \in E(\rho)$, then there exists an element $(\alpha, \beta) \in \rho$ such that $\mathfrak{c}_L\big(\operatorname{coz}(\alpha - \beta)\big) = \mathsf{O}$, which implies that $\alpha - \beta$ is a unit of $\mathcal{R}(L)$. Then

$$\begin{aligned}(\alpha, \beta) \in \rho &\Rightarrow (\alpha - \beta, \mathbf{0}) \in \rho \Rightarrow \big((\alpha - \beta)(\alpha - \beta)^{-1}, \mathbf{0}\big) \in \rho \\ &\Rightarrow (\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}) \in \rho \Rightarrow \rho = \mathcal{R}(L) \times \mathcal{R}(L),\end{aligned}$$

which is a contradiction. Hence, $\mathsf{O} \notin E(\rho)$. Let $z_1, z_2 \in E(\rho)$ be given. Then there exist $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \rho$ such that $z_1 = E(\alpha_1, \beta_1)$ and $z_2 = E(\alpha_2, \beta_2)$. Then

$$\begin{aligned}z_1 \wedge z_2 &= \mathfrak{c}_L\big(\operatorname{coz}(\alpha_1 - \beta_1)\big) \wedge \mathfrak{c}_L\big(\operatorname{coz}(\alpha_2 - \beta_2)\big) \\ &= \mathfrak{c}_L\big(\operatorname{coz}(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \alpha_2^2 - 2\alpha_1\beta_1 - 2\alpha_2\beta_2)\big)\end{aligned}$$

and $(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \alpha_2^2, 2\alpha_1\beta_1 + 2\alpha_2\beta_2) \in \rho$. Hence, $z_1 \wedge z_2 \in E(\rho)$. Let $(z_1, z_2) \in E(\rho) \times z\mathcal{S}\ell(L)$ with $z_1 \subseteq z_2$ be given. Then there exists an element $\big((\alpha_1,\beta_1),\beta\big) \in \rho \times \mathcal{R}(L)$ such that $z_1 = \mathfrak{c}_L\big(\mathrm{coz}(\alpha_1 - \beta_1)\big)$ and $z_2 = \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. From $z_2 = z_1 \vee z_2 = \mathfrak{c}_L\big(\mathrm{coz}(\beta(\alpha_1 - \beta_1)\big)$ and $(\beta\alpha_1, \beta\beta_1) \in \rho$, we conclude $z_2 \in E(\rho)$. Therefore, $E(\rho)$ is a z-filter on $\mathcal{S}\ell(L)$.

(2). Let Q be an ideal of $\mathcal{R}(L)$. If $\alpha \in Q$, then $(\alpha, \mathbf{0}) \in \rho_Q$. Thus $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \in E(\rho_Q)$. If $z \in E(\rho_Q)$, then there exists an element $(\alpha,\beta) \in \rho_Q$ such that $z = \mathfrak{c}_L\big(\mathrm{coz}(\alpha - \beta)\big) \in \mathfrak{c}_L\mathrm{coz}[Q]$. Hence, $\mathfrak{c}_L\mathrm{coz}[Q] = E(\rho_Q)$.

(3). Let $(\alpha,\beta),(\beta,\gamma) \in E^{-1}(\mathcal{F})$ be given. Then

$$\mathfrak{c}_L\big(\mathrm{coz}(\alpha-\beta)\big), \mathfrak{c}_L\big(\mathrm{coz}(\beta-\gamma) \in \mathcal{F},$$

which implies that

$$\begin{aligned}\mathfrak{c}_L\big(\mathrm{coz}(\alpha-\gamma)\big) &\geq \mathfrak{c}_L\big(\mathrm{coz}(\alpha^2+2\beta^2+\gamma^2-2\beta(\alpha-\gamma))\big)\\ &= \mathfrak{c}_L\big(\mathrm{coz}(\alpha-\beta)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta-\gamma)\big) \in \mathcal{F}.\end{aligned}$$

and so, $(\alpha,\gamma) \in E^{-1}(\mathcal{F})$. Hence, $E^{-1}(\mathcal{F})$ is an equivalence relation on $\mathcal{R}(L)$ and $E^{-1}(\mathcal{F}) \neq \mathcal{R}(L) \times \mathcal{R}(L)$. Let $(\alpha,\beta) \in E^{-1}(\mathcal{F})$ and $\gamma \in \mathcal{R}(L)$ be given. Then

$$\begin{aligned}\mathfrak{c}_L\big(\mathrm{coz}(\alpha-\beta)\big) \in \mathcal{F} &\Rightarrow \mathfrak{c}_L\big(\mathrm{coz}(\alpha+\gamma-(\beta+\gamma)\big) \in \mathcal{F} \text{ and}\\ &\qquad \mathfrak{c}_L\big(\mathrm{coz}(\gamma(\alpha-\beta)\big) \geq \mathfrak{c}_L\big(\mathrm{coz}(\alpha-\beta)\big)\\ &\Rightarrow (\alpha+\gamma,\beta+\gamma) \in E^{-1}(\mathcal{F}) \text{ and}\\ &\qquad \mathfrak{c}_L\big(\mathrm{coz}(\gamma(\alpha-\beta)\big) \in \mathcal{F}\\ &\Rightarrow (\alpha+\gamma,\beta+\gamma),(\alpha\gamma,\beta\gamma) \in E^{-1}(\mathcal{F}).\end{aligned}$$

Therefore, $E^{-1}(\mathcal{F})$ is a proper congruence on $\mathcal{R}(L)$. □

Lemma 5.3. *Let L be a frame. Then, the following statements are true:*

(1) *If ρ is a z-congruence on $\mathcal{R}(L)$, then $E^{-1}(E(\rho)) = \rho$.*

(2) *If $\mathcal{F}$ is a z-filter on $\mathcal{S}\ell(L)$, then $E^{-1}(\mathcal{F})$ is a z-congruence on $\mathcal{R}(L)$.*

Proof. (1). Let $\alpha,\beta \in \mathcal{R}(L)$ and let $(\alpha,\beta) \in E^{-1}(E(\rho))$. Then $\mathfrak{c}_L\big(\mathrm{coz}(\alpha - \beta)\big) \in E(\rho)$. Hence there exists $(\gamma,\delta) \in \rho$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha-\beta)\big) = \mathfrak{c}_L\big(\mathrm{coz}(\gamma-\delta)\big)$, and so $\mathrm{coz}(\alpha-\beta) = \mathrm{coz}(\gamma-\delta)$ and $\gamma - \delta \in Q_\rho$. Since ρ is a z-congruence, we conclude $(\alpha,\beta) \in \rho$. Therefore $E^{-1}(E(\rho)) \subseteq \rho$. The converse of inclusion is clear.

(2). Let $\alpha, \beta \in \mathcal{R}(L)$ with $\operatorname{coz}(\alpha - \beta) \in \operatorname{coz}(Q_{E^{-1}(\mathcal{F})})$ be given. Then, there exists $(f, g) \in E^{-1}(\mathcal{F})$ such that $\operatorname{coz}(\alpha - \beta) = \operatorname{coz}(f - g)$. Therefore, $\mathfrak{c}_L\big(\operatorname{coz}(f-g)\big) \in \mathcal{F}$ and $\mathfrak{c}_L\big(\operatorname{coz}(\alpha - \beta)\big) = \mathfrak{c}_L\big(\operatorname{coz}(f-g)\big)$. Hence $\mathfrak{c}_L\big(\operatorname{coz}(\alpha - \beta)\big) \in \mathcal{F}$, and so $(\alpha, \beta) \in E^{-1}(\mathcal{F})$. □

Proposition 5.4. *Let L be a completely regular frame. Then, the following statements are true:*

(1) *If M is a maximal ideal on $\mathcal{R}(L)$, then $\mathfrak{c}_L\operatorname{coz}[M]$ is a z-ultrafilter on $\mathcal{S}\ell(L)$.*

(2) *If $\mathcal{F}$ is a z-ultrafilter on $\mathcal{S}\ell(L)$, then $\big(\mathfrak{c}_L\operatorname{coz}\big)^{\leftarrow}[\mathcal{F}] = \{\alpha \in \mathcal{R}(L) \colon \mathfrak{c}_L(\operatorname{coz}\alpha) \in \mathcal{F}\}$ is a maximal ideal on $\mathcal{R}(L)$.*

Proof. The proof is similar to [15, Theorem 2.5]. □

Proposition 5.5. *Let L be a frame. Then, the following statements are true:*

(1) *If ρ is a maximal congruence relation on $\mathcal{R}(L)$, then $E(\rho)$ is a z-ultrafilter on $\mathcal{S}\ell(L)$.*

(2) *If $\mathcal{F}$ is a z-ultrafilter on $\mathcal{S}\ell(L)$, then $E^{-1}(\mathcal{F})$ is a maximal congruence on $\mathcal{R}(L)$.*

Proof. (1). Since ρ is a maximal congruence relation on $\mathcal{R}(L)$, by Proposition 4.7 and 4.9, Q_ρ is a maximal ideal and $\rho_{Q_\rho} = \rho$. Then, by Proposition 5.4, $\mathfrak{c}_L\operatorname{coz}[Q_\rho]$ is a z-ultrafilter on $\mathcal{S}\ell(L)$. Also, by Proposition 5.2,

$$\mathfrak{c}_L\operatorname{coz}[Q_\rho] = E(\rho_{Q_\rho}) = E(\rho).$$

Then, $E(\rho)$ is a z-ultrafilter on $\mathcal{S}\ell(L)$.

(2). Let $\mathcal{F}$ be a z-ultrafilter on $\mathcal{S}\ell(L)$. Then by Proposition 5.4, $M = \{\alpha \in \mathcal{R}(L) \colon \mathfrak{c}_L\big(\operatorname{coz}(\alpha) \in \mathcal{F}\}$ is a maximal ideal of $\mathcal{R}(L)$, and so ρ_M is a maximal congruence by Proposition 4.9. Moreover, it is easy to see that $\rho_M = E^{-1}(\mathcal{F})$ and so $E^{-1}(\mathcal{F})$ is a maximal congruence on $\mathcal{R}(L)$. □

Lemma 5.6. *Let $\alpha, \beta \in \mathcal{R}^+(L)$ be given. Suppose that $A := \mathfrak{c}_L(a)$ and $B := \mathfrak{c}_L(b)$, where $a := \operatorname{coz}\big((\alpha - \beta)^+\big)$ and $b := \operatorname{coz}\big((\beta - \alpha)^+\big)$. If $h \in \mathcal{R}(L)$ such that $\nu_A h(s) = h(s) \vee \operatorname{coz}\big((\alpha - \beta)^+\big)$ and $\nu_B h(s) = h(s) \vee \operatorname{coz}\big((\beta - \alpha)^+\big)$ for every $s \in \mathcal{L}(\mathbb{R})$, then $\alpha - h, \beta - h \in \mathcal{R}^+(L)$.*

Proof. It is evident that, for every $p \leq 0$,

$$\begin{aligned}\nu_B(\alpha-\beta)(p,-) &= \operatorname{coz}((\beta-\alpha)^+) \vee (\alpha-\beta)(p,-)\\ &= (\beta-\alpha)(0,-) \vee (\alpha-\beta)(p,-)\\ &= (\alpha-\beta)(-,0) \vee (\alpha-\beta)(p,-)\\ &= \top\\ &= \mathbf{0}(p,-).\end{aligned}$$

Now, let $p \leq 0$, then

$$\begin{aligned}(\alpha-h)(p,-) &= (\alpha-h)(p,-) \vee (a \wedge b)\\ &= \big((\alpha-h)(p,-) \vee a\big) \wedge \big((\alpha-h)(p,-) \vee b\big)\\ &= \big((\nu_A\alpha-\nu_A h)(p,-)\big) \wedge \big((\nu_B\alpha-\nu_B h)(p,-)\big)\\ &= \nu_A\alpha(p,-) \wedge (\nu_B\alpha-\nu_B\beta)(p,-)\\ &= \nu_A\alpha(p,-) \wedge \nu_B(\alpha-\beta)(p,-)\\ &= \top \wedge \top = \top\end{aligned}$$

and if $p > 0$, then

$$(\alpha-h)(p,-) \geq \bot = \mathbf{0}(p,-),$$

Therefore $\alpha - h \in \mathcal{R}^+(L)$. Also, from

$$\begin{aligned}p \leq 0 \Rightarrow (\beta-h)(p,-) &= (\beta-h)(p,-) \vee (a \wedge b)\\ &= \big((\beta-h)(p,-) \vee a\big) \wedge \big((\beta-h)(p,-) \vee b\big)\\ &= \big((\nu_A\beta-\nu_A h)(p,-)\big) \wedge \big((\nu_B\beta-\nu_B h)(p,-)\big)\\ &= \nu_A\beta(p,-) \wedge \big((\nu_B\beta-\nu_B\beta)(p,-)\big)\\ &= \nu_A\beta(p,-) \wedge \top\\ &= \top \wedge \top = \top\end{aligned}$$

and

$$p > 0 \Rightarrow (\beta-h)(p,-) \geq \bot = \mathbf{0}(p,-),$$

we infer that $\beta - h \in \mathcal{R}^+(L)$. □

Proposition 5.7. *Let L be a frame. Then, the following statements are true:*

(1) *If ρ be a proper cancellative congruence relation on $\mathcal{R}^+(L)$, then $E(\rho)$ is a z-filter on $\mathcal{S}\ell(L)$.*

(2) *For a k-ideal Q of $\mathcal{R}^+(L)$, $\mathfrak{c}_L\mathrm{coz}[Q] = E(k_Q^+)$.*

(3) *If $\mathcal{F}$ is a z-filter on $\mathcal{S}\ell(L)$, then*

$$E^{-1}(\mathcal{F}) := \{(\alpha, \beta) \in \mathcal{R}^+(L) \times \mathcal{R}^+(L) \colon \mathfrak{c}_L\big(\mathrm{coz}(\alpha - \beta)\big) \in \mathcal{F}\}$$

is a proper congruence on $\mathcal{R}^+(L)$.

Proof. (1). If $\mathsf{0} \in E(\rho)$, then there exists an element $(\alpha, \beta) \in \rho$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha - \beta)\big) = \mathsf{0}$, which implies that $\alpha - \beta$ is a unit of $\mathcal{R}(L)$. Since for every positive real r, $(\alpha + \mathbf{r}, \beta + \mathbf{r}) \in \rho$, without loss of generality, we may assume $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) = \mathsf{0} = \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. We set

$$\begin{cases} A := \mathfrak{c}_L\big(\mathrm{coz}((\alpha - \beta)^+)\big), \\ B := \mathfrak{c}_L\big(\mathrm{coz}((\beta - \alpha)^+)\big), \end{cases} \quad \begin{cases} k_1 := \nu_A \alpha, \\ k_2 := \nu_B \mathbf{0}, \end{cases} \quad \begin{cases} h_1 := \nu_A \mathbf{0}, \\ h_2 := \nu_B \beta. \end{cases}$$

It is evident that

$$\begin{aligned} h_1(s) \vee \mathrm{coz}\big((\alpha - \beta)^+\big) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) &= \top \\ &= h_2(s) \vee \mathrm{coz}\big((\alpha - \beta)^+\big) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) \end{aligned}$$

and that

$$\begin{aligned} k_1(s) \vee \mathrm{coz}\big((\alpha - \beta)^+\big) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) &= \top \\ &= k_2(s) \vee v\mathrm{coz}\big((\alpha - \beta)^+\big) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) \end{aligned}$$

for every $s \in \mathcal{L}(\mathbb{R})$. Then, by [6, Proposition 1.7], there exists a pair unique elements h, k in $\mathcal{R}(L)$ such that

$$\begin{cases} \nu_A h(s) = h(s) \vee \mathrm{coz}\big((\alpha - \beta)^+\big) = h_1(s), \\ \nu_B h(s) = h(s) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) = h_2(s), \end{cases} \begin{cases} \nu_A k(s) = k(s) \vee \mathrm{coz}\big((\alpha - \beta)^+\big) = k_1(s), \\ \nu_B k(s) = k(s) \vee \mathrm{coz}\big((\beta - \alpha)^+\big) = k_2(s), \end{cases}$$

for all $s \in \mathcal{L}(\mathbb{R})$. By Lemma 5.6, $\alpha - h, \beta - h, \alpha - k, \beta - k \in \mathcal{R}^+(L)$ and since ρ is a cancellative congruence, we conclude that $(\alpha - h, \beta - h), (\alpha - k, \beta - k) \in \rho$. Consequently,

$$\big((\alpha - h)(\beta - k), (\beta - h)(\alpha - k)\big) \in \rho.$$

From

$$\operatorname{coz}\big(\nu_B(\beta - h)\big) = \operatorname{coz}\big(\nu_B\beta - \nu_B h)\big) = \operatorname{coz}\big(\mathbf{0}_B\big) = \operatorname{coz}\big((\beta - \alpha)^+\big)$$

and

$$\operatorname{coz}\big(\nu_A(\alpha - k)\big) = \operatorname{coz}\big(\nu_A\alpha - \nu_A k\big) = \operatorname{coz}\big(\mathbf{0}_A\big) = \operatorname{coz}\big((\alpha - \beta)^+\big)$$

and since $\Big(\operatorname{coz}\big((\alpha-\beta)^+\big)\Big) \wedge \Big(\operatorname{coz}\big((\beta-\alpha)^+\big)\Big) = \bot$, we infer that

$$\begin{aligned}
\operatorname{coz}\big((\beta - h)(\alpha - k)\big) &= \operatorname{coz}(\beta - h) \wedge \operatorname{coz}(\alpha - k) \\
&= \big(\operatorname{coz}(\beta - h) \wedge \operatorname{coz}(\alpha - k)\big) \vee \\
&\quad \big(\operatorname{coz}\big((\alpha - \beta)^+\big) \wedge \operatorname{coz}\big((\beta - \alpha)^+\big)\big) \\
&= \operatorname{coz}\big(\nu_A(\beta - h)\big) \wedge \operatorname{coz}\big(\nu_B(\beta - h)\big) \wedge \\
&\quad \operatorname{coz}\big(\nu_A(\alpha - k)\big) \wedge \operatorname{coz}\big(\nu_B(\alpha - k)\big) \\
&= \operatorname{coz}\big((\alpha - \beta)^+\big) \wedge \operatorname{coz}\big((\beta - \alpha)^+\big) \wedge \\
&\quad \operatorname{coz}\big(\nu_A(\beta - h)\big) \wedge \operatorname{coz}\big(\nu_B(\alpha - k)\big) \\
&= \bot,
\end{aligned}$$

which implies that $(\beta - h)(\alpha - k) = \mathbf{0}$. Moreover,

$$\begin{aligned}
\operatorname{coz}\big(\nu_A(\beta - k)(\alpha - h)\big) &= \operatorname{coz}\big((\nu_A\beta - \nu_A k)(\nu_A\alpha - \nu_A h)\big) \\
&= \operatorname{coz}\big((\nu_A\beta - \nu_A\alpha)(\nu_A\alpha - \nu_A\mathbf{0})\big) \\
&= \operatorname{coz}\big(\nu_A(\beta - \alpha)\alpha\big) \\
&= \operatorname{coz}\big((\alpha - \beta)^+\big) \vee \operatorname{coz}\big((\beta - \alpha)\alpha\big) \\
&= \operatorname{coz}\big((\alpha - \beta)^+\big) \vee \top \\
&= \top.
\end{aligned}$$

A similar argument shows that $\operatorname{coz}\big(\nu_B(\beta - k)(\alpha - h)\big) = \top$. Hence,

$$\begin{aligned}
\operatorname{coz}\big((\alpha - h)(\beta - k)\big) &= \operatorname{coz}\big((\alpha - h)(\beta - k)\big) \vee \\
&\quad \big(\operatorname{coz}\big((\alpha - \beta)^+\big) \wedge \operatorname{coz}\big((\beta - \alpha)^+\big)\big) \\
&= \operatorname{coz}\big(\nu_A(\alpha - h)(\beta - k)\big) \wedge \operatorname{coz}\big(\nu_B(\alpha - h)(\beta - k)\big) \\
&= \top,
\end{aligned}$$

which implies that $(\alpha - h)(\beta - k)$ has a multiplicative inverse in $\mathcal{R}^+(L)$. Then

$$\begin{aligned}((\alpha - h)(\beta - k), \mathbf{0}) \in \rho &\Rightarrow \big((\alpha - h)(\beta - k)(\alpha - h)^{-1}(\beta - k)^{-1}, \mathbf{0}\big) \in \rho \\ &\Rightarrow (\mathbf{1}, \mathbf{0}) \in \rho \\ &\Rightarrow \rho = \mathcal{R}(L) \times \mathcal{R}(L),\end{aligned}$$

which is a contradiction. Hence, $\mathsf{O} \notin E(\rho)$. Let $z_1, z_2 \in E(\rho)$ be given. Then there exist $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \rho$ such that $z_1 = E(\alpha_1, \beta_1)$ and $z_2 = E(\alpha_2, \beta_2)$. Then

$$\begin{aligned}z_1 \wedge z_2 &= \mathfrak{c}_L\big(\mathrm{coz}(\alpha_1 - \beta_1)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\alpha_2 - \beta_2)\big) \\ &= \mathfrak{c}_L\big(\mathrm{coz}(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \alpha_2^2 - 2\alpha_1\beta_1 - 2\alpha_2\beta_2)\big)\end{aligned}$$

and $(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \alpha_2^2, 2\alpha_1\beta_1 + 2\alpha_2\beta_2) \in \rho$. Hence, $z_1 \wedge z_2 \in E(\rho)$. Let $(z_1, z_2) \in E(\rho) \times z\mathcal{S}\ell(L)$ with $z_1 \subseteq z_2$ be given. Then there exists an element $\big((\alpha_1, \beta_1), \beta\big) \in \rho \times \mathcal{R}(L)$ such that $z_1 = \mathfrak{c}_L\big(\mathrm{coz}(\alpha_1 - \beta_1)\big)$ and $z_2 = \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. From $z_2 = z_1 \vee z_2 = \mathfrak{c}_L\big(\mathrm{coz}(\beta(\alpha_1 - \beta_1)\big)$ and $(\beta\alpha_1, \beta\beta_1) \in \rho$, we conclude $z_2 \in E(\rho)$. Therefore, $E(\rho)$ is a z-filter on $\mathcal{S}\ell(L)$.

(2). Let $\alpha \in Q$. Then $\alpha + \mathbf{0} = \mathbf{0} + \alpha$ and so $(\alpha, \mathbf{0}) \in k_Q^+$, which implies that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \in E(k_Q^+)$. If $z \in E(k_Q^+)$, then there exists an element $(\alpha, \beta) \in k_Q^+$ such that $z = \mathfrak{c}_L\big(\mathrm{coz}(\alpha - \beta)\big)$. Since $(\alpha, \beta) \in k_Q^+$, $\alpha + f = \beta + g$ for some $f, g \in Q$. Then $(\alpha - \beta) + f = g$. Since Q is a k-ideal and $f, g \in Q$, we conclude $\alpha - \beta \in Q$, and so $z \in \mathfrak{c}_L\mathrm{coz}[Q]$.

(3). The proof is similar to the proof of Proposition 5.2. □

6 Coz-filters on L and minimal prime ideals in semiring $\mathcal{R}^+(L)$

In this section, we investigate relation between prime coz-filters and prime z-ideals on frame L. Also, we show that there is a bijection between the minimal prime ideals of $\mathcal{R}(L)$, and coz-ultrafilters on L.

Proposition 6.1. *The following statements are true:*

(1) *If P is a prime ideal of $\mathcal{R}(L)$, then* $\mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$ *is a coz-filter. In particular, if P is a prime z-ideal of $\mathcal{R}(L)$, then* $\mathrm{Coz}\big(\mathcal{R}(L)\big) \setminus \mathrm{Coz}\big(P\big)$ *is a prime coz-filter and*

$$\mathcal{R}(L) \setminus P = \mathrm{Coz}^{\leftarrow}\Big(\mathrm{Coz}\big(\mathcal{R}(L)\big) \setminus \mathrm{Coz}\big(P\big)\Big).$$

(2) *If P is a prime ideal of $\mathcal{R}^+(L)$, then* $\mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ *is a coz-filter.*

Proof. (1). Let $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$ be given. We claim $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$. If $\alpha\beta \in \mathcal{R}(L) \setminus P$, then $\alpha, \beta \in \mathcal{R}(L) \setminus P$, which implies that $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$. Now, suppose $\alpha\beta \notin \mathcal{R}(L) \setminus P$ and $\alpha \in P$. By our hypothesis, there exists an element γ in $\mathcal{R}(L) \setminus P$ such that $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\gamma)$, which implies $\alpha^2 + \gamma^2 \in \mathcal{R}(L) \setminus P$ and

$$\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha^2 + \gamma^2) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big),$$

This proves the claim.

Let $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$ be given. By our hypothesis, there exist $\gamma, \delta \in \mathcal{R}(L) \setminus P$ such that $\mathrm{coz}(\alpha) = \mathrm{coz}(\gamma)$ and $\mathrm{coz}(\beta) = \mathrm{coz}(\delta)$, which implies $\gamma\delta \in \mathcal{R}(L) \setminus P$ and $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\gamma\delta) \in \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$.

Let $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L)\big) \setminus \mathrm{Coz}\big(P\big)$ be given. Since P is a z-ideal, $\alpha^2 + \beta^2 \notin P$, which implies $\alpha \notin P$ or $\beta \notin P$. Hence, $\mathrm{coz}(\alpha) \in \mathrm{Coz}\big(\mathcal{R}(L)\big) \setminus \mathrm{Coz}\big(P\big)$ or $\mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}(L)\big) \setminus \mathrm{Coz}\big(P\big)$. The proof is now complete.

(2). Let $\alpha, \beta \in \mathcal{R}^+(L)$ with $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ be given. We claim $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$. If $\alpha\beta \in \mathcal{R}^+(L) \setminus P$, then $\alpha, \beta \in \mathcal{R}(L)^+ \setminus P$, which implies that $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$. Now, suppose $\alpha\beta \notin \mathcal{R}^+(L) \setminus P$ and $\alpha \in P$. By our hypothesis, there exists an element γ in $\mathcal{R}^+(L) \setminus P$ such that $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\gamma)$, which implies that $\alpha + \gamma \in \mathcal{R}^+(L) \setminus P$ and

$$\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha + \gamma) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big),$$

which proves the claim.

Let $\alpha, \beta \in \mathcal{R}^+(L)$ with $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ be given. By our hypothesis, there exist $\gamma, \delta \in \mathcal{R}^+(L) \setminus P$ such that $\mathrm{coz}(\alpha) = \mathrm{coz}(\gamma)$ and $\mathrm{coz}(\beta) = \mathrm{coz}(\delta)$, which implies $\gamma\delta \in \mathcal{R}^+(L) \setminus P$ and $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\gamma\delta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$. □

In the following remark, we show that the primeness is necessary for Proposition 6.1.

Remark 6.2. It is well known that the homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ given by $(p,q) \mapsto < p, q >$ is an isomorphism. Let B be a Boolean frame and let a, b, and c be three atoms in B. By [14, Proposition 4.1], $e_a : \mathcal{L}R \to B$ by

$$e_a(U) = \begin{cases} \top & \text{if } 0,1 \in \tau(U) \\ a' & \text{if } 0 \in \tau(U) \text{ and } 1 \notin \tau(U) \\ a & \text{if } 0 \notin \tau(U) \text{ and } 1 \in \tau(U) \\ \bot & \text{if } 0 \notin \tau(U) \text{ and } 1 \notin \tau(U), \end{cases}$$

is a continuous real valued function on B, $e_a^2 = e_a$, and $\mathrm{coz}(e_a) = a$ for every $a \in B$. We set $Q := e_a\mathcal{R}(B)$. For every $\alpha \in \mathcal{R}(B)$,

$$\alpha \in Q \Rightarrow \mathrm{coz}(\alpha) \leq \mathrm{coz}(e_a) \Rightarrow \mathrm{coz}(\alpha) = \bot \text{ or } \mathrm{coz}(\alpha) = a \Rightarrow \alpha = \mathbf{0} \text{ or } \mathrm{coz}(\alpha) = a.$$

Hence, $e_b, e_c \notin Q$ and $e_b e_c = e_\bot = \mathbf{0} \in Q$, and thus Q is not a prime ideal. Also, from $b, c \in \mathrm{Coz}\big(\mathcal{R}(B) \setminus Q\big)$ and $b \wedge c = \bot$, we infer that $\mathcal{R}(B) \setminus Q$ is not a coz-filter. This shows that primeness is needed in Proposition 6.1.

Proposition 6.3. *The following statements are true:*

1. *If $\mathcal{F}$ is a prime coz-filter on L, then $P := \mathcal{R}(L) \setminus \mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a prime z-ideal of $\mathcal{R}(L)$ and $\mathcal{F} = \mathrm{Coz}\big(\mathrm{Coz}^{\leftarrow}(\mathcal{F})\big)$.*
2. *If $\mathcal{F}$ is a prime coz-filter on L and $Q := \big\{\alpha \in \mathcal{R}^+(L) \colon \mathrm{coz}(\alpha) \in \mathcal{F}\big\}$, then $P := \mathcal{R}^+(L) \setminus Q$ is a prime z-ideal of $\mathcal{R}^+(L)$ and $\mathcal{F} = \mathrm{Coz}(Q)$.*

Proof. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\alpha, \beta \in P$ be given. Then $\mathrm{coz}(\alpha) \notin \mathcal{F}$ and $\mathrm{coz}(\beta) \notin \mathcal{F}$, which follows that $\mathrm{coz}(\alpha - \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \notin \mathcal{F}$, because $\mathcal{F}$ is prime, and this implies that $\alpha - \beta$ belongs to P. Also, since $\mathrm{coz}(\alpha\gamma) \leq \mathrm{coz}(\alpha) \notin \mathcal{F}$, we infer that $\alpha\gamma \in P$. Hence, P is an ideal of $\mathcal{R}(L)$.

Let $(\alpha, \beta) \in P \times \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. Then $\mathrm{coz}(\beta) \notin \mathcal{F}$, which implies that $\beta \in P$. It is clear $P \neq \mathcal{R}(L)$. Thus P is a z-ideal of $\mathcal{R}(L)$.

Let $\alpha, \beta \in \mathcal{R}(L)$ with $\alpha\beta \in P$ be given. Then

$$\begin{aligned} \mathrm{coz}(\alpha\beta) \notin \mathcal{F} &\Rightarrow \mathrm{coz}(\alpha) \notin \mathcal{F} \text{ or } \mathrm{coz}(\beta) \notin \mathcal{F}, \text{ since } \mathcal{F} \text{ is a coz-filter on } L \\ &\Rightarrow \alpha \in P \text{ or } \beta \in P. \end{aligned}$$

Thus, P is a prime z-ideal of $\mathcal{R}(L)$. The rest is evident.

(2). Let $\alpha, \beta \in \mathcal{R}^+(L)$ with $\alpha, \beta \in P$ be given. Then $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \notin \mathcal{F}$, which follows that $\mathrm{coz}(\alpha+\beta) = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \notin \mathcal{F}$, because $\mathcal{F}$ is prime, and this implies that $\alpha+\beta$ belongs to $\mathcal{R}^+(L) \setminus Q$. Also, since $\mathrm{coz}(\alpha\gamma) \leq \mathrm{coz}(\alpha) \notin \mathcal{F}$, we infer $\alpha\gamma \in P$. Hence, P is an ideal of $\mathcal{R}^+(L)$.

Let $(\alpha, \beta) \in P \times \mathcal{R}^+(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. Then $\mathrm{coz}(\beta) \notin \mathcal{F}$, which implies that $\beta \in P$. It is clear $P \neq \mathcal{R}^+(L)$. Thus P is a z-ideal of $\mathcal{R}^+(L)$.

Let $\alpha, \beta \in \mathcal{R}^+(L)$ with $\alpha\beta \in P$ be given. Then

$$\begin{aligned}\mathrm{coz}(\alpha\beta) \notin \mathcal{F} &\Rightarrow \mathrm{coz}(\alpha) \notin \mathcal{F} \text{ or } \mathrm{coz}(\beta) \notin \mathcal{F}, \text{ since } \mathcal{F} \text{ is a coz-filter on } L \\ &\Rightarrow \alpha \in P \text{ or } \beta \in P.\end{aligned}$$

Thus, P is a prime z-ideal of $\mathcal{R}(L)$. Now, we show $\mathcal{F} = \mathrm{Coz}(Q)$. Let $\alpha \in Q$. Then $\mathrm{coz}(\alpha) \in \mathcal{F}$, and so $\mathrm{Coz}(Q) \subseteq \mathcal{F}$. It is clear $\mathcal{F} \subseteq \mathrm{Coz}(Q)$.

□

We recalled from [23] that a **nilpotent-free semiring** is a semiring with no nontrivial multiplicative nilpotent elements. It is clear that $\mathcal{R}^+(L)$ is a nilpotent-free semiring.

Lemma 6.4. [23, Corollary 3.6] *Let S be a nilpotent-free semiring and let P be a prime ideal of S. Then P is a minimal prime ideal of S if and only if for each $x \in P$, there exist $y \notin P$ such that $xy = 0$.*

Proposition 6.5. *The following statements are true:*

1. *If $\mathcal{F}$ is a coz-ultrafilter on L, then $P := \mathcal{R}(L) \setminus \mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a minimal prime ideal of $\mathcal{R}(L)$.*
2. *If $\mathcal{F}$ is a coz-ultrafilter on L and $Q := \{\alpha \in \mathcal{R}^+(L) \colon \mathrm{coz}(\alpha) \in \mathcal{F}\}$, then $P := \mathcal{R}^+(L) \setminus Q$ is a minimal prime ideal of $\mathcal{R}^+(L)$.*

Proof. (1). Let $\alpha \in P$ be given. Then $\mathrm{coz}(\alpha) \notin \mathcal{F}$, which follows that there exists an element β in $\mathcal{R}(L)$ with $\mathrm{coz}(\beta) \in \mathcal{F}$ such that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$, that is, $\beta \notin P$ and $\alpha\beta = \mathbf{0}$, because $\mathcal{F}$ is a coz-ultrafilter on L. Thus, by [19, Corollary 2.2], P is a minimal prime ideal of $\mathcal{R}(L)$.

(2).It is evident that P is a proper ideal of $\mathcal{R}^+(L)$. Let $\alpha \in P$ be given. Then $\mathrm{coz}(\alpha) \notin \mathcal{F}$, which follows that there exists an element β in $\mathcal{R}^+(L)$

with $\mathrm{coz}(\beta) \in \mathcal{F}$ such that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$, that is, $\beta \notin P$ and $\alpha\beta = \mathbf{0}$, because $\mathcal{F}$ is a coz-ultrafilter on L. Thus, by Lemma 6.4, P is a minimal prime ideal of semiring $\mathcal{R}^+(L)$. □

Proposition 6.6. *The following statements are true:*

(1) *If P is a minimal prime ideal of $\mathcal{R}(L)$, then $\mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)$ is a coz-ultrafilter on L. In particular, the map*

$$\Psi\Big(P \mapsto \mathrm{Coz}\big(\mathcal{R}(L) \setminus P\big)\Big) : \mathrm{Min}\big(\mathcal{R}(L)\big) \to \Omega(L)$$

is a bijection map ($\Omega(L)$ is the set of all coz-ultrafilter *on L).*

(2) *If P is a minimal prime ideal of $\mathcal{R}^+(L)$, then $\mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ is a coz-ultrafilter on L.*

Proof. (1). See [1, Proposition 4.6].

(2). Let $\alpha \in \mathcal{R}^+(L)$ with $\mathrm{coz}(\alpha) \notin \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ be given. Then $\alpha \in P$, which implies from Lemma 6.4 that there exists an element $\beta \in \mathcal{R}^+(L)$ with $\beta \notin P$ such that $\alpha\beta = \mathbf{0}$. Hence $\mathrm{coz}(\beta) \in \mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ and $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$. Therefore, $\mathrm{Coz}\big(\mathcal{R}^+(L) \setminus P\big)$ is a coz-ultrafilter on L. □

Proposition 6.7. *Let L be a completely regular frame. Then, the following statements are equivalent:*

(1) *L is a P-frame.*

(2) *For every $I \in \Sigma\beta L$, $\mathbf{O}^I = \mathbf{M}^I$.*

(3) *For every $I \in \Sigma\beta L$, $\mathrm{Coz}\big(\mathcal{R}(L) \setminus \mathbf{M}^I\big)$ is a coz-ultrafilter.*

Proof. (1) $\Leftrightarrow$ (2). See [7, Proposition 3.9].

(2) $\Rightarrow$ (3). Let $I \in \Sigma\beta L$ be given. Since, by [9, Proposition 5.2], $\mathbf{M}^I$ is a minimal prime ideal of $\mathcal{R}(L)$, we infer from Proposition 6.6 that $\mathrm{Coz}\big(\mathcal{R}(L) \setminus \mathbf{M}^I\big)$ is a coz-ultrafilter.

(3) $\Rightarrow$ (2). Let $I \in \Sigma\beta L$ be given. Suppose $\alpha \in \mathbf{M}^I$. Since, by Proposition 6.5, $\mathbf{M}^I$ is a minimal prime ideal of $\mathcal{R}(L)$, we infer from [19, Corollary 2.2] that there exists an element $\beta \in \mathcal{R}(L)$ with $\beta \notin \mathbf{M}^I$ such that $\alpha\beta = \mathbf{0}$, which, from [9, Lemma 5.3], gives $\alpha \in \mathbf{O}^I$. Since $\mathbf{O}^I \subseteq \mathbf{M}^I$, we deduce that $\mathbf{O}^I = \mathbf{M}^I$. □

Let L be a completely regular frame and let $I \in \Sigma\beta L$. Throughout this paper, we define

$$C_I := \big\{\mathrm{coz}(\alpha)\colon \alpha \in \mathcal{R}(L) \text{ and } \mathfrak{c}_{\beta L}(I) \subseteq \mathrm{cl}_{\beta L}\mathfrak{o}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))\big)\big\}.$$

One can prove

$$C_I = \big\{\mathrm{coz}(\alpha)\colon \alpha \in \mathcal{R}(L) \text{ and } \mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{c}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))^*\big)\big\}.$$

Proposition 6.8. *Let L be a completely regular frame and let $I \in \Sigma\beta L$. Then, the following statements are true:*

(1) *For every $\alpha \in \mathcal{R}(L)$, $\mathrm{coz}(\alpha) \in C_I$ if and only if $\alpha \notin \mathbf{O}^I$.*

(2) *If the set $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$, then C_I is a prime coz-filter on L.*

(3) *If the set C_I is a coz-filter on L, then $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$.*

Proof. (1). *Necessity.* Let $\alpha \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) \in C_I$ be given. If $\alpha \in \mathbf{O}^I$, then

$$\mathfrak{c}_{\beta L}(I) \subseteq \mathrm{int}_{\beta L}\mathfrak{c}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))\big) = \mathfrak{o}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))^*\big),$$

which implies that

$$\mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{o}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))^*\big) \cap \mathfrak{c}_{\beta L}\big(r_L(\mathrm{coz}(\alpha))^*\big) = \mathsf{O},$$

which is a contradiction. Hence, $\alpha \notin \mathbf{O}^I$.

Sufficiency. Let $\alpha \in \mathcal{R}(L)$ with $\alpha \notin \mathbf{O}^I$ be given. Since $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\alpha)^* = \bot \in I$ and $\mathrm{coz}(\alpha) \notin I$, we deduce $\mathrm{coz}(\alpha)^* \in I$, because I is a prime ideal of L. Thus,

$$\begin{aligned} r_L(\mathrm{coz}(\alpha))^* = r_L(\mathrm{coz}(\alpha)^*) \subseteq I &\Rightarrow \mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{c}_{\beta L}(r_L(\mathrm{coz}(\alpha))^*) \\ &\Rightarrow \mathrm{coz}(\alpha) \in C_I. \end{aligned}$$

(2). Since $\mathfrak{c}_{\beta L}(I) \nsubseteq \mathsf{O} = \mathrm{cl}_{\beta L}\mathfrak{o}_{\beta L}\big(r_L(\bot)\big)$, we infer $\bot \notin C_I$. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in C_I$ be given. Then, by part (1), $\alpha, \beta \notin \mathbf{O}^I$, and since $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$, we deduce $\alpha\beta \notin \mathbf{O}^I$, which, from part (1), gives

$$\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta) \in C_I.$$

Let $\alpha, \beta \in \mathcal{R}(L)$ with $\operatorname{coz}(\alpha) \in C_I$ and $\operatorname{coz}(\alpha) \leq \operatorname{coz}(\beta)$ be given. Then

$$\mathfrak{c}_{\beta L}(I) \subseteq \operatorname{cl}_{\beta L}\mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}(\alpha))\big) \subseteq \operatorname{cl}_{\beta L}\mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}(\beta))\big),$$

which implies $\operatorname{coz}(\beta) \in C_I$. Hence, C_I is a coz-filter on L.

Let $\alpha, \beta \in \mathcal{R}(L)$ with $\operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta) \in C_I$ be given. Then, by part (1), $\alpha^2 + \beta^2 \notin \mathbf{O}^I$, which gives $\alpha \notin \mathbf{O}^I$ or $\beta \notin \mathbf{O}^I$, and hence, by part (1), $\operatorname{coz}(\alpha) \in C_I$ or $\operatorname{coz}(\beta) \in C_I$. Therefore, C_I is a prime coz-filter on L.

(3). Let $\alpha, \beta \in \mathcal{R}(L)$ with $\alpha \notin \mathbf{O}^I$ and $\beta \notin \mathbf{O}^I$ be given. Then, by part (1), $\operatorname{coz}(\alpha) \in C_I$ and $\operatorname{coz}(\beta) \in C_I$, which imply that $\operatorname{coz}(\alpha\beta) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta) \in C_I$, because C_I is a coz-filter on L, which, from part (1), gives $\alpha\beta \notin \mathbf{O}^I$. Therefore, $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$. □

We recall that a frame L is a F**-frame** (a quasi-F-frame) if the open quotient of each (dense) cozero element is a C^*-quotient. L is a F'**-frame** if $a \wedge b = \bot$ for $a, b \in \operatorname{Coz}(L)$ implies $a^* \vee b^* = \top$.

A frame L is a F-frame if and only if for all $a, b \in \operatorname{Coz}(L)$ with $a \wedge b = \bot$, there exist $c, d \in \operatorname{Coz}(L)$ such that $c \vee d = \top$ and $c \wedge a = d \wedge b = \bot$.

Proposition 6.9. *For a completely regular frame L, the following statements are equivalent:*

(1) *L is a F-frame.*

(2) *For every $I \in \Sigma\beta L$, C_I is a prime coz-filter on L.*

(3) *For every $I \in \Sigma\beta L$, C_I is a coz-ultrafilter on L.*

Proof. (1) ⇒ (2). Let L be a F-frame. Then for every $I \in \Sigma\beta L$, by [8, Proposition 3.4], $\mathbf{O}^I$ is a minimal prime ideal of $\mathcal{R}(L)$, which implies from Proposition 6.8 that C_I is a prime coz-filter on L.

(2) ⇒ (3). Let $I \in \Sigma\beta L$ be given. Then, by our hypothesis, C_I is a prime coz-filter on L, which, from Proposition 6.8, gives $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$, and so, by [8, Proposition 3.4], $\mathbf{O}^I$ is a minimal prime ideal of $\mathcal{R}(L)$. Therefore, by Propositions 6.3 and 6.8, $C_I = \operatorname{Coz}\big(\mathcal{R}(L) \setminus \mathbf{O}^I\big)$ is a coz-ultrafilter on L.

(3) ⇒ (1). Let $I \in \Sigma\beta L$ be given. Then, by our hypothesis and Proposition 6.8, $\mathbf{O}^I$ is a prime ideal of $\mathcal{R}(L)$. Hence, by [10, Proposition 4.9], L is a F-frame. □

It is well known that an ideal I of $\mathcal{R}(L)$ is fixed if and only if $\bigvee_{\alpha\in I} \mathrm{coz}(\alpha) \neq \top$ and by Lemma 4.4 in [8], for every $I \in \Sigma\beta L$,

$$\bigvee_{\alpha\in O^I} \mathrm{coz}(\alpha) = \bigvee_{\alpha\in M^I} \mathrm{coz}(\alpha) = \bigvee I.$$

In [11, Lemma 4.7], it was shown that a completely regular frame L is compact if and only if every maximal ideal of $\mathcal{R}(L)$ is fixed. Also, by [8, Proposition 4.5], a completely regular frame L is a F'-space if and only if $\mathbf{O}^I$ is a prime ideal for every $I \in \Sigma\beta L$ with $\bigvee I \neq \top$. Using these facts and Propositions 6.8 and 6.9, the following proposition holds.

Proposition 6.10. *For a compact completely regular frame L, the following statements are equivalent:*

(1) *L is a F'-frame.*

(2) *For every $I \in \Sigma\beta L$, C_I is a prime coz-filter on L.*

(3) *For every $I \in \Sigma\beta L$, C_I is a coz-ultrafilter on L.*

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