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Category of $\mathcal{M}$-relations as a quotient of the span category

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Abstract. We introduce $\mathcal{M}$-spans for a class $\mathcal{M}$ of morphisms in a category $\mathcal{C}$. Using the equivalence class of $\mathcal{M}$-spans under a given equivalence relation, we give the notion of an $\mathcal{M}$-relation in $\mathcal{C}$. We first show under what conditions, $\mathcal{C}$-objects together with $\mathcal{M}$-relations form a category, called the category of $\mathcal{M}$-relations and we construct a quotient of the span category as a byproduct. Then we investigate the connection between $\mathcal{M}$-relation categories and quotient span categories. We establish when a category of $\mathcal{M}$-relations is isomorphic to a quotient span category. Finally several illustrative examples are given.

1 Introduction and Preliminaries

Relations are defined in [4] in a category with a stable factorization structure $(\mathcal{E}, \mathcal{M})$, where the category of relations is formed and investigated structurally. In [3] relations are defined in a regular category where one has $(RegEpi, Mono)$-factorizations, and they are utilized as an approach to

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the notion of topos theory. In [1], the category of relations is defined based on the collection of monomorphisms and explained how it can be used as a model for quantum theory. Also in [2] relations are defined in a category with a stable factorization structure $(\mathcal{E}, \mathcal{M})$ and the authors show that the category of relations is isomorphic to a quotient of the span category. In [3] and [1] the collection $\mathcal{M}$ of monomorphisms is used in defining relations, however in [4] and [2] the collection $\mathcal{M}$ is the counterpart of a stable factorization structure $(\mathcal{E}, \mathcal{M})$. In all the cases, the isomorphism class of a span is used to define a relation.

In this paper we provide a two-folded generalization of a relation in a category, by utilizing an arbitrary class $\mathcal{M}$ of morphisms and instead of using the isomorphism calss of morphisms as a relation, we use the class with respect to a more general equivalence relation. In Section 2, we investigate conditions under which we actually get a category of relations, which we call the category of $\mathcal{M}$-relations. We use that to get a more general version of the span category, called the quotient span category. In Section 3, we discuss the interrelation between the category of $\mathcal{M}$-relations and the quotient span category. As the main result we show that under what conditions the category of $\mathcal{M}$-relations is isomorphic to a quotient span category. Finally in Setion 4, we give several illustrative examples, showing on the one hand that some of the previous work can be proved in a more straightforward fashion and on the other hand how we get more general categories of relations.

In the following we give briefly some of the concepts needed in the subsequent sections.

1.1 Spans A span $f = (f_1, f_2) : A \longrightarrow B$ in a category $\mathcal{C}$ is a pair of morphisms $A \xleftarrow{f_1} F \xrightarrow{f_2} B$ with the same domain, see [2]. In a category $\mathcal{C}$ with pullbacks, the composition $g \circ f$ of spans f and g is given by a pullback and so it is unique only up to isomorphism of spans. For $\mathcal{C}$ a category with binary products, there exists a one to one correspondence between spans $A \xleftarrow{f_1} F \xrightarrow{f_2} B$ and morphisms $\langle f_1, f_2 \rangle : F \longrightarrow A \times B$. For a span $f = (f_1, f_2)$ we write $\hat{f}$ for its corresponding morphism $\langle f_1, f_2 \rangle$ and for a morphism $g = \langle g_1, g_2 \rangle$ we write $\dot{g}$ for its corresponding span (g_1, g_2).

1.2 Quasi Right Factorization Structures

Definition 1.1. [5]. A quasi right factorization structure in a category $\mathcal{C}$ is a collection $\mathcal{M}$ of morphisms such that for every morphism f in $\mathcal{C}$ there is a morphism m_f in $\mathcal{M}$ satisfying:

- $f = m_f g$, for some g and
- if there is a morphism $m \in \mathcal{M}$ such that $f = mh$, for some h, then $m_f = mk$, for some k.

m_f is called a quasi right part or just an $\mathcal{M}$-part, of f.

Definition 1.2. Let f and g be morphisms in $\mathcal{C}$ with the same codomain. We say $f \leq_{\mathcal{C}} g$ if there is a morphism $\alpha \in \mathcal{C}$ such that $f = g\alpha$. And we define $f \bowtie g$ if $f \leq_{\mathcal{C}} g$ and $g \leq_{\mathcal{C}} f$.

One can easily verify that $\leq_{\mathcal{C}}$ is a preorder and that $\bowtie$ is an equivalence relation.

Remark 1.3. Using Theorem 1.2, we can restate the two parts of Theorem 1.1 as:

- $f \leq_{\mathcal{C}} m_f$ and
- if there is a morphism $m \in \mathcal{M}$ such that $f \leq_{\mathcal{C}} m$, then $m_f \leq_{\mathcal{C}} m$.

So $\mathcal{M}$-parts of a morphism are only unique up to $\bowtie$.

We state proposition 3 of [5] with a slight change of notation, as follows:

Lemma 1.4. *Let $\mathcal{M}$ be a quasi right factorization structure in $\mathcal{C}$ and m_f be an $\mathcal{M}$-part of a morphism f.*

- *If $f \in \mathcal{M}$, then $f \bowtie m_f$.*
- *m is an $\mathcal{M}$-part of f if and only if $m \in \mathcal{M}$ and $m \bowtie m_f$.*
- *If $f \bowtie g$, then $m_f \bowtie m_g$, and so m_f is an $\mathcal{M}$-part of g.*

2 $\mathcal{M}$-Relations modulo an equivalence relation

Let $\mathcal{M}$ be a collection of morphisms in a category $\mathcal{C}$ with binary products. We call a span f an $\mathcal{M}$-span whenever $\hat{f}$ belongs to $\mathcal{M}$. Throughout the paper we assume a collection $\mathcal{M}$ is given and whenever we talk about an equivalence relation on spans or on $\mathcal{M}$-spans, we mean spans or $\mathcal{M}$-spans with the same domain and codomain. In this section we construct the $\mathcal{M}$-relation category and the quotient span category. To this end we have:

Definition 2.1. For an equivalence relation $\approx$ on $\mathcal{M}$-spans, an $\mathcal{M}$-relation modulo $\approx$ from A to B is the equivalence class,

$$[f]_{\approx} = \{g | g \text{ is a } \mathcal{M}\text{-span and } g \approx f\}$$

where $f : A \longrightarrow B$ is an $\mathcal{M}$-span. $[f]_{\approx}$ is also denoted by $\bar{f}$.

We denote by $Span(\mathcal{C})(A, B)$ (respectively $\mathcal{M}Span(\mathcal{C})(A, B)$) the collection of all spans ($\mathcal{M}$-spans) with domain A and codomain B, and we assume that for each pair of objects A and B, a function

$$\mathbf{m}_{A,B} : Span(\mathcal{C})(A, B) \longrightarrow \mathcal{M}Span(\mathcal{C})(A, B)$$

which we simply write as $\mathbf{m}$, is given.

Definition 2.2. Suppose that $\mathcal{C}$ has binary products and pullbacks. An equivalence relation $\approx$ on $\mathcal{M}$-spans is said to be $\text{comp}_{\mathbf{m}}$-compatible if for $\mathcal{M}$-spans f, g, h, k, $f \approx h$ and $g \approx k$ yields $\mathbf{m}(f \circ g) \approx \mathbf{m}(h \circ k)$.

We remark that since the span composition $f \circ g$ is only unique up to isomorphism, $\text{comp}_{\mathbf{m}}$-compatibility requires that $\mathbf{m}(f \circ g)_1 \approx \mathbf{m}(f \circ g)_2$, where $(f \circ g)_1$ and $(f \circ g)_2$ are two isomorphic copies of the composition.

Definition 2.3. For composable $\mathcal{M}$-spans f and g, the composition of $\mathcal{M}$-relations is defined by $[f]_{\approx}[g]_{\approx} = [\mathbf{m}(f \circ g)]_{\approx}$.

One can easily verify that:

Proposition 2.4. *The composition of $\mathcal{M}$-relations is well-defined if and only if the equivalence relation $\approx$ is $\text{comp}_{\mathbf{m}}$-compatible.*

Definition 2.5. An equivalence relation $\sim$ on spans is called $\mathbf{m}$-proper if

- it contains the isomorphism equivalence relation $\cong$,
- it is comp-compatible, i.e., $f \sim g$ and $h \sim k$, yields $f \circ h \sim g \circ k$; and
- it is $\mathbf{m}$-compatible, i.e., for every span f, $f \sim \mathbf{m}(f)$.

The following easy consequence of $\mathbf{m}$-compatibility will be used frequently in the paper.

Lemma 2.6. *Suppose $\sim$ is an $\mathbf{m}$-compatible equivalence relation on spans. For all spans f and g with the same domain and codomain, $f \sim g$ if and only if $\mathbf{m}(f) \sim \mathbf{m}(g)$.*

Denoting the restriction of $\sim$ to $\mathcal{M}$-spans by $\sim_{|\mathcal{M}}$, we have:

Proposition 2.7. *Let $\approx$ be an equivalence relation on $\mathcal{M}$-spans. If there is an $\mathbf{m}$-proper equivalence relation $\sim$ on spans such that $\sim_{|\mathcal{M}} \subseteq \approx$, then the composition of $\mathcal{M}$-relations is associative.*

Proof. Let f, g and h be composable $\mathcal{M}$-spans. Since $\sim$ is $\mathbf{m}$-compatible, $\mathbf{m}(g \circ f) \sim g \circ f$. Comp-compatibility of $\sim$ implies that $h \circ \mathbf{m}(g \circ f) \sim h \circ (g \circ f)$. It follows that $\mathbf{m}(h \circ \mathbf{m}(g \circ f)) \sim h \circ \mathbf{m}(g \circ f) \sim h \circ (g \circ f)$. Similarly $\mathbf{m}(\mathbf{m}(h \circ g) \circ f) \sim (h \circ g) \circ f$. Now since composition of spans is associative up to isomorphism, $h \circ (g \circ f) \cong (h \circ g) \circ f$, and since $\cong \subseteq \sim$, we get $h \circ (g \circ f) \sim (h \circ g) \circ f$. It follows that $\mathbf{m}(h \circ \mathbf{m}(g \circ f)) \sim \mathbf{m}(\mathbf{m}(h \circ g) \circ f)$ and thus $\mathbf{m}(h \circ \mathbf{m}(g \circ f)) \sim_{|\mathcal{M}} \mathbf{m}(\mathbf{m}(h \circ g) \circ f)$. Since $\sim_{|\mathcal{M}} \subseteq \approx$, $\mathbf{m}(h \circ \mathbf{m}(g \circ f)) \approx \mathbf{m}(\mathbf{m}(h \circ g) \circ f)$. Hence $\bar{h}(\bar{g}\bar{f}) = (\bar{h}\bar{g})\bar{f}$ as desired. □

For each object $A \in \mathcal{C}$, setting $\delta_A = \mathbf{m}((1_A, 1_A))$, we have:

Lemma 2.8. *Suppose $\sim$ is $\mathbf{m}$-proper. For spans $f : A \longrightarrow B$ and $g : B \longrightarrow A$, $f \circ \delta_A \sim f$ and $\delta_A \circ g \sim g$.*

Proof. Since $\sim$ is $\mathbf{m}$-compatible, we have $\delta_A = \mathbf{m}((1_A, 1_A)) \sim (1_A, 1_A)$. Now by comp-compatibility we have, $f \circ \delta_A \sim f \circ (1_A, 1_A) \cong f$. Hence $f \circ \delta_A \sim f$. Similarly $\delta_A \circ g \sim g$. □

Proposition 2.9. *Let $\approx$ be an equivalence relation on $\mathcal{M}$-spans. If there is an $\mathbf{m}$-proper equivalence relation $\sim$ on spans such that $\sim_{|\mathcal{M}} \subseteq \approx$, then for each A, $\bar{\delta}_A$ acts neutral with respect to composition of $\mathcal{M}$-relations.*

Proof. By $\mathbf{m}$-compatiblity of $\sim$ and Theorem 2.8, for each $\mathcal{M}$-span $f : A \longrightarrow B$, we have $\mathbf{m}(f \circ \delta_A) \sim f \circ \delta_A \sim f$. Since $\sim_{|\mathcal{M}} \subseteq \approx$, $\mathbf{m}(f \circ \delta_A) \approx f$, implying $\bar{f}\bar{\delta}_A = \bar{f}$. Similarly for each $\mathcal{M}$-span $g : B \longrightarrow A$, $\bar{\delta}_A \bar{g} = \bar{g}$. □

Definition 2.10. Let $\approx$ (respectively $\sim$) be an equivelence relation on $\mathcal{M}$-spans (respectively spans). We say the pair $(\approx, \sim)$ is $\mathbf{m}$-consistent if $\approx$ is comp$\mathbf{m}$-compatible, $\sim$ is $\mathbf{m}$-proper and $\sim_{|\mathcal{M}} \subseteq \approx$.

We now summarize what we have done in the following theorem.

Theorem 2.11. *Suppose $\mathcal{C}$ is a category with binary products and pullbacks and let $\approx$ be an equivalence relation on $\mathcal{M}$-spans. If there is an equivalence relation $\sim$ on spans such that $(\approx, \sim)$ is an $\mathbf{m}$-consistent pair, then $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M})$ with $\mathcal{C}$-objects as objects, $\mathcal{M}$-relations as morphisms, composition and identities as in 2.3 and 2.9, is a category.*

Proof. Follows from Propositions 2.4, 2.7 and 2.9. □

We call $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M})$ the $\mathcal{M}$-Relation category.

Corollary 2.12. *Let $\mathcal{C}$ be a category with binary products and pullbacks and $\sim'$ be an equivalence relation on spans that contains the isomorphism equivalence relation and is comp-compatible. Then $\mathsf{Span}_{\sim'}(\mathcal{C})$ with $\mathcal{C}$-objects as objects and equivalence classes $[f]_{\sim'}$ of spans as morphisms is a category.*

Proof. Follows from Theorem 2.11, by letting $\mathcal{M}$ be the collection of all $\mathcal{C}$-morphisms, $\mathbf{m}$ be the identity function and $\sim' = \sim = \approx$. □

We call $\mathsf{Span}_{\sim'}(\mathcal{C})$ the quotient span category.

3 $\mathcal{M}$-Relation category as a quotient span category

In this section we establish functors between certain $\mathcal{M}$-relation and quotient span categories and we show under what conditions they are isomorphic.

Proposition 3.1. *Let $\approx$ be an equivalence relation on $\mathcal{M}$-spans. Suppose there is an equivalence relation $\sim$ on spans such that $(\approx, \sim)$ is $\mathbf{m}$-consistent. If $\sim'$ is $\mathbf{m}$-proper and $\approx \subseteq \sim'_{|\mathcal{M}}$, then the mapping*

$$\mathsf{S} : \mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M}) \longrightarrow \mathsf{Span}_{\sim'}(\mathcal{C})$$

taking $[f]_\approx$ *to* $[f]_{\sim'}$ *is a functor. And if* $\approx\, = \,\sim'_{|\mathcal{M}}$, *then* S *is faithful as well.*

Proof. The fact that $\approx\, \subseteq \,\sim'_{|\mathcal{M}}$, gives the well-definedness of the mapping S. Using $\mathbf{m}$-compatibility of $\sim'$, for each object A we have, $\mathsf{S}([\delta_A]_\approx) = [\delta_A]_{\sim'} = [\mathbf{m}(1_A, 1_A)]_{\sim'} = [(1_A, 1_A)]_{\sim'}$, showing the preservation of identities. For preservation of composition we have, $\mathsf{S}([f]_\approx [g]_\approx) = \mathsf{S}([\mathbf{m}(f \circ g)]_\approx) = [\mathbf{m}(f \circ g)]_{\sim'} = [f \circ g]_{\sim'} = [f]_{\sim'}[g]_{\sim'}$ as desired. To show faithfulness, let f and g be $\mathcal{M}$-spans such that $[f]_{\sim'} = [g]_{\sim'}$. So $f \sim' g$ and since f and g are $\mathcal{M}$-spans, we get $f \sim'_{|\mathcal{M}} g$. Since $\sim'_{|\mathcal{M}} = \approx$, $f \approx g$. Hence $[f]_\approx = [g]_\approx$. ☐

Proposition 3.2. *Let* $\approx$ *be an equivalence relation on* $\mathcal{M}$*-spans. Suppose there is an equivalence relation* $\sim$ *on spans such that* $(\approx, \sim)$ *is* $\mathbf{m}$*-consistent. If* $\sim'$ *is* $\mathbf{m}$*-proper and* $\sim'_{|\mathcal{M}} \subseteq \approx$, *then the mapping*

$$\mathsf{R} : \mathsf{Span}_{\sim'}(\mathcal{C}) \longrightarrow \mathsf{Rel}_\approx(\mathcal{C}, \mathcal{M})$$

taking $[f]_{\sim'}$ *to* $[\mathbf{m}(f)]_\approx$ *is a functor. Furthermore* R *is full.*

Proof. Lemma 2.6 and the fact that $\sim'_{|\mathcal{M}} \subseteq \approx$ yields the well-definedness of R. Identities are preserved because for each object A, $\mathsf{R}([1_A, 1_A]_{\sim'}) = [\mathbf{m}(1_A, 1_A)]_\approx = [\delta_A]_\approx$. To show preservation of composition, let f and g be composable spans. Since $\sim$ is $\mathbf{m}$-proper, $f \sim \mathbf{m}(f)$ and $g \sim \mathbf{m}(g)$. and thus $f \circ g \sim \mathbf{m}(f) \circ \mathbf{m}(g)$. Therefore $\mathbf{m}(f \circ g) \sim \mathbf{m}(\mathbf{m}(f) \circ \mathbf{m}(g))$ implying $\mathbf{m}(f \circ g) \sim_{|\mathcal{M}} \mathbf{m}(\mathbf{m}(f) \circ \mathbf{m}(g))$. Since $\sim_{|\mathcal{M}} \subseteq \approx$, $\mathbf{m}(f \circ g) \approx \mathbf{m}(\mathbf{m}(f) \circ \mathbf{m}(g))$. Now we have $R([f]_{\sim'}([g]_{\sim'}) = R([f \circ g]_{\sim'}) = [\mathbf{m}(f \circ g)]_\approx = [\mathbf{m}(\mathbf{m}(f) \circ \mathbf{m}(g))]_\approx = [\mathbf{m}(f)]_\approx [\mathbf{m}(g)]_\approx = R([f]_{\sim'}) R([g]_{\sim'})$ as desired. For the last assertion, given $[f]_\approx$, with f an $\mathcal{M}$-span, we have $\mathbf{m}(f) \sim' f$ and since f is an $\mathcal{M}$-span, we get $\mathbf{m}(f) \sim'_{|\mathcal{M}} f$ and thus $\mathbf{m}(f) \approx f$. Therefore $R([f]_{\sim'}) = [\mathbf{m}(f)]_\approx = [f]_\approx$. ☐

Theorem 3.3. *Let* $\approx$ *be an equivalence relation on* $\mathcal{M}$*-spans. Suppose there is an* $\mathbf{m}$*-proper equivalence relation* $\sim$ *on spans such that* $\sim_{|\mathcal{M}} \subseteq \approx$. *If* $\sim'$ *is* $\mathbf{m}$*-proper and* $\approx\, = \,\sim'_{|\mathcal{M}}$, *then* S *and* R *are inverse functors, so that* $\mathsf{Rel}_\approx(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\sim'}(\mathcal{C})$.

Proof. The facts that $\sim'$ is $\mathbf{m}$-proper and $\sim'_{|\mathcal{M}} = \approx$, imply that $\approx$ is comp$\mathbf{m}$-compatible. Thus $(\approx, \sim)$ is $\mathbf{m}$-consistent. So by Propositions 3.1 and 3.2, S and R are functors which act as identity on objects. On the morphisms for each $\mathcal{M}$-span f, by $\mathbf{m}$-properness of $\sim'$ and the fact that $\sim'_{|\mathcal{M}} = \approx$ we have, $\mathbf{m}(f) \sim' f$ and thus $\mathbf{m}(f) \approx f$. It follows that $R \circ S([f]_\approx) = R([f]_{\sim'}) = [\mathbf{m}(f)]_\approx = [f]_\approx$. On the other hand for each span f by $\mathbf{m}$-properness of $\sim'$ we have, $\mathbf{m}(f) \sim' f$. It follows that $S \circ R([f]_{\sim'}) = S([\mathbf{m}(f)]_\approx) = [\mathbf{m}(f)]_{\sim'} = [f]_{\sim'}$. □

In the next two lemma we discuss the connection between the equivalence relations involved in the above theorem.

Lemma 3.4. *Let $\sim$ and $\sim'$ be equivalence relations on spans, with $\sim$ $\mathbf{m}$-compatible. We have,*

(a) $\sim_{|\mathcal{M}}$ *is $\mathbf{m}$-compatible.*

(b) *if $\sim \subseteq \sim'$, then $\sim'$ is $\mathbf{m}$-compatible. This is the case if $\sim$ and $\sim'$ are equivelence relations on $\mathcal{M}$-spans.*

(c) *if $\sim'$ is $\mathbf{m}$-compatible we have, $\sim \subseteq \sim'$ if and only if $\sim_{|\mathcal{M}} \subseteq \sim'_{|\mathcal{M}}$.*

Proof. Let f be an $\mathcal{M}$-span. Since $\sim$ is $\mathbf{m}$-compatible, $f \sim \mathbf{m}(f)$. Now
(a) since f and $\mathbf{m}(f)$ are $\mathcal{M}$-spans, $f \sim_{|\mathcal{M}} \mathbf{m}(f)$ and
(b) since $\sim \subseteq \sim'$, $f \sim' \mathbf{m}(f)$.
(c) The direct implication is obvious. For the converse, let f and g be spans such that $f \sim g$. Since $\sim$ is $\mathbf{m}$-compatible, by 2.6 we get $\mathbf{m}(f) \sim \mathbf{m}(g)$. Since $\sim_{|\mathcal{M}} \subseteq \sim'_{|\mathcal{M}}$, $\mathbf{m}(f) \sim' \mathbf{m}(g)$. Now since $\sim'$ is $\mathbf{m}$-compatible, by 2.6 we get $f \sim' g$, as desired. □

For an equivalence relation $\approx$ on $\mathcal{M}$-spans, one can easily verify that the relation $\approx_e$ defined by

$$f \approx_e g \overset{def}{\equiv} \mathbf{m}(f) \approx \mathbf{m}(g)$$

and called the extension of $\approx$ to spans, is an equivalence relation.

Lemma 3.5. *Suppose $\sim$ (respectively, $\approx$) is an equivalence relation on spans (respectively $\mathcal{M}$-spans). If $\sim$ is $\mathbf{m}$-compatible and $\sim_{|\mathcal{M}} \subseteq \approx$, then*

(a) $\sim \subseteq \approx_e$.

(b) $\approx_e$ *contains $\cong$ and is $\mathbf{m}$-compatible.*

(c) $\approx_{e|\mathcal{M}} = \approx$.

(d) $\approx_e$ *is the unique* $\mathbf{m}$*-compatible equivalence relation on spans with* $\approx_{e|_\mathcal{M}} = \approx$.

Proof. (a) Let f, g be spans and $f \sim g$. Since $\sim$ is $\mathbf{m}$-compatible, by 2.6, $\mathbf{m}(f) \sim \mathbf{m}(g)$. Since $\sim_{|_\mathcal{M}} \subseteq \approx$, $\mathbf{m}(f) \approx \mathbf{m}(g)$ and so $f \approx_e g$.

(b) Since $\sim$ contains $\cong$, by (a), so does $\approx_e$. Since $\sim$ is $\mathbf{m}$-compatible, by (a) and 3.4(b), so is $\approx_e$.

(c) Let f, g be $\mathcal{M}$-spans. We have $f \approx_{e|_\mathcal{M}} g$ if and only if $f \approx_e g$ if and only if $\mathbf{m}(f) \approx \mathbf{m}(g)$. By 3.4(a) and (b), $\approx$ is $\mathbf{m}$-compatible. So by 2.6, $\mathbf{m}(f) \approx \mathbf{m}(g)$ if and only if $f \approx g$. This proves $f \approx_{e|_\mathcal{M}} g$ if and only if $f \approx g$, as desired.

(d) Suppose $\sim'$ is an $\mathbf{m}$-compatible equivalence relation on spans with $\sim'_{|_\mathcal{M}} = \approx$. By part (c), $\sim'_{|_\mathcal{M}} = \approx_{e|_\mathcal{M}}$. Since $\sim'$ and $\approx_e$ (by part (b)) are $\mathbf{m}$-compatible, by 3.4(c), $\sim' = \approx_e$. □

Theorem 3.6.

(a) *Suppose an* $\mathbf{m}$*-proper equivalence relation* $\sim$ *on spans is given. Then*

$$\mathsf{Rel}_{\sim_{|_\mathcal{M}}}(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_\sim(\mathcal{C})$$

(b) *Suppose an equivalence relation* $\approx$ *on* $\mathcal{M}$*-spans is given for which there is an equivalence relation* $\sim$ *on spans making* $(\approx, \sim)$ $\mathbf{m}$*-consistent. If* $\approx_e$ *is comp-compatible, then it is the only* $\mathbf{m}$*-proper equivalence relation on spans such that* $\approx_{e|_\mathcal{M}} = \approx$*. In this case*

$$\mathsf{Rel}_\approx(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\approx_e}(\mathcal{C})$$

Proof. (a) Follows from 3.3 by setting $\sim' = \sim$.

(b) Since $\sim$ is $\mathbf{m}$-compatible and $\sim_{|_\mathcal{M}} \subseteq \approx$, by 3.5(b) $\approx_e$ contains $\cong$ and is $\mathbf{m}$-compatible. By hypothesis $\approx_e$ is comp-compatible, so it is $\mathbf{m}$-proper. The uniqueness follows by part (d) of 3.5. So by 3.3, $\mathsf{Rel}_\approx(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\approx_e}(\mathcal{C})$. □

Let us rematk that in part (b) of the above theorem, if $\approx_e$ is not comp-compatible, then by 3.5 there is no $\mathbf{m}$-proper equivalence relation $\sim'$ such that $\sim'_{|_\mathcal{M}} = \approx$, because otherwise $\approx_e = \sim'$ is comp-comatible.

4 Examples

We give several examples in this section. The following first example is intended to provide a more concise and simpler proof of a known result given in Theorem 2.3 of [2].

Example 4.1. Suppose $(\mathcal{E}, \mathcal{M})$ is a pullback-stable factorization structure in $\mathcal{C}$. Define

$$\mathbf{m} : Span(\mathcal{C})(A, B) \longrightarrow \mathcal{M}Span(\mathcal{C})(A, B)$$

by $\mathbf{m}(f) = \dot{m}_{\hat{f}}$, where $m_{\hat{f}}$ is an $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $\hat{f}$, see Subsection 1.1. Let us remark that since a right $\mathcal{M}$-part of a morphism is not unique, in defining $\mathbf{m}$, we are choosing one such representative. Let $\sim_{\mathcal{E}}$ be the equivalence relation given in Definition 2.1 of [2]. By Definition 2.1 and Proposition 2.2 of [2], $\sim_{\mathcal{E}}$ contains $\cong$ and is comp-compatible and the comments on page 1181 of the same article show that $\sim_{\mathcal{E}}$ is $\mathbf{m}$-compatible. Thus $\sim_{\mathcal{E}}$ is $\mathbf{m}$-proper. It can be easily verified that $\sim_{\mathcal{E}|_{\mathcal{M}}} = \cong$. Hence by 3.6(a), $\mathsf{Rel}_{\cong}(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\sim_{\mathcal{E}}}(\mathcal{C})$. One can easily see that $\mathsf{Rel}_{\cong}(\mathcal{C}, \mathcal{M})$ and $\mathsf{Span}_{\sim_{\mathcal{E}}}(\mathcal{C})$ are the categories $Rel_{\mathcal{M}}(\mathcal{C})$ and $Span_{\mathcal{E}}(\mathcal{C})$ given in [2] and the isomorphism between them is just the result given in Theorem 2.3 of the same article.

The following example generalizes the above example.

Example 4.2. In this example we just suppose $\mathcal{M}$ is a quasi right factorization structure in $\mathcal{C}$, see Subsection 1.2. Define

$$\mathbf{m} : Span(\mathcal{C})(A, B) \longrightarrow \mathcal{M}Span(\mathcal{C})(A, B)$$

by $\mathbf{m}(f) = \dot{m}_{\hat{f}}$, where $m_{\hat{f}}$ is a quasi right $\mathcal{M}$-part of $\hat{f}$. Set

$$\mathcal{M}_{\perp} = \{e \in \mathcal{C} \ : \ \exists m \in \mathcal{M}, \forall n \in \mathcal{M}(me \leq_{\mathcal{C}} n \Rightarrow m \leq_{\mathcal{C}} n)\}$$

and note that e belongs to $\mathcal{M}$ provided that there is $m \in \mathcal{M}$ such that m is a quasi right part of the morphism me. Let $\mathcal{E}$ be a stable collection of morphisms in $\mathcal{C}$ that contains $\mathcal{M}_{\perp}$. As mentioned in the previous example, $\sim_{\mathcal{E}}$ contains $\cong$ and is comp-compatible. Now for a span f, we have $\hat{f} = m_{\hat{f}}e$, for some $e \in \mathcal{C}$. It follows that $e \in \mathcal{M}_{\perp} \subseteq \mathcal{E}$. Thus $f \leq_{\mathcal{E}} \dot{m}_{\hat{f}}$, implying

$f \sim_{\mathcal{E}} \dot{m}_{\hat{f}}$. Hence $\sim_{\mathcal{E}}$ is $\mathbf{m}$-proper and so by 3.6(a), $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\sim_{\mathcal{E}}}(\mathcal{C})$, where $\approx \, = \, \sim_{\mathcal{E}|_{\mathcal{M}}}$.

Note that here $(\mathcal{E}, \mathcal{M})$ is not necessarily a factorization structure and $\approx$ is not necessarily equal to $\cong$.

Example 4.3. In this example we give an equivalence relation $\approx$ on $\mathcal{M}$-spans whose extension $\approx_e$ is not comp-compatible. Let $\mathcal{M}$, $\mathbf{m}$, $\mathcal{M}_{\perp}$ and $\mathcal{E}$ be as in Example 1.1. Define $\approx$ on $\mathcal{M}$-spans by $f \approx g$ if there are $\mathcal{M}$-spans $h_1, h_2, \cdots, h_n$ for $n \geq 1$ such that $f = h_1 \geq_{\mathcal{E}} h_2 \leq_{\mathcal{E}} h_3 \cdots \geq_{\mathcal{E}} h_n = g$ (note that the spans h_i are assumed to be $\mathcal{M}$-spans). One can show that $\approx$ is an equivelance relation on $\mathcal{M}$-spans whose extension $\approx_e$ contains $\cong$, is $\mathbf{m}$-compatible, but it is not comp-compatible.

Example 4.4. Let $\mathcal{M}$ and $\mathbf{m}$ be as in Example 1.1. Since the collection $\mathcal{C}_1$ (also denoted by $\mathcal{C}$) of all the morphisms in $\mathcal{C}$ is a stable class, the relation $\sim_{\mathcal{C}}$ can be shown to be an $\mathbf{m}$-proper equivalence relation. With $\approx \, = \, \sim_{\mathcal{C}|_{\mathcal{M}}}$, by 3.6(a), $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\sim_{\mathcal{C}}}(\mathcal{C})$. Since the category $\mathcal{C}$ is assumed to have products, one can easily verify that for any span $f : A \to B$, $f \leq_{\mathcal{C}} \pi$, where π is the product span. Thus $[f]_{\sim_{\mathcal{C}}} = [\pi]_{\sim_{\mathcal{C}}}$, implying there is a unique morphism from A to B in $\mathsf{Span}_{\sim_{\mathcal{C}}}(\mathcal{C})$. Hence $\mathsf{Span}_{\sim_{\mathcal{C}}}(\mathcal{C})$ is equivalent to a partially ordered class and therefore so is $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M})$.

Example 4.5. Let for each pair of objects A, B in a category $\mathcal{C}$ with products and pullbacks, a morphism $r_{A,B} : R_{A,B} \to A \times B$ be given and set $\mathsf{R} = \{r_{A,B} : A, B \in \mathcal{C}\}$. For a span $f : A \to B$, let d_f de the diagonal of the following pullback,

$$\begin{array}{ccc} F^* & \xrightarrow{f^*_{A,B}} & R_{A,B} \\ {\scriptstyle r^*_{A,B}}\downarrow & \searrow^{d_f} & \downarrow{\scriptstyle r_{A,B}} \\ F & \xrightarrow[\hat{f}]{} & A \times B \end{array}$$

and set $\mathcal{M} = \{d_f : f \text{ is a span}\}$. By taking $f : A \to B$ to be a product span, we have $\hat{f}$ is an isomorphism and so d_f can be chosen to be $r_{A,B}$, thus $\mathsf{R} \subseteq \mathcal{M}$. Define

$$\mathbf{m}_{A,B} : Span(\mathcal{C})(A, B) \longrightarrow \mathcal{M}Span(\mathcal{C})(A, B)$$

by $\mathbf{m}(f) = \dot{d}_f$.

We have g is an $\mathcal{M}$-span if and only if $\hat{g} \in \mathcal{M}$ if and only if there is a span f such that $\hat{g} = d_f$ if and only if there is a span f such that $g = \dot{d}_f$ if and only if there is a span f such that $g = \mathbf{m}(f)$. This yields $\mathbf{m}$ is a surjection.

Let $\mathcal{E}$ be a stable system containg $\mathcal{M}$. Since $d_f = \hat{f} r^*_{A,B}$ and $r^*_{A,B} \in \mathcal{E}$, we get $\hat{f} \geq_{\mathcal{E}} d_f$. Therefore $f \geq_{\mathcal{E}} \dot{d}_f$, i.e. $f \geq_{\mathcal{E}} \mathbf{m}(f)$ and thus $f \sim_{\mathcal{E}} \mathbf{m}(f)$. It follows that $\sim_{\mathcal{E}}$ is $\mathbf{m}$-proper. Therefore by 3.6(a), $\mathsf{Rel}_{\sim_{|\mathcal{M}}}(\mathcal{C}, \mathcal{M}) \cong \mathsf{Span}_{\sim}(\mathcal{C})$.

Conclusion

A concept of relations (called $\mathcal{M}$-relations) in a category $\mathcal{C}$ based on a collection $\mathcal{M}$ of morphisms, a function $\mathbf{m}$ from spans to $\mathcal{M}$-spans and an equivalence relation $\approx$ on $\mathcal{M}$-spans is given. The category $\mathsf{Rel}_{\approx}(\mathcal{C}, \mathcal{M})$ of $\mathcal{M}$-relations is formed and the interconnection between this category and quotient span categories is investigated. In particular conditions under which a category of $\mathcal{M}$-relations is isomorphic to a quotient span category is given.

The motivation for this work is that by varying the collection $\mathcal{M}$, the function $\mathbf{m}$ and the equivalence relation $\approx$ or by imposing conditions on these entities, one can come up with a category of relations that satisfies the desired conditions, such as being a dagger category, a Mal'cev category, an allegory, or the like. Besides as shown in Example 4.1, this gives us a tool to provide easier and more concise proofs for some related results.

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