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On the homomorphisms of ∩-structure spaces

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Abstract. In [5], the concept of ∩-structure space is defined and it is studied from an algebraic and topological points of view. Indeed, the ∩-structure is considered as a model for all algebraic substructures such as subgroups, subrings and submodules, ideals, etc. Moreover, the elements of these ∩-structures are seen as an open set, and from this point of view, another goal is to relate some algebraic properties to some topological properties. The present article follows the same points of view of [5]. In particular, similar to algebraic homomorphisms, ∩-structural homomorphisms are defined and investigated in ∩-structure spaces. In addition, we examine some classical results related to homomorphisms. In this regard, similar to lattice theory, we define the congruence relation on ∩-structure spaces and give some facts about them, and then we generalize the isomorphism theorems of algebraic structure to ∩-structure spaces.

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1 Introduction

Let X be a nonempty set. An intersection structure (briefly, $\cap$-structure) on X is a nonempty family $\mathcal{M}_X$ of subsets of X which is closed under arbitrary intersection. In this case we say $(X, \mathcal{M}_X)$ is a $\cap$-structure space (in this article, we say cap-structure instead of $\cap$-structure); some times we say "X is a cap-structure space" when there is no ambiguity about $\mathcal{M}_X$. Clearly, if $(X, \mathcal{M}_X)$ is a cap-structure space, then $\mathcal{M}_X$ is a complete lattice in which for each family $\{A_i\}_{i\in I}$ of $\mathcal{M}_X$:

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i \quad , \quad \bigvee_{i\in I} A_i = \bigcap\{B \in \mathcal{M}_X : \bigcup_{i\in I} A_i \subseteq B\}.$$

It is clear that X is the top element of $\mathcal{M}_X$. The least element of this complete lattice is denoted by $\circ_X$, and for short, if there is no ambiguity, it is denoted by $\circ$.

Let L be a complete lattice. We call $S \subseteq L$ a meet-structure if $\bigwedge A \in S$ for every $A \subseteq S$. It is well-known that every cap-structure is a meet-structure. Conversely, suppose that S is a meet-structure of complete lattice L. Now, if we define $f : L \to P(L)$ with $f(a) = \downarrow a$, then f embeds S to the cap-structure $f(S)$ in $P(L)$. Therefore, cap-structure and meet-structure are the same. Also, it is well-known that the concepts of cap-structures and closure operators are two sides of the same coin. Many studies have been done on closure operators, for more informations about these concepts, see [1], [2], [3], [4], [6]. In these studies, the elements of cap-structure generated by a closure operator are considered as closed sets of a topology. Whereas when we look at the topic from a topological perspective, we consider the elements of a cap-structure as an open set. The reader should realize the difference between these two topological views. More importantly, in this article, our main focus is on algebraic substructures such as subgroups, subrings and submodules, ideals, etc., and we see cap-structure spaces mainly from these points of view. In particular, we want to generalize the isomorphism theorems of algebraic structure to cap-structure spaces.

Also in this article, the homomorphism of cap-structure spaces is defined and studied. But first, some of notations and definitions related to cap-structure spaces are stated here.

Let $(X, \mathcal{M}_X)$ be a cap-structure space and $A \subseteq X$. The intersection of all elements of $\mathcal{M}_X$ that contain A is denoted by $\langle A\rangle_X$ and if there is no ambiguity, we simply show it by $\langle A\rangle$. In the case that A is the finite set $\{a_1, a_2, \ldots, a_n\}$, $\langle A\rangle$ is written as $\langle a_1, a_2, \ldots, a_n\rangle$, and is referred as the element of $\mathcal{M}_X$ which is generated by $a_1, a_2, \ldots, a_n$. Assuming $x \in X$, we denote the set $\{u \in \mathcal{M}_X : \ x \in u\}$ by $\mathcal{M}_x$. If cl : $P(X) \to P(X)$ is a function with

$$\forall A \in P(X), \ \ \mathrm{cl}(A) = \bar{A} = A \cup \{x \in X : \ \forall u \in \mathcal{M}_x, \ u \cap A \nsubseteq \circ\},$$

then clearly, "cl" is a closure operator on X and for any collection $\{A_i\}_{i\in I}$ of subsets of X, $\overline{\cup_{i\in I} A_i} = \cup_{i\in I}\bar{A}_i$. It is clear that

$$\bar{A} = A \cup \{x \in X : \ \langle x\rangle \cap A \nsubseteq \circ\}.$$

"cl" induces a unique topology on X that we denote it by $\mathcal{T}_c(X)$, or briefly, $\mathcal{T}_c$. Similarly, let int : $P(X) \to P(X)$ be a function with

$$\forall A \in P(X), \ \ \mathrm{int}(A) = A^\circ = \{x \in X : \ \langle x\rangle \subseteq A\},$$

then "int" is an interior operation and for any collection $\{A_i\}_{i\in I}$ of subsets of X, $(\cap_{i\in I} A_i)^\circ = \cap_{i\in I} A_i^\circ$. We denoted the topology induced by interior maps, by $\mathcal{T}_{\mathcal{M}}(X)$ (or briefly, $\mathcal{T}_{\mathcal{M}}$). It is clear that $\mathcal{M}_X$ is a base for this topology.

If $(X, \mathcal{M}_X)$ is a cap-structure space and $Y \subseteq X$, the collection $\mathcal{M}_Y = \{u \cap Y : \ u \in \mathcal{M}_X\}$ is an intersection structure on Y, so $(Y, \mathcal{M}_Y)$ is a cap-structure space. In this case, we say Y is a subspace of X.

This article is organized in such a way that in section 2, we will define the homomorphism of cap-structure spaces and study its properties. Also, in this section, we will introduce types of isomorphism of cap-structure spaces and study their relationship. In section 3, we define the quotient of cap-structure spaces and using it, we generalize algebraic isomorphism theorems for cap-structure spaces.

We use the symbols and notations as follows. For any set X, the symbol $|X|$ represents the cardinality of the set X. For an integer $n \geq 2$, the symbol $\mathbb{Z}_n$ stands for the ring of integers modulo n. If $f : X \to Y$, $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$. We denote $\{f(A) : \ A \in \mathcal{A}\}$ and $\{f^{-1}(B) : \ B \in \mathcal{B}\}$ by $f(\mathcal{A})$ and $f^{-1}(\mathcal{B})$, respectively. The algebraic and topological concepts used in this article are well-known and can be found in [7, 8].

2 Homomorphisms in cap-structure spaces

In this section, after defining the concept of cap-structural homomorphism, we will see that this concept is a suitable generalization of the algebraic homomorphisms, in the subject of cap-structure spaces.

Definition 2.1. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be a cap-structure spaces. We call $f : X \to Y$ is:

(i) an $\mathcal{M}$-continuous function if $f^{-1}(\mathcal{M}_Y) \subseteq \mathcal{M}_X$;

(ii) a cap-structural homomorphism (shortly, homomorphism) whenever $f(\mathcal{M}_X) \subseteq \mathcal{M}_{f(X)}$ and it is $\mathcal{M}$-continuous. Moreover, if for each $m, n \in K_f$, the equality $f(m) = f(n)$ implies that $m = n$, where $\ker(f) = f^{-1}(\circ)$ and $K_f = \{m \in \mathcal{M}_X : \ \ker(f) \subseteq m\}$, then we call f a strong homomorphism.

Remark 2.2. Suppose that $f : X \to Y$ is a homomorphism.

(a) For each $u \in \mathcal{M}_Y$, $f^{-1}(u) \in K_f$ and $f(f^{-1}(u)) = u \cap f(X) \in \mathcal{M}_{f(X)}$. So $f(K_f) = \mathcal{M}_{f(X)}$. In particular, if $f : X \to Y$ is a homomorphism, then $f(\mathcal{M}_X) = \mathcal{M}_{f(X)}$.

(b) f is a strong homomorphism if and only if $\phi : K_f \to \mathcal{M}_{f(X)}$ with $\phi(m) = f(m)$, is an order isomorphism.

Lemma 2.3. *Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be cap-structure spaces and $f : X \to Y$ a homomorphism. Then the following statements hold:*

(a) *f is a strong homomorphism if and only if $f^{-1}f(m) = m$ for each $m \in K_f$.*

(b) *f is onto if and only if $f(\mathcal{M}_X) = \mathcal{M}_Y$.*

(c) *$f(\mathcal{M}_X) \subseteq \mathcal{M}_Y$ if and only if $f(X) \in \mathcal{M}_Y$.*

(d) *If $f(X) \in \mathcal{M}_Y$ and $g : Y \to Z$ is a homomorphism, then the function $gf : X \to Z$ is also a homomorphism.*

(e) *$f(\circ_X) = \circ_Y \cap f(X)$. In particular, if $\circ_X \neq \emptyset$ and $\circ_Y = \emptyset$, then there is no homomorphism from X to Y.*

Proof. (a) First note that for every $m \in \mathcal{M}_X$, $f^{-1}f(m) \in K_f$. So if f is a strong homomorphism, and since, always, $ff^{-1}f(m) = f(m)$, it follows that $f^{-1}f(m) = m$ for every $m \in K_f$. The proof of the converse is obvious.

According to part (a) of Remark 2.2, and the fact that $\mathcal{M}_{f(X)} \subseteq \mathcal{M}_Y$ if and only if $f(X) \in \mathcal{M}_Y$, (b) and (c) are easily obtained. The poof of (d) is also easily obtained.

(e). Since $f^{-1}(\circ_Y) \in \mathcal{M}_X$, we conclude that $\circ_X \subseteq f^{-1}(\circ_Y)$ and therefore $f(\circ_X) \subseteq \circ_Y \cap f(X)$. Now according to $f(\circ_X) \in \mathcal{M}_{f(X)}$ and also $\circ_{f(X)} = \circ_Y \cap f(X)$, we have $\circ_Y \cap f(X) \subseteq f(\circ_X)$. Therefore, the equality is proved. □

Corollary 2.4. *Assume that $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ are cap-structure spaces and $f : X \to Y$ is a homomorphism. Then $f(\circ_X) = \circ_Y$ if and only if $\circ_Y \subseteq f(X)$.*

Proof. The proof, according to part (e) of Lemma 2.3, is clear. □

Next example shows that a homomorphism of cap-structure spaces need not be a strong homomorphism in general.

Example 2.5. For all $n \in \mathbb{N}$, let $A_n = \{1, 2, \cdots, n\}$ and consider the cap structure space $(\mathbb{N}, \mathcal{M})$ where $\mathcal{M} = \{A_n : n \in \mathbb{N}\} \cup \{\mathbb{N}\}$. Now consider $f : \mathbb{N} \to \mathbb{N}$ with $f(1) = 1$ and $f(2n) = f(2n+1) = n+1$. Then one can easily see that $f(A_1) = A_1$, $f(A_{2n}) = f(A_{2n+1}) = A_{n+1}$, $f^{-1}(A_n) = A_{2n-1}$ for all $n \in \mathbb{N}$. Therefore f is a homomorphism. But f is not a strong homomorphism, because $\ker(f) = A_1 \subseteq A_2$ but $f^{-1}f(A_2) = A_3$.

The following example shows that for a homomorphism of cap-structure spaces, say $f : X \to Y$, it may be $f(\circ_X) = \circ_Y$ but $\ker(f) \neq \circ_X$ and vise versa.

Example 2.6. (a). Let X and Y be rings and let $\mathcal{M}_X$, $\mathcal{M}_Y$ be the sets of all ideals of X and Y, respectively. If $f : X \to Y$ is an onto ring homomorphism which is not a ring isomorphism, then $f(\circ_X) = \circ_Y$ while $f^{-1}(\circ_Y) \neq \circ_X$.
(b). Let $X = Y = \mathbb{N}$, $\mathcal{M}_X = \{A_1, A_2, A_3, \cdots\} \cup \{\mathbb{N}\}$ and $\mathcal{M}_Y = \{A_2, A_3, A_4, \cdots\} \cup \{\mathbb{N}\}$, where $A_k = \{1, 2, \cdots k\}$, for each $k \in \mathbb{N}$. Now let $f : X \to Y$ with $f(x) = x + 1$, then one can easily see that $f^{-1}(\circ_Y) = \circ_X$ but $f(\circ_X) \neq \circ_Y$.

Remark 2.7. As we already mentioned in Lemma 2.3, if $f : X \to Y$ and $g : Y \to Z$ are two homomorphisms such that f is onto or $f(X) \in \mathcal{M}_Y$, then gf is a homomorphism. The following example shows, the composition of two homomorphisms need not be a homomorphism in general.

Example 2.8. Let Y be a cap-structure space such that $\circ_Y \neq \emptyset$ and $|\mathcal{M}_Y| \geq 2$. So there exists a $y_0 \in Y \setminus \circ_Y$. If $Z = \{y_0\}$ and $\mathcal{M}_Z = \{Z\}$, then clearly $g : Y \to Z$ with $g(y) = y_0$ is a homomorphism. Now, consider $X = Y \setminus \circ_Y$ as a subspace of Y. Then it is easy to see that the inclusion map $f : X \to Y$ is a homomorphism, but gf is not a homomorphism, by Corollary 2.4.

Before we discuss other properties of homomorphisms, we state the following definition with the motivation of the previous remark and example.

Definition 2.9. Let X, Y, Z be cap-structure spaces and $g : Y \to Z$ a homomorphism. We say g is left combinational, if gf is a homomorphism for any homomorphism $f : X \to Y$. The right combinational homomorphism is defined similarly.

Here, the natural question is whether the restriction of a homomorphism is also a homomorphism? The following proposition shows that the answer is negative.

Proposition 2.10. *Let $g : Y \to Z$ be a homomorphism. Then g is left combinational if and only if any restriction of g is a homomorphism.*

Proof. ($\Rightarrow$) Suppose that $B \subseteq Y$ and $\mathcal{M}_B = \mathcal{M}_Y \cap B$. In this case, it is clear that the inclusion function $f : B \to Y$ is a homomorphism. So, by assumption, $gf : B \to Z$ is also a homomorphism. It is clear that $gf = g|_B$.

($\Leftarrow$) Let $f : X \to Y$ be a homomorphism. By assumption the function $g|_{f(X)}$ is a homomorphism. It is clear that $f : X \to f(X)$ is an onto homomorphism, so $gf = g|_{f(X)}f$ is also a homomorphism. □

Lemma 2.11. *Suppose that $A \subseteq X$. The following statements hold:*

(a) *$f : X \to Y$ is a homomorphism if and only if $f^{-1}(m') \in \mathcal{M}_X$ for every $m' \in \mathcal{M}_Y$ and $f(m) = \langle f(m)\rangle \cap f(X)$ for each $m \in \mathcal{M}_X$.*

(b) *The restriction of a homomorphism $f : X \to Y$ to A is a homomorphism if and only if $f(m\cap A) = \langle f(m\cap A)\rangle \cap f(A)$ for each $m \in \mathcal{M}_X$.*

Proof. (a) Suppose that $m \in \mathcal{M}_X$. By assumption, there exists $m' \in \mathcal{M}_Y$ such that $f(m) = m' \cap f(X)$. It is clear that $f(m) \subseteq m'$, and so we can write:

$$f(m) \subseteq \langle f(m)\rangle \cap f(X) \subseteq m' \cap f(X) = f(m).$$

Thus $f(m) = \langle f(m)\rangle \cap f(X)$. The proof of ($\Leftarrow$) is obvious.

(b) Using part (a), it is clear. $\square$

Proposition 2.12. *Suppose that* Y *is a cap-structure space and* $o_Y \neq \emptyset$*. Then every homomorphism* $g : Y \to Z$ *is a left combinational if and only if* $\mathcal{M}_Y = \{Y\}$*.*

Proof. ($\Rightarrow$) Let, on the contrary, there exists $y_0 \in Y \setminus \circ_Y$ and Z a cap-structure space such that $o_Z = \{z_0\}$. In this case, we consider the constant function $g : Y \to Z$ with $g(y) = z_0$. It is clear that g is a homomorphism. But, if $Y_0 = Y \setminus \circ_Y$, then the restriction function $g|_{Y_0}$ cannot be a homomorphism. Because for each $m' \in \mathcal{M}_Z$ we can write:

$$g(\circ_Y \cap Y_0) = g(\emptyset) =\neq \{z_0\} = m' \cap \{z_0\} = m' \cap g(Y_0).$$

Therefore, by Proposition 2.10, g cannot be left combinational.

($\Leftarrow$) It is obvious. $\square$

Remark 2.13. In the above proposition, the condition $o_Y \neq \emptyset$ is necessary. To see this fact, first notice that if Y has a trivial cap-structure space, that is, $\mathcal{M}_Y = \{\emptyset, Y\}$, then the function $g : Y \to Z$ is a homomorphism if and only if $g(Y)$ has trivial cap-structure. Therefore, the restriction of any homomorphism on such structures is a homomorphism.

The next proposition shows that if $f : X \to Y$ is a homomorphism (strong homomorphism), then $f : \mathcal{M}_X \to \mathcal{M}_{f(X)}$ ($f^{-1} : \mathcal{M}_Y \to \mathcal{M}_X$) is a complete join-homomorphism.

Proposition 2.14. *If* $f : X \to Y$ *is a homomorphism and* $m_i \in \mathcal{M}_X$ *for each* $i \in I$*, then* $f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} f(m_i)$*, in which, the second join is regarded within* $\mathcal{M}_{f(X)}$*. In addition, if* f *is a strong homomorphism then* $f^{-1}(\bigvee_{i\in I} m_i^{'}) = \bigvee_{i\in I} f^{-1}(m_i^{'})$*, for each family* $\{m_i^{'}\}_{i\in I}$ *of* $\mathcal{M}_Y$*.*

Proof. It is clear that $\bigvee_{i\in I} f(m_i) \subseteq f(\bigvee_{i\in I} m_i)$. Conversely, let $\bigvee_{i\in I} f(m_i) = u$, then $f(m_i) \subseteq u$ for each $i \in I$, so $m_i \subseteq f^{-1}(u)$ for each $i \in I$. Hence, $\bigvee_{i\in I} m_i \subseteq f^{-1}(u)$, consequently $f(\bigvee_{i\in I} m_i) \subseteq u$ and so $f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} f(m_i)$.

To prove the last part of this proposition, it is clear that $f^{-1}(\vee_{i\in I} m_i^{'})$ is an upper bound for the family $\{f^{-1}(m_i^{'}\}_{i\in I}$. Now let n be any upper bound of this family. Then by the hypothesis $m_i^{'} \subseteq f(n)$ for each $i \in I$, and therefore $\bigvee_{i\in I} m_i^{'} \subseteq f(n)$. Thus $f^{-1}(\bigvee_{i\in I} m_i^{'}) \subseteq f^{-1}f(n) = n$ (note that $n \in K_f$), and the proof is complete. □

Proposition 2.15. *Let $f : X \to Y$ be a homomorphism. Then*

(a) $f(\langle A\rangle_X) = \langle f(A)\rangle_{f(X)}$ *for any* $A \subseteq X$*;*

(b) $\langle f(A)\rangle_{f(X)} = \langle f(A)\rangle_Y \cap f(X)$ *for any* $A \subseteq X$*;*

(c) $f(\langle x\rangle_X) = \langle f(x)\rangle_{f(X)}$ *for any* $x \in X$*;*

(d) $\langle f(x)\rangle_{f(X)} = \langle f(x)\rangle_Y \cap f(X)$ *for any* $x \in X$*;*
In addition, if f is an onto strong homomorphism with $\ker(f) = \circ$*, then*

(e) $f^{-1}(\langle B\rangle_Y) = \langle f^{-1}(B)\rangle_X$*, for every* $B \subseteq Y$*;*

(f) $f^{-1}(\langle y\rangle_Y) = \langle f^{-1}\{y\}\rangle_X$*, for every* $y \in Y$*.*

Proof. (a) It is obvious that $\langle f(A)\rangle_{f(X)} \subseteq f(\langle A\rangle_X)$. For the reverse inclusion suppose that $n = \langle f(A)\rangle_{f(X)}$, then we have

$$f(A) \subseteq n \Rightarrow A \subseteq f^{-1}(n) \Rightarrow \quad \langle A\rangle_X \subseteq f^{-1}(n) \Rightarrow f(\langle A\rangle_X) \subseteq ff^{-1}(n) = n.$$

(b)

$$\begin{aligned}\langle f(A)\rangle_{f(X)} &= \bigcap \{m' \cap f(X) :\ m' \in \mathcal{M}_Y,\ f(A) \subseteq m' \cap f(X)\}\\ &= \bigcap \{m' \in \mathcal{M}_Y :\ f(A) \subseteq m'\} \cap f(X)\\ &= \langle f(A)\rangle_Y \cap f(X).\end{aligned}$$

(c) By part (a), it is clear.
(d) By part (b), it is clear.
(e) Since f is an onto strong homomorphism, by part (a), we can write:

$$ff^{-1}\langle B\rangle_Y = \langle B\rangle_Y = \langle ff^{-1}B\rangle_Y = f(\langle f^{-1}B\rangle_X).$$

On the other hand, clearly, $\ker(f) = \circ \subseteq f^{-1}(\langle B\rangle_Y)$ and $\ker(f) = \circ \subseteq \langle f^{-1}B\rangle_X$. Since f is a strong homomorphism, we conclude that $\langle f^{-1}(B)\rangle_X = f^{-1}(\langle B\rangle_Y)$.

(f) By part (e), it is clear. □

In parts (e) and (f) of the above proposition, the mentioned conditions are necessary. To see this, suppose that $X = Y = \mathbb{Z}$, $\mathcal{M}_X$ is the set of all ideals of $\mathbb{Z}$ and $\mathcal{M}_Y = \{\{0\}, \mathbb{Z}\}$. Now, let $f : X \to Y$ be a function such that $f(0) = 0$ and $f(n) = 1$ for every $n \in \mathbb{Z} \setminus \{0\}$. It is clear that f is a homomorphism but the parts (e) and (f) is not true for f.

In the next proposition, we state the necessary and sufficient conditions for establishing the equality $\langle f(x) \rangle_{f(X)} = \langle f(x) \rangle_Y$, where $f : X \to Y$ is a homomorphism.

Proposition 2.16. *Let $f : X \to Y$ be a homomorphism such that $\circ \subseteq f(X)$ and let $x \in X$. The following statements are equivalent:*

(a) $f(x) \notin \overline{Y - f(X)}$.

(b) $f(x) \notin \overline{\{y\}}$ *for any* $y \in Y - f(X)$,

(c) $\langle f(x) \rangle_{f(X)} = \langle f(x) \rangle_Y$.

(d) $\langle f(x) \rangle_Y \subseteq f(X)$.

(e) $f(x) \in (f(X))^\circ$.

Proof. (a) $\Leftrightarrow$ (b) By the fact that the closure operator is distributive over arbitrary union, it is clear.

(c) $\Leftrightarrow$ (d) By Proposition 2.15, it is evident.

(b) $\Rightarrow$ (d) Suppose that $y \notin f(X)$, then $f(x) \notin \overline{\{y\}}$ and so by the definition of closure, $\langle f(x) \rangle_Y \cap \{y\} \subseteq \circ$. By the hypothesis, $\circ \subseteq f(X)$ and $y \notin \circ$, so we must have $\langle f(x) \rangle_Y \cap \{y\} = \emptyset$, and therefore $y \notin \langle f(x) \rangle_Y$.

(d) $\Rightarrow$ (b) If $y \notin f(X)$, then $y \notin \langle f(x) \rangle$. Therefore, $\langle f(x) \rangle \cap \{y\} \subseteq \circ$ and this implies that $f(x) \notin \overline{\{y\}}$.

(e) $\Leftrightarrow$ (d) It is evident. □

The following corollary is immediate.

Corollary 2.17. *Let $f : X \to Y$ be a homomorphism such that $\circ \subseteq f(X)$. The following statements are equivalent:*

(a) $f(x) \notin \overline{Y - f(X)}$ *for every* $x \in X$.

(b) $f(X) \cap \overline{Y - f(X)} = \emptyset$.

(c) $\langle f(x) \rangle_{f(X)} = \langle f(x) \rangle_Y$ *for every* $x \in X$.

(d) $\langle f(x) \rangle_Y \subseteq f(X)$ *for every* $x \in X$.

(e) $f(X) \in \mathcal{T}_{\mathcal{M}}(Y)$ *and consequently* $f(X) \in \mathcal{T}_c(Y)$.

Definition 2.18. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be cap-structure spaces. We say $f : X \to Y$ is:

i) an $\mathcal{M}$-embedding if f is a strong homomorphism and $\ker(f) = \circ$;

ii) an $\mathcal{M}$-isomorphism if f is an onto $\mathcal{M}$-embedding;

iii) an embedding if f is a one-one homomorphism. An embedding which is onto is called an isomorphism. In this case, we write $X \simeq Y$.

It is clear that every isomorphism is an $\mathcal{M}$-isomorphism but, as the following example shows, the converse is not true in general.

Example 2.19. Let $X = \mathbb{Z}_{10}$, $Y = \mathbb{Z}_6$, and $\mathcal{M}_X$, $\mathcal{M}_Y$ be the sets of all ideals of X and Y, respectively. Now define $f : X \to Y$ with $f(0) = 0, f(1) = f(7) = f(9) = 1, f(2) = f(6) = f(8) = 2, f(5) = 3, f(4) = 4$, and $f(3) = 5$. We can easily see that f is an $\mathcal{M}$-isomorphism but it is not an isomorphism.

Proposition 2.20. *Let $f : X \to Y$ be a homomorphism. The following are equivalent:*

(a) f *is an* $\mathcal{M}$*-isomorphism.*

(b) $f : \mathcal{M}_X \to \mathcal{M}_Y$ *is an order isomorphism.*

(c) $f : \mathcal{M}_X \to \mathcal{M}_Y$ *is an isomorphism of complete lattices.*

(d) $f : X \to Y$ *is onto and* $f^{-1}f(m) = m$, *for each* $m \in \mathcal{M}_X$.

Proof. (a) $\Rightarrow$ (b) Since f is a strong homomorphism, by part (b) of Remark 2.2, f is an order embedding. On the other hand, since $f : X \to Y$ is onto, so is $f : \mathcal{M}_X \to \mathcal{M}_Y$. Therefore, f is an order isomorphism.

(b) $\Rightarrow$ (c) It is sufficient to show that $\bigcap_{i\in I} f(m_i) = f(\bigcap_{i\in I} m_i)$ for any family $\{m_i\}_{i\in I}$ of $\mathcal{M}_X$. To do this, we just have to prove that $\bigcap_{i\in I} f(m_i) \subseteq f(\bigcap_{i\in I} m_i)$. So let $u = \bigcap_{i\in I} f(m_i)$. Then $u \in \mathcal{M}_Y$ and $u \subseteq f(m_i)$ for any $i \in I$. Therefore $f^{-1}(u) \subseteq m_i$, for any $i \in I$, hence $f^{-1}(u) \subseteq \bigcap_{i\in I} m_i$ and it implies that $u \subseteq f(\bigcap_{i\in I} m_i)$.

(c) $\Rightarrow$ (d) Clearly, $f^{-1}f(m) = m$ for each $m \in \mathcal{M}_X$ and since $f(X) \in \mathcal{M}_Y$, it follows that $f : X \to Y$ is onto.

(d) $\Rightarrow$ (a) Since $f : X \to Y$ is onto, we have $f(\circ) = \circ$, so $\ker f = f^{-1}(\circ) = f^{-1}f(\circ) = \circ$. Therefore, f is an onto $\mathcal{M}$-embedding, that is, f is an $\mathcal{M}$-isomorphism. □

Proposition 2.21. *Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be two cap-structure spaces. If $f : X \to Y$ is an isomorphism, then f as a function from $(X, \mathcal{T}_c(X))$ to $(Y, \mathcal{T}_c(Y))$ is a homeomorphism.*

Proof. It suffices to show that $\overline{f(E)} = f(\overline{E})$. For this, let $t = f(x) \notin f(\overline{E})$. Thus, $x \notin \overline{E}$ and so there is $u \in \mathcal{M}_X$ such that $x \in u$ and $u \cap E \subseteq \circ$. Therefore, $f(u) \cap f(E) \subseteq \circ$, and since $f(u) \in \mathcal{M}_Y$ and $t \in f(u)$, we conclude that $t \notin \overline{f(E)}$. Thus, $\overline{f(E)} \subseteq f(\overline{E})$. Now, suppose that $t = f(x) \notin \overline{f(E)}$. Thus, there is $v \in \mathcal{M}_Y$ such that $t \in v$ and $v \cap f(E) \subseteq \circ$. Therefore, $f^{-1}(v) \cap E = f^{-1}(v) \cap f^{-1}f(E) \subseteq \circ$. Since $f^{-1}(v) \in \mathcal{M}_X$ and $x \in f^{-1}(v)$, it follows that $x \notin \overline{E}$ and so $t = f(x) \notin f(\overline{E})$, Therefore, $f(\overline{E}) \subseteq \overline{f(E)}$. □

In the previous proposition, if we substitute $\mathcal{T}_{\mathcal{M}}$ for $\mathcal{T}_c$, the statement is still true. In addition, the converse of the above proposition is not true; to see this, assume that

$$X = Y = \mathbb{R},\ \mathcal{M}_X = P(X),\ \mathcal{M}_Y = \{(a, b) :\ a, b \in \mathbb{R}\} \cup \{\{x\} :\ x \in \mathbb{R}\} \cup \{\mathbb{R}\}.$$

Then $(X, \mathcal{T}_c(X))$ and $(Y, \mathcal{T}_c(Y))$ are discrete spaces and so homeomorphic by identity map, but $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ are not isomorphic. Note that in this example $\mathcal{T}_c(X) = \mathcal{T}_{\mathcal{M}}(X) = \mathcal{T}_c(Y) = \mathcal{T}_{\mathcal{M}}(Y) = P(X)$.

3 Quotient of cap-structure spaces

In this section we define the quotient space and its related homomorphism theorems for the cap-structure spaces. In particular, our aim is to generalize algebraic isomorphism theorems to cap-structure spaces. Suppose that $(X, \mathcal{M}_X)$ is a cap-structure space and R is an equivalence relation on X. Assume that $P : X \to X/R$ is defined by $x \to [x]$, which is called the natural quotient map. We want to define an intersection structure on X/R, which is induced naturally by $\mathcal{M}_X$. It is natural that in this case we should expect that:

(a) the function P holds the arbitrary intersection and

$$\mathcal{M}_{X/R} = \{P(m) :\ m \in \mathcal{M}_X\}$$

has an intersection structure on X/R;

(b) $P^{-1}P(m) \in \mathcal{M}_X$ for each $m \in \mathcal{M}_X$.

We will see later that the condition (b) alone is sufficient for this purpose.

Definition 3.1. Let $(X, \mathcal{M}_X)$ be a cap-structure space, θ an equivalence relation on X and $P : X \to X/\theta$ the natural quotient map. We call θ is a cap-structural congruence whenever $P^{-1}P(m) \in \mathcal{M}_X$ for each $m \in \mathcal{M}_X$. In addition, a cap-structural congruence θ is called a strong cap-structural congruence when $P^{-1}P(m) = m$ for each $m \supseteq P^{-1}P(\circ)$. The set of all cap-structural congruence (strong cap-structural congruence) on a cap-structure space is denoted by $con(X)$ $(scon(X))$.

Lemma 3.2. *Let f be a function from X onto Y and $\mathcal{S} \subsetneq P(X)$ be a complete lattice under inclusion relation such that $f^{-1}f(S) \in \mathcal{S}$, for each $S \in \mathcal{S}$. Then the following statements hold:*

(a) $f(\bigvee \mathcal{A}) = \bigvee f(\mathcal{A})$, *for each $\mathcal{A} \subseteq \mathcal{S}$, where the first and second symbols "$\vee$" are considered within $\mathcal{S}$ and $f(\mathcal{S})$, respectively.*

(b) $f(\bigwedge \mathcal{A}) = \bigwedge f(\mathcal{A})$, *for each $\mathcal{A} \subseteq \mathcal{S}$, where the first and second symbols "$\wedge$" are considered within $\mathcal{S}$ and $f(\mathcal{S})$, respectively.*

(c) *$f(\mathcal{S})$ with the above operations is a complete lattice.*

(d) *If $\mathcal{S}$ is a cap-structure, then so is $f(\mathcal{S})$.*

Proof. (a) It is clear that $f(\bigvee \mathcal{A})$ is an upper bound of $f(\mathcal{A})$. Now suppose that $S \in \mathcal{S}$ and $f(S)$ is an upper bound of $f(\mathcal{A})$. Then $f(A) \subseteq f(S)$, for each $A \in \mathcal{A}$. Hence $A \subseteq f^{-1}f(S)$ for each $A \in \mathcal{A}$, so $\bigvee \mathcal{A} \subseteq f^{-1}f(S)$. Hence $f(\bigvee \mathcal{A}) \subseteq ff^{-1}f(S) = f(S)$. Therefore $f(\bigvee \mathcal{A})$ is a least upper bound of $f(\mathcal{A})$.

The proof of (b) is similar to (a), and (c) is obvious.

(d) Let $S_i \in \mathcal{S}$ for every $i \in I$. Since f is onto, for each $i \in I$, we can write:

$$\bigcap_{i \in I} f(S_i) = ff^{-1} \bigcap_{i \in I} f(S_i) = f(\bigcap_{i \in I} f^{-1}f(S_i)) \in f(\mathcal{S}).$$

□

Proposition 3.3. *Let $(X, \mathcal{M}_X)$ be a cap-structure space, θ a cap-structural congruence on X and $P : X \to X/\theta$ the natural quotient map. Then the following statements hold:*

(a) $P(\bigcap_{i\in I} m_i) = \bigcap_{i\in I} P(m_i)$, *therefore* P *preserves arbitrary intersections.*

(b) P *preserves arbitrary joins.*

(c) $\mathcal{M}_{X/\theta} = \{P(m) :\ m \in \mathcal{M}_X\}$ *is a cap-structure on* X/θ.

(d) *The map* $P : X \to X/\theta$ *is an onto homomorphism.*

Proof. By Definition 3.1, and Lemma 3.2, the proof is clear. □

According to above proposition, we have the following definition:

Definition 3.4. Assume that the conditions of Proposition 3.3, are satisfied. In this case, we call the space $(X/\theta, \mathcal{M}_{X/\theta})$ is a quotient of the space X corresponding to the cap-structural congruence θ.

Remark 3.5. Let X and Y be cap-structure spaces and $f : X \to Y$ an arbitrary function. For every $a, b \in X$, define

$$a\theta_f b :\Leftrightarrow\ f(a) = f(b).$$

Then the following statements hold:

(a) θ_f is an equivalence relation on X and if f is a homomorphism, then $\theta_f \in con(X)$ (in this case, we call θ_f the congruence kernel of f).

(b) Assuming that f is a homomorphism, $\theta_f \in scon(X)$ if and only if f is a strong homomorphism.

Proof. (a) Obviously, θ_f is an equivalence relation on X. Now, assume that f is a homomorphism and P_f is the natural quotient map induced by θ_f. $P_f^{-1}P_f(m) = f^{-1}f(m) \in \mathcal{M}_X$ for every $m \in \mathcal{M}_X$. Thus, $\theta_f \in con(X)$

(b $\Rightarrow$) It is clear, since $P_f^{-1}P_f = f^{-1}f$.

(b $\Leftarrow$) By part (a), $\theta_f \in con(X)$. Now, assume that $P_f^{-1}P_f(0) \subseteq m$, since f is strong homomorphism, we can write

$$f^{-1}f(0) = P_f^{-1}P_f(0) \subseteq m \quad \Rightarrow \quad P_f^{-1}P_f(m) = f^{-1}f(m) = m.$$

Therefore, $\theta_f \in scon(X)$. □

Note that the converse of part (a) is not true, in general; that is, if $f : X \to Y$, then it may be $\theta_f \in con(X)$ whereas f is not a homomorphism. To see this, let $f : X \to Y$ be a homomorphism, then $\theta_f \in con(X)$.

Now, it is enough to change the cap-structure on Y such that f is not a homomorphism with respect to this new cap-structure space. The following proposition shows that there is a close relation between $con(X)$ and the set of homomorphism on X.

Proposition 3.6. *An equivalence relation θ on a cap-structure space X is a cap-structural congruence (strong cap-structural congruence) if and only if there exists a cap-structure space Y and a homomorphism (strong homomorphism) $f : X \to Y$ such that $\theta = \theta_f$.*

Proof. ($\Rightarrow$) If θ is a cap-structural congruence on X, then clearly $\theta = \theta_P$ where P is the natural quotient map from X to X/θ.

($\Leftarrow$) By Remark 3.5, it is clear. □

The following is a counterpart of the first isomorphism theorem of algebraic structures.

Theorem 3.7. *Let $(X, \mathcal{M}_X)$, $(Y, \mathcal{M}_Y)$ be cap-structure spaces and $f : X \to Y$ an onto homomorphism. In this case $g : X/\theta_f \to Y$ which is defined by $g([x]) = f(x)$, is an isomorphism.*

Proof. It is clear that g is a bijection. So we just have to show that g is a homomorphism. For this purpose, suppose that $v \in \mathcal{M}_Y$. It is clear from the definition of g that $gP_f = f$, so considering that P_f is onto, it follows that

$$g^{-1}(v) = P_f P_f^{-1} g^{-1}(v) = P_f f^{-1}(v) \in \mathcal{M}_{X/\theta_f}.$$

Now let $P_f(u) \in \mathcal{M}_{X/\theta_f}$. In this case, it is clear that $g(P_f(u)) = f(u) \in \mathcal{M}_Y$. Hence, g is a homomorphism and the proof is complete. □

Definition 3.8. Let $(X, \mathcal{M}_X)$ be a cap-structure space and $n \in \mathcal{M}_X$. We say that n is *quotientable* (respectively, *strong quotientable*) when there exists a cap-structure space $(Y, \mathcal{M}_Y)$ and an onto homomorphim (respectively, strong homomorphism) $f : X \to Y$ such that $\ker(f) = n$.

The following example shows that it is not the case that in a cap-structure space $(X, \mathcal{M})$ every element of $\mathcal{M}$ is quotientable.

Example 3.9. Let $X = \{e, a, b, c, d\}$ and $\mathcal{M}_X = \{\{e\}, n_1 = \{e, a\}, n_2 = \{e, b\}, n = \{e, a, b\}, m_1 = \{e, a, c\}, m_2 = \{e, b, d\}, X\}$. It is clear that

$(X, \mathcal{M})$ is a cap-structure space. We claim that n is not quotientable. To this end, let there exists a cap-structure space $(Y, \mathcal{M}_Y)$ and let $f : X \to Y$ be an onto homomorphism with $\ker(f) = n$. With these assumbtion $P : X \to X/\theta_f$ is a homomorphism and $\ker(P) = n$. Since $P(n) \subseteq P(m_1)$, we have $n \subseteq P^{-1}P(n) \subseteq P^{-1}P(m_1)$. Therefore, $m_1 \cup n \subseteq P^{-1}P(m_1)$ and we conclude that $P^{-1}P(m_1) = X$. Similarly, $P^{-1}P(m_2) = X$. Now, we have $n = P^{-1}P(m_1 \cap m_2) = P^{-1}P(m_1) \cap P^{-1}P(m_2) = X$, which is a contradiction.

Proposition 3.10. *Let $(X, \mathcal{M}_X)$ be a cap-structure space. Then the following statements hold:*

(a) *Let $\circ_X = \emptyset$ and $n \in \mathcal{M}_X$. Then n is quotientable if and only if $n = \emptyset$.*

(b) *Let $\circ_X \neq \emptyset$ and $n \in \mathcal{M}_X$ be such that $m \cup n \in \mathcal{M}_X$ for every $m \in \mathcal{M}_X$, then n is a strong quotientable element.*

Proof. (a) In the case $n = \emptyset$, it suffices to consider identity map on X. Now, suppose that $n \neq \emptyset$, $(Y, \mathcal{M}_Y)$ is an arbitrary structure space and f is a homomorphism from X to Y. Clearly, $\circ_Y = f(\circ_X) = f(\emptyset) = \emptyset$ and so $\ker(f) = f^{-1}(\circ_Y) = f^{-1}(\emptyset) = \emptyset \neq n$.

(b) Let $Y = (X \setminus n) \cup \{e\}$ where $e \notin X$. It is easy to see that $M_Y = \{(m \setminus n) \cup \{e\} : m \in M_X\}$ is a cap-structure on Y and $\circ_Y = \{e\}$. Now we define the map $f : X \to Y$ by

$$f(x) = \begin{cases} e & x \in n \\ x & x \in X \setminus n \end{cases}$$

To complete the proof, we show that f is an onto strong homomorphism with the desired property. First note that f is onto and $f^{-1}(\circ_Y) = n$, and so $f(n) = \circ_Y$. Also, note that $f(m) = (m \setminus n) \cup \circ_Y \in M_Y$ and $f^{-1}((m \setminus n) \cup \circ_Y) = f^{-1}(m \setminus n) \cup n = (m \setminus n) \cup n = m \cup n \in M_X$, for any $m \in M_X$. This implies that f is a homomorphism. Finally, if $n = \ker(f) \subseteq m_1, m_2$ and $f(m_1) = f(m_2)$, then we can write:

$$\begin{aligned} (m_1 \setminus n) \cup \circ_Y = (m_2 \setminus n) \cup \circ_Y &\Rightarrow (m_1 \setminus n) \cup \circ_X = (m_2 \setminus n) \cup \circ_X \\ &\Rightarrow m_1 = n \cup ((m_1 \setminus n) \cup \circ_X) = n \cup ((m_2 \setminus n) \cup \circ_X) = m_2. \end{aligned}$$

□

Example 3.11. Suppose that $(X, \mathcal{M}_X)$ is a cap-structure space and $n \in \mathcal{M}_X$. We define

$$\forall a, b \in X \quad a\,\theta_n\,b : \quad \Leftrightarrow \quad \langle a\rangle \vee n = \langle b\rangle \vee n.$$

In this case, θ_n is clearly an equivalence relation. Now if θ_n is a cap-structural congruence; in other words, if P_n, the natural quotient map associated with θ_n, is a homomorphism, then we have the following:

(a) $\ker(P_n) = n$.

(b) $P_n^{-1}P_n(m) = m \vee n$, for each $m \in \mathcal{M}_X$.

(c) P_n is a strong onto homomorphism, that is n is strong quotientable.

Proof. (a) $P_n(\circ) = \circ_{X/\theta_n}$ and so we can write:

$$\ker(P_n) = P_n^{-1}(\circ_{X/\theta_n}) = (P_n^{-1}P_n(\circ_X) = \{x \in X : \exists a \in \circ_X, \ [x] = [a]\}$$

$$= \{x \in X : \exists a \in \circ_X, \ \langle x\rangle \vee n = \langle a\rangle \vee n\} = \{x \in X : \langle x\rangle \vee n = n\} = n.$$

(b) For each $m \in \mathcal{M}_X$, $m \subseteq P_n^{-1}P_n(m)$ and by part (a), $n \subseteq P_n^{-1}P_n(m)$. Therefore, $m \vee n \subseteq P_n^{-1}P_n(m)$ and so for each $m \in \mathcal{M}_X$, we have:

$$\begin{aligned} m \vee n \subseteq P_n^{-1}P_n(m) &= \{x \in X : \exists a \in m, \ [x] = [a]\} \\ &= \{x \in X : \exists a \in m, \ \langle x\rangle \vee n = \langle a\rangle \vee n\} \\ &\subseteq \{x \in X : \exists a \in m, \ \langle x\rangle \vee n \subseteq m \vee n\} \subseteq m \vee n. \end{aligned}$$

Therefore, $P_n^{-1}P_n(m) = m \vee n$.

(c) If $m \in K_{P_n}$, then $n = \ker(P_n) \subseteq m$ and according to (b), we have $P_n^{-1}P_n(m) = m \vee n = m$ and so $P_n^{-1}P_n(m) = m$. □

Note that according to the proof of part (a), in the above example, if $\circ_X = \emptyset$ and $n \neq \emptyset$, then θ_n cannot be a cap-structural congruence.

Remark 3.12. Suppose that $f : X \to Y$ and $g : X \to Z$ be two onto functions. We say f and g are equivalent (denoted by $f \simeq g$) if $\theta_f = \theta_g$. Clearly, $\simeq$ is an equivalence relation. Also, if θ is an equivalence relation on X then by $[\theta]$ we mean the class of all functions f from X onto an arbitrary Y such that $\theta_f = \theta$. It is easy to see that if we define $P_\theta : X \to X/\theta$ with $P_\theta(x) = [x]_\theta$, then $P_\theta \in [\theta]$.

Proposition 3.13. *Suppose that θ is an equivalence relation on X. Then the following statements are equivalent:*

(a) $\theta \in con(X)$ $(\theta \in scon(X))$.

(b) P_θ *is a homomorphism* (*strong homomorphism*).

(c) *There exists* $f \in [\theta]$ *such that* f *is a homomorphism* (*strong homomorphism*).

Proof. (a) $\Rightarrow$ (b) is clear, by definition. (b) $\Rightarrow$ (c) is evident and (c) $\Rightarrow$ (a) is obvious, by Remark 3.5. □

Proposition 3.14. *Let $f : X \to Y$ and $g : Y \to Z$ be two onto functions. Then the following statements hold:*

(a) *If we define the function $P : X \to Y/\theta_g$ with $P(x) = [f(x)]_{\theta_g}$, then $P = P_g f$.*

(b) *If f is a homomorphism (strong homomorphism), then P_g is a homomorphism (strong homomorphism) if and only if P is too.*

(c) *If f and g are homomorphisms, then $X/\theta_P \simeq Y/\theta_g$.*

Proof. (a) It is evident.

(b $\Rightarrow$) Suppose that f and P_g are homomorphisms (respectively, strong homomorphisms). Since f is also an onto homomorphism, by (a), P is homomorphism (respectively, strong homomorphism).

(b $\Leftarrow$) Let P is a homomorphism. It suffices to show that P_g is an $\mathcal{M}$-continuous. To see this, by (a), we have $P^{-1}P(m) = f^{-1}P_g^{-1}P_g f(m)$ for every $m \in \mathcal{M}_X$. Now, suppose that $P_g(n) \in \mathcal{M}_{Y/\theta_g}$ where $n \in \mathcal{M}_Y$. Thus, we can write

$$f^{-1}P_g^{-1}(P_g(n)) = f^{-1}P_g^{-1}(P_g(ff^{-1}(n))) = P^{-1}P(f^{-1}(n)) \in \mathcal{M}_X$$
$$\Rightarrow \quad P_g^{-1}(P_g(n)) = ff^{-1}P_g^{-1}(P_g(n)) \in \mathcal{M}_Y.$$

Now, assume that f and P are strong homomorphism and $P_g^{-1}(\circ) \subseteq f(m) \in \mathcal{M}_Y$. To complete the proof, it suffices to show that $P_g^{-1}P_g(f(m)) = f(m)$. It is clear that $\ker(P_g f) = f^{-1}P_g^{-1}(\circ) \subseteq f^{-1}f(m)$. Thus, we can write

$$f^{-1}P_g^{-1}P_g f(f^{-1}f(m)) = P^{-1}P(f^{-1}f(m)) = f^{-1}f(m)$$
$$\Rightarrow \quad P_g^{-1}P_g(f(m)) = P_g^{-1}P_g f(f^{-1}f(m)) = ff^{-1}f(m) = f(m).$$

(c) By part (b), $P = P_g f : X \to Y/\theta_g$ is an onto homomorphism. Hence, by Theorem 3.7, $X/\theta_P \simeq Y/\theta_g$. □

Lemma 3.15. *Let θ_1 and θ_2 be two equivalence relations on X, and also P_1 and P_2 be the quotient maps induced by θ_1 and θ_2, respectively. Then the following statements are equivalent:*
(a) $\theta_1 \subseteq \theta_2$.
(b) $P_1^{-1}P_1(x) \subseteq P_2^{-1}P_2(x)$ *for every* $x \in X$.
(c) *For every* $x \in X$, *we have* $P_2^{-1}P_2(x) = \bigcup_{a \in P_2^{-1}P_2(x)} P_1^{-1}P_1(a)$.

Proof. The proof is straightforward. □

Now, we are ready to state the following theorem, which can be considered as the counterpart of the last isomorphism theorem of algebraic structures.

Theorem 3.16. *Let θ_1 and θ_2 be two equivalence relation on X, $\theta_1 \subseteq \theta_2$, and P_1 and P_2 the quotient maps induced by θ_1 and θ_2, respectively. For any $P_1(a), P_1(b) \in X/\theta_1$, define*

$$P_1(a)\ \theta_2/\theta_1\ P_1(b) :\ \Leftrightarrow\ P_1^{-1}\{P_1(a)\} \times P_1^{-1}\{P_1(b)\} \subseteq \theta_2.$$

Then the following statements hold:
(a) *If* $\theta_1, \theta_2 \in con(X)$, *then* $\theta_2/\theta_1 \in con(X/\theta_1)$.
(b) *If* $\theta_1, \theta_2 \in scon(X)$, *then* $\theta_2/\theta_1 \in scon(X/\theta_1)$.
(c) $\frac{X/\theta_1}{\theta_2/\theta_1} \simeq X/\theta_2$.

Proof. (a) For simplicity, we denote P_{θ_2/θ_1} by P. First, we show that $P_2 \simeq PP_1$; that is, for every $x \in X$, we have $P_2^{-1}P_2(x) = g^{-1}g(x)$, where $g = PP_1$. To see this, suppose that $a \in g^{-1}g(x)$. Thus, $PP_1(a) = g(a) = g(x) = PP_1(x)$ and hence $P_1^{-1}P_1(a) \times P_1^{-1}P_1(x) \subseteq \theta_2$. Therefore, $(a, x) \in \theta_2$ and so $a \in P_2^{-1}P_2(x)$. Thus, $g^{-1}g(x) \subseteq P_2^{-1}P_2(x)$. Conversely, assume that $a \in P_2^{-1}P_2(x)$. Hence, by Lemma 3.15, there exists $b \in P_2^{-1}P_2(x)$ such that $a \in P_1^{-1}P_1(b)$ and so $P_1(a) = P_1(b)$. Therefore, since $P_2^{-1}P_2(b) = P_2^{-1}P_2(x)$, we can write

$$\begin{aligned} P_1(a) = P_1(b) &\Rightarrow P_1^{-1}P_1(a) \times P_1^{-1}P_1(b) \subseteq \theta_1 \subseteq \theta_2 \\ &\Rightarrow P_1^{-1}P_1(a) \times P_1^{-1}P_1(x) \subseteq \theta_2 \\ &\Rightarrow (P_1(a), P_1(x)) \in \theta_2/\theta_1 \\ &\Rightarrow \quad g(a) = PP_1(a) = PP_1(x) = g(x) \end{aligned}$$

$$\Rightarrow \quad a \in g^{-1}g(x).$$

Thus, $P_2^{-1}P_2(x) \subseteq g^{-1}g(x)$. Therefore, $P_2^{-1}P_2(x) = g^{-1}g(x)$ for every $x \in X$ and so $P_2 \simeq PP_1$. Now, since P_1 and P_2 is homomorphism (strong homomorphism), P is so; that is, $\theta_2/\theta_1 \in con(X)$ ($\theta_2/\theta_1 \in scon(X)$). Now, suppose that $\theta_1, \theta_1 \in con(X)$; that is, P_1 and P_2 are homomorphism. It suffices to show that P is $\mathcal{M}$-continuous. Suppose that $k \in \mathcal{M}_{\frac{X/\theta_1}{\theta_2/\theta_1}}$. Clearly, $m \in \mathcal{M}_X$ exists such that $k = PP_1(m)$. Therefore, $P_1^{-1}P^{-1}(k) = P_1^{-1}P^{-1}(PP_1(m)) = P_2^{-1}P_2(P_1(m)) \in \mathcal{M}_X$, which implies that $P^{-1}(k) \in \mathcal{M}_{X/\theta_1}$.

(b) Assume that $\ker(P) \subseteq n$. Clearly, there exists $m \supseteq \ker(P_1)$ such that $P_1(m) = n$. So, we can write

$$\begin{aligned} P^{-1}(\circ) \subseteq P_1(m) &\Rightarrow P_2^{-1}(\circ) = P_1^{-1}P^{-1}(\circ) \subseteq P_1^{-1}P_1(m) = m \\ &\Rightarrow P_1^{-1}P^{-1}PP_1(m) = P_2^{-1}P_2(m) = m \\ &\Rightarrow P^{-1}P(P_1(m)) = P_1(m). \end{aligned}$$

Therefore, P is strong homomorphism; that is, $\theta_2/\theta_1 \in scon(X/\theta_1)$.

(c) By Theorem 3.7 and the fact that $P_2 \simeq PP_1$, the proof is straightforward. □

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References

[1] Blyth, T., "Lattices and Ordered Algebraic Structures", Springer, 2005.

[2] Davey, B.A. and Priestly, H.A., "Introduction to Lattices and Order", Cambridge University Press, 2002.

[3] Dikranjan, D. and Tholen, W., "Categorical Structure of Closure Operators: with Applications to Topology, Algebra and Discrete Mathematics", Springer Science & Business Media 346, 2013.

[4] Grätzer, G., "Lattice Theory", Foundation, Springer Basel AG, 2011.

[5] Hashemi, J., *On the $\cap$-structure spaces*, J. Math. Extension, 14(3) (2020), 225-236.

[6] Roman, S., "Lattices and Ordered Sets", Springer, 2008.

[7] Sharp, R.Y., "Steps in Commutative Algebra", Cambridge University Press, 2000.

[8] Willard, S., "General Topology", Addison Wesley, Reading, Mass., 1970.

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