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with Applications

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Ideals and congruences in L-algebras and pre-L-algebras

Alberto Facchini, Marino Gran*, and Mara Pompili

Abstract. We link the recent theory of L-algebras to previous notions of Universal Algebra and Categorical Algebra concerning subtractive varieties, commutators, multiplicative lattices, and their spectra. We show that the category of L-algebras is subtractive and normal in the sense of Zurab Janelidze, but neither the category of L-algebras nor that of pre-L-algebras are Mal'tsev categories, hence in particular they are not semi-abelian. Therefore L-algebras are a rather peculiar example of an algebraic structure.

1 Introduction

The aim of this paper is to link the recent fruitful theory of L-algebras [15, 16, 18] to previous articles of Universal Algebra and Categorical Algebra concerning subtractive varieties [1, 11, 12, 19], commutators and multiplicative lattices, and their spectra [6].

L-algebras are related to right ℓ-groups, projection lattices of von Neumann algebras, quantum Yang–Baxter equation, MV-algebras, braidings,

* Corresponding author

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and non-commutative logic. In this paper we prove that the category of L-algebras is subtractive [11, 19] and normal (in the sense of [12]), but neither the category of L-algebras nor that of pre-L-algebras are Mal'tsev categories, hence in particular they are not semi-abelian. This shows that L-algebras are a rather peculiar example of algebraic structure. In general, in a pre-L-algebra X, there is a monotone Galois connection between the lattice $\mathcal{C}(X)$ of congruences of X, and the lattice $\mathcal{I}(X)$ of ideals of X. We show that L-algebras do not form a variety in the sense of Universal Algebra, but their category is a normal subtractive quasivariety. The lattice $\mathcal{I}(X)$ is a distributive lattice, as was proved in [18]. Thus the commutator of two ideals (congruences) turns out to be the intersection of the two ideals (congruences).

After uploading this article on the arXiv (arXiv:2305.19042) we have noticed that some results similar to those presented in Section 4 were later published in [17].

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2 Basic notions

Recall that an *L-algebra* [15] is a set X with a binary operation $(x, y) \mapsto x \cdot y$ and a 0-ary operation $1 \in X$ such that

$$x \cdot x = x \cdot 1 = 1, \;\; 1 \cdot x = x, \tag{2.1}$$

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z), \text{ and} \tag{2.2}$$

$$x \cdot y = y \cdot x = 1 \implies x = y \tag{2.3}$$

for every $x, y, z \in X$.

Similarly, we define a *pre-L-algebra* assuming that only Properties (2.1) and (2.2) hold. These pre-L-algebras are called *unital cycloids* in [15]. It is easily seen that in a pre-L-algebra the element 1 with Property (2.1) is unique (this follows from $x \cdot x = 1$.) It is called the *logical unit* of the pre-L-algebra. Equation (2.2) holds in most generalizations of classical logic, including intuitionistic, many-valued, and quantum logic.

On any pre-L-algebra X there is a natural preorder $\leq$, that is, a reflexive and transitive relation, defined by $x \leq y$ if $x \cdot y = 1$. A pre-L-algebra is an L-algebra if and only if this natural preorder is a partial order, that is, if and only if it is also an antisymmetric relation. Clearly, pre-L-algebras form a variety of algebras in the sense of Universal Algebra. (We will see in Example 3 that this is not the case for L-algebras.) In particular, a pre-L-algebra morphism is any mapping f between two pre-L-algebras such that $f(1) = 1$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all x, y. But notice that the first condition $f(1) = 1$ follows from the second, because

$$f(1) = f(1 \cdot 1) = f(1) \cdot f(1) = 1$$

by (2.1).

Let us now consider congruences on a pre-L-algebra X. We will see that they correspond to suitably defined ideals of X. A subset I of a pre-L-algebra X is an *ideal* of X [15, Definition 1] if

$$1 \in I, \tag{2.4}$$

$$x \in I \text{ and } x \cdot y \in I \implies y \in I, \tag{2.5}$$

$$x \in I \implies (x \cdot y) \cdot y \in I, \tag{2.6}$$

$$x \in I \implies y \cdot x \in I, \tag{2.7}$$

$$x \in I \implies y \cdot (x \cdot y) \in I \tag{2.8}$$

for every $x, y \in X$. Clearly, $\{1\}$ and X are ideals in any pre-L-algebra X.

For instance, it is easily seen that if $f\colon X \to Y$ is a pre-L-algebra morphism, then its *kernel*, that is, the inverse image $f^{-1}(1)$ of the logical unit 1 of Y, is an ideal of X.

Also, if $\sim$ is a congruence on a pre-L-algebra X, the equivalence class $[1]_\sim$ of the logical unit 1 of X is an ideal of X.

Recall that if $(A, \leq)$ and $(B, \leq)$ are two partially ordered sets, a *monotone Galois connection* between A and B consists of two order-preserving mappings $f\colon A \to B$ and $g\colon B \to A$ such that $f(a) \leq b$ if and only if $a \leq g(b)$ for every $a \in A$ and $b \in B$. A *closure operator* on the partially ordered set A is a mapping $c\colon A \to A$ such that $x \leq c(y)$ if and only if $c(x) \leq c(y)$ for every $x, y \in A$.

Proposition 2.1. *Let X be a pre-L-algebra, $\mathcal{C}(X)$ its lattice of congruences, and $\mathcal{I}(X)$ the lattice of ideals of X. Define two mappings*

$$\varphi\colon \mathcal{C}(X) \to \mathcal{I}(X), \qquad \varphi\colon\ \sim\, \in \mathcal{C}(X) \mapsto [1]_\sim,$$

that maps any congruence $\sim$ of X to the equivalence class modulo $\sim$ of the logical unit 1*, and*

$$\psi\colon \mathcal{I}(X) \to \mathcal{C}(X), \qquad \varphi\colon I \in \mathcal{I}(X) \mapsto \sim_I,$$

where $\sim_I$ is the congruence of X defined, for every $x, y \in X$, by $x \sim_I y$ if both $x \cdot y$ and $y \cdot x$ belong to I. Then:

(a) *φ and ψ are well-defined.*

(b) *$\varphi\psi = id_{\mathcal{I}(X)}$;*

(c) *φ and ψ form a monotone Galois connection.*

(d) *$\psi\varphi$ is a closure operator on $\mathcal{C}(X)$.*

(e) *The image of ψ is the set of all congruences $\sim$ of X for wchich $X/{\sim}$ is an L-algebra.*

(f) *For every $\sim \in \mathcal{C}(X)$, $\psi\varphi(\sim)$ is the smallest congruence $\equiv$ on X that contains $\sim$ and is such that $X/{\equiv}$ is an L-algebra.*

Proof. (a) This is proved in [15, Proposition 1]. Notice that here "φ is a well-defined mapping" means that $\varphi(\sim) := [1]_\sim$ is an ideal of I for every congruence $\sim$ on X. As far as "ψ is well-defined" is concerned, we mean that, for every ideal I of X, the relation $\sim_I$ on X, defined, for every $x, y \in X$, by $x \sim_I y$ if $x \cdot y \in I$ and $y \cdot x \in I$, is a congruence on X.

(b) is trivial.

(c) φ and ψ are clearly order-preserving. In order to show that they form a monotone Galois connection, we must prove that if $\sim$ is any congruence and I is any ideal, then $[1]_\sim \subseteq I$ if and only if $\sim \subseteq \sim_I$. Suppose $[1]_\sim \subseteq I$. Fix $x, y \in X$ with $x \sim y$. Then $x \cdot y \sim y \cdot y = 1 \in I$. Similarly $y \cdot x = 1 \in I$. Therefore $x \sim_I y$. The converse is easy.

(d) follows immediately from (c).

(e) Clearly, if I is an ideal of X, then $X/{\sim_I}$ is an L-algebra. Conversely, let $\sim$ be a congruence with $L/\sim$ an L-algebra, and I be the ideal $\varphi(\sim)$. We must prove that $\sim\ =\ \sim_I$, that is, that for every $x, y \in X$, $x \sim y$ if and only if $x \cdot y \in I$ and $y \cdot x \in I$. Now $x \sim y$ implies $x \cdot y = y \cdot y = 1$, so $x \cdot y \in I$. Similarly, $y \cdot x \in I$. Conversely, $x \cdot y \in I$ and $y \cdot x \in I$ are equivalent to $x \cdot y \sim 1$ and $y \cdot x \sim 1$. But $L/\sim$ is an L-algebra, hence we have that $[x]_\sim = [y]_\sim$, that is, $x \sim y$, as desired.

(f) From (d) and (e) we know that $\psi\varphi(\sim)$ is a congruence on X that contains $\sim$ and that $X/\psi\varphi(\sim)$ is an L-algebra. If $\equiv$ is any other congruence on X that contains $\sim$ and is such that $X/\equiv$ is an L-algebra, then $\equiv = \sim_I$ for some ideal I by (e), so that $\sim_I \supseteq \sim$. It follows that $\varphi(\sim_I) \supseteq \varphi(\sim)$, that is, $I \supseteq \varphi(\sim)$. Therefore $\equiv \ = \ \sim_I \ = \psi(I) \supseteq \psi\varphi(\sim)$, as desired. $\square$

As a trivial consequence of the previous proposition, we have that for any pre-L-algebra X there is a one-to-one correspondence between ideals of X and congruences $\sim$ on X for which $X/\sim$ is an L-algebra. Cf. [15, Corollary 1].

Clearly, the category of all L-algebras is a full reflective subcategory of the category of all pre-L-algebras. The left adjoint of the inclusion associates with any pre-L-algebra X the L-algebra $X/\sim$, where $\sim$ is the congruence on X defined, for every $x, y \in X$, by $x \sim y$ if $x \cdot y = 1$ and $y \cdot x = 1$.

Also notice that for every pre-L-algebra morphism between two L-algebras, the kernel pair always corresponds to an ideal of the domain. This occurs because any pre-L-subalgebra of an L-algebra is an L-algebra.

3 L-algebras do not form a variety

Example 3.1. Let $X = \{x, y, z, 1\}$ be the L-algebra given by Table 1 and let $Y = \{a, b, 1\}$ be the magma given by Table 2.

	x	y	z	1
x	1	y	z	1
y	1	1	x	1
z	1	x	1	1
1	x	y	z	1

Table 1: An L-algebra.

	a	b	1
a	1	1	1
b	1	1	1
1	a	b	1

Table 2: A pre-L-algebra that is not an L-algebra.

One checks that Y is not an L-algebra, as $a \cdot b = b \cdot a = 1$ but $a \neq b$. The surjective map $f\colon X \to Y$, $f(x) = f(1) = 1$, $f(y) = a$ and $f(z) = b$, satisfies $f(u \cdot v) = f(u) \cdot f(v)$ for all $u, v \in X$ and $f(1) = 1$. Since the image of f is not an L-algebra, it follows from Birkhoff's theorem (see for example [5, Theorem 3.1]) that the class of L-algebras is not a variety.

4 Subtractive and normal categories

Let us recall some definitions that are of interest in categorical algebra. A variety of universal algebras $\mathbb{V}$ is *subtractive* [19] if its algebraic theory contains a constant 0 and a binary term $s(x, y)$ with the properties that $s(x, x) = 0$ and $s(x, 0) = x$.

Lemma 4.1. *The variety* $\mathsf{PreLAlg}$ *of pre-L-algebras is a subtractive variety.*

Proof. It suffices to choose the term $s(x, y) = y \cdot x$ in the theory of pre-L-algebras: this term is such that

$$s(x, x) = 1$$

and

$$s(x, 1) = 1 \cdot x = x.$$

□

The definition of subtractive variety was extended to a categorical context by Z. Janelidze in [11]. When a category $\mathcal{C}$ is *pointed*, that is it has a zero object 0, the property of subtractivity can be defined as follows. In a pointed category consider a reflexive relation (R, r_1, r_2, e) on an object X, where r_1, r_2 are the projections and $e\colon X \to R$ is the morphism giving the reflexivity: $r_1 \circ e = 1_X = r_2 \circ e$. One says that the relation (R, r_1, r_2, e) is *right* (respectively, *left*) *punctual* if there is a morphism $t\colon X \to R$ (respectively, $s\colon X \to R$) such that $r_2 \circ t = 1_X$ and $r_1 \circ t = 0$ (respectively, $r_2 \circ s = 0$ and $r_1 \circ s = 1_X$).

Definition 4.2. [11] A finitely complete pointed category is *subtractive* if any right punctual reflexive relation is left punctual.

As shown in [11], a pointed variety is subtractive in the sense of Definition 4.2 if and only if is subtractive in the sense of [19]. In particular the variety $\mathsf{PreLAlg}$ is a subtractive category, and this implies the following:

Lemma 4.3. *The category* LAlg *is subtractive.*

Proof. This follows from the fact that the inclusion functor $U\colon \mathsf{LAlg} \to \mathsf{PreLAlg}$ is full, faithful, preserves all limits (as any right adjoint), hence it preserves and reflects punctual right (respectively, left) reflexive relations, and the zero object. $\square$

The next result we shall prove concerns L-algebras again. They turn out to form a normal category in the following sense:

Definition 4.4. [12] Let $\mathcal{C}$ be a finitely complete category with a zero object 0. Then $\mathcal{C}$ is a *normal* category if

(1) any morphism $f\colon A \to B$ admits a factorization $f = m \cdot q$, where q is a normal epimorphism (=a cokernel) and m is a monomorphism:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & B \\ & C \arrow[ur, tail, "m"'] & \end{tikzcd}$$

(2) normal epimorphisms are stable under pullbacks, that is, given any pullback

$$\begin{tikzcd} A \times_C B \arrow[r, two heads, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

in $\mathcal{C}$, where f is a normal epimorphism, then p_2 is a normal epimorphism.

Equivalently, one can define a normal category as a regular category [2] with a zero object with the property that any regular epimorphism is a cokernel.

Observe then that LAlg is a quasivariety, since it is the class of algebras determined by adding a finite number of implications to the theory of $\mathsf{PreLAlg}$. Example 3 shows that these implications cannot be "transformed" into an equivalent set of identities, since in that case LAlg would then be a subvariety of $\mathsf{PreLAlg}$ and then it would be stable in it under quotients, and this is not the case. As a consequence LAlg is not a variety of algebras.

Proposition 4.5. *The category* LAlg *of* L*-algebras is a normal subtractive category.*

Proof. Let us check that in LAlg every surjective homomorphism is a cokernel. The terms $t_1(x,y) = x \cdot y$ and $t_2(x,y) = y \cdot x$ have the property that

$$(\forall i \in \{1,2\}\ t_i(a,b) = 1) \Leftrightarrow a = b.$$

These terms will attest the 1-regularity (in the sense of Beutler [3]) of LAlg (also see [9, Theorem 2.2]).

Note that the terms $t_1(x,y) = x \cdot y$ and $t_2(x,y) = y \cdot x$ also satisfy the identities $t_i(x,x) = 1$, for $i \in \{1,2\}$. Let $f\colon A \to B$ be a surjective homomorphism, $\kappa\colon K \to A$ be the inclusion of its kernel K in A (that we shall omit to simplify the notations), and $g\colon A \to C$ be any homomorphism such that $(g \circ \kappa)(k) = g(k) = 1$, for every $k \in K$. Now, for any $b \in B$ there is an $a \in A$ such that $f(a) = b$. Let us prove that, by setting $\phi(b) = g(a)$, we get a well-defined function. To this end, we will show that if a and a' are such that $f(a) = f(a')$ one always has that $g(a) = g(a')$. First observe that

$$f(t_i(a,a')) = t_i(f(a), f(a')) = t_i(f(a), f(a)) = 1,$$

hence $t_i(a,a') \in K$. This implies that

$$t_i(g(a), g(a')) = g(t_i(a,a')) = 1,$$

so that $g(a) = g(a')$, and ϕ is well-defined. It remains to prove that ϕ is a homomorphism: this is easy, because $\phi \circ f = g$ is a homomorphism by assumption and f is a surjective homomorphism. The uniqueness of the factorization ϕ is clear, and f is then necessarily the cokernel of its kernel κ.

To conclude that LAlg is a normal category, now it suffices to observe that any quasivariety is a regular category (see [14], for instance) since surjective homomorphisms are always stable under pullbacks in a quasivariety. From Lemma 4.3 it follows that LAlg is indeed a normal subtractive category. □

The subtractive variety $\mathsf{PreLAlg}$ of pre-L-algebras then contains the normal subtractive quasivariety LAlg of L-algebras. One might then wonder whether LAlg is also a Mal'tsev category [4]. The answer to this question is negative, as we are now going explain. For this, let us fix some notations.

The *kernel pair* $(Eq(f), p_1, p_2)$ of a morphism $f\colon A \to B$ is the (effective) equivalence relation obtained by the pullback of f along itself:

$$\begin{tikzcd} Eq(f) \arrow[r, two heads, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\ A \arrow[r, two heads, "f"'] & B. \end{tikzcd}$$

In the case of a quasivariety of universal algebras, then the kernel pair of a homomorphism $f\colon A \to B$ is the congruence

$$Eq(f) = \{(a, a') \in A \times A \mid f(a) = f(a')\}.$$

We will be interested in the following two elements L-algebra X whose multiplication is defined in the following Table:

	0	1
0	1	1
1	0	1

Consider then the relation $R = \{(0,1), (1,0), (1,1)\}$ on X: this is easily seen to be a subalgebra of the product L-algebra $X \times X$.

Write $p_1\colon R \to X$ and $p_2\colon R \to X$ for the first and the second projections, and $Eq(p_1)$ and $Eq(p_2)$ for the congruences associated with these homomorphisms, namely

$$Eq(p_1) = \{((0,1),(0,1)), ((1,0),(1,0)), ((1,1),(1,1)), ((1,0),(1,1))\}$$

and

$$Eq(p_2) = \{((0,1),(0,1)), ((1,0),(1,0)), ((1,1),(1,1)), ((0,1),(1,1))\}.$$

Then, clearly,

$$(1,0)Eq(p_1)(1,1)Eq(p_2)(0,1)$$

showing that

$$((1,0),(0,1)) \in Eq(p_2) \circ Eq(p_1).$$

However,

$$((1,0),(0,1)) \notin Eq(p_1) \circ Eq(p_2),$$

hence

$$Eq(p_1) \circ Eq(p_2) \neq Eq(p_2) \circ Eq(p_1).$$

This shows that:

Proposition 4.6. *The categories* LAlg *and* $\mathsf{PreLAlg}$ *are not Mal'tsev categories.*

Proof. The fact that LAlg is not a Mal'tsev category follows from the fact that the two congruences $Eq(p_1)$ and $Eq(p_2)$ on the L-algebra X above do not permute in the sense of composition of relations. Since the category LAlg is stable in $\mathsf{PreLAlg}$ under subalgebras and products in LAlg, the same counter-example also shows that $\mathsf{PreLAlg}$ is not a Mal'tsev category. □

In particular, the above proposition implies that LAlg and $\mathsf{PreLAlg}$ are not semi-abelian categories.

Remark 4.7. It is well-known that any subtractive variety $\mathbb{V}$ (with a constant 1) is "permutable at 1", which means that, for any pair of congruences R and S on any algebra X in $\mathbb{V}$, the following implication holds:

$$(x,1) \in S \circ R \quad \Leftrightarrow \quad (x,1) \in R \circ S$$

In the case of the variety $\mathsf{PreLAlg}$ we have the term $s(x,y) = y \cdot x$. Accordingly, when $(x,1) \in S \circ R$, from the existence of a y such that $xRyS1$, one deduces that

$$x = s(x,1)Ss(x,y)Rs(y,y) = 1,$$

that is, $(x,1) \in R \circ S$. The variety $\mathsf{PreLAlg}$ is then "permutable at 1", even though it is not congruence permutable.

5 Commutators

In Proposition 2.1 we have considered, for a pre-L-algebra X, the lattice of congruences $\mathcal{C}(X)$, and the lattice of ideals $\mathcal{I}(X)$. These are complete lattices because any intersection of congruences (of ideals) is a congruence (an ideal). Let us focus onto the case of ideals. Notice that ideals of X are pre-L-subalgebras of X (essentially because, for a congruence $\sim$, $x \sim 1$ and $y \sim 1$ imply $x \cdot y \sim 1$). Since the lattice $\mathcal{I}(X)$ is complete, there is an obvious notion of ideal of X generated by a subset of X.

Let us consider the notion of commutator of two ideals of an L-algebra X. In Group Theory, if we have two normal subgroups M and N of a group G, the commutator $[M, N]$ is the smallest normal subgroup of G for which group multiplication $\mu\colon M \times N \to G/[M, N]$, $\mu(m, n) = mn[M, N]$, where $mn[M, N]$ is the coset of mn in the quotient $G/[M, N]$, is a group homomorphism. This argument can be repeated for L-algebras, as follows.

Let X be an L-algebra and I, J be two ideals of X. Define their *commutator* $[I, J]$ as the smallest ideal of X for which the multiplication $\cdot$ in X, that is, the mapping $\mu\colon I \times J \to X/[I, J]$, $\mu(i, j) = [i \cdot j]_{\sim_{[I,J]}}$, is an L-algebra morphism. Notice that the ideal $[I, J]$ is always contained in $I \cap J$. This follows from the remark that the mapping $\mu\colon I \times J \to X/I \cap J$ is clearly an L-algebra morphism. One actually has the following

Proposition 5.1. *For every pair I, J of ideals of an L-algebra X, one has*

$$[I, J] = I \cap J.$$

Proof. We only need to prove that $I \cap J \subseteq [I, J]$. For this it will suffice to show that, for any $x \in I \cap J$, its equivalence class $[x]_{\sim_{I \cap J}}$, that will be simply written $[x]$, is the neutral element in the quotient $X/[I, J]$:

$$[x] = [1].$$

By assumption, for any $i \in I$, $j \in J$, one has the equality

$$([x] \cdot [x]) \cdot ([i] \cdot [j]) = ([x] \cdot [i]) \cdot ([x] \cdot [j]).$$

By choosing $i = 1$ and $j = x$ we get

$$([x] \cdot [x]) \cdot ([1] \cdot [x]) = ([x] \cdot [1]) \cdot ([x] \cdot [x]),$$

from which it follows that $[x] = [1]$, as desired. □

In particular, this result implies that the only abelian algebras in the quasivariety of L-algebras are the trivial ones:

Corollary 5.2. *Let X be an abelian L-algebra. Then $|X| = 1$.*

Proof. By Proposition 5.1, the condition $[X, X] = \{1\}$ gives $X \cap X = X = \{1\}$. □

Remark 5.3. The fact that the commutator $[I, J]$ of two ideals is simply their intersection $I \cap J$ is not surprising, since the category LAlg is congruence distributive, as it follows from Proposition 2.1 and the fact that the lattice of ideals on each L-algebra is distributive [18]. It would then be interesting to revisit the results on commutators of congruences in terms of pseudogroupoids in varieties [10] in the more general context of quasivarieties (also see [13]).

We now consider the multiplicative lattice $(\mathcal{I}(X), \cap)$ in the sense of [6], which has been implicitly considered in [18]. Now, primes ideals studied in [18] agree with the notion of prime elements of [6]. Notice that an ideal P of an L-algebra X is prime if and only if P is a $\wedge$-irreducible element of the lattice $\mathcal{I}(X)$.

Recall that an ideal I of X is *semiprime* if, for every ideal J, $[J, J] \subseteq I$ implies $J \subseteq I$. Hence Proposition 5.1 trivially implies that:

Corollary 5.4. *In an L-algebra, every ideal is semiprime.*

Thus, in an L-algebra, every ideal is an intersection of prime ideals [6]. Now, in any multiplicative lattice, the lattice of all semiprime elements is isomorphic to the lattice of all open subsets of the Zariski spectrum. The Zariski spectrum is always a sober space [6, 18]. Hence we find that:

Proposition 5.5. [18] *For an L-algebra X, the lattice $\mathcal{I}(X)$ of all ideals of X is isomorphic to the lattice of all open subsets of the sober topological space* $\mathrm{Spec}(X)$. *In particular, the lattice $\mathcal{I}(X)$ is a complete distributive lattice.*

In view of Proposition 5.1, several classical notions of Algebra trivialize for L-algebras. For instance, solvable L-algebras, nilpotent L-algebras, L-algebras with empty Zariski spectrum are only those with one element, the centralizer of any nontrivial ideal is the trivial ideal, the center of any L-algebra is the trivial ideal, and the central series and the derived series are always stationary.

Notice that maximal ideals are prime, because the zero element is $\wedge$-irreducible in the lattice of two elements. It would be interesting to describe simple L-algebras.

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Alberto Facchini *Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, 35121 Padova, Italy.*

Email: facchini@math.unipd.it

Marino Gran *Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, 1348 Louvain-la-Neuve, Belgique.*

Email: marino.gran@uclouvain.be

Mara Pompili *Department of Mathematics and Scientific Computing, University of Graz, 8010 Graz, Austria*

Email: mara.pompili@uni-graz.at