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Slicing points in a pointfree adjunction for T_D partial spaces

John Frith* and Anneliese Schauerte

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Abstract. The T_D axiom, a low order separation axiom between T_0 and T_1, has been of interest to classical topologists for some time; latterly it has also proved interesting to pointfree topologists. Here we investigate it in the context of partial spaces and partial frames (think: σ-frames, κ-frames, frames, bounded distributive lattices). We establish an adjunction between the category of T_D partial spaces with continuous maps and the category of partial frames with D-homomorphisms. Several standard tools (covered primes, right adjoints, point closures) are not appropriate in our setting; we use linked pairs and slicing points instead. Of particular interest are the slicing points of free frames and congruence frames.We examine the fixed objects of the adjunction; both similarities and differences to the classical situation become clear. In particular, there are compact Hausdorff partial spaces that are not T_D. We introduce sharp partial frames, those for which all points are slicing and characterize these as well as the T_D spatial and strongly T_D spatial partial frames. We conclude with a comparison of sober and T_D partial spaces.

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* Corresponding author
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1 Introduction and Preliminaries

The appearance of the text "Separation in Point-Free Topology" [27] shows the substantial interest in this topic by frame and locale theorists. One of the themes in the book concerns the function of the T_D axiom in the study of topological spaces and frames. Partial spaces and partial frames serve to elucidate the role of separation axioms more generally; here we focus on the T_D axiom specifically. In particular, some of the classical relations between the various separation properties are seen not to hold in this more general setting.

We are particularly indebted to the authors of two papers: Banaschewski and Pultr for [7], and Arrieta and Suarez for [4]. For further work on this topic see, for instance, [26], [2] and [3]. Our presentation of this work is modelled primarily on [7], but substantially different tools are required. To be specific, on the frame side, the notions of prime elements, completely prime elements, covered primes, sublocales and right adjoints of maps are inappropriate tools for this particular paper. On the topological side, closures in partial spaces are ill-behaved, making point closures not the right thing to use here.

The essential idea for a partial frame is that it should be "frame-like" but that not all joins need exist; only certain joins have guaranteed existence and binary meets should distribute over these joins. The guaranteed joins are specified in a global way on the category of meet-semilattices by specifying what is called a selection function. Partial spaces are to partial frames what topological spaces are to frames; the categories are adjoint on the right with the expected open set and spectrum functors. The latter makes use of maps into the two-element object, or filters, rather than prime elements.

In order to define the T_D spectrum we make use of special, so-called *slicing* points, each of which is associated with a *linked pair*. We call a pair of distinct elements $a < b$ a linked pair if there are no elements between them; an associated slicing point then maps a to 0 and b to 1. We make use of these to define D-homomorphisms, which, in the frame case, are seen to be the same as the D-homomorphisms of [7]. Of particular interest in our context are the embeddings of partial frames into their free frames and congruence frames, and the slicing points of these.

We move on to the spatial side by defining T_D partial spaces and characterizing these in terms of slicing points. They are also characterized using certain congruences, which echoes their behaviour in the full topological setting. However we provide an example of a compact Hausdorff partial space which is not T_D showing that, unlike in the topological case, T_1 objects need not be T_D.

Next we describe the open set and T_D spectrum functors needed to establish the adjunction between the category of T_D partial spaces with continuous maps and the category of partial frames with D-homomorphisms. As usual, the adjunction gives rise to a category equivalence between the fixed objects on either side. On the space side all objects are fixed, but on the frame side the fixed objects are the so-called T_D spatial ones. Of interest are those partial frames all of whose points are slicing; we call these *sharp* and provide a description of these using D-homomorphisms. With these definitions, sharp spatial partial frames correspond to the strongly T_D spatial frames of [4].

We conclude with an examination of the relationship between sober and T_D partial spaces.

2 Background and Preliminaries

This entire section summarises some prerequisites on partial frames from our previous work. New material begins in Section 3.

See [25] and [19] as references for frame theory; see [6] and [5] for σ-frames; see [22] and [23] for κ-frames; see [21] and [1] for general category theory.

The basics of our approach to partial frames can be found in [8], [9] and [10]. For earlier work by other authors in this field see [24], [29], [30] and [28]. For those interested in a comparison of the various approaches, see [9].

A *meet-semilattice* is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We do not insist that a meet-semilattice should have a bottom element, which, if it exists, we denote by 0. A function between meet-semilattices $f : L \to M$ is a *meet-semilattice map* if it preserves finite meets, including the top element. A *sub meet-semilattice* is a subset for which the inclusion map is a meet-semilattice map.

Definition 2.1. A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):

(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.

(S2) If $G, H \in \mathcal{S}A$ then $\{x \wedge y : x \in G, y \in H\} \in \mathcal{S}A$.

(S2)′ If $G, H \in \mathcal{S}A$ then $\{x \vee y : x \in G, y \in H\} \in \mathcal{S}A$.

(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

(SSub) For any sub meet-semilattice B of meet-semilattice A, if $G \subseteq B$ and $G \in \mathcal{S}A$, then $G \in \mathcal{S}B$.

(SFin) If F is a finite subset of A, then $F \in \mathcal{S}A$.

(SCov) If $G \subseteq H$ and $H \in \mathcal{S}A$ with $\bigvee H = 1$ then $G \in \mathcal{S}A$. (Such H are called $\mathcal{S}$-*covers*.)

(SRef) Let $X, Y \subseteq A$. If $X \leq Y$ with $X \in \mathcal{S}A$ there is a $C \in \mathcal{S}A$ such that $X \leq C \subseteq Y$. (By $X \leq Y$ we mean, as usual, that for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.)

Of course (SFin) implies (S1) but there are situations where we do not impose (SFin) but insist on (S1). Note that we always have $\emptyset \in \mathcal{S}A$.
Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.

Definition 2.2. An $\mathcal{S}$-*frame* L is a meet-semilattice in which every designated subset has a join and for any such designated subset B of L and any $a \in L$

$$a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b.$$

Of course such an $\mathcal{S}$-frame has both a top and a bottom element which we denote by 1 and 0 respectively.
A meet-semilattice map $f : L \to M$, where L and M are $\mathcal{S}$-frames, is an *$\mathcal{S}$-frame map* if $f(\bigvee B) = \bigvee_{b \in B} f(b)$ for any designated subset B of L. In particular such an f preserves the top and bottom element.
A *sub $\mathcal{S}$-frame* T of an $\mathcal{S}$-frame L is a subset of L such that the inclusion map $i : T \to L$ is an $\mathcal{S}$-frame map.
The category $\mathcal{S}$**Frm** has objects $\mathcal{S}$-frames and arrows $\mathcal{S}$-frame maps.

We use the terms "partial frame" and "$\mathcal{S}$-frame" interchangeably, especially if no confusion about the selection function is likely. We also use the term *full frame* in situations where we wish to emphasize that all joins exist.

Note 2.3. Here are some examples of different selection functions and their corresponding $\mathcal{S}$-frames.

(1) In the case that all joins are specified, we are of course considering the notion of a frame.

(2) In the case that (at most) countable joins are specified, we have the notion of a σ-frame.

(3) In the case that joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.

(4) In the case that only finite joins are specified, we have the notion of a bounded distributive lattice.

The next results come from [10] on $\mathcal{H}_{\mathcal{S}}L$, [11], [12] and [15] on $\mathcal{C}_{\mathcal{S}}L$.

Definition 2.4. Let L be an $\mathcal{S}$-frame.

(1) A subset J of L is an *$\mathcal{S}$-ideal of* L if J is a non-empty downset closed under designated joins (the latter meaning that if $X \subseteq J$, for X a designated subset of L, then $\bigvee X \in J$).

(2) The collection of all $\mathcal{S}$-ideals of L will be denoted $\mathcal{H}_{\mathcal{S}}L$, and called the *$\mathcal{S}$-ideal frame of* L. It is in fact the free full frame over L, with embedding map $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ given by $\downarrow x = \{t \in L : t \leq x\}$.

(3) We call $\theta \subseteq L \times L$ an *$\mathcal{S}$-congruence on* L if it satisfies the following:
(C1) θ is an equivalence relation on L.
(C2) $(a, b), (c, d) \in \theta$ implies that $(a \wedge c, b \wedge d) \in \theta$.

(C3) If $\{(a_\alpha, b_\alpha) : \alpha \in \mathcal{A}\} \subseteq \theta$ and $\{a_\alpha : \alpha \in \mathcal{A}\}$ and $\{b_\alpha : \alpha \in \mathcal{A}\}$ are designated subsets of L, then $(\bigvee_{\alpha\in\mathcal{A}} a_\alpha, \bigvee_{\alpha\in\mathcal{A}} b_\alpha) \in \theta$.

(4) The collection of all $\mathcal{S}$-congruences on L is denoted by $\mathcal{C}_\mathcal{S}L$; we refer to it as the congruence frame of L. It is in fact a full frame with meet given by intersection.

(5) (i) For $A \subseteq L \times L$, $\langle A\rangle$ denotes the smallest $\mathcal{S}$-congruence containing A.
  (ii) For $a \in L$ we define $\nabla_a = \{(x, y) : x \vee a = y \vee a\}$ and $\Delta_a = \{(x, y) : x \wedge a = y \wedge a\}$; these are $\mathcal{S}$-congruences on L.
  (iii) It is easily seen that $\nabla_a = \langle(0, a)\rangle$ and that $\Delta_a = \langle(a, 1)\rangle$.
  (iv) For $a \leq b$, it follows that $\Delta_a \cap \nabla_b = \langle(a, b)\rangle$ and $\Delta_a \cap \nabla_a = \Delta$.

(6) For any $I \in \mathcal{H}_\mathcal{S}L$, $\bigvee_{x\in I} \nabla_x = \bigcup_{x\in I} \nabla_x$, the point being that this union is indeed an $\mathcal{S}$-congruence.

(7) The function $\nabla : L \to \mathcal{C}_\mathcal{S}L$ given by $\nabla(a) = \nabla_a$ is an $\mathcal{S}$-frame embedding. It has the universal property that if $f : L \to M$ is an $\mathcal{S}$-frame map into a frame M with complemented image, then there exists a unique frame map $\bar{f} : \mathcal{C}_\mathcal{S}L \to M$ such that $f = \bar{f} \circ \nabla$.

We note that partial spaces are to partial frames as spaces are to frames with the appropriate open and spectrum functors and fixed objects. Some details appear below.

Definition 2.5. Let $\mathcal{S}$ be a selection function.

(1) An *$\mathcal{S}$-topological space* (or *$\mathcal{S}$-space*) is a pair $(X, \mathcal{O}X)$ where X is a set, $\mathcal{O}X \subseteq \mathcal{P}X$, the power set of X and $\mathcal{O}X$ is a sub $\mathcal{S}$-frame of $\mathcal{P}X$, with binary meet given by intersection and designated join by union.

(2) Let $(X, \mathcal{O}X)$ and $(Y, \mathcal{O}Y)$ be $\mathcal{S}$-spaces; a function $f : X \to Y$ is *$\mathcal{S}$-continuous* if, for each $U \in \mathcal{O}Y$, $f^{-1}(U) \in \mathcal{O}X$.

(3) The category $\mathcal{S}$**Top** has objects $\mathcal{S}$-spaces and arrows $\mathcal{S}$-continuous functions.

Definition 2.6. (1) The *open set* functor $\mathcal{O} : \mathcal{S}\mathbf{Top} \to \mathcal{S}\mathbf{Frm}$ assigns to each $\mathcal{S}$-space $(X, \mathcal{O}X)$ the set of opens $\mathcal{O}X$. For an $\mathcal{S}$-continuous map $f : (X, \mathcal{O}X) \to (Y, \mathcal{O}Y)$ define $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ by $\mathcal{O}f(V) = f^{-1}(V)$

for each $V \in \mathcal{O}Y$. This easily provides a contravariant functor from $\mathcal{S}$**Top** to $\mathcal{S}$**Frm**.

(2) Let L be an $\mathcal{S}$-frame. Set $\Sigma L = \hom(L, \mathbf{2})$, the set of all $\mathcal{S}$-frame maps from L to $\mathbf{2}$, where $\mathbf{2}$ is the 2-chain viewed as an $\mathcal{S}$-frame. These are the "points" of L, also called $\mathcal{S}$-points. For each $a \in L$, set $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$. The collection $\mathcal{T}_L = \{\Sigma_a : a \in L\}$ is an $\mathcal{S}$-topology on ΣL. The assignment $L \to (\Sigma L, \mathcal{T}_L)$ provides the object part of the contravariant *spectrum functor* $\Sigma : \mathcal{S}\mathbf{Frm} \to \mathcal{S}\mathbf{Top}$. For the maps, let $h : L \to M$ be an $\mathcal{S}$-frame map. Define $\Sigma h : (\Sigma M, \mathcal{T}_M) \to (\Sigma L, \mathcal{T}_L)$ by $\Sigma h(\xi) = \xi \circ h$. Then Σh is $\mathcal{S}$-continuous.

(3) We have $\mathcal{O}$ and Σ adjoint on the right. For an $\mathcal{S}$-space X and an $\mathcal{S}$-frame L the adjunction maps $\eta_L : L \to \mathcal{O}\Sigma L$ and $\epsilon_X : X \to \Sigma\mathcal{O}X$ are given as follows, where $a \in L$ and $U \in \mathcal{O}X$:

$$\eta_L(a) = \Sigma_a \text{ and } \epsilon_X(x)(U) = 1 \text{ iff } x \in U.$$

In this paper, we will denote the map $\epsilon_X(x)$ by ξ_x, so for $U \in \mathcal{O}X$,

$$\xi_x(U) = 1 \iff x \in U.$$

A *spatial* $\mathcal{S}$-frame L is one for which the adjunction map η_L is an isomorphism; similarly a *sober* $\mathcal{S}$-space X is one for which ϵ_X is an isomorphism. Spatial $\mathcal{S}$-frames and sober $\mathcal{S}$-spaces are the fixed objects of the adjunction maps. The functors $\mathcal{O}$ and Σ restrict to a category equivalence on the fixed objects. (See [13] and [14] for details.)

3 Linked pairs and slicing points

Our main technical tool throughout this paper will be the notion of a slicing point; this depends on the idea of a linked pair. We use slicing points to define D-homomorphisms, which are needed to establish an adjunction between T_D partial spaces and a non-full subcategory of partial frames. (See Proposition 5.7.) We conclude this section with a discussion of linked pairs and slicing points in the context of free frames and congruence frames over partial frames.

Definition 3.1. If a and b are elements of a partial frame L, we say a is *directly below* b and write $a \lessdot b$ if $a < b$ and $a \leq x \leq b$ implies $x = a$ or $x = b$. We call such $a \lessdot b$ a *linked pair.*
Grätzer [18] and Banaschewski and Pultr [7] say *a is covered by b.*

Lemma 3.2. *Suppose $a \lessdot b$ in an $\mathcal{S}$-frame L. For any $x \in L$,*

$$x \wedge a = x \wedge b \iff x \vee a \neq x \vee b.$$

Proof. ($\Rightarrow$) If $x \wedge a = x \wedge b$ and $x \vee a = x \vee b$ then distributivity gives $a = b$, a contradiction.

($\Leftarrow$) Suppose $x \vee a \neq x \vee b$. Clearly $a \leq (x \wedge b) \vee a \leq b$, so using $a \lessdot b$ gives us two cases.

Case 1 $(x \wedge b) \vee a = b$: then $(x \vee a) \wedge (b \vee a) = b$, so $(x \vee a) \wedge b = b$, so $b \leq x \vee a$, so $x \vee b \leq x \vee a$, so $x \vee b = x \vee a$, a contradiction.

Case 2 $(x \wedge b) \vee a = a$: then $x \wedge b \leq a$, so $x \wedge b \leq x \wedge a$, so $x \wedge b = x \wedge a$, as required. □

Note 3.3. The statement of Lemma 3.2 could be rephrased as:
if $a \lessdot b$ then $(a, b) \in \Delta_x \iff (a, b) \notin \nabla_x$, for all x. So $(a, b) \in \Delta_x$ or $(a, b) \in \nabla_x$ but $(a, b) \notin \Delta_x \cap \nabla_x$.

Proposition 3.4. *Suppose $a \lessdot b$ in an $\mathcal{S}$-frame L. Define $\lambda : L \to \mathbf{2}$ by $\lambda(x) = 0 \iff x \wedge a = x \wedge b$. Then λ is an $\mathcal{S}$-point of L (that is, λ is an $\mathcal{S}$-frame map).*

Proof. By Lemma 3.2, $\lambda(x) = 1 \iff x \vee a = x \vee b$. Clearly $\lambda(0) = 0$ and $\lambda(1) = 1$. The remainder of the proof is routine and we prove some of it for illustrative purposes. We first show that λ is order preserving: suppose that $c \leq d$ and $\lambda(c) = 1$. Then $c \vee a = c \vee b$, so $d \vee a = d \vee b$ and $\lambda(d) = 1$. We also prove that if S is a designated subset of L, then $\lambda(\bigvee S) \leq \bigvee\{\lambda(s) : s \in S\}$: suppose that $\bigvee\{\lambda(s) : s \in S\} = 0$. Then, for all $s \in S$, $\lambda(s) = 0$, so $s \wedge a = s \wedge b$; now

$$a \wedge \bigvee S = \bigvee\{s \wedge a : s \in S\} = \bigvee\{s \wedge b : s \in S\} = b \wedge \bigvee S,$$

so $\lambda(\bigvee S) = 0$ as needed.

We note further that $\lambda(a) = 0$ and $\lambda(b) = 1$. □

Proposition 3.5. *Suppose $a \lessdot b$ in an $\mathcal{S}$-frame L, and $\xi : L \to \mathbf{2}$ is an $\mathcal{S}$-point satisfying $\xi(a) = 0$ and $\xi(b) = 1$. Then ξ is the unique such.*

Proof. We show that $\xi(x) = 0 \iff x \wedge a = x \wedge b$.

($\Rightarrow$) Suppose $x \in L$ and $\xi(x) = 0$. Clearly $a \leq (a \vee x) \wedge b \leq b$ so there are two cases:

Case 1 $(a \vee x) \wedge b = b$: then $b \leq a \vee x$, so $\xi(b) \leq \xi(a) \vee \xi(x) = 0$ which is a contradiction.

Case 2 $(a \vee x) \wedge b = a$: then $(a \wedge b) \vee (x \wedge b) = a$, so $x \wedge b \leq a$ giving $x \wedge b \leq x \wedge a$ and so $x \wedge a = x \wedge b$ as needed.

($\Leftarrow$) Suppose $x \in L$ and $x \wedge a = x \wedge b$. Then $\xi(x) \wedge \xi(a) = \xi(x) \wedge \xi(b)$, so $\xi(x) \wedge 0 = \xi(x) \wedge 1$, giving $\xi(x) = 0$. $\square$

Definition 3.6. We call an $\mathcal{S}$-point $\xi : L \to \mathbf{2}$ a *slicing point* of L if there exists a linked pair $a \lessdot b$ in L for which $\xi(a) = 0$ and $\xi(b) = 1$. We say that ξ *slices* the pair $a \lessdot b$.

Note 3.7. Propositions 3.4 and 3.5 show that any linked pair $a \lessdot b$ in an $\mathcal{S}$-frame has a unique associated slicing point. This is given, equivalently, by:

$$\lambda(x) = 0 \iff a \wedge x = b \wedge x \quad \text{or}$$
$$\lambda(x) = 1 \iff a \vee x = b \vee x.$$

Example 3.8. In [17] we discuss in some detail closed and open maps in categories of partial frames. In particular, we consider closed and open points. These are indeed examples of slicing points:

A *closed* point of an $\mathcal{S}$-frame L is an $\mathcal{S}$-frame map $\zeta_a : L \to \mathbf{2}$ where a is a co-atom of L and $\zeta_a(x) = 0 \Leftrightarrow x \leq a$. Here $a \lessdot 1$ and ζ_a is the slicing point corresponding to this linked pair.

Similarly, an *open* point of L is an $\mathcal{S}$-frame map $\rho_a : L \to \mathbf{2}$ where a is an atom of L and $\rho_a(x) = 1 \Leftrightarrow x \geq a$. Here $0 \lessdot a$ and ρ_a is the slicing point corresponding to this linked pair.

Note 3.9. Alternative using filters: As mentioned in [13] the spectrum ΣL of a partial frame L can be described equally well using maps from L to $\mathbf{2}$ or using certain kinds of filters, the $\mathcal{S}$-prime ones, on L. The filter corresponding to $\xi : L \to \mathbf{2}$ is $F = \{x \in L : \xi(x) = 1\}$. We naturally call a filter corresponding to a slicing point a *slicing filter* of L: these are the

$\mathcal{S}$-prime filters F of L for which there exists a linked pair $a \lessdot b$ with $a \notin F$ but $b \in F$. There is then a one-one correspondence between the slicing points of L and the slicing filters of L. We mention that slicing filters are introduced and used in [7].

We will frequently use the following, from [18].

Lemma 3.10. *Grätzer's Lemma.*
Suppose a, b, c are elements of a distributive lattice and $a \lessdot b$.

$$\textit{Then } a \vee c \lessdot b \vee c \textit{ or } a \vee c = b \vee c.$$
$$\textit{Also } a \wedge c \lessdot b \wedge c \textit{ or } a \wedge c = b \wedge c.$$

Definition 3.11. Let $h : L \to M$ be an $\mathcal{S}$-frame map between $\mathcal{S}$-frames. We call h a *D-homomorphism* if it satisfies the following condition:

Whenever $\xi : M \to \mathbf{2}$ is a slicing point of M
then $\xi \circ h : L \to \mathbf{2}$ is a slicing point of L.

We note that, in the above definition, $\xi \circ h$ is automatically a point of L; the substance of the definition is that it should be a *slicing* point of L. These D-homomorphisms will play an essential role in the adjunction between T_D partial spaces and partial frames. (See Proposition 5.7.)

We now justify this choice of terminology by showing that, in the category of full frames, the morphisms that we call "D-homomorphisms" are exactly the ones called "D-homomorphisms" by Banaschewski and Pultr in [7]. Note that there a frame map $h : L \to M$ with right adjoint r is called a D-homomorphism if, whenever p is a covered prime of M, then $r(p)$ is a covered prime of L, where p is a covered prime if it is prime and $p \lessdot t$ for some t.

Proposition 3.12. *Let $h : L \to M$ be a frame map between* (*full*) *frames, and let r be its right adjoint. Then h is a D-homomorphism* (*according to Definition* 3.11) *if and only if, whenever p is a covered prime of M, then $r(p)$ is a covered prime of L.*

Proof. ($\Rightarrow$) Suppose p is a covered prime of M; so $p \lessdot t$ for some $t \in M$ and p is prime. We note that $r(p)$ is automatically prime (see [7] 1.3); we show that $r(p)$ is covered. Let $\xi : M \to \mathbf{2}$ be the point of M associated with p, that is

$$\xi(m) = \begin{cases} 1 \text{ if } m \not\leq p \\ 0 \text{ if } m \leq p \end{cases}$$

Then $\xi(p) = 0$ and $\xi(t) = 1$, so ξ is a slicing point of M. By assumption, $\xi \circ h$ is a slicing point of L. So there exists $a \lessdot b$ in L with $\xi h(a) = 0$ and $\xi h(b) = 1$. Then $h(a) \leq p$ and so $a \leq r(p)$. Also $h(b) \not\leq p$ and so $b \not\leq r(p)$. By Grätzer's Lemma, $a \vee r(p) = b \vee r(p)$ or $a \vee r(p) \lessdot b \vee r(p)$. If $a \vee r(p) = b \vee r(p)$ then $r(p) = b \vee r(p)$ so $b \leq r(p)$, a contradiction. So $r(p) = a \vee r(p) \lessdot b \vee r(p)$ as needed.

($\Leftarrow$) Suppose that $\xi : M \to \mathbf{2}$ is a slicing point of M. Then there exists $s \lessdot t$ in M with $\xi(s) = 0$ and $\xi(t) = 1$. Let p be the associated prime element of M, that is,

$$p = \bigvee\{x \in M : \xi(x) = 0\}.$$

Then $s \leq p$ and $t \not\leq p$. By assumption, $r(p)$ is a covered prime of L so there exists $c \in L$ with $r(p) \lessdot c$. Then $\xi h(r(p)) \leq \xi(p) = 0$ and $\xi h(c) = 1$ (this latter because $\xi h(c) = 0$ would imply that $h(c) \leq p$, so $c \leq r(p)$, a contradiction).

So ξh is indeed a slicing point of L as required. □

Banaschewski and Pultr ([7] 3.1) give a substantial list of D-homomorphisms, a justification of their claim that this is not a particularly strong requirement.

Lemma 3.13. *Let $\xi : L \to \mathbf{2}$ be an $\mathcal{S}$-point of the $\mathcal{S}$-frame L. Then ξ is a D-homomorphism if and only if ξ is a slicing point of L.*

Proof. ($\Rightarrow$) Use the fact that the identity function id : $\mathbf{2} \to \mathbf{2}$ is a slicing point of $\mathbf{2}$ to conclude that $\mathrm{id} \circ \xi : L \to \mathbf{2}$ is a slicing point of L.

($\Leftarrow$) Use the fact that the identity function $id : \mathbf{2} \to \mathbf{2}$ is the only slicing point of $\mathbf{2}$ to deduce that $\xi : L \to \mathbf{2}$ is a D-homomorphism. □

In the remainder of this section, we consider linked pairs and slicing points in two interesting and important contexts: free frames and congruence frames over partial frames.

Remark 3.1. We note that if $h : L \to M$ is a one-one $\mathcal{S}$-frame map between $\mathcal{S}$-frames then, for $a, b \in L$, $h(a) \lessdot h(b) \Rightarrow a \lessdot b$, as a direct calculation shows.

Proposition 3.14. *Let L be an $\mathcal{S}$-frame and $\downarrow: L \to \mathcal{H}_{\mathcal{S}}L$ its embedding into its free frame. For $a, b \in L$, $a \lessdot b \Leftrightarrow \downarrow a \lessdot \downarrow b$.*

Proof. ($\Rightarrow$): Suppose that $\downarrow a \subseteq I \subseteq \downarrow b$ for some $I \in \mathcal{H}_{\mathcal{S}}L$, and $I \neq \downarrow b$. We show that $\downarrow a = I$. Suppose $i \in I$. Then $a \vee i \in I$, so $a \leq a \vee i \leq b$. By assumption, $a = a \vee i$ or $a \vee i = b$. In the former case $i \leq a$ making $I = \downarrow a$. In the latter case, $b \in I$, a contradiction.

($\Leftarrow$): The embedding map $\downarrow$ is one-one, so this follows from Remark 3.1. □

Proposition 3.15. *Let L be an $\mathcal{S}$-frame and $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ its embedding into its congruence frame. For $a, b \in L$, $a \lessdot b \Leftrightarrow \nabla_a \lessdot \nabla_b$.*

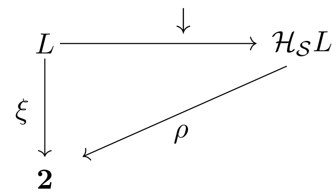
Proof. ($\Rightarrow$): Suppose that $\nabla_a \subseteq \theta \subseteq \nabla_b$ for some $\theta \in \mathcal{C}_{\mathcal{S}}L$, and $\theta \neq \nabla_b$. We prove that $\nabla_a = \theta$. To this end, suppose that $(s, t) \in \theta, s \leq t$. We show that $s \vee a = t \vee a$.

Case 1 $(s \vee a) \wedge b = (t \vee a) \wedge b$.
Here $(s \vee a, t \vee a) \in \Delta_b$. However, $(s, t) \in \theta \subseteq \nabla_b$, so $(s \vee a, t \vee a) \in \nabla_b$ also. So $(s \vee a, t \vee a) \in \Delta_b \cap \nabla_b = \Delta$, giving $s \vee a = t \vee a$, as desired.

Case 2 $(s \vee a) \wedge b < (t \vee a) \wedge b$.
Here $a \leq (s \vee a) \wedge b < (t \vee a) \wedge b \leq b$. Since $a \lessdot b$, we have two deductions.
First $a = (s \vee a) \wedge b$. Distributing gives $b \wedge s = a \wedge s$, so $(a, b) \in \Delta_s$.
Next $b = (t \vee a) \wedge b$. Distributing gives $b \vee t = a \vee t$, so $(a, b) \in \nabla_t$.
Together this yields $(a, b) \in \nabla_t \cap \Delta_s \subseteq \theta$, since $\nabla_t \cap \Delta_s$ is the congruence generated by the pair (s, t). Finally, since $(0, a) \in \theta$, we have $(0, b) \in \theta$, a contradiction.

($\Leftarrow$): Automatic, since the embedding $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ is one-one; see Remark 3.1. □

We recall that, for any $\mathcal{S}$-frame L, the $\mathcal{S}$-points of L are in one-one correspondence with the frame points of its free frame $\mathcal{H}_{\mathcal{S}}L$. Given any $\mathcal{S}$-point $\xi : L \to \mathbf{2}$, there exists a unique frame point $\rho : \mathcal{H}_{\mathcal{S}}L \to \mathbf{2}$ making the diagram below commute:



The behaviour of slicing points is different, as the following result shows. We use the notation above.

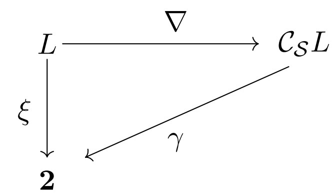
Proposition 3.16. *Let L be an $\mathcal{S}$-frame. If $\xi : L \to \mathbf{2}$ is a slicing point of L, then the corresponding $\rho : \mathcal{H}_{\mathcal{S}}L \to \mathbf{2}$ is a slicing point of $\mathcal{H}_{\mathcal{S}}L$; but not conversely.*

Proof. ($\Rightarrow$) Suppose $\xi : L \to \mathbf{2}$ slices the linked pair $a \lessdot b$. By Proposition 3.14 ${\downarrow}a \lessdot {\downarrow}b$. Further $\rho({\downarrow}a) = \xi(a) = 0$ and $\rho({\downarrow}b) = \xi(b) = 1$, as required.

($\not\Leftarrow$) See Example 3.17 below. □

Example 3.17. Let L be the σ-frame consisting of all countable subsets of $\mathbb{R}$ with $\mathbb{R}$ itself as top element. The σ-frame map $\xi : L \to \mathbf{2}$ given by $\xi(A) = 0 \Leftrightarrow A$ is countable, is a a σ-point of L which is not slicing, since there is no countable subset of A of $\mathbb{R}$ with $A \lessdot \mathbb{R}$. Now let K consist of all countable subsets of $\mathbb{R}$. Then K is indeed a σ-ideal, so a member of $\mathcal{H}_{\mathcal{S}}L$, and the frame point corresponding to ξ is given by $\rho(I) = 0 \Leftrightarrow I \subseteq K$, for all $I \in \mathcal{H}_{\mathcal{S}}L$. Since K is a co-atom of $\mathcal{H}_{\mathcal{S}}L$, ρ is indeed slicing.

We now present a very similar result for the congruence frame $\mathcal{C}_{\mathcal{S}}L$. Again, for any $\mathcal{S}$-frame L, the $\mathcal{S}$-points are in one-one correspondence with the frame points of $\mathcal{C}_{\mathcal{S}}L$. Given any $\mathcal{S}$-point $\xi : L \to \mathbf{2}$, there exists a unique frame point $\gamma : \mathcal{C}_{\mathcal{S}}L \to \mathbf{2}$ making the diagram below commute:



Proposition 3.18. *Let L be an $\mathcal{S}$-frame. If $\xi : L \to \mathbf{2}$ is a slicing point of L then the corresponding $\gamma : \mathcal{C}_{\mathcal{S}}L \to \mathbf{2}$ is a slicing point of $\mathcal{C}_{\mathcal{S}}L$; but not conversely.*

Proof. ($\Rightarrow$) Similar to that of Proposition 3.16.

($\not\Leftarrow$) The σ-frame of Example 3.17 applies once more, with the same $\xi : L \to \mathbf{2}$. The corresponding frame point $\gamma : \mathcal{C}_{\mathcal{S}}L \to \mathbf{2}$ is slicing, because $\gamma(\bigcup\{\nabla_A : A \text{ is countable}\}) = 0$ and $\gamma(\nabla_{\mathbb{R}}) = 1$ and there is no congruence strictly between these two. We note that $\bigcup\{\nabla_A : A \text{ is countable}\}$ is in fact an $\mathcal{S}$-congruence of L, using Definition 2.4 (6). □

4 T_D partial spaces

We now turn from partial frames to partial spaces; in particular we introduce T_D partial spaces. The definition is modelled directly on that of T_D topological spaces, but the two situations are significantly different, as Example 4.7 shows. In [7] heavy use is made of closures of singletons in the characterization of T_D spaces. In [14] closures of subsets of partial spaces are defined, but need not be closed in the given $\mathcal{S}$-topology, though in fact, they are closed in the generated full topology. As a result, closures are not a useful tool in this context of partial spaces. In place of point closures, we make substantial use of slicing points.

Definition 4.1. We say that an $\mathcal{S}$-space $(X, \mathcal{O}X)$ is S_0 if for any $x, y \in X$ there is $U \in \mathcal{O}X$ such that $x \in U$ but $y \notin U$, or conversely. (See Definition 4.11 of [13].)

Clearly the S_0 property just generalizes the T_0 property to partial spaces. So we make the same restrictions as in [7] (1.6): since the restriction to S_0 spaces has no impact on the pointfree aspects of spaces, that is, their partial frames of open sets, *we assume throughout that all spaces are* S_0.
Recall that we defined $\xi_x : \mathcal{O}X \to \mathbf{2}$ by $\xi_x(U) = 1 \iff x \in U$.

Lemma 4.2. *Let X be an S_0 $\mathcal{S}$-space and $U, V \in \mathcal{O}X$. Then:*
(1) *$U \lessdot V$ in $\mathcal{O}X \Leftrightarrow U = V\backslash\{z\}$ for some $z \in X$.*
(2) *Every slicing point of $\mathcal{O}X$ has the form ξ_x, for some $x \in X$.*

Proof. (1) ($\Rightarrow$) Since $U \neq V$ there is some $z \in V\backslash U$. Suppose now that $z_1, z_2 \in V\backslash U$. Using the S_0 property, there exists $W \in \mathcal{O}X$ with, say, $z_1 \in W$ and $z_2 \notin W$.Then $U \subseteq U\cup(V\cap W) \subseteq V$ and of course $U\cup(V\cap W) \in \mathcal{O}X$. Also $U \neq U \cup (V \cap W)$ (use z_1) and $U \cup (V \cap W) \neq V$ (use z_2). This contradicts $U \lessdot V$.

($\Leftarrow$) Clear.

(2) Suppose $\rho : \mathcal{O}X \to \mathbf{2}$ is a slicing point of $\mathcal{O}X$. Then there exists a linked pair $U \lessdot V$ in $\mathcal{O}X$ with $\rho(U) = 0$ and $\rho(V) = 1$. By (a) there exists $x \in V$ with $V = U \cup \{x\}$. We show that, for $W \in \mathcal{O}X$, $\rho(W) = 1 \Leftrightarrow \xi_x(W) = 1$, that is, $x \in W$.
Suppose $\rho(W) = 1$. If $x \notin W$ then $U \cap W = V \cap W$. But $\rho(U \cap W) = \rho(U) \wedge \rho(W) = 0$ and $\rho(V \cap W) = \rho(V) \wedge \rho(W) = 1$, a contradiction. So $x \in W$. Conversely, suppose $x \in W$: then

$$\rho(W) = \rho(W) \vee \rho(U) = \rho(W \cup U) = \rho(W \cup V) = \rho(W) \vee \rho(V) = 1.$$

□

Definition 4.3. Let $(X, \mathcal{O}X)$ be an $\mathcal{S}$-space. We call $x \in X$ a T_D-*point* of X if there exists $V \in \mathcal{O}X$ with $x \in V$ and $V \backslash \{x\} \in \mathcal{O}X$. We call $(X, \mathcal{O}X)$ a T_D *space* (or T_D $\mathcal{S}$-space) if every $x \in X$ is a T_D-point.

In [27] the authors state when describing the role of T_D: "Here is one of its features that is particularly important for point-free topology, namely that this is precisely the condition under which subspaces are correctly represented by frame congruences." This holds equally well in the partial setting as Proposition 4.5 shows.

Definition 4.4. Let $(X, \mathcal{O}X)$ be an $\mathcal{S}$-space and Y a subset of X. We define an $\mathcal{S}$-congruence on $\mathcal{O}X$ by

$$E_Y = \{(U, V) \in \mathcal{O}X \times \mathcal{O}X : U \cap Y = V \cap Y\}$$

That E_Y is in fact an $\mathcal{S}$-congruence is straightforward to check.

Proposition 4.5. *The $\mathcal{S}$-space X is T_D iff $E_Y \neq E_Z$ for any two distinct $Y, Z \subseteq X$.*

Proof. The proof is identical to that provided in Chapter I Section 6.3 of [27]. □

Note 4.6. We note that any T_D partial space is S_0: Suppose $x \neq y$ and U_x, U_y are open neighbourhoods of x and y respectively with $U_x \backslash \{x\}$ and $U_y \backslash \{y\}$ open. If $y \in U_x \backslash \{x\}$, the S_0 condition is satisfied, since $x \notin U_x \backslash \{x\}$. If $y \notin U_x \backslash \{x\}$ then $y \notin U_x$ since $x \neq y$. Again the S_0 condition is satisfied, since $x \in U_x$.

In (full) topological spaces, $T_1 \Rightarrow T_D \Rightarrow T_0$, making T_D a weak separation axiom. The situation in partial spaces is rather different, as the example below shows. The notions Boolean, normal, regular, compact and Hausdorff referred to below will not be used again in this paper but have familiar definitions (see [9, 16]).

Example 4.7. The Finite Extended Sorgenfrey Line.
Let $\mathcal{S}$ be the selection function designating finite subsets, so $\mathcal{S}$-frames are bounded distributive lattices. Let $X = \mathbb{R}$ and $\mathcal{O}X$ consist of finite unions of intervals of the form $[a, b), [a, \infty), (-\infty, b)$. Then $(X, \mathcal{O}X)$ is an $\mathcal{S}$-space. Here $\mathcal{O}X$ is Boolean, so normal and regular, $(X, \mathcal{O}X)$ is S_2 (Hausdorff) and (vacuously) compact; yet $(X, \mathcal{O}X)$ is not T_D.

Note 4.8. The open sets of any $\mathcal{S}$-space can be used to generate a topological space (see [13]). If X is a T_D $\mathcal{S}$-space, the topological space generated by it is still T_D; this is clear. The converse does not hold, as Example 4.7 shows, since the Sorgenfrey topology on the real line is T_D.

Note 4.9. A T_0 topological space X is T_D if and only its Skula topology (generated by the open and closed sets of X) is discrete. The corresponding result for $\mathcal{S}$-spaces does not hold, as Example 4.10 shows.

Example 4.10. Let $X = \mathbb{R}$ and $\mathcal{O}X = \{A \subseteq \mathbb{R} : A \text{ is countable}\} \cup \{\mathbb{R}\}$, with $\mathcal{S}$ designating countable subsets. Then $(X, \mathcal{O}X)$ is a T_D σ-space. Its Skula σ-space is $(X, \mathcal{U}X)$ where

$$\mathcal{U}X = \{A \subseteq \mathbb{R} : A \text{ is countable or cocountable}\}$$

not the power set of X.

We can now characterize T_D spaces in terms of slicing points.

Proposition 4.11. *Let X be an S_0 $\mathcal{S}$-space.*

(1) *For all $x \in X$, x is a T_D-point of X if and only if ξ_x is a slicing point of $\mathcal{O}X$.*

(2) *X is a T_D space if and only if ξ_x is a slicing point of $\mathcal{O}X$ for all $x \in X$.*

(3) *If X is a T_D space and $x \in X$, then ξ_x slices every linked pair $W \lessdot Z$ in $\mathcal{O}X$ for which $Z = W \cup \{x\}$, and ξ_x is the only slicing point of $\mathcal{O}X$ to do so.*

Proof. (1) We have

x is a T_D-point of X
$\Leftrightarrow$ there exists $V \in \mathcal{O}X$ with $x \in V$ and $V \backslash \{x\} \in \mathcal{O}X$
$\Leftrightarrow$ there exist $U, V \in \mathcal{O}X$ with $U \lessdot V$ and $V = U \cup \{x\}$
$\Leftrightarrow$ there exist $U, V \in \mathcal{O}X$ with $U \lessdot V$ and $x \notin U, x \in V$
$\Leftrightarrow$ there exist $U, V \in \mathcal{O}X$ with $U \lessdot V$ and $\xi_x(U) = 0, \xi_x(V) = 1$
$\Leftrightarrow$ ξ_x is a slicing point of $\mathcal{O}X$

(2) This follows immediately from (1).

(3) If $W \lessdot Z$ in $\mathcal{O}X$ and $Z = W \cup \{x\}$, then $x \notin W, x \in Z$, so $\xi_x(W) = 0$ and $\xi_x(Z) = 1$. Uniqueness follows from Proposition 3.5. □

Note 4.12. If $x \in X$ is a T_D-point of an S_0 space X, and Y is a subspace of X with $x \in Y$, then x is a T_D-point of Y. So the subspace $Y = \{x \in X : x \text{ is a } T_D\text{-point of } X\}$ of X is a T_D space.

5 The adjunction between $\mathcal{S}\mathbf{Top}_D$ and $\mathcal{S}\mathbf{Frm}_D$

Technically, this section provides a generalization of the adjunction given by Banaschewski and Pultr in [7], between the category of T_D spaces and continuous functions and that of frames with D-homomorphisms. However, while our results are similar, the tools we use are different.

Prime elements, covered primes, completely prime elements and right adjoints of frame maps are not available in our setting. This is in contrast to the work of [4, 7, 20]. The notions of T_D partial spaces and D-homomorphisms between partial frames were defined and characterized in terms of slicing points in Sections 3 and 4. In this section we see their use in the construction of the open set functor and the T_D spectrum functor.

Definition 5.1. We denote by $\mathcal{S}\mathbf{Top}_D$ the category with objects all $\mathcal{S}$-spaces that are T_D and morphisms all continuous functions between them.

We denote by $\mathcal{S}\mathbf{Frm}_D$ the category with objects all $\mathcal{S}$-frames and morphisms all D-homomorphisms between them.

The Open Set Functor We restrict the usual open set functor $\mathcal{O}$: $\mathcal{S}\mathbf{Top} \to \mathcal{S}\mathbf{Frm}$ to $\mathcal{O} : \mathcal{S}\mathbf{Top}_D \to \mathcal{S}\mathbf{Frm}_D$. (See Definition 2.5.)

To this end we need check only that, if $f : X \to Y$ is a continuous function between T_D spaces, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is a D-homomorphism. That $\mathcal{O}f$ is an $\mathcal{S}$-frame map is already established; it remains to show that if $\rho : \mathcal{O}X \to \mathbf{2}$ is a slicing point of $\mathcal{O}X$, then $\rho \circ \mathcal{O}f$ is a slicing point of $\mathcal{O}Y$:

Proof. By Lemma 4.2, ρ has the form ξ_x, for some $x \in X$. Since $f(x)$ is a T_D-point of Y, by assumption $\xi_{f(x)}$ is a slicing point of $\mathcal{O}Y$, by Proposition 4.11. Further $\xi_x \circ \mathcal{O}f = \xi_{f(x)}$:
For $W \in \mathcal{O}Y$,

$$\begin{aligned}(\xi_x \circ \mathcal{O}f)(W) = 1 &\Leftrightarrow \xi_x(f^{-1}(W)) = 1\\ &\Leftrightarrow x \in f^{-1}(W)\\ &\Leftrightarrow f(x) \in W\\ &\Leftrightarrow \xi_{f(x)}(W) = 1\end{aligned}$$

□

The T_D spectrum functor Let L be an $\mathcal{S}$-frame. The *spectrum* ΣL of L consists of all points of L, i.e. all $\mathcal{S}$-frame maps $\xi : L \to \mathbf{2}$. We denote by ΦL the set of all slicing points of L, with ΦL regarded as a subspace of ΣL.

For $a \in L$, let $\Phi_a = \{\xi \in \Phi L : \xi(a) = 1\}$. Since $\Phi_a = \Sigma_a \cap \Phi L$, all open subsets of ΦL have the form Φ_a for some $a \in L$. Further:

$\Phi_0 = \emptyset$ and $\Phi_1 = \Phi L$
$\Phi_{a \wedge b} = \Phi_a \cap \Phi_b$ for all $a, b \in L$
$\Phi_{\bigvee S} = \cup\{\Phi_x : x \in S\}$ for all designated subsets S of L.

Lemma 5.2. *For any $\mathcal{S}$-frame L, ΦL is a T_D $\mathcal{S}$-space.*

Proof. Since ΣL is S_0, so is ΦL. We show that, for all $\xi \in \Phi L$, there exists Φ_b such that $\xi \in \Phi_b$ and $\Phi_b \backslash \{\xi\} = \Phi_a$ for some $a \in L$:
Since ξ is a slicing point of L, there exists a linked pair $a \lessdot b$ in L with $\xi(a) = 0$ and $\xi(b) = 1$. Then

- $\xi \in \Phi_b$ because $\xi(b) = 1$

- $\xi \notin \Phi_a$ because $\xi(a) = 0$
- $\Phi_b = \Phi_a \cup \{\xi\}$; the inclusion $\Phi_b \subseteq \Phi_a \cup \{\xi\}$ following because if ρ is a slicing point of L, $\rho(b) = 1$ and $\rho \neq \xi$, then $\rho(a) = 1$ also, by Proposition 3.5.

$\square$

The T_D spectrum functor $\Phi : \mathcal{S}\mathbf{Frm}_D \to \mathcal{S}\mathbf{Top}_D$ acts on objects as given above, and on morphisms by restricting the action of the spectrum functor to slicing points. To be specific:

Suppose $h : L \to M$ is a D-homomorphism between $\mathcal{S}$-frames. Then $\Phi h : \Phi M \to \Phi L$ is given as follows:

For $\xi \in \Phi M$, $\Phi h(\xi) = \xi \circ h$.

We note that $\xi \circ h$ is a slicing point of L precisely because ξ is a slicing point of M and h is a D-homomorphism.

That Φh is continuous now follows in routine fashion. For $a \in L$

$$\begin{aligned}(\Phi h)^{-1}(\Phi_a) &= \{\rho \in \Phi M : (\Phi h)(\rho) \in \Phi_a\} \\ &= \{\rho \in \Phi M : \rho \circ h \in \Phi_a\} \\ &= \{\rho \in \Phi M : \rho(h(a)) = 1\} \\ &= \Phi_{h(a)}\end{aligned}$$

Lemma 5.3. *For any $\mathcal{S}$-frame L, the function $\delta_L : L \to \mathcal{O}\Phi L$ given by $\delta_L(a) = \Phi_a$ is a D-homomorphism.*

Proof. That δ_L is an $\mathcal{S}$-frame map follows from the properties of the Φ_a listed earlier.

To show that δ_L is a D-homomorphism, begin with a slicing point $\xi : \mathcal{O}\Phi L \to \mathbf{2}$. This means that there exists a linked pair $\Phi_a \lessdot \Phi_b$ in $\mathcal{O}\Phi L$ such that $\xi(\Phi_a) = 0$ and $\xi(\Phi_b) = 1$. Also, by Lemma 4.2 $\Phi_b = \Phi_a \cup \{\rho\}$ for some $\rho \in \Phi_b$. Clearly $\rho(a) = 0$ and $\rho(b) = 1$. Since ρ is a slicing point of L, there exists a linked pair $c \lessdot d$ in L with $\rho(c) = 0$ and $\rho(d) = 1$.

We show that $(\xi \circ \delta_L)(c) = 0$ and $(\xi \circ \delta_L)(d) = 1$. This will show that $\xi \circ \delta_L$ is a slicing point of L, and also, in the process, that $\xi \circ \delta_L = \rho$.

By definition $(\xi \circ \delta_L)(c) = \xi(\Phi_c)$ and by Note 3.7, $\xi(\Phi_c) = 1 \Leftrightarrow \Phi_a \cup \Phi_c = \Phi_b \cup \Phi_c$. Since $\rho \notin \Phi_a \cup \Phi_c$ and $\rho \in \Phi_b \cup \Phi_c$ we conclude that $\xi(\Phi_c) = 0$. Assuming $\xi(\Phi_d) = 0$ leads to a similar contradiction, and so $\xi(\Phi_d) = 1$. $\square$

Lemma 5.4. *The maps $\delta_L : L \to \mathcal{O}\Phi L$ provide a natural transformation $\delta : Id \to \mathcal{O}\Phi$.*

Proof. The claim is that, for any D-homomorphism $h : L \to M$ between $\mathcal{S}$-frames, the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\delta_L} & \mathcal{O}\Phi L \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{O}\Phi h} \\ M & \xrightarrow[\delta_M]{} & \mathcal{O}\Phi M \end{array}$$

A routine calculation shows that, indeed, for all $a \in L$, $\mathcal{O}\Phi h(\Phi_a) = \Phi_{h(a)}$, as needed. □

Lemma 5.5. *For any T_D $\mathcal{S}$-space X, the function $\pi_X : X \to \Phi\mathcal{O}X$ given by $\pi_X(x) = \xi_x$, is a homeomorphism.*

Proof. We note that Proposition 4.11 shows that ξ_x is indeed a slicing point of $\mathcal{O}X$, for any $x \in X$ since X is T_D. To show that π_X is one-one, take $x \neq y$ in X. There exists $U \lessdot V$ in $\mathcal{O}X$ with $V = U \cup \{x\}$. If $y \in U$, then $\xi_y(U) = 1$ while $\xi_x(U) = 0$. If $y \notin U$ then $y \notin V$, so $\xi_y(V) = 0$ but $\xi_x(V) = 1$. In either case $\xi_x \neq \xi_y$.

We note that Lemma 4.2 shows that π_X is onto.

To show π_X is a homeomorphism use the fact that $\pi_X^{-1}(\Phi_U) = U$, for all $U \in \mathcal{O}X$. □

Lemma 5.6. *The maps $\pi_X : X \to \Phi\mathcal{O}X$ provide a natural equivalence $\pi : Id \to \Phi\mathcal{O}$.*

Proof. The claim is that, for any continuous function $f : X \to Y$ between T_D $\mathcal{S}$-spaces, the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & \Phi\mathcal{O}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Phi\mathcal{O}f} \\ Y & \xrightarrow[\pi_Y]{} & \Phi\mathcal{O}Y \end{array}$$

For $x \in X$, $\Phi\mathcal{O}f(\pi_X(x)) = \Phi\mathcal{O}f(\xi_x) = \xi_x \circ \mathcal{O}f$ and $\pi_Y(f(x)) = \xi_{f(x)}$. These are equal as seen earlier in the section on the Open Set Functor. □

Proposition 5.7. *The contravariant functors $\mathcal{O} : \mathcal{S}\mathbf{Top}_D \to \mathcal{S}\mathbf{Frm}_D$ and $\Phi : \mathcal{S}\mathbf{Frm}_D \to \mathcal{S}\mathbf{Top}_D$ are adjoint on the right.*

Proof. The adjunction maps are given in Lemmas 5.3 and 5.5. □

6 T_D spatiality of partial frames

As usual the adjunction between $\mathcal{S}\mathbf{Top}_D$ and $\mathcal{S}\mathbf{Frm}_D$ gives rise to a dual equivalence between the fixed objects on either side. By Lemma 5.5, all objects of $\mathcal{S}\mathbf{Top}_D$ are fixed. We now consider the corresponding fixed objects of $\mathcal{S}\mathbf{Frm}_D$, namely, the T_D spatial partial frames, and we characterize these.

Definition 6.1. An $\mathcal{S}$-frame L is called T_D *spatial* if the $\mathcal{S}$-frame map $\delta_L : L \to \mathcal{O}\Phi L$ is an isomorphism.

Corollary 6.2. *The category $\mathcal{S}\mathbf{Top}_D$ is dually equivalent to the full subcategory of $\mathcal{S}\mathbf{Frm}_D$ consisting of the T_D spatial partial frames.*

We note that an $\mathcal{S}$-frame L is T_D spatial if and only if $L \cong \mathcal{O}X$ for some T_D $\mathcal{S}$-space X.

In [7] an example is given showing that T_D spatiality is strictly stronger for frames than spatiality; this example also applies in our situation.

Proposition 6.3. *For any $\mathcal{S}$-frame L, the following are equivalent:*

(1) *L is T_D spatial.*

(2) *Whenever $s < t$ in L, there is a slicing point ρ of L with $\rho(s) = 0$ and $\rho(t) = 1$.*

(3) *Every proper interval $[s, t]$ in L contains a linked pair; that is, there exists $a \lessdot b$ with $s \leq a \lessdot b \leq t$.*

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that $\delta_L : L \to \mathcal{O}\Phi L$, always being an onto D-homomorphism is an isomorphism if and only if it is one-one.

(2) ⇒ (3) Suppose $s < t$ and $\rho \in \Phi L$ satisfies $\rho(s) = 0$ and $\rho(t) = 1$. Since ρ is a slicing point of L, there exists a linked pair $c \lessdot d$ with $\rho(c) = 0$ and $\rho(d) = 1$. Now let $a = (c \vee s) \wedge t$ and $b = (d \vee s) \wedge t$. Then $a \neq b$ and by Grätzer's Lemma 3.10, $a \lessdot b$, and $s \leq a \lessdot b \leq t$ as needed.

(3) ⇒ (2) Given $s \leq a \lessdot b \leq t$, let ρ be the slicing point of L associated with the linked pair $a \lessdot b$. (See Note 3.7.) Then $\rho(s) = 0$ and $\rho(t) = 1$. □

A natural condition on a partial frame is that of having all its points slicing: we call such a partial frame *sharp*. We note that such (full) frames have been called T_D-frames in, for instance, [3].

Definition 6.4. (1) An $\mathcal{S}$-frame L is called *sharp* if each $\mathcal{S}$-point of L is slicing; that is, $\Phi L = \Sigma L$.
(2) An $\mathcal{S}$-frame L is called *strongly T_D spatial* if it is sharp and spatial.

We will see in Lemma 6.7 that a strongly T_D spatial partial frame is indeed T_D spatial.

Lemma 6.5. *The following are equivalent for an $\mathcal{S}$-frame L.*

(1) *L is sharp.*

(2) *Every $\mathcal{S}$-frame map $h : L \to M$ to an $\mathcal{S}$-frame M is a D-homomorphism.*

(3) *Every onto $\mathcal{S}$-frame map $h : L \to M$ to an $\mathcal{S}$-frame M is a D-homomorphism.*

Proof. (1) ⇒ (2) We must show that, for every slicing point $\xi : M \to \mathbf{2}$ of M, the composite $\xi \circ h : L \to \mathbf{2}$ is slicing. However, $\xi \circ h$ is certainly an $\mathcal{S}$-point of of L, so, by assumption, is a slicing point.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Use Lemma 3.13. □

In Section 3.1 of [7] the authors show that every frame map with regular domain is a D-homomorphism; in our terminology, regular frames are sharp. The corresponding result for $\mathcal{S}$-frames does not hold, as the following example shows. We refer the reader to [9] for definitions of regularity and Booleanness for $\mathcal{S}$-frames.

Example 6.6. Let $\mathcal{L}$ consist of the countable and cocountable subsets of $\mathbb{R}$, and let $\mathcal{S}$ designate the countable subsets. The σ-frame $\mathcal{L}$ is Boolean, hence regular. However, the map $\xi : \mathcal{L} \to \mathbf{2}$ given by $\xi(A) = 0 \Leftrightarrow A$ is countable, is a σ-frame point that is not slicing, since there are no subsets C, D of $\mathbb{R}$ with C countable, D cocountable and $C \lessdot D$.

Lemma 6.7. *The following are equivalent for any $\mathcal{S}$-frame L.*

(1) *L is strongly T_D spatial.*

(2) *$L \cong \mathcal{O}Y$ for some $\mathcal{S}$-space Y that is T_D and sober.*

(3) *L is spatial and ΣL is a T_D $\mathcal{S}$-space.*

Proof. (1) $\Rightarrow$ (3) If L is T_D spatial, L is clearly spatial. Moreover $\Phi L = \Sigma L$ and since ΦL is automatically T_D, ΣL is T_D.

(3) $\Rightarrow$ (2) Since L is spatial, $L \cong \mathcal{O}\Sigma L$ and use $Y = \Sigma L$.

(2) $\Rightarrow$ (1) Since $L \cong \mathcal{O}Y$, L is clearly spatial. Suppose $\rho \in \Sigma\mathcal{O}Y$. By sobriety of Y, there exists $y \in Y$ with $\rho(U) = 1 \Leftrightarrow y \in U$, for $U \in \mathcal{O}Y$. This means $\rho = \xi_y$. By Proposition 4.11(2), Y being T_D gives ξ_y slicing, as required. $\square$

We note that the concept of strong T_D spatiality was introduced for frames by Arrieta and Suarez in [4]. Lemma 6.7 shows that their notion and ours correspond for frames.

7 T_D and sober partial spaces

The study of T_D and sober topological spaces has a long history. There is an informal duality between these two notions that is presented clearly in [7]. In this section we see that the very same ideas extend, using somewhat different methods, to the partial setting. The informal duality in question is given in Proposition 7.3.

Sober partial spaces were studied in [14] where it is shown that they are not as seemingly plentiful as in the topological case: even regular S_2 (Hausdorff) partial spaces need not be sober. In fact, Example 6.6 above is a case in point.

Lemma 7.1. (1) *Let X be an $\mathcal{S}$-space, $x \in X$ and $j : X\backslash\{x\} \to X$ the identical embedding. Then $\mathcal{O}j$ is an isomorphism if and only if x is not a T_D-point of X.*

(2) *For $Z \subseteq Y \subseteq X$, let $k : Z \to Y$ and $j : Y \to Z$ be the identical embeddings and let $l = j \circ k$. If $\mathcal{O}l$ is an isomorphism, then $\mathcal{O}j$ and $\mathcal{O}k$ are isomorphisms.*

Proof. The proof is as it appears in [7] Lemma 4.2 and Note 4.2. We remind the reader that $\mathcal{O}j(U) = U\backslash\{x\}$ for $U \in \mathcal{O}X$. □

Definition 7.2. For $\mathcal{S}$-spaces A and B we consider the relation:

$R(A, B)$:
- A is a proper subspace of B
- the identical embedding $j : A \to B$ makes $\mathcal{O}j : \mathcal{O}B \to \mathcal{O}A$ an isomorphism

Proposition 7.3. *Let X be an $\mathcal{S}$-space.*

(1) *X is sober iff there is no Y with $R(X, Y)$.*

(2) *X is T_D iff there is no Y with $R(Y, X)$.*

Proof. (1) We show that X is not sober iff there exists Y with $R(X, Y)$.

($\Rightarrow$) If X is not sober, let $Y = \Sigma\mathcal{O}X$, the sobrification of X. Then $\epsilon_X : X \to \Sigma\mathcal{O}X$ is not onto, but $\mathcal{O}\epsilon_X : \mathcal{O}\Sigma\mathcal{O}X \to \mathcal{O}X$ is an isomorphism.

($\Leftarrow$) Suppose X is sober and there exists Y with $R(X, Y)$. Let $j : X \to Y$ be an embedding that is not onto, with $\mathcal{O}j : \mathcal{O}Y \to \mathcal{O}X$ an isomorphism. Consider this diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & Y \\ {\scriptstyle \cong\, \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \Sigma\mathcal{O}X & \xrightarrow[\cong]{\Sigma\mathcal{O}j} & \Sigma\mathcal{O}Y \end{array}$$

We note that $\epsilon_Y : Y \to \Sigma\mathcal{O}Y$ is one-one because Y is S_0. Since the diagram commutes, $\epsilon_Y \circ j$ is onto, making j onto as well; a contradiction.

(2) We show that X is not T_D iff there exists Y with $R(Y, X)$.

($\Rightarrow$) If X is not T_D, it has a non-T_D-point x. Lemma 7.1 (1) shows that $X\backslash\{x\}$ can be chosen for Y.

($\Leftarrow$) Suppose $j : Y \to X$ is an identical embedding that is not onto, with $\mathcal{O}j : \mathcal{O}X \to \mathcal{O}Y$ an isomorphism. For $x \in X\backslash Y$, consider the identical embeddings

$$Y \to X\backslash\{x\} \to X.$$

By Lemma 7.1 (2) $\mathcal{O}X \to \mathcal{O}(X\backslash\{x\})$ is an isomorphism. By Lemma 7.1 (1), x is not a T_D-point of X. □

Corollary 7.4. (1) *If $j : X \to Y$ is an embedding that is not onto and $\mathcal{O}j : \mathcal{O}Y \to \mathcal{O}X$ is an isomorphism, then X is not sober and Y is not T_D.*

(2) *A non-trivial sobrification is never T_D.*

(3) *If, for each $x \in X$, $X \backslash \{x\}$ is sober, then X is T_D.*

Proof. (3) If x were a non-T_D-point of X, then $R(X\backslash\{x\}, X)$ making $X\backslash\{x\}$ not sober, by (1). □

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John Frith *Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag Rondebosch, 7701, South Africa.*
Email: johnlfrith@gmail.com

Anneliese Schauerte *Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag Rondebosch, 7701, South Africa.*
Email: anneliese.schauerte@uct.ac.za