Categories and
General Algebraic Structures
with Applications

In press.

$\mathcal{H}$-Fibrations: Fibrations in homotopy category

Ali Pakdaman*, Saba Dehrooye, Mehdi Tajik, and Behrooz Mashayekhy

Abstract. In this paper we generalize fibrations by $\mathcal{H}$-fibrations, the maps which homotopically lift homotopies. We replace the equalities in the definition of covering homotopy property with the homotopy relation so that we can first get an expression of the concept of covering homotopy property in the homotopy category. After introducing $\mathcal{H}$-fibrations, we will have a homotopy expression of some concepts related to fibration, such as path lifting, lifting function and unique path lifting property, to generalize some results in fibration. In particular, we show that an $\mathcal{H}$-fibration has homotopical path lifting property and also prove that a map is an $\mathcal{H}$-fibration if and only if it has a homotopical lifting function.

1 Introduction

1.1 Motivation Fibrations are maps with the covering homotopy property. A map $p : E \to B$ has the covering homotopy property if for every space X, every map $\widetilde{f} : X \to E$ and every homotopy $F : X \times I \to B$ with

* Corresponding author
Keywords: Fiber homotopy, h-fibration, $\mathcal{H}$-fibration, homotopical path lifting.
Mathematics Subject Classification [2020]: 57M10, 57M12, 54D05, 55Q05.
Received: 13 March 2024, Accepted: 1 July 2024.
ISSN: Print 2345-5853, Online 2345-5861.
© Shahid Beheshti University

$p\circ \widetilde{f} = F\circ J_0$, there exists a homtopy $\widetilde{F} : X\times I \to E$ such that $p\circ \widetilde{F} = F$ and $\widetilde{F}\circ J_0 = \widetilde{f}$, where $J_0 : X \to X\times I$ is $J_0(x) = (x, 0)$.

Putting this definition in diagrammatic form, we have that p has the covering homotopy property if and only if for every commutative square

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle J_o}\downarrow & \overset{\widetilde{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ X\times I & \xrightarrow{F} & B, \end{array}$$

there exists a map $\widetilde{F}$, as indicated by the dashed arrow, that makes the two triangles commute in the $\mathcal{TOP}$ category: Category of topological spaces and continuous maps.

A homotopical version of the covering homotopy property, that is, the weak covering homotopy property introduced by K. Fuchs [2]. A map $p : E \to B$ has the weak covering homotopy property if in the definition of the covering homotopy property, $\widetilde{F}\circ J_0 = \widetilde{f}$ is replaced by the fiber homotopy $\widetilde{F}\circ J_0 \simeq_p \widetilde{f}$. We recall that two maps $f_0, f_1 : X \to E$ are said to be fiber homotopic with respect to p, denoted by $f_0 \simeq_p f_1$ if there is a homotopy $F : f_0 \simeq f_1$ such that $p\circ F = p\circ f_0 = p\circ f_1$ [7].

In diagrammatic form, we have that p has the weak covering homotopy property if and only if for every commutative square (in $\mathcal{TOP}$ category) there exists a map $\widetilde{F}$, as indicated by the dashed arrow, that makes the lower triangle commutes in the $\mathcal{TOP}$ category and the upper triangle commutes to fiber homotopy, i.e. commutes in the homotopy category $\mathcal{HTOP}$: The category that has topological spaces for objects and homotopy equivalence classes of continuous maps for morphisms. In fact $\mathcal{HTOP}$ is the quotient category of $\mathcal{TOP}$ by the homotopy relation on morphisms.

A. Dold, et.al [1, 2, 4], studied the maps which have the weak covering homotopy property, called h-fibrations (or Dold fibrations). They proved that the weak covering homotopy property is invariant under the fiber homotopy equivalence, the fibers of an h-fibration have the same homotopy type, for every h-fibration there exists the long exact sequence, and etc (see [1, 2, 4]).

We introduce $\mathcal{H}$-fibrations as another homotopical generalization of fibrations. An $\mathcal{H}$-fibration is a map which has the homotopical covering homotopy property: **in the definition of the covering homotopy property, $p \circ \widetilde{F} = F$ is replaced by $p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$.**

In diagrammatic form, the dashed arrow makes the upper triangle commute in the $\mathcal{TOP}$ category and the lower triangle commute in the $\mathcal{HTOP}$. This approach led us to use the name $\mathcal{H}$-fibration.

It is well known that fibrations by the unique path lifting property have interesting results (see [7]). A map $p : E \to B$ has the unique path lifting property if for given paths $\widetilde{\alpha}$ and $\widetilde{\beta}$ in E with the same initial point, $p \circ \widetilde{\alpha} = p \circ \widetilde{\beta}$ implies that $\widetilde{\alpha} = \widetilde{\beta}$. In [6] it is shown that an h-fibration with the unique path lifting property is a fibration and therefore do not give a new result. Here, we consider unique path homotopical lifting property, that is a homotopical analogue of unique path lifting property, but stronger of it.

A map $p : E \to B$ has the unique path homotopical lifting property if for given paths $\widetilde{\alpha}$ and $\widetilde{\beta}$ in E with the same initial point, $p \circ \widetilde{\alpha} \simeq p \circ \widetilde{\beta}$ rel $\dot{I}$ implies that $\widetilde{\alpha} \simeq \widetilde{\beta}$ rel $\dot{I}$ (see [5]). We show that an $\mathcal{H}$-fibration even with unique path homotopical lifting necessarily is not a fibration, while some important results of fibrations with unique path lifting property are satisfied for $\mathcal{H}$-fibrations with unique path homotopical lifting property.

As main results, we show that an $\mathcal{H}$-fibration has homotopical path lifting property and also prove that a map is an $\mathcal{H}$-fibration if and only if it has a homotopical lifting function.

1.2 Preliminaries Throughout this article, all spaces are path connected, unless otherwise stated. A *map* $f : X \longrightarrow Y$ means a continuous function and $I := [0, 1]$. The map $\alpha : I \longrightarrow X$ is called a path from $x_0 = \alpha(0)$ to $x_1 = \alpha(1)$.

For given maps $p : E \to B$ and $f : X \to B$, a map $\widetilde{f} : X \to E$ is called a lifting of f if $p \circ \widetilde{f} = f$, and p has unique lifting property, if every two lifts $\widetilde{f}, \overline{f}$ of f with the same image on some points of X, are equal. When $F : X \times I \longrightarrow Y$ is a map, we say that F is a homotopy from F_0 to F_1 and write $F : F_0 \simeq F_1$, where $F_i : X \longrightarrow Y$ is $F_i(x) = F(x, i)$, for $i = 0, 1$. The constant map from X to Y which sends all points to $y \in Y$ is denoted by C_y.

2 $\mathcal{H}$-fibrations

In this section we introduce $\mathcal{H}$-fibrations, the maps which have homotopical covering homotopy property.

Definition 2.1. A map $p : E \to B$ is said to have homotopical covering homotopy property, if for every space X, every map $\widetilde{f} : X \to E$ and every homotopy $F : X \times I \to B$ with $p \circ \widetilde{f} = F \circ J_0$, there exists a homotopy $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$, where $\dot{I} = \{0, 1\}$.
A map $p : E \to B$ is said to be an $\mathcal{H}$-fibration if it has homotopical covering homotopy property.

Clearly every fibration is an $\mathcal{H}$-fibration. The following example shows that an $\mathcal{H}$-fibration necessarily is not a fibration.

Example 2.2.
(i) Let $E = I \times I - \{(0, \frac{1}{2})\}$, $B = I$ and p be the projection on the first component. Moreover, let $F : X \times I \to B$, $\widetilde{f} : X \to E$ be maps with $p \circ \widetilde{f} = F \circ J_0$. Let $e = (1, \frac{1}{2})$, and define a homotopy $\widetilde{F} : X \times I \to E$ by

$$\widetilde{F}(x,t) = \begin{cases} \widetilde{f}(x) + 2(e - \widetilde{f}(x))t & t \in [0, \frac{1}{2}], \\ (F(x,t),0) + 2(e - (F(x,t),0))(1-t) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Define $H : X \times I \times I \to B$ by $H(x,t,s) = (1-s)p \circ \widetilde{F}(x,t) + sF(x,t)$. Then $H : p \circ \widetilde{F} \simeq F$ rel $X \times \dot{I}$ because

$$\begin{cases} H(x,t,0) = p \circ \widetilde{F}(x,t), \\ H(x,t,1) = F(x,t), \\ H(x,i,s) = (1-s)p \circ \widetilde{F}(x,i) + sF(x,i) = p \circ \widetilde{F}(x,i) = F(x,i), \text{ for } i = 0, 1. \end{cases}$$

Also, $\widetilde{F} \circ J_0 = \widetilde{f}$. Note that $\widetilde{F}(x,t)$ is a line (consisting of two line segments) from $\widetilde{f}(x)$ to $(F(x,t),0)$ which passes through the point e at $t = \frac{1}{2}$. Hence for every $x \in X$, $p \circ \widetilde{F}(x, \frac{1}{2}) = p(e) = 1$ while $F(x, \frac{1}{2})$ is not necessarily 1. This shows that $p \circ \widetilde{F} \neq F$.

(ii) Let $E = \{(t,0) | t \in I\} \cup \{(t,t) | t \in I - \{1\}\}$, $B = I$ and $p : E \to B$ be the projection on the first component. Let $\widetilde{f} : X \to E$ and $F : X \times I \to B$

be maps with $p \circ \widetilde{f} = F \circ J_0$. Define,

$$\widetilde{F}(x,t) = \begin{cases} (1-2t)\widetilde{f}(x) & t \in [0, \frac{1}{2}], \\ \big((2t-1)F(x,t), 0\big) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Therefore, by the gluing lemma $\widetilde{F}$ is continuous. Define $H : X \times I \times I \to B$ by $H(x,t,s) = (1-s)p \circ \widetilde{F}(x,t) + sF(x,t)$. Then for every $x \in X$ and every $t, s \in I$ we have

$$\begin{aligned} &H(x,t,0) = p \circ \widetilde{F}(x,t), \\ &H(x,t,1) = F(x,t), \\ &H(x,0,s) = (1-s)p \circ \widetilde{F}(x,0) + sF(x,0) = p \circ \widetilde{F}(x,0) = F(x,0), \\ &H(x,1,s) = (1-s)p \circ \widetilde{F}(x,1) + sF(x,1) = p \circ \widetilde{F}(x,1) = F(x,1). \end{aligned}$$

Moreover, $\widetilde{F} \circ J_0 = \widetilde{f}$. Note that similar to the part (i), $\widetilde{F}(x,t)$ is a line (consisting of two line segments) from $\widetilde{f}(x)$ to $(F(x,t), 0)$ which passes through the point $(0,0)$ at $t = \frac{1}{2}$. Hence for every $x \in X$, $p \circ \widetilde{F}(x, \frac{1}{2}) = 0$ while $F(x, \frac{1}{2})$ is not necessarily 0. This shows that $p \circ \widetilde{F} \neq F$.

A map $p : E \to B$ has path lifting property if for a given $b \in B$, $e \in p^{-1}(b)$ and a path α in B beginning at b, there exists a path $\widetilde{\alpha}$ in E such that $\widetilde{\alpha}(0) = e$ and $p \circ \widetilde{\alpha} = \alpha$ (see [7]). Also by replacing $p \circ \widetilde{\alpha} = \alpha$ by $p \circ \widetilde{\alpha} \simeq \alpha$, rel $\dot{I}$, it is said that p has homotopical path lifting property and $\widetilde{\alpha}$ is a homotopical lifting of α (see [6]). We know that fibrations and h-fibrations have path lifting property and homotopical path lifting property (see [6, 7]).

Proposition 2.3. *If $p : E \to B$ is an $\mathcal{H}$-fibration, then p has homotopical path lifting property.*

Proof. If α is a path in B and $e \in p^{-1}(\alpha(0))$, we show that α has a homotopical lifting at e. Let $F : \{*\} \times I \to B$ be the homotopy defined by $F(*, t) = \alpha(t)$ and $\widetilde{f} : \{*\} \to E$ be the map $\widetilde{f}(*) = e$. Then $p \circ \widetilde{f} = F \circ J_0$ and since p is an $\mathcal{H}$-fibration, there exist two homotopies $\widetilde{F} : \{*\} \times I \to E$ and $H : \{*\} \times I \times I \to B$ such that $H : p \circ \widetilde{F} \simeq F$ rel $\{*\} \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$. Let $\widetilde{\alpha}(t) = \widetilde{F}(*, t)$ and define $\overline{H} : I \times I \to B$ by $\overline{H}(s,t) = H(*, s, t)$. Therefore we have $\widetilde{\alpha}(0) = \widetilde{F}(*, 0) = \widetilde{F} \circ J_0(*) = \widetilde{f}(*) = e$ and $\overline{H} : p \circ \widetilde{\alpha} \simeq \alpha$ rel $\dot{I}$, because for every $s, t \in I$ we have

$$\begin{aligned}
\overline{H}(s,0) &= H(*,s,0) = p \circ \widetilde{F}(*,s) = p \circ \widetilde{\alpha}(s), \\
\overline{H}(s,1) &= H(*,s,1) = F(*,s) = \alpha(s), \\
\overline{H}(0,t) &= H(*,0,t) = p \circ \widetilde{F}(*,0) = p \circ \widetilde{\alpha}(0) = F(*,0) = \alpha(0), \\
\overline{H}(1,t) &= H(*,1,t) = p \circ \widetilde{F}(*,1) = p \circ \widetilde{\alpha}(1) = F(*,1) = \alpha(1)
\end{aligned}$$

□

If $p : E \to B$ is an $\mathcal{H}$-fibration and $A \subseteq B$, then $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is not necessarily an $\mathcal{H}$-fibration. Let p be the $\mathcal{H}$-fibration in Example 2.2, (ii), and $A = \{(t,0) | t \in [\frac{1}{2}, 1]\}$. Since $p|_{p^{-1}(A)}$ has not the homotopical path lifting property, by Proposition 2.3, $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is not an $\mathcal{H}$-fibration. This shortcoming is fixed when A is a path component.

Proposition 2.4. *Let $p : E \to B$ be a map.*
(i) If p is an $\mathcal{H}$-fibration, then for every path component A of E, $p(A)$ is a path component of B and $p|_A : A \to p(A)$ is an $\mathcal{H}$-fibration.
(ii) If E is locally path connected and for every path component A of E, $p|_A : A \to p(A)$ is an $\mathcal{H}$-fibration, then p is an $\mathcal{H}$-fibration.

Proof. (i) $p(A)$ be path connected. Let α be a path in B such that $\alpha(0) \in p(A)$. If e is a point of A such that $p(e) = \alpha(0)$, then by Proposition 2.3, there is a homotopical lifting $\widetilde{\alpha}$ of α beginning at e. Since A is a path component and $e \in A$, $\widetilde{\alpha}$ belongs to A, so $p(\widetilde{\alpha})$ is a path in $p(A)$ and $p \circ \widetilde{\alpha} \simeq \alpha$, rel $\dot{I}$. Then α is a path in $p(A)$ which implies that $p(A)$ is path component.
Now, let $\widetilde{f} : X \to A$ and $F : X \times I \to p(A)$ be two maps such that $p \circ \widetilde{f} = F \circ J_0$. Since p is an $\mathcal{H}$-fibration, there exists a homotopy $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$. For every $x \in X$, $\widetilde{F}(x,-)$ and $F(x,-)$ are paths beginning at $\widetilde{F} \circ J_0(x)$ and $F(x,0) = p \circ \widetilde{f}(x)$, respectively. Since, $\widetilde{F} \circ J_0(x) = \widetilde{f}(x) \in A$ and $F \circ J_0 = p \circ \widetilde{f} \in p(A)$, we have $\widetilde{F}(x,t) \in A$ and $F(x,t) \in p(A)$.
(ii) Let $\widetilde{f} : Y \to E$ and $F : Y \times I \to B$ be maps such that $F \circ J_0 = p \circ \widetilde{f}$. If $\{A_j | j \in J\}$ are the path components of E, then $\{A_j | j \in J\}$ are disjoint open subsets of E. Let $V_j = \widetilde{f}^{-1}(A_j)$, $\widetilde{f}_j = \widetilde{f}|V_j : V_j \to A_j$ and $F_j = F|V_j \times I$. For every $y \in V_j$, $F(\{y\} \times I)$ is contained in the path component of B containing $F \circ J_0(y) = p \circ \widetilde{f}(y)$. Also, since $p \circ \widetilde{f}(y) = p \circ \widetilde{f}_j(y) \in p \circ \widetilde{f}_j(V_j) = p(A_j)$

and $p(A_j)$ is a path component of B, $ImF_j = F(V_j \times I) \subseteq p(A_j)$ for all j. Because $p|A_j : A_j \to p(A_j)$ is an $\mathcal{H}$-fibration, there exists a homotopy $\widetilde{F}_j : V_j \times I \to A_j$ with homotopy $H_j : p \circ \widetilde{F}_j \simeq F_j$, rel $\{V_j\} \times \dot{I}$ and $\widetilde{F}_j \circ J_0 = \widetilde{f}_j$.
Define $\widetilde{F} : Y \times I \to E$ and $H : Y \times I \to B$ by $\widetilde{F}|_{V_j \times I} = \widetilde{F}_j$ and $H|_{V_j \times I} = H_j$. Since $\{V_j | j \in J\}$ is an open cover of Y and V_j's are disjoint, $\widetilde{F}$ and H are well-defined. We claim that $H : p \circ \widetilde{F} \simeq F$, rel $Y \times \dot{I}$. For every $y \in Y$, there exists a unique $j \in J$ such that $y \in V_j$. Then for every $s, t \in I$ we have

$$\begin{aligned}
H(y,s,0) &= H_j(y,s,0) = p \circ \widetilde{F}_j(y,s) = p \circ \widetilde{F}(y,s),\\
H(y,s,1) &= H_j(y,s,1) = F_j(y,s) = F(y,s),\\
H(y,0,t) &= H_j(y,0,t) = p \circ \widetilde{F}_j(y,0) = F_j(y,0) = p \circ \widetilde{F}(y,0) = F(y,0),\\
H(y,1,t) &= H_j(y,1,t) = p \circ \widetilde{F}_j(y,1) = F_j(y,1) = p \circ \widetilde{F}(y,1) = F(y,1).
\end{aligned}$$

Moreover, $\widetilde{F} \circ J_0 = \widetilde{f}$ and so p is an $\mathcal{H}$-fibration. □

Theorem 2.5. *Composition of two $\mathcal{H}$-fibrations is an $\mathcal{H}$-fibration.*

Proof. Let $p : E \longrightarrow E'$ and $p' : E' \longrightarrow B$ be two $\mathcal{H}$-fibrations, $\widetilde{f} : X \longrightarrow E$ and $F : X \times I \longrightarrow B$ be two maps such that $(p' \circ p) \circ \widetilde{f} = F \circ J_0$. Since p' is an $\mathcal{H}$-fibration and $p' \circ (p \circ \widetilde{f}) = F \circ J_0$, there exists a homotopy $G : X \times I \longrightarrow E'$ such that $G \circ J_0 = p \circ \widetilde{f}$ and $p' \circ G \simeq F$, rel $X \times \dot{I}$. Since p is an $\mathcal{H}$-fibration, there exists $\widetilde{F} : X \times I \longrightarrow E$ such that $\widetilde{F} \circ J_0 = \widetilde{f}$ and $p \circ \widetilde{F} \simeq G$, rel $X \times \dot{I}$. Thus $p' \circ (p \circ \widetilde{F}) \simeq p' \circ G$, rel $X \times \dot{I}$ which implies that $(p' \circ p) \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$ because $p' \circ G \simeq F$, rel $X \times \dot{I}$. □

Definition 2.6. [6, Definition 2.21]
Let $p : E \to B$ be a map and $\overline{B} = \{(e, \omega) \in E \times B^I | \omega(0) = p(e)\}$. A homotopical lifting function for p is a continuous function $\lambda : \overline{B} \to E^I$ which assigns to each point $e \in E$ and path ω in B starting at $p(e)$, a path $\lambda(e, \omega)$ in E starting at e such that it is a homotopical lifting of ω.

Note that the topology on B^I is the compact-open topology and the topology on $\overline{B}$ is the subspace topology induced by the product topology on $E \times B^I$.

Theorem 2.7. *A map $p : E \to B$ is an $\mathcal{H}$-fibration if and only if it has a homotopical lifting function.*

Proof. Assume that p is an $\mathcal{H}$-fibration. Define two maps $\widetilde{f} : \overline{B} \to E$ and $F : \overline{B} \times I \to B$ by $\widetilde{f}(e, \omega) = e$ and $F((e, \omega), t) = \omega(t)$ which are continuous. Since $F((e, \omega), 0) = \omega(0) = p(e) = p \circ \widetilde{f}(e, \omega)$ and p is an $\mathcal{H}$-fibration, there exists a homotopy $\widetilde{F} : \overline{B} \times I \to E$ such that $p \circ \widetilde{F} \simeq F$, rel $\overline{B} \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$. Define $\lambda : \overline{B} \to E^I$ by $\lambda(e, \omega)(t) = \widetilde{F}((e, \omega), t)$. Then,

$$\lambda(e, \omega)(0) = \widetilde{F}((e, \omega), 0) = \widetilde{F} \circ J_0(e, \omega) = \widetilde{f}(e, \omega) = e$$

$$p \circ \lambda(e, \omega) = p \circ \widetilde{F}((e, \omega), -) \simeq F((e, \omega), -) = \omega, \ \mathit{rel} \ \dot{I}.$$

Continuity of λ comes from the continuity of $\widetilde{F}$. Therefore λ is a homotopical lifting function for p.

For the converse, let $\lambda : \overline{B} \to E^I$ be a homotopical lifting function for p. Also, let $\widetilde{f} : X \to E$ and $F : X \times I \to B$ be the maps such that $p \circ \widetilde{f} = F \circ J_0$. Define $g : X \to B^I$ by $g(x)(t) = F(x, t)$. By assumption for the path $g(x) : I \to B$ starting at $p \circ \widetilde{f}(x)$, there exists a homotopical lifting $\lambda_x := \lambda(\widetilde{f}(x), g(x)) : I \to E$ of $g(x)$. Then there exists a homotopy $H : I \times I \to B$ such that $H : p \circ \lambda_x \simeq g(x)$, rel $\dot{I}$. Therefore we can define a map $\widetilde{F} : X \times I \to E$ by $\widetilde{F}(x, t) = \lambda_x(t)$ which is continuous because λ is continuous. Hence $\widetilde{F}(x, 0) = \lambda_x(0) = \widetilde{f}(x)$. Moreover, by defining $H' : X \times I \times I \to B$ by $H'(x, t, s) = H(t, s)$. We have $H' : p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$ since for every $x \in X$ and every $s, t \in I$ we have

$$\begin{aligned}
H'(x, t, 0) &= H(t, 0) = p \circ \lambda_x(t) = p \circ \widetilde{F}(x, t), \\
H'(x, t, 1) &= H(t, 1) = g(x)(t) = F(x, t), \\
H'(x, 0, s) &= H(0, s) = p \circ \lambda_x(0) = p \circ \widetilde{F}(x, 0) = p \circ \widetilde{f}(x) = F(x, 0), \\
H'(x, 1, s) &= H(1, s) = p \circ \lambda_x(1) = p \circ \widetilde{F}(x, 1) = g(x)(1) = F(x, 1).
\end{aligned}$$

Thus, p is an $\mathcal{H}$-fibration. □

Remark 2.8. Every fibration $p : E \to B$ with contractible base space B is fibre-homotopy equivalent to the trivial fibration $pr : B \times F \to B$ (see [3, Proposition 2.1]). But this is not true for $\mathcal{H}$-fibrations. For instance, Example 2.2 is an $\mathcal{H}$-fibration with contractible base $B := I$ that is not fibre-homotopy equivalent to the trivial fibration. Because, the fibers of $pr : I \times F \to I$ have the same homotopy while in Example 2.2 this is not true. In fact, in part (i), $p^{-1}(0)$ has two path components but the other fibers are path connected and in part (ii), $p^{-1}(0)$ and $p^{-1}(1)$ are singletons but the other fibers have two points.

For a map $f : X \to E$, by $\check{f} : X \times I \to E$ we mean $\check{f}(x,t) = f \circ pr_1(x,t)$, for every $x \in X$ and any $t \in I$.

Definition 2.9. Let $p : E \to B$ be a map. The maps $f_0, f_1 : X \to E$ are said to be homotopically fiber homotopic, with respect to p denoted by $f_0 \simeq_{hp} f_1$, if there is a homotopy $F : f_0 \simeq f_1$ such that $p \circ F \simeq p \circ \check{f}_0 = p \circ \check{f}_1$, rel $X \times \dot{I}$.

Note that the above definition guarantees $p \circ f_0 = p \circ f_1$. Also, by definitions of fiber homotopy and Definition 2.9, if two maps are fiber homotopic, they are also homotopically fiber homotopic. Moreover,

Proposition 2.10. *Let $p : E \to B$ be a map. The homotopically fiber homotopy with respect to p is an equivalence relation on the set of maps from X to E.*

Proof. Clearly, it is reflexive and symmetric. Now, consider the maps $f_0, f_1, f_2 : X \to E$ with $f_0 \simeq_{hp} f_1$ and $f_1 \simeq_{hp} f_2$. By definition, there exist homotopies $F : f_0 \simeq f_1$, $G : f_1 \simeq f_2$, $T : p \circ F \simeq p \circ \check{f}_0 = p \circ \check{f}_1$ rel $X \times \dot{I}$ and $K : p \circ G \simeq p \circ \check{f}_1 = p \circ \check{f}_2$ rel $X \times \dot{I}$. Define, $H : X \times I \to E$ and $H' : X \times I \times I \to B$ by

$$H(x,s,t) = \begin{cases} F(x,2s) & s \in [0,\frac{1}{2}], \\ G(x,2s-1) & s \in [\frac{1}{2},1], \end{cases}$$

$$H'(x,s,t) = \begin{cases} T(x,2s,t) & s \in [0,\frac{1}{2}], \\ K(x,2s-1,t) & s \in [\frac{1}{2},1]. \end{cases}$$

Then $H : f_0 \simeq f_2$. T and K are continuous and

$$T(x,1,t) = p \circ \check{f}_1(x,1) = p \circ f_1(x) = p \circ \check{f}_1(x,0) = K(x,0,t).$$

Hence by gluing lemma H' is continuous. Also, $H' : p \circ H \simeq p \circ \check{f}_0 = p \circ \check{f}_2$ rel $X \times \dot{I}$ because

$$H'(x,s,0) = \begin{cases} T(x,2s,0) = p \circ F(x,2s) & s \in [0,\frac{1}{2}], \\ K(x,2s-1,0) = p \circ G(x,2s-1) & s \in [\frac{1}{2},1], \end{cases}$$

which is $p \circ H$ and

$$H'(x,s,1) = \begin{cases} T(x,2s,1) = p \circ \check{f}_1(x,2s) & s \in [0,\frac{1}{2}], \\ K(x,2s-1,1) = p \circ \check{f}_1(x,2s-1) & s \in [\frac{1}{2},1], \end{cases}$$

which is $p \circ \check{f}_1 = p \circ \check{f}_0 = p \circ \check{f}_2$. Since

$$H'(x,0,t) = T(x,0,t) = p \circ F(x,0) = p \circ \check{f}_0(x,0) = p \circ \check{f}_1(x,0),$$

we have

$$H'(x,0,t) = p \circ H(x,0) = p \circ \check{f}_0(x,0) = p \circ \check{f}_2(x,0)$$

and since

$$H'(x,1,t) = K(x,1,t) = p \circ G(x,1) = p \circ \check{f}_1(x,1) = p \circ \check{f}_2(x,1),$$

we have

$$H'(x,1,t) = p \circ H(x,1) = p \circ \check{f}_1(x,1) = p \circ \check{f}_2(x,1).$$

□

Now, we have the following proposition which is a homotopical version of [7, Theorem 2.8.10]. We recall that for a homotopy $F : X \times I \to E$, by F_0 and F_1 we mean $F|X \times \{0\}$ and $F|X \times \{1\}$, respectively.

Proposition 2.11. *Let $p : E \to B$ be an $\mathcal{H}$-fibration and $F, F' : X \times I \to E$ be homotopies. Given homotopies $H : p \circ F \simeq p \circ F'$ and $G : F_0 \simeq F'_0$ with $H|X \times \{0\} \times I = p \circ G$, there exists an extension $\widetilde{H}$ of G from F to F' with $p \circ \widetilde{H} \simeq H$, rel $X \times I \times \dot{I}$.*

Proof. Let $A = (I \times \{0\}) \cup (\{0\} \times I) \cup (I \times \{1\}) \subseteq I \times I$. We know that there exists a homeomorphism $\delta : I \times I \to I \times I$ such that $\delta(A) = I \times \{0\}$ (see [7, Theorem 2.8.10]). So we have the homeomorphism $\theta : X \times I \times I \to X \times I \times I$ defined by $\theta(x, r, s) = (x, \delta(r, s))$. Define $f : X \times A \to E$ by $f|X \times I \times \{0\} = F$, $f|X \times \{0\} \times I = G|X \times \{0\} \times I$, $f|X \times I \times \{1\} = F'$. Since $H|X \times A = p \circ f$,

$$H \circ \theta^{-1}|X \times I \times \{0\} = p \circ f \circ \theta^{-1}|X \times I \times \{0\}.$$

Since p is an $\mathcal{H}$-fibration, there exists $\widetilde{G} : X \times I \times I \to E$ with the homotopy $T : p \circ \widetilde{G} \simeq H \circ \theta^{-1}$ rel $X \times I \times \dot{I}$ and $\widetilde{G} \circ J_0 = f \circ \theta^{-1}|X \times I \times \{0\}$. Define $\widetilde{H} : X \times I \times I \to E$ by $\widetilde{H} = \widetilde{G} \circ \theta$.

Let $T' : X \times I \times I \times I \to B$ be the map $T'(x, r, s, t) = T(\theta(x, r, s), t)$. Therefore, $T' : p \circ \widetilde{H} \simeq H$ rel $X \times I \times \dot{I}$ because

$$\begin{aligned} T'(x, r, s, 0) &= T(\theta(x, r, s), 0) = p \circ \widetilde{G}(\theta(x, r, s)) \\ &= p \circ \widetilde{G} \circ \theta(x, r, s) = p \circ \widetilde{H}(x, r, s), \\ T'(x, r, s, 1) &= T(\theta(x, r, s), 1) = H \circ \theta^{-1}(\theta(x, r, s)) = H(x, r, s). \end{aligned}$$

For $(x, r, 0) \in X \times A$, By definition of δ and θ, there exists $r' \in I$ such that $\theta(x, r, 0) = (x, r', 0)$. Since $T : p \circ \widetilde{G} \simeq H \circ \theta^{-1}$ rel $X \times I \times \dot{I}$, we have

$$T'(x, r, 0, t) = T(x, r', 0, t) = p \circ \widetilde{G}(x, r', 0) = p \circ \widetilde{G} \circ \theta(x, r, 0) = p \circ \widetilde{H}(x, r, 0),$$

$$T'(x, r, 0, t) = T(x, r', 0, t) = H \circ \theta^{-1}(x, r', 0) = H(x, r, 0).$$

Also, for $(x, r, 1) \in X \times A$, there exists $r'' \in I$ such that $\theta(x, r, 1) = (x, r'', 0)$, then

$$T'(x, r, 1, t) = T(x, r'', 0, t) = p \circ \widetilde{G}(x, r'', 0) = p \circ \widetilde{G} \circ \theta(x, r, 1) = p \circ \widetilde{H}(x, r, 1),$$

$$T'(x, r, 1, t)) = T(x, r'', 0, t) = H \circ \theta^{-1}(x, r'', 0) = H(x, r, 1).$$

Moreover, $\widetilde{H} : F \simeq F'$ and $\widetilde{H}$ is a extension of G because

$$\begin{aligned} \widetilde{H}(x, r, 0) &= \widetilde{G} \circ \theta(x, r, 0) = f \circ \theta^{-1} \circ \theta(x, r, 0) = f(x, r, 0) = F(x, r), \\ \widetilde{H}(x, r, 1) &= \widetilde{G} \circ \theta(x, r, 1) = f \circ \theta^{-1} \circ \theta(x, r, 1) = f(x, r, 1) = F'(x, r), \\ \widetilde{H}(x, 0, s) &= \widetilde{G} \circ \theta(x, 0, s) = f \circ \theta^{-1} \circ \theta(x, 0, s) = f(x, 0, s) = G(x, 0, s). \end{aligned}$$

□

By using this proposition, we can show that liftings of a homotopy with the same starting map by an $\mathcal{H}$-fibration is uniqe up to homotopically fiber homotopy.

Theorem 2.12. *Let $p : E \to B$ be an $\mathcal{H}$-fibration and $F, F' : X \times I \to E$ be two homotopies in which $F_0 = F'_0$ and $p \circ F = p \circ F'$. Then $F \simeq_{hp} F'$.*

Proof. Let $H : X \times I \times I \to B$ and $G : X \times \{0\} \times I \to E$ be the corresponding constant homotopies $H : p \circ F = p \circ F'$ and $G : F_0 = F'_0$. Since, $H(x, 0, s) = p \circ F(x, 0) = p \circ G(x, 0, s)$, by Proposition 2.11 there exists a homotopy $\widetilde{H}$ from F to F' such that $p \circ \widetilde{H} \simeq H$ rel $X \times I \times \dot{I}$. By the definitions, $H = p \circ \check{F} = p \circ \check{F}'$. Then, $p \circ \widetilde{H} \simeq p \circ \check{F} = p \circ \check{F}'$ rel $X \times I \times \dot{I}$ and hence the result holds. □

Acknowledgement

The authors would like to thank the referee for his/her careful reading and useful suggestions.

References

[1] Dold, A., *Partitions of unity in the theory of fibrations*, Annal. Math. 78 (1963), 223-255.

[2] Fuchs, K., "Homotopiefaserungen", Dissertation, Universität des Saarbrücken.

[3] Pavešić, P., *A note on trivial fibrations*, Glas. Mat. 46 (2011), 513-519.

[4] Pavešić, P. and Piccinini, R.A., "Fibrations and their Classification", Heldermann Verlag, 2013.

[5] Tajik, M., Mashayekhy, B., and Pakdaman, A., *Unique path lifting from homotopy point of view*, Acta Math. Vietnam. 43(2) (2018), 257-273.

[6] Tajik, M., Mashayekhy, B., and Pakdaman, A., *h-fibrations*, Hacet. J. Math. Stat. 48(3) (2019), 732-742.

[7] Spanier, E.H., "Algebraic Topology", McGraw-Hill, 1966.

Ali Pakdaman *Department of Mathematics, Faculty of Science, Golestan university, P.O.Box*

155, Gorgan, Iran.
Email: a.pakdaman@gu.ac.ir

Saba Dehrooye *Department of Mathematics, Faculty of Science, Golestan university, P.O.Box 155, Gorgan, Iran.*
Email: sabadehrooye@gmail.com

Mehdi Tajik *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran.*
Email: azi−1392@yahoo.com

Behrooz Mashayekhy *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran.*
Email: bmashf@um.ac.ir