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On homological classification of monoids by Condition $(\mathrm{P_{sc}})$ and new classification on Condition $(\mathrm{P_E})$

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Abstract. In 1997, Golchin and Renshaw introduced Condition (P_E) and showed that this condition implies weak flatness, although the converse is not generally valid. In this paper, we present Condition (P_{sc}) as a generalization of Condition (P_E). We also see that Condition (P_{sc}) implies weak flatness, but the converse is not necessarily true. However, for left PSF monoids the converse is holds. Moreover, we discuss some general properties and provide a homological classification of monoids by comparing Condition (P_{sc}) with some other properties. Furthermore, a new homological classification of monoids is presented by comparing Condition (P_E) with other properties.

1 Introduction and Preliminaries

In this paper, we refer to a monoid as S, with 1 representing its identity element. A non-empty set A is called a *right S-act*, usually denoted by A_S (or simply A), when S acts on A unitarily from the right. This means there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions

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$a1 = a$ and $(as)t = a(st)$ for all $a \in A$ and all $s, t \in S$. Left S-act can be defined dually. Hereafter, when we mention S-act, we are referring to a right S-act. For further details and definitions related to semigroups and acts over monoids, we recommend referring to [8, 11].

As per the definition provided in [11], an S-act A is called *flat* if the functor $A \otimes_S -$ preserves all monomorphisms of left S-acts. If the functor $A \otimes_S -$ preserves all embeddings of (principal) left ideals into S, then A is called (*principally*) *weakly flat*. An S-act A satisfies *Condition* (E) if for all $a \in A$ and $s, s' \in S$, $as = as'$ implies the existence of $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = us'$. It satisfies *Condition* (P) if for all $a, a' \in A$ and $s, s' \in S$, $as = a's'$ implies the existence of $a'' \in A$ and $u' \in S$ such that $a = a''u, a' = a''u'$, and $us = u's'$. An S-act A is considered *strongly flat* if it satisfies both Conditions (P) and (E).

Moreover, an S-act A satisfies Condition (P') if for all $a, a' \in A$ and $s, t, z \in S$, $as = a't$ and $sz = tz$ imply the existence of $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$, and $us = vt$. It is obvious that Condition (P) implies Condition (P'), but the converse is not necessarily true. For more comprehensive information, refer to [6].

Recall from [12], [7], [13], [5], and [15] that an S-act A satisfies Condition (WP) if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$ and all $a, a' \in A$, when $af(s) = a'f(t)$, then there exist $a'' \in A$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes_S (Ss \cup St)$, and $f(us') = f(vt')$. Furthermore, it satisfies *Condition* (P_E) if for all $a, a' \in A$, and $s, s' \in S$, $as = a's'$ implies the existence of $a'' \in A$, $u, u' \in S$ and $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''u'f, es = s, fs' = s'$ and $us = u's'$. An S-act A satisfies *Condition* (PWP) if for all $a, a' \in A$ and $s \in S$, $as = a's$ implies the existence of $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. Moreover, an S-act A satisfies Condition (PWP_E) if $as = a's$ with $a, a' \in A$ and $s \in S$, implies the existence of $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf, es = s = fs$ and $us = vs$, and it satisfies Condition (PWP_{sc}) if $as = a's$, with $a, a' \in A$ and $s \in S$, implies the existence of $a'' \in A$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$ and $us = vs$.

An S-act A is called *torsion free* if for any $a, b \in A$ and for any right cancellable element $u \in S$, the equality $au = bu$ implies $a = b$. It is also called $\mathfrak{R}$-*torsion free*, if for any $a, b \in A$ and any right cancellable element

$c \in S$, $ac = bc$ and $a\mathfrak{R}b$ ($\mathfrak{R}$ denotes Green's equivalence, as described in [8]), imply that $a = b$. It is evident that torsion freeness implies $\mathfrak{R}$-torsion freeness, however, the converse is not generally true.

An element s of S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$ (where ρ_x represents the right translation on S, for every $x \in S$, defined as $\rho_x : S \longrightarrow S, t \mapsto tx$, for every $t \in S$). A monoid S is called *left PP* if every principal left ideal of S is projective as a left S-act. This is equivalent to saying that every element $s \in S$ is right e-cancellable for an idempotent $e \in S$. Furthermore, S is called *left PSF* if every principal left ideal of S is strongly flat as a left S-act. Equivalently, this implies that S is right semi-cancellative, meaning that whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$ (refer to [1, 14]).

2 General properties

In this section, we present Condition (P_{sc}) and show that this condition for acts can be transferred to their coproduct and vice versa. Additionally, we show that a retract or coproduct of any act satisfying Condition (P_{sc}) also satisfies Condition (P_{sc}). Furthermore, we observe that Condition (P_{sc}) implies weak flatness, although the converse is not necessarily valid. For left PSF monoids, we establish that the converse holds true as well.

Definition 2.1. An S-act A satisfies *Condition* (P_{sc}) if $as = a't$ for $a, a' \in A$ and $s, t \in S$, implies the existence of $a'' \in A$ and $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$.

As a reminder from [11], S is called *right reversible* if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$. In the following proposition, all statements are straightforward consequences of the definition.

Proposition 2.2. *The following statements are true:*

(1) S_S *satisfies Condition* (P_{sc}).

(2) Θ_S *satisfies Condition* (P_{sc}) *if and only if* S *is right reversible.*

(3) *For an idempotent monoid, Conditions* (P_E) *and* (P_{sc}) *are equivalent.*

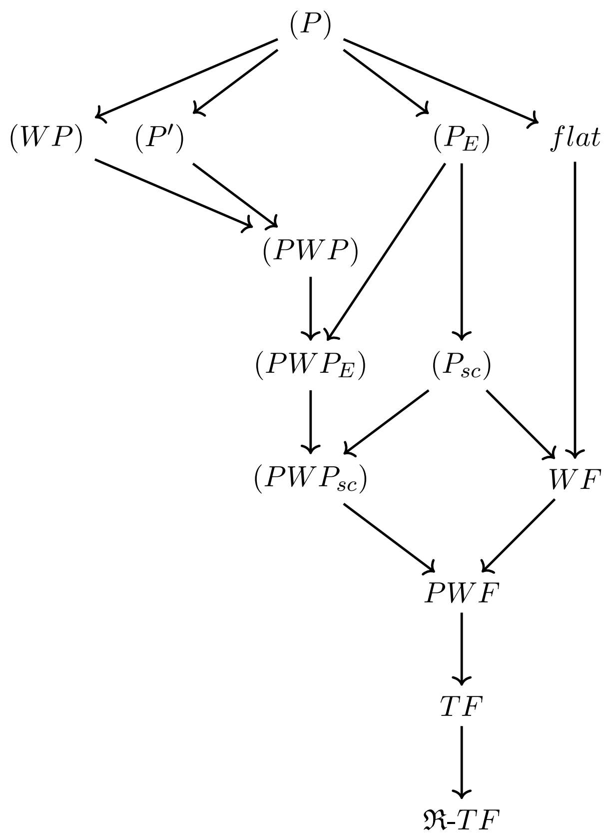
(4) *Let* $A = \coprod_{i\in I} A_i$, *where each* A_i *is an* S*-act. Then* A *satisfies Condition* (P_{sc}) *if and only if each* A_i *satisfies Condition* (P_{sc}).

(5) *Let* $\{B_i \mid i \in I\}$ *is a chain of subacts of* A *and every* B_i*,* $i \in I$*, satisfies Condition* (P_{sc})*, then* $\bigcup_{i\in I} B_i$ *satisfies Condition* (P_{sc})*.*

(6) *If* A *satisfies Condition* (P_{sc})*, then every retract of* A *satisfies Condition* (P_{sc})*.*

In items 4, 5 and 6 of the above theorem, Condition (P_{sc}) can be replaced by Condition (P_E).

The following diagram illustrates how the conditions are related to the properties already studied.



WF=weak flatness, PWF=principal weak flatness, TF=being torsion-free, $\mathfrak{R}$-TF=being $\mathfrak{R}$-torsion free

Now, an equivalent condition for a cyclic S-act satisfying Condition (P_{sc}) is given.

Theorem 2.3. *Let* ρ *be a right congruence on* S. *Then the* S*-act* S/ρ *satisfies Condition* (P_{sc}) *if and only if for all* $x, y, t, t' \in S$ *with* $(xt)\rho(yt')$, *there exist* $u, v, r, r' \in S$ *such that* $ut = vt'$, $(xr)\rho(ur)$, $(yr')\rho(vr')$, $rt = t$ *and* $r't' = t'$.

Proof. Necessity. Let $(xt)\rho(yt')$, for $x, y, t, t' \in S$, then $[x]_\rho t = [y]_\rho t'$. Therefore, there exist $u_1, v_1, w, r, r' \in S$ such that $u_1 t = v_1 t'$, $[x]_\rho r = [w]_\rho u_1 r$, $[y]_\rho r' = [w]_\rho v_1 r'$, $rt = t$ and $r't' = t'$. By letting $u = wu_1$ and $v = wv_1$, we get $(xr)\rho(ur)$, $(yr')\rho(vr')$ and $ut = vt'$.
Sufficiency. Suppose that $[x]_\rho t = [y]_\rho t'$ for $x, y, t, t' \in S$, then $(xt)\rho(yt')$. By the assumption, there exist $u, v, r, r' \in S$ such that $(xr)\rho(ur)$, $(yr')\rho(vr')$, $ut = vt'$, $rt = t$ and $r't' = t'$. This implies $[x]_\rho r = [1]_\rho ur$, $[y]_\rho r' = [1]_\rho vr'$, $rt = t$, $r't' = t'$ and $ut = vt'$. Therefore, S/ρ satisfies Condition (P_{sc}). $\square$

Corollary 2.4. *For* $z \in S$, zS *satisfies Condition* (P_{sc}) *if and only if* $zxt = zyt'$, *for* $x, y, t, t' \in S$, *implies that there exist* $u, v, r, r' \in S$ *such that* $ut = vt'$, $zxr = zur$, $zyr' = zvr'$, $rt = t$ *and* $r't' = t'$.

Proof. Since $zS \cong S/ker\lambda_z$, it suffices to put $\rho = ker\lambda_z$. $\square$

Theorem 2.5. *The following statements are true:*

(1) *Condition* (P_{sc}) *implies weak flatness.*

(2) *For a left* PSF *monoid* S, *Condition* (P_{sc}) *and weak flatness property are equivalent.*

(3) *If* S *is left* PP, *then for every* S*-act we have:*

$$(P_E) \Longleftrightarrow (P_{sc}) \Longleftrightarrow \textit{weakly flat}.$$

Proof. (1). Assume that A satisfies Condition (P_{sc}) and $as = a't$, for $a, a' \in A$ and $s, t \in S$. Then there exist $a'' \in A$ and $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$. This implies

$$a \otimes s = a \otimes rs = ar \otimes s = a''ur \otimes s = a'' \otimes urs = a'' \otimes vr't = a''vr' \otimes t = a'r' \otimes t = a' \otimes r't = a' \otimes t,$$

in $A_S \otimes {}_S(Ss \cup St)$. Therefore, A is weakly flat by [11, Lemma 3.11.1].

(2). Let A is weakly flat and $as = a't$, for $a, a' \in A$ and $s, t \in S$. The last equality implies the existence of $a'' \in A$ and $u' \in Ss \cap St$ such that $as = a't = a''u'$, by [11, Theorem 3.11.4]. Let $u' = xs = yt$. According to [17, Lemma 1.3], the equality $as = a''xs$ implies the existence of $r \in S$, such that $rs = s$ and $ar = a''xr$. Similarly, the equality $a't = a''yt$, implies the existence of $r' \in S$, such that $r't = t$ and $a'r' = a''yr'$. By setting $u = x$ and $v = y$, the result follows.

(3). Since left PP implies left PSF, it is straightforward by part (2) and [7, Theorem 2.5]. □

It is well known that Condition $(P) \Rightarrow$ flat $\Rightarrow$ weakly flat and it is clear that Condition $(P) \Rightarrow$ Condition (P_{sc}).

We recall from [11] that a right ideal K of S satisfies Condition (LU) if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

In the following examples, we show that Condition (P_{sc}) is incomparable with flatness.

Example 2.6. [flatness $\not\Rightarrow$ Condition (P_{sc})] For a proper right ideal I of S, and any $a, b, c \notin S$, we set $A(I) := (\{a, b\} \times (S \setminus I)) \cup (\{c\} \times I)$, and we define a right S-action on $A(I)$ by

$$(a, u)s = \begin{cases} (a, us), & us \notin I \\ (c, us), & us \in I \end{cases},$$

$$(b, u)s = \begin{cases} (b, us), & us \notin I \\ (c, us), & us \in I \end{cases},$$

$$(c, u)s = (c, us).$$

Then $A(I)$ is a right S-act. According to [11, Proposition 3.12.19], $A(I)$ is flat if and only if I satisfies Condition (LU).

Let's consider the monoid S with following multiplication table

.	0	1	e	x
0	0	0	0	0
1	0	1	e	x
e	0	e	e	0
x	0	x	x	0

and let $I = eS = \{0, e\}$. It is straightforward to verify that $A(I)$ is flat. Next, we show that $A(I)$ does not satisfy Condition (P_{sc}). Since $(a, x)x = (b, x)x$, there must exist $w \in \{a, b, c\}, t \in S$, and $u, v, r, r' \in S$, such that $(a, x)r = (w, t)ur$, $(b, x)r' = (w, t)vr'$, $rx = x$, $r'x = x$ and $ux = vx$. Hence $r = r' = 1$, implying $w = a = b$, which is a contradiction. Therefore, $A(I)$ does not satisfy Condition (P_{sc}).

From the above example, we can deduce that weak flatness does not imply Condition (P_{sc}).

Example 2.7. [7, Example 2] [Condition $(P_{sc}) \not\Rightarrow$ flatness] Let $U = \{a, b\}$, $V = \{c, d\}$ be left zero semigroups and let $S = U\dot{\cup}V$. Extend the multiplications in U and V to S by defining a and b as left zero elements for S and $cU = \{a\}$, $dU = \{b\}$. It has been demonstrated in [7] that all right S^1-acts satisfy Condition (P_E) but not all right S^1-acts are flat. On the other hand, Condition (P_E) implies Condition (P_{sc}). Consequently, all right S^1-acts satisfy Condition (P_{sc}) but not all right S^1-acts are flat.

It is important to observe that Condition (P_{sc}) does not imply Condition (P), as otherwise, Condition (P_{sc}) would imply flatness, which is contradicted by Example 2.7.

The following example illustrates that Condition (P_{sc}) does not imply Condition (P_E).

Example 2.8. [Condition $(P_{sc}) \not\Rightarrow$ Condition (P_E)] Consider the commutative monoid $S = \{x_i^m | i \in \mathbb{R}, m \in \mathbb{N}\} \cup \{1\}$ such that

$$x_i^m x_j^n = \begin{cases} x_j^n & i < j \\ x_i^{m+n} & i = j. \end{cases}$$

Let $K = \{x_i^m | i \in \mathbb{R}, m \in \mathbb{N}\}$. It is evident that K is an ideal of S. Let $x_i^m \in K$ and $j < i$. Then $x_j^m x_i^m = x_i^m$, and so K satisfies Condition (LU).

Hence, by [11, Proposition 3.12.19], $A = S \coprod^{K} S$ is weakly flat. Since S is left PSF (refer to [17, Example 1.6]), according to Theorem 2.5(2), A satisfies Condition (P_{sc}). Now, we proceed to show that A does not satisfy Condition (P_E). Since $(1,x)x_i^m = (1,y)x_i^m$ and $e = 1$ is the only idempotent such that $ex_i^m = x_i^m$, there must exist $a'' \in A$ and $u, u' \in S$ such that $(1,x) = a''u$, $(1,y) = a''u'$ and $ux_i^m = u'x_i^m$. Notice that $(1,x) = a''u$ implies $a'' = (1,x)$ and $u = 1$, but there is no element $u' \in S$ such that $(1,y) = (1,x)u'$.

Now, in the following example, we show that Condition (PWP_{sc}) does not imply Condition (P_{sc}).

Example 2.9. [Condition $(PWP_{sc}) \not\Rightarrow$ Condition (P_{sc})] Let S is not right reversible (for example, consider free monoid generated by two elements). Then Θ_S does not satisfy Condition (P_{sc}) according to Proposition 2.2(2), but it satisfies Condition (PWP_{sc}), as proven in [15, Theorem 2.2].

3 Classification of monoids by Condition (P_{sc})

In this section, we present some results on homological classifications. We start with questions where some properties imply Condition (P_{sc}) for finitely generated, cyclic, and monocyclic acts. Additionally, we provide a classification of monoids for which acts with this property have some other flatness properties.

Theorem 3.1. *The following statements are equivalent:*

(1) *All S-acts satisfy Condition (P_{sc}).*

(2) *S is regular and satisfies Condition (R).*
(R) : for any elements $s, t \in S$, there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$.

Proof. (1) $\Rightarrow$ (2). By part (1) of Theorem 2.5, all S-acts are weakly flat. Consequently, based on [11, Theorem 4.7.5], it can be deduced that S is regular and satisfies Condition (R).

(2) $\Rightarrow$ (1). The result follows from the fact that every regular monoid is left PP, by [11, Theorem 4.7.5] and part (3) of Theorem 2.5. □

It is worth noting that the above theorem holds true for finitely generated (monocyclic) S-acts.

Theorem 3.2. *The following statements are equivalent:*

(1) *All* S*-acts satisfying Condition* (P_{sc}) *are projective generator.*

(2) $S = \{1\}$.

Proof. (1) $\Rightarrow$ (2). Since Condition (P) implies Condition (P_{sc}), by the assumption, all right S-acts satisfying Condition (P) are projective generator. Therefore, by [11, Theorem 4.12.8], $S = \{1\}$.

(2) $\Rightarrow$ (1). If $S = \{1\}$, then all S-acts are free, and so the result follows. □

It is noted that in the above theorem, "projective generator" can be substituted with"free" without impacting the validity of the statement.

We recall from [3], [2], and [12] that:
The S-act A satisfies *Condition* (EP), if for all $a \in A, s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's'),$$

A satisfies *Condition* $(E'P)$, if for all $a \in A, s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's'),$$

and A satisfies *Condition* (E'), if for all $a \in A, s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \text{ and } us = us').$$

Theorem 3.3. *The following statements are equivalent:*

(1) *All* S*-acts satisfying Condition* $(E'P)$ *satisfy Condition* (P_{sc}).

(2) *All* S*-acts satisfying Condition* (E') *satisfy Condition* (P_{sc}).

(3) *All* S*-acts satisfying Condition* (EP) *satisfy Condition* (P_{sc}).

(4) *All* S*-acts satisfying Condition* (E) *satisfy Condition* (P_{sc}).

(5) S *is regular.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$, which implies $sxs = s$ and s is regular. On the other hand, if $sS \neq S$, then we can consider

$$A = S \coprod^{sS} S = \big\{(l,x)|\ l \in S \setminus sS\big\} \dot{\cup} sS \dot{\cup} \{(t,y)|\ t \in S \setminus sS\}.$$

Indeed,

$$B = \{(l,x)|\ l \in S \setminus sS\} \dot{\cup} sS \cong S \cong \{(t,y)|\ t \in S \setminus sS\} \dot{\cup} sS = C.$$

B and C are subacts of A generated by $(1,x)$ and $(1,y)$ respectively, implying that A is generated by $(1,x)$ and $(1,y)$, because $A = B \cup C$. Consequently, B and C satisfy Condition (E) through the isomorphisms, leading to A satisfies Condition (E) and subsequently Condition (P_{sc}), by the assumption. Thus $(1,x)s = (1,y)s$ implies that there exist $a \in A$ and $u, v, r, r \in S$, such that $(1,x)r = aur$, $(1,y)r = avr'$, $rs = s = r's$ and $us = vs$. From $(1,x)r = aur$ and $(1,y)r' = avr'$, we deduce that at least one of r and r' belongs to sS. If $r \in sS$, then there exists $s' \in S$ such that $r = ss'$. Then $s = rs = ss's$. Similarly, $r' \in sS$ implies there exists $s'' \in S$ such that $s = ss''s$ and so, s is regular. Thus S is regular.

(5) $\Rightarrow$ (1). Since S is regular, all S-acts satisfying Condition $(E'P)$ are weakly flat, according to [2, Theorem 2.8]. Furthermore, S being left PP due to its regularity implies that the result follows as per part (3) of Theorem 2.5. □

It is important to note that based on the above theorem, if S is not regular, there exists an S-act that satisfies Condition (E), but does not satisfy Condition (P_{sc}). Hence, it is evident that Condition (E) does not imply Condition (P_{sc}) generally. Similarly, Condition (P_{sc}) does not imply Condition (E). Otherwise, Condition (P) implies Condition (E) and so Condition (P) implies strong flatness, which is not true in general.

The validity of the above theorem extends to finitely generated S-acts and S-acts generated by at most two elements, as shown in the proof of Theorem 3.3. Additionally, Condition (P_{sc}) can be substituted with Condition (P_E).

Condition (P_{sc}) implies weak flatness and consequently torsion freeness. However, it is important to note that the converse implication is not generally valid. If it were, flatness would imply Condition (P_{sc}), which is not true by Example 2.6.

Theorem 3.4. *The following statements are equivalent:*

(1) *All S-acts satisfy Condition (P_{sc}).*

(2) *All $\mathfrak{R}$-torsion free S-acts satisfy Condition (P_{sc}).*

(3) *S is regular and satisfies Condition (R).*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). According to [19, Proposition 1.2], every S-act satisfying Condition (E) also satisfies Condition (P_{sc}). Then by Theorem 3.3, S is regular. Consequently, all S-acts are principally weakly flat, by [11, Theorem 4.6.6]. Since

$$\text{principally weakly flat} \Rightarrow \text{torsion free} \Rightarrow \mathfrak{R}\text{-torsion free},$$

all S-acts are $\mathfrak{R}$-torsion free. Therefore, by the assumption, all S-acts satisfy Condition (P_{sc}), and so S is regular and satisfying Condition (R), by Theorem 3.1.

(3) $\Rightarrow$ (1). It is straightforward, by Theorem 3.1. □

By the proof of Theorem 3.4, we conclude that the above theorem is true for finitely generated S-acts as well as for S-acts generated by at most two elements. Additionally, Condition (P_{sc}) can be replaced by Condition (P_E), by [4, Theorem 2.1] and this fact that Condition (P_E) implies Condition (P_{sc}).

An S-act A is defined as *strongly torsion free* (STF) if the equality $as = bs$ for any $a, b \in A$ and any $s \in S$ implies that $a = b$ (see [18]).

It is obvious that $STF \Rightarrow$ Condition $(PWP) \Rightarrow$ principally weakly flat $\Rightarrow$ torsion free. Since Condition (P_{sc}) $((P_E))$ implies torsion free, and S_S satisfies Condition (P_{sc}) $((P_E))$, so by [18, Theorem 3.1], we have the following theorem.

Theorem 3.5. *The following statements are equivalent:*

(1) *All S-acts satisfying Condition (P_{sc}) $((P_E))$ are STF.*

(2) *All finitely generated S-acts satisfying Condition (P_{sc}) $((P_E))$ are STF.*

(3) *All cyclic S-acts satisfying Condition (P_{sc}) $((P_E))$ are STF.*

(4) *S is right cancellative.*

We recall from [11] that an act A is called *divisible* if $Ac = A$, for any left cancellable element $c \in S$.

Theorem 3.6. *The following statements are equivalent:*

1. *All S-acts are divisible.*
2. *All S-acts satisfying Condition (P_{sc}) are divisible.*
3. *All left cancellable elements of S are left invertible.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$. Since $S/\rho(x,x) = S/\Delta_S \cong S_S$, for every $x \in S$ and S_S satisfies Condition (P_{sc}), by the assumption S_S is divisible and so by [11, Proposition 3.2.2], the result follows.

$(3) \Rightarrow (1)$. It is straightforward by [11, Proposition 3.2.2]. □

The conclusion reached by the proof of Theorem 3.6 affirms the validity of the theorem for finitely generated S-acts and cyclic (monocyclic) S-acts. Additionally, Condition (P_{sc}) can be substituted with Condition (P_E).

We recall from [11] that A is (*strongly*) *faithful* if for $s,t \in S$ the equality $as = at$ for (some) all $a \in A$, implies $s = t$. It is straightforward that every strongly faithful S-act is faithful, but the converse implication does not hold in general.

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S. It is clear that C_l (C_r) is not empty, because $1 \in C_l$ (C_r)

Theorem 3.7. *The following statements are equivalent:*

1. *All strongly faithful S-acts satisfy Condition (P_{sc}).*
2. *All strongly faithful S-acts generated by exactly two elements satisfy Condition (P_{sc}).*
3. *S is not left cancellative or it is a group.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$. If S is not left cancellative, then (3) is satisfied. Let S be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Consequently, $sxs = s$ indicating s is regular. Now, consider the case where $sS \neq S$. Put

$$A = S \coprod^{sS} S = \big\{(l,x) \mid l \in S \setminus sS\big\} \dot\cup sS \dot\cup \{(t,y) \mid t \in S \setminus sS\}.$$

We have

$$B = \{(l,x)|\ l \in S \setminus sS\}\dot{\cup} sS \cong S_S \cong \{(t,y)|\ t \in S \setminus sS\}\dot{\cup} sS = C$$

and

$$A =< (1,x),(1,y) >= B \cup C.$$

Since S is left cancellative, S_S is strongly faithful, as shown in [10, Lemma 3.7]. Through the isomorphisms mentioned, B and C are also strongly faithful as subacts of A. Consequently, A is strongly faithful. Since A is generated by $(1,x)$ and $(1,y)$, by the assumption, it satisfies Condition (P_{sc}). Following the proof of part (4) $\Rightarrow$ (5) of Theorem 3.3, s is regular, implying S is regular. Thus, for every $s \in S$, there exists $x \in S$, such that $sxs = s$. Since S is left cancellative, $xs = 1$. Therefore, every element in S has a left inverse, making S a group.

(3) $\Rightarrow$ (1). If S is not left cancellative, then there is no strongly faithful S-act, as stated in [10, Lemma 3.7], thus (1) is satisfied. On the other hand, if S is left cancellative, then there is at least one strongly faithful S-act, according to [10, Lemma 3.7]. Since S is group, all S-acts satisfy Condition (P), by [11, Theorem 4.9.10]. Consequently, all S-acts satisfy Condition (P_{sc}), leading to the desired outcome. $\square$

According to the proof of Theorem 3.7, we can affirm that the aforementioned theorem holds true for finitely generated S-acts and for S-acts generated by at most two elements. Furthermore, Condition (P_{sc}) can be substituted with Condition (P_E).

In [16] and [11], it is mentioned that A is *almost weakly flat* if it is principally weakly flat and satisfies Condition
(W') If $as = a't$, and $Ss \cap St \neq \emptyset$, for $a, a' \in A, s, t \in S$, then there exists $a'' \in A$, $u \in Ss \cap St$ such that $as = a't = a''u$.
An object G in category $\mathcal{C}$ is called a *generator* in $\mathcal{C}$ if the functor $Mor_{\mathcal{C}}(G,-)$ is faithful, i.e, for any $X, Y \in \mathcal{C}$ and any $f, g \in Mor_{\mathcal{C}}(X,Y)$ where $f \neq g$, there exists $\alpha \in Mor_{\mathcal{C}}(G,X)$ such that $f\alpha \neq g\alpha$. By [11, Theorem 2.3.16], G is a generator if and only if there exists an epimorphism $\pi : G \longrightarrow S$. Consequently, S is a generator in **Act-S**. It has been established in [16, Theorem 3.4] that all generators are weakly flat if and only if all S-acts are almost weakly flat.

Theorem 3.8. *The following statements are equivalent:*

(1) *All generator S-acts satisfy Condition (P_{sc}).*

(2) *$S \times A$ satisfies Condition (P_{sc}) for every S-act A.*

(3) *The S-act A satisfies Condition (P_{sc}) if $Hom(A, S_S) \neq \emptyset$.*

(4) *All S-acts are almost weakly flat.*

(5) *S is regular and satisfies the following condition:*

$$(\forall s, t \in S)(Ss \cap St \neq \emptyset \Rightarrow (\exists w \in Ss \cap St \quad s.t \quad 1(\ker \lambda_s \vee \ker \lambda_t)w)).$$

Proof. (1) $\Rightarrow$ (2). Since $S \times A$ is a generator, the conclusion is straightforward.

(2) $\Rightarrow$ (3). Let A be an S-act such that $Hom(A, S_S) \neq \emptyset$ and $as = a't$, for $a, a' \in A$ and $s, t \in S$. Since $Hom(A, S_S) \neq \emptyset$, there exists a homomorphism $f : A \to S_S$. Therefore, the equality $as = a't$ in A implies that $(f(a), a)s = (f(a'), a')t$ in $S \times A^\star$, where $A^\star = aS \cup a'S$. Consequently, there exists $(w, a'') \in S \times A^\star$ and $u, v, r, r' \in S$ such that

$$\begin{cases} (f(a), a)r = (w, a'')ur \\ (f(a'), a')r' = (w, a'')vr' \end{cases}, \; rs = s, \; r't = t, \; us = vt.$$

Thus, $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$. This means that A satisfies Condition (P_{sc}). (3) $\Rightarrow$ (1). Let A be a generator such that $as = a't$, for $a, a' \in A, s, t \in S$ and $A^\star = aS \cup a'S$. Since $A^\star$ is a subact of A and $A^\star$ is a generator S-act, then by assumption, $A^\star$ satisfies Condition (P_{sc}). Therefore, the equality $as = a't$ in $A^\star$ implies that there exist $a'' \in A^\star \subseteq A$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vs$. Thus, the result follows. (1) $\Rightarrow$ (4). According to part (1) of Theorem 2.5, Condition (P_{sc}) implies weakly flat. Therefore, all generator S-acts are weakly flat, based on the assumption. Consequently, by [16, Theorem 3.4], all S-acts are almost weakly flat.

(4) $\Rightarrow$ (1). By [16, Theorem 3.4], all generator S-acts are weakly flat, which implies that S is regular, according to [16, Theorem 3.8]. Therefore, S is left PP, and the result follows from part (3) of Theorem 2.5.

(4) $\Leftrightarrow$ (5) It is straightforward, by [16, Theorem 3.8]. $\square$

By the proof of Theorem 3.8, we conclude that the above theorem is true for finitely generated (generator) S-acts as well as for (generator) S-acts generated by at most (tree) two elements. Also, since every regular monoid is left PP, by part (3) of Theorem 2.5, Conditions (P_{sc}) and (P_E) are equivalent, and so Condition (P_{sc}) can be replaced by Condition (P_E) in Theorem 3.8.

For fixed elements $u, v \in S$, a binary relation $P_{u,v}$ can be defined as follows:

$$(x, y) \in P_{u,v} \Leftrightarrow ux = vy \ (x, y \in S).$$

For $s, t \in S$, let $\mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ and for any right ideal I of S, let ρ_I denote the right Rees congruence on S, i.e., for $x, y \in S$,

$$(x, y) \in \rho_I \Leftrightarrow (x = y) \vee (x, y \in I).$$

For $x, y \in S$

$$L(x, y) = \{(a, b) \in S \times S |\ ax = by\}$$

is either empty or a subact of ${}_S(S \times S)$. Similarly, we define

$$R(x, y) = \{(a, b) \in S \times S |\ xa = yb\}.$$

Therefore $P_{u,v} = R(u, v)$, for every $u, v \in S$.

Recall from [11] that an act is called *cofree* if it is isomorphic to the act $X^S = \{f | f \text{ is a mapping from } S \text{ to } X\}$, for some nonempty set X, where fs is defined by $fs(t) = f(st)$, for $f \in X^S$ and $s, t \in S$.
An S-act Q is called injective if for any homomorphism $\iota : A \to B$ and any homomorphism $f : A \to Q$ there exists a homomorphism $\bar{f} : B \to Q$ such that $f = \bar{f}\iota$. It is called *(fg-) weakly injective*, if it is injective relative to all embeddings of (finitely generated) right ideals into S.

Theorem 3.9. *The following statements are equivalent:*

(1) *All fg-weakly injective S-acts satisfy Condition (P_{sc}).*

(2) *All cofree S-acts satisfy Condition (P_{sc}).*

(3) *For all* $s, t \in S$, *there exist* $u, v, r, r' \in S$ *such that* $rs = s$, $r't = t$, $(s,t) \in P_{u,v}$ *(or* $(s,t) \in P_{ur,vr'}$*) and the following conditions hold:*

(i) $P_{ur,vr'} \subseteq P_{r,s} \circ \mu_{s,t} \circ P_{t,r'}$

(ii) $\ker \lambda_u \cap (rS \times rS) \subseteq \rho_{sS}$

(iii) $\ker \lambda_v \cap (r'S \times r'S) \subseteq \rho_{tS}$.

Proof. The implication (1) ⇒ (2) is obvious, because cofree ⇒ fg-weakly injective.

(2) ⇒ (3). Let $s, t \in S$, S_1, S_2 be two sets such that $|S_1| = |S_2| = |S|$ and $\alpha : S \to S_1$, $\beta : S \to S_2$ are bijections. Put $X = S/\mu_{s,t} \dot\cup S_1 \dot\cup S_2$. Define the mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = sy \\ \\ \alpha(x) & \text{if } x \in S \setminus sS \end{cases}$$

and

$$g(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = ty \\ \\ \beta(x) & \text{if } x \in S \setminus tS. \end{cases}$$

If there exist $y_1, y_2 \in S$, such that $sy_1 = sy_2$, then

$$(y_1, y_2) \in \ker \lambda_s \subseteq \ker \lambda_s \vee \ker \lambda_t = \mu_{s,t}.$$

Thus $f(sy_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = f(sy_2)$, and so f is well- defined. Similarly, it follows that g is well-defined. According to our definition of f and g, we clearly have $fs = gt$. By the assumption, the cofree S-act X^S satisfies Condition (P_{sc}), and so, there exist $u, v, r, r' \in S$ and a map $h : S \to X$, such that $fr = hur, gr' = hvr', rs = s, r't = t, us = vt$. Clearly from $us = vt$, we have $(s,t) \in P_{u,v}$ (or by $rs = s, r't = t$, and $us = vt$ we have $(s,t) \in P_{ur,vr'}$). Now we show that the statements $(i), (ii)$ and (iii) are true.

(i): Let $(l_1, l_2) \in P_{ur,vr'}$, $l_1, l_2 \in S$. Then $url_1 = vr'l_2$ and so, $f(rl_1) = (fr)(l_1) = (hur)(l_1) = h(url_1) = h(vr'l_2) = (hvr')(l_2) = (gr')(l_2) = g(r'l_2)$. By definition f and g, there exist $y_1, y_2 \in S$ such that $rl_1 = sy_1$ and $r'l_2 = ty_2$. Thus, $[y_1]_{\mu_{s,t}} = f(sy_1) = f(rl_1) = g(r'l_2) = g(ty_2) = [y_2]_{\mu_{s,t}}$. Now

$rl_1 = sy_1$, $[y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}}$ and $ty_2 = r'l_2$ imply $(l_1, y_1) \in P_{r,s}$, $(y_1, y_2) \in \mu_{s,t}$ and $(y_2, l_2) \in P_{t,r'}$, respectively. Therefore, $(l_1, l_2) \in P_{r,s} \circ \mu_{s,t} \circ P_{t,r'}$. Thus $P_{ur,vr'} \subseteq P_{r,s} \circ \mu_{s,t} \circ P_{t,r'}$ and so, (i) is satisfied.

(ii): Let $(t_1, t_2) \in \ker \lambda_u \cap (rS \times rS)$, $t_1, t_2 \in S$. Then $ut_1 = ut_2$ and there exist $w_1, w_2 \in S$, such that $t_1 = rw_1$ and $t_2 = rw_2$. Thus $urw_1 = ut_1 = ut_2 = urw_2$, which implies $f(rw_1) = (fr)(w_1) = (hur)(w_1) = h(urw_1) = h(urw_2) = (hur)(w_2) = (fr)(w_2) = f(rw_2)$. Having in mind the definition of f, we consider two cases as follows.

Case 1. If $rw_1, rw_2 \in S \backslash sS$, then $f(rw_1) = f(rw_2)$ implies $\alpha(rw_1) = \alpha(rw_2)$. Then $t_1 = rw_1 = rw_2 = t_2$ and so, $(t_1, t_2) \in \rho_{sS}$.

Case 2. If $rw_1, rw_2 \in sS$, then there exist $y_1, y_2 \in S$ such that $rw_1 = sy_1$ and $rw_2 = sy_2$. Thus $(t_1, t_2) = (rw_1, rw_2) = (sy_1, sy_2) \in (sS \times sS) \cup \Delta_S = \rho_{sS}$. Hence $\ker \lambda_u \cap (rS \times rS) \subseteq \rho_{sS}$ and (ii) is satisfied.

The proof of (iii) is similar to that of (ii).

(3) $\Rightarrow$ (1). Suppose that A is fg-weakly injective, and that $as = a't$, for $a, a' \in A$ and $s, t \in S$. By the assumption, there exist $u, v, r, r' \in S$ such that $rs = s, r't = t, us = vt$ and conditions $(i), (ii), (iii)$ are true. Define a mapping $\varphi : urS \cup vr'S \to A$ by

$$\varphi(x) = \begin{cases} arp & \exists p \in S : \ x = urp \\ \\ a'r'q & \exists q \in S : \ x = vr'q. \end{cases}$$

First, we show that φ is well-defined. If there exist $p, q \in S$ such that $urp = vr'q$, then $(p, q) \in P_{ur,vr'}$. By (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{r,s}, (y_1, y_2) \in \mu_{s,t}, (y_2, q) \in P_{t,r'}$. Thus, $rp = sy_1$, $(y_1, y_2) \in \mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ and $ty_2 = r'q$. The relation $(y_1, y_2) \in \mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ implies that there exist $z_1, ..., z_n \in S$ such that

$$\begin{array}{ccccc} sy_1 = sz_1 & sz_2 = sz_3 & ... & sz_{n-1} = sz_n & \\ & tz_1 = tz_2 & ... & & tz_n = ty_2. \end{array}$$

Then $arp = asy_1 = asz_1 = a'tz_1 = a'tz_2 = ... = a'tz_n = a'ty_2 = a'r'q$.
If there exist $p_1, p_2 \in S$ such that $urp_1 = urp_2$, then $(rp_1, rp_2) \in \ker \lambda_u \cap (rS \times rS)$. Now, by (ii), $rp_1 = rp_2$ or there exist $y_1, y_2 \in S$ such that $rp_1 = sy_1$ and $rp_2 = sy_2$. If $rp_1 = rp_2$, then $arp_1 = arp_2$. If $rp_1 = sy_1$ and $rp_2 = sy_2$, then $usy_1 = urp_1 = urp_2 = usy_2 = vty_2$. Thus $ursy_1 = vr'ty_2$

and so, $(sy_1, ty_2) \in P_{ur,vr'}$. Therefore by (i), there exist $l_1, l_2 \in S$ such that $(sy_1, l_1) \in P_{r,s}$, $(l_1, l_2) \in \mu_{s,t}$ and $(l_2, ty_2) \in P_{t,r'}$. Hence $sy_1 = rsy_1 = sl_1$, $(l_1, l_2) \in \mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ and $tl_2 = r'ty_2 = ty_2$. The relation $(l_1, l_2) \in \mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ implies that there exist $z'_1, ..., z'_m \in S$ such that

$$\begin{array}{ccccccc} sl_1 = sz'_1 & & sz'_2 = sz'_3 & \dots & sz'_{m-1} = sz'_m & & \\ & tz'_1 = tz'_2 & & \dots & & tz'_m = tl_2, & \end{array}$$

and so, $arp_1 = asy_1 = asl_1 = asz'_1 = a'tz'_1 = a'tz'_2 = ... = a'tz'_m = a'tl_2 = a'ty_2 = asy_2 = arp_2$.
If there exist $q_1, q_2 \in S$ such that $vr'q_1 = vr'q_2$, then, using a similar argument as in the previous case, by (i) and (iii), we have $a'r'q_1 = a'r'q_2$. Thus, φ is well-defined. It is clear that φ is an S-homomorphism. Since A is fg-weakly injective, there exists an S-homomorphism $\psi : S_S \to A$ such that $\psi|_{urS \cup vr'S} = \varphi$. Put $a'' = \psi(1)$. Then $ar = \varphi(ur) = \psi(ur) = \psi(1)ur = a''ur$ and $a'r' = \varphi(vr') = \psi(vr') = \psi(1)vr' = a''vr'$, indicating that A satisfies Condition (P_{sc}). □

Based on the proof of Theorem 3.9, we can deduce that the aforementioned theorem holds for (weakly) injective S-acts.

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