Categories and
General Algebraic Structures
with Applications

In press.

Picard group of dual categories

Adriana Mejía Castaño

Abstract. We provide an explicit description of the Picard group (the group of isomorphism classes of invertible objects, those that have an inverse under the tensor product) of the dual category of the category of comodules over a supergroup algebra, by using the description of this group for group-theoretical categories. In fact we prove that there is a subgroup relation between these groups. As an interest application of this group in a modular context, it can be used to construct examples of symmetric special Frobenius algebras. They also plays an important role in the theory of braided tensorcategories for the classification of group extensions of fusion categories.

1 Introduction

Given a monoidal category $\mathcal{C}$ the Picard group $\mathbf{Pic}(\mathcal{C})$ of $\mathcal{C}$ is the group of isomorphism classes of invertible objects, those that have an inverse under the tensor product. This group has been calculated in several contexts, for example in derived categories [6], autoequivalence categories [7], stable module categories [14] and tensor categories [2], [3], [10]. It is the last context which interests us in this article.

Keywords: Picard group, category of comodules, group theoretical categories.
Mathematics Subject Classification [2020]: 18D10, 18M99.
Received: 12 January 2024, Accepted: 8 May 2024.
ISSN: Print 2345-5853, Online 2345-5861.

As an interest application of this group in a modular context, it can be used to construct examples of symmetric special Frobenius algebras A [5]. This group also plays an important role in the theory of braided tensor categories, in particular, it is used in the classification of group extensions of fusion categories [4]. In this article we explicitly describe the Picard group of certain dual categories. Let G be a finite group, F a subgroup of G, $\omega \in Z(G, \mathbf{k}^\times)$ a 3-cocycle and $\psi \in C^2(F, \mathbf{k}^\times)$ a 2-cochain such that $d\psi = \omega_{F\times F\times F}$. The Picard group for a group-theoretical category $\mathcal{C}(G, \omega, F, \psi)$ was described in [13], [9]; and each bimodule, when G is abelian and ω is trivial, is parametrized by the following set

$$L = \{(g, \rho) : g \in R, \rho \in \hat{F}\}$$

where R is a representative set of $F\backslash G/F$ and $\hat{F}$ is the group of 1-dimensional F-representations.

Consider now the following category: Let $H := \mathcal{A}(V, G, u)$ be a finite supergroup algebra, where $u \in G$ be an element of order 2 and V a finite-dimensional G-module such that

$$u \cdot v = -v, \text{ for any } v \in V.$$

We want to describe the Picard group of the dual category of the category of $\mathcal{A}(V, G, u)$-comodules. To this end, are necessary all the exact indecomposable $\mathcal{C}omod(H)$-module categories, which are parametrized by tuple (W, β, F, ψ) where $F \subseteq G$ be a subgroup, $\psi \in Z^2(F, \mathbf{k}^\times)$ be a 2-cocycle, $W \subseteq V$ be an F-invariant subspace, and $\beta : W \times W \to \mathbf{k}$ be a symmetric F-invariant [12]. Consider the following subgroup

$$\mathcal{L} = \{(g, \rho) \in L : g \cdot W \subset W, \beta(g^{-1}v, g^{-1}w) = \beta(v, w)\}.$$

Let $\mathcal{M}$ be the module category parametrized by (W, β, F, ψ), we prove that

$$\mathbf{Pic}(\mathcal{C}omod(\mathcal{A}(V, G, u))^*_{\mathcal{M}_{(W,\beta,F,\psi)}}) \simeq \mathcal{L}$$

which is a subgroup of $\mathbf{Pic}(\mathcal{C}(G, 1, F, \psi))$.

2 Preliminaries and notations

2.1 Hopf algebras Throughout this paper $\mathbf{k}$ will denote an algebraic field of characteristic 0 and $\mathbf{k}^\times = \mathbf{k} - \{0\}$. If H is a finite-dimensional Hopf algebra, then

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$$

will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra, the associated graded algebra $\operatorname{gr} H$ is a coradically graded Hopf algebra. If (M, λ) is a left H-comodule, the coradical filtration on H induces a filtration on M, given by $M_n = \lambda^{-1}(H_n \otimes_{\mathbf{k}} M)$. This filtration is called the *Loewy series* on M. It is known that M_0 is the socle of M.

If the coradical H_0 is a Hopf subalgebra, the associated graded vector space $\operatorname{gr} M = \oplus_{i=0} M(i)$, where $M(i) = M_i/M_{i-1}$, inherits a left $\operatorname{gr} H$-comodule structure. If $\mathcal{K}$ is a left H-comodule algebra, and $M \in {}^H\mathcal{M}_{\mathcal{K}}$ then $\operatorname{gr} M \in {}^{\operatorname{gr} H}\mathcal{M}_{\operatorname{gr} \mathcal{K}}$. We shall say that a left H-comodule algebra $(\mathcal{K}, \lambda)$ is *H-simple* if it has no non-trivial costable ideals.

A *Hopf 2-cocycle* for H is a map $\sigma : H \otimes_{\mathbf{k}} H \to \mathbf{k}$, invertible with respect to convolution, such that for all $x, y, z \in H$

$$\sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sigma(y_1, z_1)\sigma(x, y_2 z_2), \sigma(x, 1) = \varepsilon(x) = \sigma(1, x).$$

There is a new Hopf algebra structure constructed over the same coalgebra H with the product described by

$$x._{[\sigma]} y = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})\ x_{(2)} y_{(2)}, \qquad x, y \in H. \tag{2.1}$$

This new Hopf algebra is denoted by $H^{[\sigma]}$.

2.2 Module categories over Hopf algebras Let H be a finite dimensional Hopf algebra. In this section we recall some facts about representation of $\operatorname{Rep}(H)$.

If K is a finite-dimensional left H-comodule algebra, then the category of finite-dimensional left K-modules ${}_K\mathcal{M}$ is a $\operatorname{Rep}(H)$-module. The action $\overline{\otimes} : \operatorname{Rep}(H) \times {}_K\mathcal{M} \to {}_A\mathcal{M}$, is given by $X \overline{\otimes} M = X \otimes_{\mathbf{k}} M$, for all $X \in \operatorname{Rep}(H), M \in {}_K\mathcal{M}$. The left K-module structure on $X \otimes_{\mathbf{k}} M$ is given by λ, that is, if $r \in K$, $x \in X, m \in M$ then $r \cdot (x \otimes m) = r\_{-1} \cdot x \otimes r\_0 \cdot m$.

We shall denote by ${}^H_K\mathcal{M}_K$, and respectively ${}^H\mathcal{M}_K$, the category of finite dimensional left H-comodules that are K-bimodules, respectively right K-modules, such that the coaction is a morphism of K-bimodules, respectively right K-modules. The category ${}^H_K\mathcal{M}_K$ is a finite tensor category with monoidal product $M \otimes N = M \otimes_K N$ if $M, N \in {}^H_K\mathcal{M}_K$. The H-comodule structure on $M \otimes_K N$ is

$$\lambda : M \otimes_K N \to H \otimes_{\mathbf{k}} M \otimes_K N, \lambda(m \otimes n) = m\_{-}1 n\_{-}1 \otimes m\_0 \otimes n\_0.$$

Proposition 2.1. [11, Prop. 1.20, 1.23] *Under the above assumptions*

1. *If K is H-simple then ${}_K\mathcal{M}$ is an exact indecomposable* $\mathrm{Rep}(H)$*-module.*
2. *There is a monoidal equivalence* $\mathrm{Rep}(H)^*_{{}_K\mathcal{M}} \simeq {}^H_K\mathcal{M}_K$.

Let V be a Yetter-Drinfeld module over $\mathbf{k}G$, $H = \mathfrak{B}(V)\#\mathbf{k}G$, $\mathcal{K}$ be a right H-simple left H-comodule algebra such that $\mathcal{K}_0 = \mathbf{k}_\psi F$, for some subgroup $F \subseteq G$ and a 2-cocycle $\psi \in Z^2(F, \mathbf{k}^\times)$. Let $K \subseteq \mathfrak{B}(V)$ be an homogeneous coideal subalgebra such that $\operatorname{gr}\mathcal{K} = K\#\mathbf{k}_\psi F$. Consider another $\mathcal{K}'$ with associated F', ψ' as before.

Proposition 2.2. *Let $M \in {}^H_{\mathcal{K}'}\mathcal{M}_{\mathcal{K}}$ be an invertible object. Under the above hypothesis,*

1. [8, Lemma 4.1] *If $\alpha \in Z^2(G, \mathbf{k}^\times)$ is a 2-cocycle, then there exists a Hopf 2-cocycle $\varsigma : H \otimes_{\mathbf{k}} H \to \mathbf{k}$ such that $\varsigma_{|G \times G} = \alpha$.*
2. [8, Theorem 4.2] $\dim M = \dim \mathcal{K}$.
3. [8, Claim 4.1] *There exists $m \in M_0$ such that $M = m \cdot \mathcal{K}$.*
4. *The object $M_0 \in {}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_{\psi'} F'}$ is invertible.*

Proof. (1) Since M is an invertible object, there exists $N \in {}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}'}$ such that

$$M \otimes_{\mathcal{K}} N \simeq \mathcal{K}', \quad N \otimes_{\mathcal{K}'} M \simeq \mathcal{K},$$

as objects in the corresponding category. The graded object $\operatorname{gr} M$ with respect to the Loewy filtration, see subsection 2.1, is an object in ${}^H\mathcal{M}_{K\#\mathbf{k}_\psi F}$. In particular $\operatorname{gr} M \in {}^H\mathcal{M}_K$. Since $K/K^+ = \mathbf{k}$, it follows from Theorem [8, Theorem 3.1] that $\operatorname{gr} M \simeq P \otimes_{\mathbf{k}} K$, where $P = \operatorname{gr} M / (\operatorname{gr} M \cdot K^+)$. Let us denote $\lambda : \operatorname{gr} M \to H \otimes_{\mathbf{k}} \operatorname{gr} M$ the comodule structure. Since $K = K^+ \oplus \mathbf{k}$, then

$$\lambda(\operatorname{gr} M \cdot K^+) \subset (H \otimes_{\mathbf{k}} \operatorname{gr} M)(K^+ \otimes_{\mathbf{k}} 1 + H \otimes_{\mathbf{k}} K^+).$$

Whence, the map λ induces a map $\overline{\lambda} : P \to Q \otimes_{\mathbf{k}} P$, where $Q = H/HK^+H$. Observe that Q is a pointed Hopf algebra with coradical $\mathbf{k}G$, since H is coradically graded and the ideal HK^+H is homogeneous and does not intersect H_0. The space P has a right $\mathbf{k}_\psi F$-module structure with action given by $\overline{p}\cdot f = \overline{p\cdot f}$, for all $\overline{p}\in P$ and $f\in F$. This action is well-defined and makes the object P an object in the category ${}^{Q}\mathcal{M}_{\mathbf{k}_\psi F}$.

Let $\Psi \in Z^2(G,\mathbf{k}^\times)$ be a 2-cocyle such that $\Psi\mid_{F\times F}=\psi$, and let $\varsigma : H\otimes_{\mathbf{k}} H \to \mathbf{k}$ the Hopf 2-cocycle obtained as in Lemma 2.2, such that $\varsigma_{|G\times G}=\Psi$. It follows from [11, Lemma 2.1] that there exists an equivalence of categories

$$ {}^{Q^{[\varsigma]}}\mathcal{M}_{\mathbf{k}F} \simeq {}^{Q}\mathcal{M}_{\mathbf{k}_\psi F}. $$

Again using Theorem [8, Theorem 3.1], any object in ${}^{Q}\mathcal{M}_{\mathbf{k}_\psi F}$ is a free $\mathbf{k}_\psi F$-module. Thus, there exists an object $Z\in {}^{Q}\mathcal{M}_{\mathbf{k}_\psi F}$ such that $\operatorname{gr} M\simeq Z\otimes_{\mathbf{k}}\mathbf{k}_\psi F\otimes_{\mathbf{k}} K$ as vector spaces, and $\dim M=\dim\operatorname{gr} M=\dim Z\dim\mathcal{K}$. Analogously, there exists $s\in\mathcal{N}$ such that $\dim N = s\dim\mathcal{K}'$.

By [8, Theorem 3.1] there exists $t\in\mathcal{N}$ such that M^t is a free right $\mathcal{K}$-module, i.e. there exists a vector space E such that $M^t\simeq E\otimes_{\mathbf{k}}\mathcal{K}$, hence

$$ \dim E = t\dim Z. \tag{2.2} $$

Since $M\otimes_{\mathcal{K}} N\simeq\mathcal{K}$, then $M^t\otimes_{\mathcal{K}} N\simeq E\otimes_{\mathbf{k}} N\simeq\mathcal{K}^t$, thus $\dim E\dim N = t\dim\mathcal{K}$. Using (2.2), we obtain that $\dim Zs=1$, hence $\dim Z=1$ and

$$ \dim M = \dim\mathcal{K}. $$

Analogously $\dim N=\dim\mathcal{K}'$.

(2) Since M_0 is the socle of M, then $M_0\neq 0$. Let $g\in G$ and $m\in M_0$ be elements such that $\lambda(m)=g\otimes m$. Let $\phi:\mathcal{K}\to M$ be the map defined as $\phi(k)=m\cdot k$. Let $J\subseteq\mathcal{K}$ be the kernel of ϕ, then J is a right ideal in $\mathcal{K}$. Let $x\in J$, and write $\lambda(x)=\sum_i x^i\otimes x_i$ in such a way that the set $\{x^i\}\subseteq H$ is linearly independent. Thus, the set $\{gx^i\}\subseteq H$ is also linearly independent. Since

$$ 0=\lambda(m\cdot x)=\sum_i gx^i\otimes m\cdot x_i, $$

then $m\cdot x_i=0$ for any i, thus $\lambda(x)\in H\otimes J$ and the ideal J is H-costable. Since $J\neq\mathcal{K}$ then $J=0$, and ϕ is injective. By part (1) $\dim M=\dim\mathcal{K}$, thus ϕ is surjective and this proves that $M=m\cdot\mathcal{K}$. Analogously, there exist $n\in N_0$ such that $=n\cdot\mathcal{K}'$.

(3) Since $M \otimes_{\mathcal{K}} N \simeq \mathcal{K}'$, then $(M \otimes_{\mathcal{K}} N)_0 \simeq \mathbf{k}_{\psi'}F'$ as objects in the category ${}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_{\psi'}F'}$. Consider the not null map

$$\theta : M_0 \otimes_{\mathbf{k}_\psi F} N_0 \to (M \otimes_{\mathcal{K}} N)_0, \quad \overline{m \otimes n} \mapsto \overline{m \otimes n}.$$

Since $(M \otimes_{\mathcal{K}} N)_0 \simeq \mathbf{k}_{\psi'}F'$, and the object $\mathbf{k}_{\psi'}F'$ is simple, then θ is surjective. Part (2) of the Proposition implies that

$$\dim(M_0 \otimes_{\mathbf{k}_\psi F} N_0) = \mid F' \mid = \dim \mathbf{k}_{\psi'}F' = \dim((M \otimes_{\mathcal{K}} N)_0).$$

Hence θ is injective. □

2.3 Finite supergroup algebras Let G be a finite abelian group, $u \in G$ be an element of order 2 and V a finite-dimensional G-module such that

$$u \cdot v = -v, \text{ for any } v \in V.$$

The space V is a $\mathbf{k}G$-Yetter-Drinfeld module where the G-comodule structure $\delta : V \to \mathbf{k}G \otimes_{\mathbf{k}} V$ is given by $\delta(v) = u \otimes v$, for all $v \in V$. The Nichols algebra of V is the exterior algebra $\mathfrak{B}(V) = \wedge(V)$. The bosonization $\wedge(V)\#\mathbf{k}G$ is called, in [1], a *finite supergroup algebra*, and it is denoted by $\mathcal{A}(V, G, u)$. Hereafter we shall denote the element $v\#g$ simply by vg, for all $v \in V, g \in G$.

The algebra $\mathcal{A}(V, G, u)$ is generated by elements $v \in V, g \in G$ subject to relations

$$vw + wv = 0, \quad gv = (g \cdot v)g, \text{ for all } v, w \in V, g \in G.$$

The coproduct and antipode are determined for all $v \in V, g \in G$ by

$$\Delta(v) = v \otimes 1 + u \otimes v, \quad \Delta(g) = g \otimes g, \quad S(v) = -uv, \quad S(g) = g^{-1}.$$

2.4 Algebras over $\mathcal{C}omod(\mathcal{A}(V, G, u))$ Let $F \subseteq G$ be a subgroup, $\psi \in Z^2(F, \mathbf{k}^\times)$ be a 2-cocycle, $W \subseteq V$ be an F-invariant subspace, and $\beta : W \times W \to \mathbf{k}$ be a symmetric F-invariant bilinear form. With this data we define $\mathcal{K}(W, \beta, F, \psi)$ the algebra generated by elements $w \in W, e_f, f \in F$ subject to relations, for any $w_1, w_2 \in W, f, h \in F$,

$$w_1w_2 + w_2w_1 = \beta(w_1, w_2)1, \quad e_f w = (f \cdot w)e_f, \quad e_f e_h = \psi(f, h)e_{fh}. \tag{2.3}$$

The algebra $\mathcal{K}(W,\beta,F,\psi)$ is a left $\mathcal{A}(V,G,u)$-comodule algebra with coaction $\lambda:\mathcal{K}(W,\beta,F,\psi)\to\mathcal{A}(V,G,u)\otimes\mathcal{K}(W,\beta,F,\psi)$ determined by

$$\lambda(w)=w\otimes 1+u\otimes w,\quad \lambda(e_f)=f\otimes e_f,\quad w\in W, f\in F.$$

When F,ψ are trivial $\mathcal{K}(W,\beta,1,1)$ is know as the Clifford algebra

$$Cl(W,\beta)=T(W)/\langle v\otimes w+w\otimes v=\beta(v,w)1\rangle.$$

Then $Cl(W,\beta)\subset\mathcal{K}(W,\beta,F,\psi)$ is a subalgebra.

Proposition 2.3. [12, Theorem 4.6] *The following assertions hold:*

1. *The categories $\mathcal{C}omod(\mathcal{A}(V,G,u))_{\mathcal{K}(W,\beta,F,\psi)}$ are exact indecomposable $\mathcal{C}omod(\mathcal{A}(V,G,u))$-modules.*
2. *A exact indecomposable $\mathcal{C}omod(\mathcal{A}(V,G,u))$-module category is equivalent to $\mathcal{C}omod(\mathcal{A}(V,G,u))_{\mathcal{K}(W,\beta,F,\psi)}$, for some (W,β,F,ψ) as in Section 2.4.*
3. *Let (W,β,F,ψ), (W',β',F',ψ') be two data as above. There is an equivalence of module categories*

$$\mathcal{C}omod(\mathcal{A}(V,G,u))_{\mathcal{K}(W,\beta,F,\psi)}\simeq\mathcal{C}omod(\mathcal{A}(V,G,u))_{\mathcal{K}(W',\beta',F',\psi')}$$

if and only if there exists $g\in G$ such that for all $w,v\in W$ $W'=g\cdot W$, $F=F'$, $\psi=\psi'$, and $\beta'(g\cdot w,g\cdot v)=\beta(w,v)$.

2.5 Invertible objects in group-theoretical fusion categories

Let G be a finite group, F a subgroup of G, $\omega\in Z(G,\mathbf{k}^\times)$ a 3-cocycle and $\psi\in C^2(F,\mathbf{k}^\times)$ a 2-cochain such that $d\psi=\omega_{F\times F\times F}$. We denote by Vec_G^ω the category of finite dimensional G-graded $\mathbf{k}$-vector spaces with associativity determined by ω. The twisted group algebra $\mathbf{k}_\psi[F]$ is an associative algebra in Vec_G^ω, so it is possible to define the category of $\mathbf{k}_\psi[F]$-bimodules in Vec_G^ω. This fusion category is denoted by $\mathcal{C}=\mathcal{C}(G,\omega,F,\psi)$ and it is said to be *group-theoretical.*

The classification of invertible objects in $\mathcal{C}(G,\omega,F,\psi)$ was given in [9]. For our purposes, we only need the classification when the 3-cocycle ω is trivial. Let $R:=\{u(X):X\in F\backslash G/F\}$, where $u(X)\in G$ is a representative of X in the double coset of F in G and assume $u(F1_GF)=1_G$. For each

$g \in G$, let $F^g := FCpgFg^{-1}$. In the group F^g we have the following 2-cocycle:

$$\psi^g(f_1, f_2) := \psi(f_1, f_2)\psi(g^{-1}f_2^{-1}g, g^{-1}f_1^{-1}g).$$

Theorem 2.4. [9, Theorem 5.1] *There is a bijection between:*

- *isomorphism classes of simple objects in $\mathcal{C}(G, 1, F, \psi)$, and*
- *isomorphism classes of pairs (g, ρ), where $g \in R$ and ρ is an irreducible projective representation of F^g with 2-cocycle ψ^g,*

where two pairs (g, ρ), (g, ρ') are isomorphic if ρ and ρ' are isomorphic as irreducible projective representations of F^g with 2-cocycle ψ^g.

We explicitly describe this correspondence: Let B be in $\mathcal{C}(G, 1, F, \psi)$, since it is a G-graded vector space, $B = \oplus_{g\in G} B_g$. Because B is a $\mathbf{k}_\psi[F]$-bimodule, for each $g \in G$, $f \in F$ we have isomorphisms $l_{f,g} : B_g \to B_{fg}$ and $r_{g,f} : B_g \to B_{gf}$, and they satisfy the following identities:

$$\psi(f_1, f_2)l_{f_1f_2,g} = l_{f_1,f_2g} \circ l_{f_2,g}, \psi(f_1, f_2)r_{g,f_1f_2} = r_{gf_1,f_2} \circ r_{g,f_1},$$

$$l_{f_1,gf_2} \circ r_{g,f_2} = r_{f_1g,h_2} \circ l_{f_1,g}.$$

We can write $B = \oplus_{X\in F\backslash G/F} \oplus_{g\in X} B_g$, where each $B_X := \oplus_{g\in X} B_g$ is a sub-bimodule of B.

Suppose that B is a simple object in $\mathcal{C}(G, 1, F, \psi)$, then there exist $g \in G$ such that $B = B_X = \oplus_{l\in X} B_l$, with $u(X) = g$. Then one gets a projective representation $\rho : F^g \to GL(B_g)$ with 2-cocycle ψ^g where

$$\rho(f) := r_{fg,g^{-1}f^{-1}g} \circ l_{f,g}, \quad f \in F.$$

Coversely, let (g, ρ) be a pair, where $g \in R$ and $\rho : F^g \to GL(V)$ is an irreducible projective representation with 2-cocycle ψ^g. With this data we construct a simple object B in $\mathcal{C}$ as follows. Let T be a set of representatives of F/F^g with $1 \in T$. Let $B := \oplus_{t\in T,k\in F} B_{tgk}$, where $B_{tgk} = V$ as vector spaces. The right and left module structure r and l, respectively, on B are given by:

$$r_{tgk,f} : B_{tgk} \to B_{tgkf}, v \mapsto \psi(k, f)v, \tag{2.4}$$

$$l_{f,tgk} : B_{tgk} \to B_{sg(g^{-1}pg)k}, v \mapsto \frac{\psi(f, t)}{\psi(s, p)\psi(g^{-1}p^{-1}g, g^{-1}pgk)}\rho(p)(v), \tag{2.5}$$

where $s \in T$, and $p \in F^g$ is determined by the equation $ft = sp$. Moreover, we explicitly know the group structure of this pairs.

Theorem 2.5. [9, Theorem 5.2] *The group $G(\mathcal{C}(G,1,F,\psi))$ of isomorphism clases of invertible objects of $\mathcal{C}$ is isomorphic to the group*

$$L = \{(g,\rho) \mid g \in K, \rho : F \to \mathbf{k}^{\times} \text{ such that } d\rho = \psi^g\}$$

where $K := \{g \in R \mid g \in N_G(F)$ and ψ^g is cohomologically trivial$\}$, with product given by

$$(g_1,\rho_1)\cdot(g_2,\rho_2) = (g_1\cdot g_2, \gamma(g_1,g_2)\rho_1({}^{g_1}\rho_2)),$$

with $({}^{g_1}\rho_2)(f) = \rho_2(g_1^{-1}fg_1), f \in F^{g_2}$ and $\gamma(g_1,g_2) : F \to \mathbf{k}^{\times}$

$$\gamma(g_1,g_2)(f) = \frac{\psi(g_2^{-1}g_1^{-1}fg_1g_2k, g_3^{-1}f^{-1}g_3)}{\psi(g_1^{-1}f^{-1}g_1, g_1^{-1}fg_1)\psi(g_2^{-1}g_1^{-1}f^{-1}g_1g_2, g_2^{-1}g_1^{-1}fg_1g_2k)},$$

for $g_3 = g_1g_2k, k \in F$.

Example 2.6. *G abelian and ω trivial.* For any $g \in G$ we have that $F^g = F$ and $\psi^g(f_1,f_2) = \psi(f_1,f_2)\psi(f_2^{-1},f_1^{-1})$, with $f_1,f_2 \in F$, then ψ^g is always cohomologically trivial, hence $K = R = F\backslash G/F$.

Invertible objects B in $\mathcal{C}(G,\omega,F,\psi) = {}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F}$ are parametrized by pairs $(g,\rho) \in L$, with $g \in R$ and ρ a 1-dimensional F-representation, are given by $B = \oplus_{f\in F}\mathbf{k}_{gf}$, where each component is equal to the field $\mathbf{k}$. The right and left module structures on B are given by 2.4 and 2.5, explicitly

$$\cdot : \mathbf{k}_\psi F \times B \to B \qquad\qquad \cdot : B \times \mathbf{k}_\psi F \to B$$

$$e_f \cdot p_{gk} = \psi((fg)^{-1}g, g^{-1}gfk)\rho(f)p_{gf}, \qquad p_{gk}\cdot e_f = \psi^{-1}(k,f)p_{gf}.$$

The product in L is $(g_1,\rho_1)\cdot(g_2,\rho_2) = (g_1\cdot g_2, \beta(g_1,g_2)\rho_1\rho_2)$, where $\gamma(g_1,g_2)(f) = \frac{\psi(fk,f^{-1})}{\psi(f^{-1},fk)}$, with $f \in F$ and $k = (g_1g_2)^{-1}u(Fg_1g_2F)$.

3 Picard group and the group of invertible objects in $\mathcal{C}omod(\mathcal{A}(V,G,u))^*_{\mathcal{M}}$

Given a monoidal category $\mathcal{C}$ the Picard group $\mathbf{Pic}(\mathcal{C})$ of $\mathcal{C}$ is the group of isomorphism classes of invertible objects (have an inverse under the tensor product). By Theorem 2.5, $\mathbf{Pic}(\mathcal{C}(G,1,F,\psi)) \simeq L$. If $H = \mathcal{A}(V,G,u)$ is the Hopf algebra presented in 2.3, then the comodule algebra $\mathcal{K} = \mathcal{K}(W,\beta,F,\psi)$, presented in 2.4, satisfies the hipotesis of Theorem 2.2. In Theorem 2.2(4) we show how to obtain an invertible object in ${}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F} = \mathcal{C}(G,1,F,\psi)$ from an invertible object in $\mathcal{C}omod(H)^*_{\mathcal{K}\mathcal{M}} \simeq {}^H_K\mathcal{M}_K$.

Let P be an invertible object in ${}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F}$. In the following we will construct an invertible object $I(P)$ in the category ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$. There exists an element $g \in G$ and ρ an 1-dimensional F-representation such that $P = \oplus_{f\in F}\mathbf{k}_{gf}$, see Example 2.6. Suppose that $g\cdot W \subseteq W$ and that $\beta(g^{-1}\cdot v, g^{-1}\cdot w) = \beta(v,w)$ for all $v, w \in W$. Define

$$I(P) = P \otimes_{\mathbf{k}} Cl(W,\beta),$$

where $Cl(W,\beta)$ is the Clifford algebra. The $\mathcal{K}$-bimodule structure in $I(P)$ is given as follows, for $f, h \in F$, $p \in P$ such that $w_1, \ldots, w_r, w \in W$:

$$\begin{aligned}
(p \otimes w_1 \ldots w_r) \cdot w &= p \otimes w_1 \ldots w_r w,\\
(p \otimes w_1 \ldots w_r) \cdot e_h &= p \cdot h \otimes (h^{-1}\cdot w_1)\ldots(h^{-1}\cdot w_r),\\
w \cdot (p \otimes w_1 \ldots w_r) &= p \otimes (g^{-1}\cdot w) w_1 \ldots w_r,\\
e_h \cdot (p \otimes w_1 \ldots w_r) &= (h \cdot p) \otimes w_1 \ldots w_r.
\end{aligned}$$

These actions are well defined by the previous assumptions. The H-comodule structure in $I(P)$ has to be a morphism of left $\mathcal{K}$-modules, thus it is determined by declaring $\lambda(p \otimes 1) = p_{-1} \otimes (p_0 \otimes 1)$, for any $p \in P$. In particular, for any $w \in W$ we have that $\lambda(p\otimes w) = p_{-1}w \otimes (p_0 \otimes 1) + p_{-1}u \otimes (p_0 \otimes w)$. Clearly, we have that $I(\mathbf{k}_\psi F) = \mathcal{K}$.

Lemma 3.1. *$I(P)$ is an object in the category ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$.*

Proof. By definition $I(P)$ is a right $\mathcal{K}$-module and a left $\mathcal{K}$-module.

$I(P)$ is a $\mathcal{K}$-bimodule: let $p \in P$,$w_1, \ldots, w_r, w, w' \in W$, $h, l \in F$

$$\begin{aligned}
(w \cdot (p \otimes w_1 \ldots w_r)) \cdot w' &= (p \otimes (g^{-1}\cdot w) w_1 \ldots w_r) \cdot w'\\
&= p \otimes (g^{-1}\cdot w) w_1 \ldots w_r w'\\
&= w \cdot (p \otimes w_1 \ldots w_r w') = w \cdot ((p \otimes w_1 \ldots w_r) \cdot w'),\\
(w \cdot (p \otimes w_1 \ldots w_r)) \cdot e_h &= (p \otimes (g^{-1}\cdot w) w_1 \ldots w_r) \cdot e_h\\
&= p \cdot h \otimes (h^{-1}\cdot g^{-1}\cdot w)(h^{-1}\cdot w_1)\ldots(h^{-1}\cdot w_r)\\
&= w \cdot (p \cdot h \otimes (h^{-1}\cdot w_1)\ldots(h^{-1}\cdot w_r))\\
&= w \cdot ((p \otimes w_1 \ldots w_r) \cdot e_h),\\
(e_h \cdot (p \otimes w_1 \ldots w_r)) \cdot w &= (h \cdot p \otimes w_1 \ldots w_r) \cdot w\\
&= h \cdot p \otimes w_1 \ldots w_r w = e_h \cdot ((p \otimes w_1 \ldots w_r) \cdot w),
\end{aligned}$$

$$\begin{aligned}(e_h \cdot (p \otimes w_1 \dots w_r)) \cdot e_l &= (h \cdot p \otimes w_1 \dots w_r) \cdot e_l \\ &= h \cdot p \cdot l \otimes (l^{-1} \cdot w_1) \dots (l^{-1} \cdot w_r) \\ &= e_h \cdot (p \cdot l \otimes (l^{-1} \cdot w_1) \dots (l^{-1} \cdot w_r)), \\ &= e_h \cdot ((p \otimes w_1 \dots w_r) \cdot e_l)\end{aligned}$$

$I(P)$ is an H-Comodule:

$$\begin{aligned}&((\Delta \otimes Id) \circ \lambda)(p \otimes w) = (\Delta \otimes Id)(p_{-1}w \otimes (p \otimes 1) + p_{-1}u \otimes (p_0 \otimes w)) \\ &\quad = (p_{-1} \otimes p_{-1})(w \otimes 1 + u \otimes w) \otimes (p \otimes 1) + (p_{-1}u \otimes p_{-1}u) \otimes (p_0 \otimes w) \\ &\quad = (p_{-1}w \otimes p_{-1} + p_{-1}u \otimes p_{-1}w) \otimes (p \otimes 1) + (p_{-1}u \otimes p_{-1}u) \otimes (p_0 \otimes w), \\ &((Id \otimes \lambda) \circ \lambda)(p \otimes w) = (Id \otimes \lambda)(p_{-1}w \otimes (p \otimes 1) + p_{-1}u \otimes (p_0 \otimes w)) \\ &\quad = p_{-1}w \otimes (p_{-1} \otimes (p_0 \otimes 1)) + p_{-1}u \otimes ((p_{0-1}w \otimes (p_{00} \otimes 1) \\ &\quad + p_{0-1}u \otimes (p_{00} \otimes w)), \\ &\ 1 \otimes (\varepsilon(p_{-1}u) \otimes p_0 \otimes w) = \varepsilon(p_{-1})\varepsilon(u) \otimes p_0 \otimes w = p \otimes w.\end{aligned}$$

□

Lemma 3.2. *Let (P, g) and (Q, r) be invertible objects in ${}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F}$ such that $g \cdot W \subseteq W$, $\beta(g^{-1} \cdot v, g^{-1} \cdot w) = \beta(v, w)$ for all $v, w \in W$ and $r \cdot W \subseteq W$, $\beta(r^{-1} \cdot v, r^{-1} \cdot w) = \beta(v, w)$ for all $v, w \in W$. Then, there is an isomorphism in the category ${}^{H}_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$*

$$I(P \otimes_{\mathbf{k}_\psi F} Q) \simeq I(P) \otimes_{\mathcal{K}} I(Q).$$

In particular, $I(P)$ is an invertible object in ${}^{H}_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$.

Proof. Recall that

$$I(P) \otimes_{\mathcal{K}} I(Q) = (P \otimes_{\mathbf{k}} Cl(W, \beta)) \otimes_{\mathcal{K}} (Q \otimes_{\mathbf{k}} Cl(W, \beta)),$$

$$I(P \otimes_{\mathbf{k}_\psi F} Q) = (P \otimes_{\mathbf{k}_\psi F} Q) \otimes_{\mathbf{k}} Cl(W, \beta).$$

Since associators are trivial $P \otimes_{\mathbf{k}_\psi F} (Q \otimes_{\mathbf{k}} Cl(W, \beta)) \simeq (P \otimes_{\mathbf{k}_\psi F} Q) \otimes_{\mathbf{k}} Cl(W, \beta)$, we induce the morphism

$$\begin{aligned}(P \otimes_{\mathbf{k}} Cl(W, \beta)) \otimes_{\mathcal{K}} (Q \otimes_{\mathbf{k}} Cl(W, \beta)) &\xrightarrow{F} (P \otimes_{\mathbf{k}_\psi F} Q) \otimes_{\mathbf{k}} Cl(W, \beta) \\ (p \otimes w) \otimes_{\mathcal{K}} (q \otimes v) &\mapsto (p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)v.\end{aligned}$$

F is well defined: Since the $\mathbf{k}G$-comodule structure over Q is given by $q_{-1} \otimes q_0 = r \otimes q$ then for $w' \in \mathcal{K}$

$$\begin{aligned} F((p \otimes w) \cdot w' \otimes_{\mathcal{K}} (q \otimes v)) =& (p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot (ww'))v \\ =& (p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)(q_{-1}^{-1} \cdot w')v \\ =& (p \otimes_{\mathbf{k}_\psi F} q) \otimes (r^{-1} \cdot w)(r^{-1} \cdot w')v \\ =& F((p \otimes w) \otimes_{\mathcal{K}} (q \otimes (r^{-1} \cdot w')v)) \\ =& F((p \otimes w) \otimes_{\mathcal{K}} w' \cdot (q \otimes v)) \end{aligned}$$

F is a $\mathcal{K}$-bimodule morphism: Let $z \in W$ and $f \in G$.

$$\begin{aligned} F([(p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)] \cdot z) &= F((p \otimes w) \otimes_{\mathcal{K}} [(q \otimes v) \cdot z]) \\ &= F((p \otimes w) \otimes_{\mathcal{K}} (q \otimes vz)) \\ &= (p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)vz \\ &= [(p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)v)] \cdot z \\ &= F((p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)) \cdot z, \\ F([(p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)] \cdot e_f) &= F((p \otimes w) \otimes_{\mathcal{K}} ((q \otimes v) \cdot e_f)) \\ &= F((p \otimes w) \otimes_{\mathcal{K}} ((q \cdot f) \otimes (f^{-1} \cdot v))) \\ &= (p \otimes_{\mathbf{k}_\psi F} (q \cdot f)_0) \otimes ((q \cdot f)_{-1}^{-1} \cdot w)(f^{-1} \cdot v) \\ &= (p \otimes_{\mathbf{k}_\psi F} (q_0 \cdot f) \otimes (f^{-1} \cdot (q_{-1}^{-1} \cdot w))(f^{-1} v) \\ &= [(p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)v] \cdot e_f \\ &= F((p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)) \cdot e_f. \end{aligned}$$

$$\begin{aligned} F(z \cdot [(p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)]) &= F((z \cdot (p \otimes w)) \otimes_{\mathcal{K}} (q \otimes v)) \\ &= F((p_0 \otimes (p_{-1}^{-1} \cdot z)w) \otimes_{\mathcal{K}} (q \otimes v)) \\ &= (p_0 \otimes_{\mathbf{k}_\psi F} q_{00}) \otimes (q_{0-1}^{-1} \cdot (p_{-1}^{-1} \cdot z))(q_{-1}^{-1} \cdot w)v \\ &= z \cdot [(p \otimes_{\mathbf{k}_\psi F} q_0) \otimes (q_{-1}^{-1} \cdot w)v] \\ &= z \cdot F((p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)), \\ F(e_f \cdot [(p \otimes w) \otimes_{\mathcal{K}} (q \otimes v)]) &= F((e_f \cdot (p \otimes w)) \otimes_{\mathcal{K}} (q \otimes v)) \\ &= F(((f \cdot p) \otimes w) \otimes_{\mathcal{K}} (q \otimes v)) \end{aligned}$$

$$
\begin{aligned}
&= ((f\cdot p)\otimes_{\mathbf{k}_\psi F} q_0)\otimes(q_{-1}^{-1}\cdot w)v\\
&= f\cdot(p\otimes_{\mathbf{k}_\psi F} q_0)\otimes(q_{-1}^{-1}\cdot w)v\\
&= e_f\cdot[(p\otimes_{\mathbf{k}_\psi F} q_0)\otimes(q_{-1}^{-1}\cdot w)v]\\
&= e_f\cdot F((p\otimes w)\otimes_{\mathcal{K}}(q\otimes v)).
\end{aligned}
$$

F is a H-comodule morphism: Let $h, f \in G$ such that $\lambda(p) = h\otimes p$, $\lambda(q) = f\otimes q$:

$$
\begin{aligned}
&((\mathrm{id}_H\otimes F)\circ\lambda)((p\otimes w)\otimes_{\mathcal{K}}(q\otimes v)) = (\mathrm{id}_H\otimes F)[(hw)(fv)\otimes((p\otimes 1)\otimes\\
&\quad q\otimes 1)) + (hw)(fu)\otimes((p\otimes 1)\otimes(q\otimes v)+\\
&\quad (hu)(fv)\otimes((p\otimes w)\otimes(q\otimes 1)) + (hu)(fu)\otimes((p\otimes w)\otimes(q\otimes v))]\\
&= (hw)(fv)\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes 1) + (hw)(fu)\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes v)+\\
&\quad (hu)(fv)\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)) + (hu)(fu)\\
&\quad \otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)v)]\\
&= (h\cdot w)((hf)\cdot v)hf\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes 1) + (h\cdot w)hfu\otimes((p\otimes_{\mathbf{k}_\psi F} q)\\
&\quad \otimes v) + ((huf)\cdot v)huf\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)) + hfu^2\otimes\\
&\quad ((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)v)].\\
&(\lambda\circ F)((p\otimes w)\otimes_{\mathcal{K}}(q\otimes v)) = \lambda((p\otimes_{\mathbf{k}_\psi F} q)\otimes((f^{-1}\cdot w)v)\\
&= (hf)(f^{-1}\cdot w)v\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes 1) + (hf)(f^{-1}\cdot w)u\otimes((p\otimes_{\mathbf{k}_\psi F} q)\\
&\quad \otimes v) + ((hf)uv\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)) + hfu^2\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes\\
&\quad (f^{-1}\cdot w)v)]\\
&= (h\cdot w)((hf)\cdot v)hf\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes 1) + (h\cdot w)hfu\otimes((p\otimes_{\mathbf{k}_\psi F} q)\\
&\quad \otimes v) + ((huf)\cdot v)huf\otimes((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)) + hfu^2\otimes\\
&\quad ((p\otimes_{\mathbf{k}_\psi F} q)\otimes(f^{-1}\cdot w)v)].
\end{aligned}
$$

Since $F(p\otimes 1\otimes 1\otimes v) = p\otimes q\otimes v$, F is an epimorphism and

$$\dim\ (I(P)\otimes_{\mathcal{K}} I(Q)) = \dim\ \mathcal{K}, \dim\ I(P\otimes_{\mathbf{k}_\psi F} Q) = \dim\ \mathcal{K},$$

using the fact that $\dim I(P) = \dim\mathcal{K} = \dim I(Q)$. We conclude that F is an isomorphism in ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$. □

Now, let M be an invertible object in the category ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$. Since M_0, the socle of M see Section 2.1, is an invertible object in ${}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F}$, there exists an element $g \in G$ and a representation ρ such that $M_0 = \oplus_{f \in F}\mathbf{k}_{gf} = m \cdot \mathbf{k}_\psi F$, where $\lambda(m) = g \otimes m$ (Proposition 2.2(2)).

Lemma 3.3. *Under the above condition,*

1. $g \cdot W \subseteq W$ *and* $\beta(g^{-1} \cdot v, g^{-1} \cdot w) = \beta(v, w)$ *for all* $v, w \in W$.
2. $M \simeq I(M_0)$ *in* ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$.

Proof. (1) Let $w \in W$, then $w \cdot m \in M_1$, by definition. Because $M_1 = m \cdot K_1$, there exists $k \in K_1$ such that $w \cdot m = m \cdot k$. Now, $K_1 = (W \otimes \mathbf{k}_\psi F) \oplus \mathbf{k}_\psi F$, so we can write $k = \sum ve_f + \sum e_h$. Hence, we have the following identity

$$w \cdot m = m \cdot \sum ve_f + m \cdot \sum e_h.$$

Applying the coaction we get:

$$wg \otimes m + ug \otimes w \cdot m = \sum(gvf \otimes m \cdot e_f + guf \otimes m \cdot (ve_f)) + \sum gh \otimes m \cdot e_h.$$

Note that $wg \otimes m \in H(1) \otimes M_0, ug \otimes w \cdot m \in H(0) \otimes M_1, gvf \otimes m \cdot e_f \in H(1) \otimes M_0, guf \otimes m \cdot (ve_f)) \in H(0) \otimes M_1, gh \otimes m \cdot c_h e_h \in H(0) \otimes M_0$. Hence

$$wg \otimes m = \sum_f gvf \otimes m \cdot e_f \in H(1) \otimes M_0,$$

$$ug \otimes w \cdot m = \sum_f guf \otimes m \cdot (ve_f) + \sum_h gh \otimes m \cdot c_h e_h \in H(0) \otimes M_1.$$

If we think the last equation in $H(0) \otimes M(1)$, we have that

$$ug \otimes \overline{w \cdot m} = \sum_f guf \otimes \overline{m \cdot (ve_f)}.$$

But $\{guf\}_f$ are linearly independent, so the sum has only one term, then $f = e$ and

$$wg \otimes m = gv \otimes m, \quad wg = gv,$$

which implies that $wg = (g \cdot v)g$, so $w = g \cdot v$ and $v = g^{-1} \cdot w$. And that proves that $g \cdot W \subseteq W$. In particular

$$w \cdot m = m \cdot (g^{-1} \cdot w) + m \cdot \sum e_h. \tag{3.1}$$

Let $\{w_i\}_i$ be a base for W. For each $f \in F$ we define

$$P_{i,f} = \{y \in M_1 : \lambda(y) = \mu w_i f g \otimes f \cdot m + u f g \otimes y, \mu \in \mathbf{k}\}.$$

The sets $P_{i,f}$ are non-trivial vector spaces, since $(w_i f) \cdot m \in P_{i,f}$, hence dim $P_{i,f} \geq 1$. Note that $P_{i,f} \mathcal{C}p P_{i',f'} = \{0\}$, and since $M_0 \mathcal{C}p P_{i,f} = 0$, for all i, f and dim M_1=$|F|$dim$W + |F| > \dim(M_0) + \sum_{i,f} \dim(P_{i,f})$ then $|F|$dim$W + |F| > \sum_{i,f} \dim(P_{i,f})$ and dim $P_{i,f} = 1$. Now,

$$\lambda(w_i \cdot m) = w_i g \otimes m + ug \otimes w_i \cdot m,$$

$$\lambda(m \cdot (g^{-1} \cdot w_i)) = w_i g \otimes m + gu \otimes m \cdot (g^{-1} \cdot w_i),$$

then $w_i \cdot m, m \cdot (g^{-1} \cdot w_i) \in P_{i,e}$. So $w_i \cdot m = \xi_i m \cdot (g^{-1} \cdot w_i)$ for some $\xi_i \in \mathbf{k}$ but Equation (3.1) implies $\xi_i = 1$ and for all $w \in W$.

$$w \cdot m = m \cdot (g^{-1} \cdot w) \tag{3.2}$$

Now, let $v, w \in W$. We have that $vw + wv = \beta(v, w)1$, then

$$\beta(v, w)1 \cdot m = (vw + wv) \cdot m = m \cdot ((g^{-1} \cdot v)(g^{-1} \cdot w) + (g^{-1} \cdot w)(g^{-1} \cdot v)) =$$

$$m \cdot \beta(g^{-1} \cdot v, g^{-1} \cdot w)1,$$

hence $\beta(v, w) = \beta(g^{-1} \cdot v, g^{-1} \cdot w)$, for all $v, w \in W$.

(2) Defined $E : I(M_0) \to M$ for $p \in M$, $w_i \in W$

$$E(p \otimes w_1 \dots w_r) = p \cdot (w_1 \dots w_r).$$

E is a left $\mathcal{K}$-module morphism: Since $p_{-1} \otimes p_0 = g \otimes p$ and $w \cdot p = p \cdot (g^{-1} \cdot w)$, then

$$\begin{aligned} E(w \cdot (p \otimes w_1 \dots w_r)) &= E(p_0 \otimes (p_{-1}^{-1} \cdot w) w_1 \dots w_r) \\ &= p_0 \cdot [(p_{-1}^{-1} \cdot w) w_1 \dots w_r] \\ &= w \cdot [p \cdot (w_1 \dots w_r)] \\ &= w \cdot E(p \otimes w_1 \dots w_r). \end{aligned}$$

Similar calculations prove that E is a $\mathcal{K}$-bimodule morphism. Since E is induced by an action, E is a H-comodule morphism, so E is a morphism in ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$. Now, since $M = m \cdot \mathcal{K}$ and $M_0 = m \cdot \mathbf{k}_\psi F$, E is an epimorphism. But M and $I(M_0)$ have the same dimension, hence E is an isomorphism in ${}^H_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$. □

In other words, we conclude that as groups

$$\mathbf{P}({}^{H}_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}) \simeq \mathcal{L} \subset \mathbf{P}({}^{\mathbf{k}G}_{\mathbf{k}_\psi F}\mathcal{M}_{\mathbf{k}_\psi F}),$$

where each invertible bimodule in ${}^{H}_{\mathcal{K}}\mathcal{M}_{\mathcal{K}}$ parametrized by (g, ρ) is, as vector space, $M = (\oplus_{f\in F}\mathbf{k}_{gf}) \otimes Cl(W, \beta)$ with structure given in Lemma 3.1.

Proposition 3.4. *The Picard group of $\mathcal{C}omod(H)^*_{\mathcal{M}}$ is isomorphic to $\mathcal{L}$.*

Therefore, for ψ, ψ' such that $\mathcal{K}(W, \beta, F, \psi), \mathcal{K}(W, \beta, F, \psi')$ are comodule alegras, we obtain that

$$\mathbf{Pic}(\mathcal{C}omod(\mathcal{A}(V, G, u))^*_{\mathcal{M}_{(W,\beta,F,\psi)}}) \simeq \mathbf{Pic}(\mathcal{C}omod(\mathcal{A}(V, G, u))^*_{\mathcal{M}_{(W,\beta,F,\psi')}}).$$

Corollary 3.5. *The Picard group of $\mathcal{C}omod(H)^{\mathbf{op}}$ is isomorphic to $\hat{G}$.*

Proof. Since $\mathcal{C}omod(H)^{\mathbf{op}} \simeq \mathcal{C}omod(H)^*_H = \mathcal{C}omod(H)^*_{\mathcal{M}_{(V,0,G,1)}}$, then the quotient $F\backslash G/F$ is trivial and $R = \{1\}$. □

References

[1] Andruskiewitsch N., Etingof P., and Gelaki, S., *Triangular Hopf algebras with the Chevalley property*, Michigan Math. J. 49(2) (2001), 277-298.

[2] Davydov A. and Nikshych D., *The Picard crossed module of a braided tensor category*, Algebra and Number Theory 7(6) (2013), 1365-1403.

[3] Davydov A. and Nikshych D., *Braided Picard groups and graded extensions of braided tensor categories*, Selecta Math. 27(4) (2021), article no 65.

[4] Etingof P., Nikshych D., and Ostrik V., *Fusion categories and homotopy theory*, Quantum Topol. 1(3) (2010), 209-273.

[5] Frohlich J., Fuchs J., Runkel I., and Schweigert C., *Picard groups in rational conformal field theory*, Contemp. Math. 391 (2005), 1-16.

[6] Fausk H., *Picard groups of derived categories*, J. Pure Appl. Algebra 180(3) (2003), 251-261.

[7] Fuchs L. and Vámos P., *Picard groups of some module categories*, J. Commut. Algebra 11(4) (2019), 547-572.

[8] García Iglesias A. and Mombelli M., *Representations of the category of modules over pointed Hopf algebras over* $\mathbb{S}_3$ *and* $\mathbb{S}_4$, Pacific J. Math. 252(2) (2011), 343-378.

[9] Gelaki S. and Naidu N., *Some properties of group-theoretical categories*, J. Algebra 322(8) (2009), 2631-2641.

[10] Mejia Castaño A., *Morita equivalence of certain crossed product*, Categ. Gen. Algebr. Struct. Appl. 19(1) (2023), 103-125.

[11] Mombelli M., *Module categories over pointed Hopf algebras*, Math. Z. 266(2) (2010), 319-344.

[12] Mombelli M., *Representations of tensor categories coming from quantum linear spaces*, J. London Math. Soc. 2(83) (2011), 19-35.

[13] Ostrik V., *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups 8 (2003), 177-206.

[14] Rodríguez J., "On the Picard Group of the Stable Module Category for Infinite Groups", Ph.D. Thesis, Centro de Investigación en Matemáticas, 2023.

Adriana Mejía Castaño *Department of Mathematics and Statistics, Universidad del Norte, Barranquilla, Colombia.*

Email: mejiala@uninorte.edu.co