Categories and
General Algebraic Structures
with Applications
Volume 23, Number 1, July 2025, 61-77.
https://doi.org/10.48308/cgasa.23.1.61

Characterizing cogenerating and finitely cogenerated S-acts

Roghaieh Khosravi*, Xingliang Liang, and Mohammad Roueentan

Abstract. In this paper, we introduce cogenerating classes of S-acts as those that can be used to cogenerate S-acts in an appropriate sense. Next, finitely cogenerated S-acts are characterized by the property that their socle is finitely cogenerated and large in the S-act. Further, we investigate the S-acts cogenerating S_S, or generating the injective envelope $E(S)$ of S_S. This leads us to introduce the classes of cofaithful and subgenerator S-acts as the dual notions of faithful S-acts, which lie strictly between the classes of generator and faithful S-acts. Ultimately, we study relations between the cogenerating classes, finitely cogenerated S-acts, and the recently introduced new classes of S-acts.

1 Introduction and Preliminaries

The important, categorical concepts of generating and cogenerating objects play a crucial role in every concrete category. The notion of a cogenerator, as the dual concept of a generator, is of great importance in category theory.

* Corresponding author
Keywords: S-acts, cogenerator, subgenerator, finitely cogenerated, cofaithful.
Mathematics Subject Classification [2010]: 20M30.
Received: 12 May 2024, Accepted: 26 July 2024.
ISSN: Print 2345-5853, Online 2345-5861.

For instance, in most categories, each required injective object can be written as a product of some cogenerator, or more generally, as a cogenerating set of objects.

Many books have discussed categories and functors also covering the concept of cogenerators. See [8, 9] for example. Moreover, several monographs, including [16], have investigated these concepts in the category of modules over rings. In [14], Normak studied cogenerator S-acts in the category of S-acts. Additionally, Knauer and Normak found a relation between cogenerators and subdirectly irreducible S-acts in [12]. Such investigations have been continued later, see [3, 13, 15] for more information. For basic definitions and terminology related to acts over monoids, we refer the reader to [9].

In this paper, we concentrate on the concepts of cogenerating S-acts and finitely cogenerated S-acts. To do so, we begin with the definition of cogenerators in arbitrary categories. Then, we derive some special properties of cogenerators in the category of S-acts.

Let $\mathcal{C}$ be a concrete category. Recall that an object C of $\mathcal{C}$ is called a *cogenerator* (or a *coseparator*) in $\mathcal{C}$ if the functor $Mor_{\mathcal{C}}(-, C)$ is faithful. This means that for any $X, Y \in \mathcal{C}$ and any $f, g \in Mor_{\mathcal{C}}(X, Y)$ with $f \neq g$, there exists $\beta \in Mor_{\mathcal{C}}(Y, C)$ such that $\beta f \neq \beta g$. Equivalently, an object K is a cogenerator in $\mathcal{C}$ if and only if for every $X \in \mathcal{C}$, there exists a monomorphism $X \longrightarrow \prod K$.

Now, we define the categorically generalized notion, namely, cogenerator in arbitrary categories. Let $\mathcal{U}$ be a non-empty set (class) of objects of a category $\mathcal{C}$. An object B in $\mathcal{C}$ is said to be *cogenerated by* $\mathcal{U}$, or $\mathcal{U}$-cogenerated, if for every pair of distinct morphisms $f, g : A \longrightarrow B$ in $\mathcal{C}$, there exists a morphism $h : B \longrightarrow U$ with $U \in \mathcal{U}$ and $hf \neq hg$. Then, $\mathcal{U}$ is called a set (class) of cogenerators for B.

From now on, we focus on the category of S-acts. Throughout the paper, A_S are used to denote a right S-act over a monoid S. Let $\mathcal{U}$ be a class of S-acts. An S-act A_S is (finitely) cogenerated by $\mathcal{U}$ in case there is a (finite) indexed set $(U_i)_{i \in I}$ in $\mathcal{U}$ and a monomorphism $A \longrightarrow \prod_{i \in I} U_i$. If $\mathcal{U} = \{U\}$ is a singleton, then we simply say that U (finitely) cogenerates A.

In the remainder of this section, we introduce the notion of socle for S-acts. To do this, we need to recall some concepts of S-acts. An S-act is called *simple* if it contains no subacts other than itself, and *θ-simple* if it contains no subacts other than itself and the one element subact Θ. An equivalence relation ρ on an S-act A is called a *congruence* on A_S, if $a\rho a'$ implies $(as)\rho(a's)$ for $a, a' \in A,\ s \in S$. The set of all congruences on A_S is denoted by $Con(A)$. Clearly, $\Delta_A = \{(a,a) \mid a \in A\}, \nabla_A = A \times A \in Con(A)$. Recall from [3] that a monomorphism $f : A \longrightarrow B$ of S-acts is said to be *essential* if for each homomorphism $g : B \longrightarrow C$, g is a monomorphism whenever gf is. If f is an inclusion map, then B is said to be an *essential extension* of A, or A is called *large* in B. In this situation, we write $A \subseteq' B$. It follows from [9, Lemma 3.1.15] that $A \subseteq' B$ if and only if for every non-trivial $\theta \in Con(B)$, $\theta \cap \rho_A \neq \Delta_B$. Recall from [6] that if S contains a zero, a non-zero subact B of A_S is called *intersection large* if for all non-zero subacts C of A_S, $B \cap C \neq \Theta$, and will be denoted by B is $\cap$-large in A_S. It follows from [6, Proposition 4.7] that every large subact of A_S is $\cap$-large, but the converse is not true.

In module theory (see [2]), the socle of a module is defined to be the sum of the minimal nonzero its submodules, equivalently, the intersection of its essential submodules. For S-acts, we define two notions as follows.

Definition 1.1. Let A be an S-act.

(i) $\mathrm{Soc}(A) = \bigcap\{L \subseteq A \mid L \subseteq' A\}$.

(ii) If a monoid S contains a zero, we define

$$S(A) = \bigcap\{L \subseteq A \mid L \text{ is } \cap-\text{large in } A\}.$$

If $\mathrm{Soc}(A) \neq \emptyset$, then $\mathrm{Soc}(A)$ is a subact of A. By an argument closely resembling the proof in module theory, one can show that if S contains a zero,

$$S(A) = \bigcup\{L \subseteq A \mid L \text{ is a } \theta-\text{simple subact of } A\}.$$

Obviously, $S(A) \subseteq \mathrm{Soc}(A)$. However, unlike the case of module theory, the converse can not be valid in general. For instance, if $S = (\mathbb{N}, max) \cup \{\infty\}$, it is not difficult to see that $S(S) = \{\infty\} \subsetneq \mathrm{Soc}(S) = S$.

2 Cogenerating S-acts

In this section, we consider cogenerators for a class of S-acts as a common generalization of cogenerators in the category of S-acts, and obtain some characterizations of cogenerators.

Definition 2.1. Let $\mathcal{C}$ be a class of S-acts. An S-act A is (*finitely*) *cogenerated* by $\mathcal{C}$ (or $\mathcal{C}$ (finitely) cogenerates A) in case there is a (finite) indexed set $(C_i)_{i\in I}$ in $\mathcal{C}$ and a monomorphism $A \longrightarrow \prod_{i\in I} C_i$.

We make the obvious adjustments in terminology if $\mathcal{C} = \{C\}$ is a singleton. The class of all S-acts cogenerated by $\mathcal{C}$ is denoted by $Cog(\mathcal{C})$. Also, $FCog(\mathcal{C})$ denotes the class of all S-acts which are finitely cogenerated by $\mathcal{C}$. An S-act C is a cogenerator for $Cog(\mathcal{C})$ in case $Cog(\mathcal{C}) = Cog(C)$. A cogenerator for the class of all S-acts is simply called a *cogenerator*, without any reference to the class.

The proof of the following proposition is similar to that of [9, Theorem 2.4.18].

Proposition 2.2. *Let A_S and B_S be S-acts. The following are equivalent.*

(i) *B_S (finitely) cogenerates A_S.*

(ii) *There exists a (finite) subset H of* $\mathrm{Hom}(A, B)$ *with* $\bigcap_{h\in H} \ker h = \Delta_A$.

(iii) *For any S-act X_S and any $f, g \in \mathrm{Hom}(X, A)$ with $f \neq g$, there exists $\beta \in \mathrm{Hom}(A, B)$ such that $\beta f \neq \beta g$.*

Now we give a definition of the cotrace of a class of S-acts which will be useful to characterize cogenerators later on.

Definition 2.3. Let $\mathcal{C}$ be a class of S-acts. The *cotrace* of $\mathcal{C}$ in an S-act A_S is defined by
$$cotr_{A_S}(\mathcal{C}) = \bigcap\{\ker g \mid g : A_S \longrightarrow C, \text{ for some } C \in \mathcal{C}\} = \bigcap_{C\in\mathcal{C}} Cog(C).$$

In particular, when $\mathcal{C} = \{C\}$ is a singleton, Definition 2.3 is the definition of the cotrace of C in A as mentioned in [9, Definition 2.4.16], which is denoted by
$$cotr_{A_S}(C_S) = \bigcap_{g\in \mathrm{Hom}(A_S, C_S)} \ker g.$$
Note that $cotr_{A_S}(\mathcal{C})$ is a subact of $A_S \prod A_S$. By [9, Theorem 2.4.18], an S-act C is a *cogenerator* in the category of S-acts if and only if for every S-act A_S, $cotr_{A_S}(C_S) = \Delta_A$.

Proposition 2.4. *Let $\mathcal{C}$ be a class of S-acts, and let A_S be an S-act. Then, $cotr_{A_S}(\mathcal{C})$ is the unique smallest congruence ρ of A_S such that A/ρ is cogenerated by $\mathcal{C}$.*

Proof. Let $\{C_i\}_{i\in I}$ be an indexed set in $\mathcal{C}$ such that

$$\rho = cotr_{A_S}(\mathcal{C}) = \bigcap\{\ker g_i \mid g_i : A_S \longrightarrow C_i,\ i \in I\}.$$

Define $g = \prod_{i\in I} g_i : A \longrightarrow \prod_{i\in I} C_i$ by $g(a) = (g_i(a))_{i\in I}$. It can be easily checked that $\rho = \ker g$. Using the homomorphism theorem for S-acts, we find a monomorphism $g' : A/\rho \longrightarrow \prod_{i\in I} C_i$ with $g'([a]_\rho) = g(a)$. Then $A/\rho \in Cog(\mathcal{C})$. Now, suppose that σ is a congruence on A such that $A/\sigma \in Cog(\mathcal{C})$. Then, there exists a monomorphism $f : A/\sigma \longrightarrow \prod_{j\in J} C_j$. So, $f\pi : A \longrightarrow \prod_{j\in J} C_j$ and $\sigma = \ker f\pi$. This implies $f_j = \pi_j f\pi : A \longrightarrow C_j$, and that $\sigma = \ker f\pi = \bigcap_{j\in J} \ker f_j$. Thus, $\rho = cotr_{A_S}(\mathcal{C}) \subseteq \bigcap_{j\in J} \ker f_j = \sigma$, and the result follows. □

Using the previous proposition, we obtain the following result.

Corollary 2.5. *Let A_S be an S-act, and $\mathcal{C}$ be a class of S-acts. The following hold.*

(i) *$\mathcal{C}$ cogenerates A_S if and only if $cotr_{A_S}(\mathcal{C}) = \Delta_A$.*

(ii) *Let σ be a congruence on A_S. Then, $\sigma = cotr_{A_S}(\mathcal{C})$ if and only if $\sigma \subseteq cotr_{A_S}(\mathcal{C})$ and $cotr_{A_S/\sigma}(\mathcal{C}) = \Delta_{A/\sigma}$.*

Lemma 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of S-acts. If $\mathcal{D} \subseteq Cog(\mathcal{C})$, then $Cog(\mathcal{D}) \subseteq Cog(\mathcal{C})$ and $cotr_{A_S}(\mathcal{C}) \subseteq cotr_{A_S}(\mathcal{D})$ for each S-act A_S.*

Proof. The first part is obvious. To prove the second part, suppose that $(a, a') \notin cotr_{A_S}(\mathcal{D})$. Then there exists a homomorphism $f : A \longrightarrow D$ with $(a, a') \notin \ker f$ for some $D \in \mathcal{D}$, that is, $f(a) \neq f(a')$. Since $D \in Cog(\mathcal{C})$, there exists a homomorphism $h : D \longrightarrow C$ with $(f(a), f(a')) \notin \ker h$ for some $C \in \mathcal{C}$. Now, we obtain $hf : A \longrightarrow C$ with $(a, a') \notin \ker hf$. So, $(a, a') \notin cotr_{A_S}(\mathcal{C})$. □

Proposition 2.7. *Let $\mathcal{C}$ be a class of S-acts.*

(i) *If $A \in Cog(\mathcal{C})$ $(FCog(\mathcal{C}))$ and $g : A' \longrightarrow A$ is a monomorphism, then $A' \in Cog(\mathcal{C})$ $(FCog(\mathcal{C}))$.*

(ii) *If $(A_i)_{i\in I} \in Cog(\mathcal{C})$ $(FCog(\mathcal{C}))$, then $\prod_{i\in I} A_i$ is in $Cog(\mathcal{C})$ $(FCog(\mathcal{C}))$.*

Proof. (i). Let $A \in Cog(\mathcal{C})$ and $g : A' \longrightarrow A$ be a monomorphism. Then there exists a monomorphism $f : A \longrightarrow \prod_{i\in I} C_i$, where $C_i \in \mathcal{C}$ for each $i \in I$. So, $fg : A' \longrightarrow \prod_{i\in I} C_i$ is a monomorphism.

(ii). Let $(A_i)_{i\in I} \in Cog(\mathcal{C})$. Then for each $i \in I$, there exists a monomorphism $f_i : A_i \longrightarrow \prod_{j_i\in J_I} C_{j_i}$, where $C_{j_i} \in \mathcal{C}$ for each $j_i \in J_I$. Therefore,

$$\prod_{i\in I} f_i : \prod_{i\in I} A_i \longrightarrow \prod_{i\in I}(\prod_{j_i\in J_I} C_{j_i})$$

is a monomorphism, and the result follows. □

The following results can be proved similar to Proposition 2.7.

Proposition 2.8. *If $\mathcal{C}$ is the set $\{C_i \mid i \in I\}$ of S-acts, then the following hold.*

(i) $Cog(\prod_{i\in I} C_i) \subseteq Cog(\mathcal{C}) \subseteq Cog(\coprod_{i\in I} C_i)$.

(ii) *If* $\mathrm{Hom}(C_i, C_j) \neq \emptyset$ *for any* $i, j \in I$, *then* $\prod_{i\in I} C_i$ *and* $\coprod_{i\in I} C_i$ *are cogenerators for* $Cog(\mathcal{C})$.

Proposition 2.9. *Let C be a cogenerator for $Cog(\mathcal{C})$. Then for each S-act A_S, $cotr_{A_S}(C) = cotr_{A_S}(\mathcal{C})$. In particular, if $(C_i)_{i\in I}$ is an indexed set of S-acts such that* $\mathrm{Hom}(C_i, C_j) \neq \emptyset$, *then* $cotr_{A_S}(\prod_{i\in I} C_i) = \bigcap_{i\in I} cotr_{A_S}(C_i) = cotr_{A_S}(\coprod_{i\in I} C_i)$.

3 Finitely cogenerated S-acts

In this section, we focus on finitely cogenerated S-acts. In [5], for a monoid S with zero, an S-act A_S is called *finitely cogenerated* provided that for every non-empty collection $\{A_i \mid i \in I\}$ of subacts of A_S with $\bigcap_{i\in I} A_i = \Theta$, there exists a finite subset J of I such that $\bigcap_{j\in J} A_j = \Theta$. As we know, the importance of congruences is more than subacts in characterizing the structure of S-acts. So, on an arbitrary monoid S, we define finitely cogenerated S-acts based on congruences and cogenerating sets.

Definition 3.1. An S-act A_S is called *finitely cogenerated* if for every monomorphism $A \xrightarrow{f} \prod_{i\in I} A_i$,

$$A \xrightarrow{f} \prod_{i\in I} A_i \xrightarrow{\pi} \prod_{j\in J} A_j$$

is also a monomorphism for some finite subset J of I.

Clearly, if A_S is finitely cogenerated, then every class $\mathcal{C}$ of S-acts that cogenerates A_S finitely cogenerates A_S.

Proposition 3.2. *For any S-act A_S, the following are equivalent.*

(i) *A_S is finitely cogenerated.*

(ii) *For every family of homomorphisms $\{f_i : A \longrightarrow A_i\}$ in S-acts with $\bigcap_{i\in I} \ker f_i = \Delta_A$, there is a finite subset J of I with $\bigcap_{j\in J} \ker f_j = \Delta_A$.*

(iii) *For any family of congruences $\{\rho_i \mid i \in I\}$ on A_S, if $\bigcap_{i\in I} \rho_i = \Delta_A$, then $\bigcap_{j\in J} \rho_j = \Delta_A$ for some finite subset J of I.*

(iv) *Every subact of A_S is finitely cogenerated.*

Proof. (i)$\Rightarrow$(ii): Let $\{f_i : A \longrightarrow A_i\}$ be a family of homomorphisms in S-acts with $\bigcap_{i\in I} \ker f_i = \Delta_A$. Then, $f = \prod_{i\in I} f_i : A \longrightarrow \prod_{i\in I} A_i$ is defined by $f(a) = (f_i(a))_{i\in I}$ and $\ker f = \bigcap_{i\in I} \ker f_i = \Delta_A$. So, f is a monomorphism, and $\pi f : A \longrightarrow \prod_{j\in J} A_j$ is a monomorphism for some finite subset J of I, by our assumption. Thus, $\bigcap_{j\in J} \ker f_j = \ker \pi f = \Delta_A$.

The implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) can be proved in a similar way. The implication (iv)$\Rightarrow$(i) is clear.

(ii)$\Rightarrow$(iv): Let B_S be a subact of A_S, and $\{\rho_i \mid i \in I\}$ be a family of congruences on B_S such that $\bigcap_{i\in I} \rho_i = \Delta_B$. It is clear that $\sigma_i = \rho_i \cup \Delta_A$ is also a congruence on A_S, for each $i \in I$. Since $\bigcap_{i\in I} \sigma_i = \bigcap_{i\in I} \rho_i \cup \Delta_A = \Delta_A$, by our assumption, $\bigcap_{j\in J} \sigma_j = \Delta_A$ for some finite subset J of I. Thus $\bigcap_{j\in J} \rho_j \subseteq \Delta_A \cap (B \times B) = \Delta_B$, and the result follows. $\square$

In the following definition, we use Rees congruences instead of congruences to define a weaker notion. Recall that the Rees congruence $\rho_B = (B \times B) \cup \Delta_A$ for any subact B_S of A_S.

Definition 3.3. An S-act A_S is called *finitely Rees cogenerated* whenever for any family of Rees congruences $\{\rho_{B_i} \mid i \in I\}$ on A_S, if $\bigcap_{i\in I} \rho_{B_i} = \Delta_A$, then $\bigcap_{j\in J} \rho_{B_j} = \Delta_A$ for some finite subset J of I.

Clearly, A_S is finitely Rees cogenerated if and only if for any family $\{B_i \mid i \in I\}$ of subacts of A_S, if $|\bigcap_{i\in I} B_i| \leq 1$, then $|\bigcap_{j\in J} B_j| \leq 1$ for some finite subset J of I. Also, for a monoid S with zero, this is equivalent to the following statement. If $\bigcap_{i\in I} B_i = \Theta$, then $\bigcap_{j\in J} B_j = \Theta$ for some finite subset J of I, as defined in [5]. Moreover, every subact of a finitely Rees cogenerated S-act is finitely Rees cogenerated.

Note that every finitely cogenerated S-act is finitely Rees cogenerated, but the following example shows that the converse is not true.

Example 3.4. Let $S = (\mathbb{N}, \min)^{\varepsilon} = (\mathbb{N}, \min)\dot{\cup}\{\varepsilon\}$, where ε denotes the externally adjoint identity, and denote $\min\{m, n\}$ by $m * n$. Then, $K_S = S \setminus \{\varepsilon\}$ is a right ideal of S. The subacts of K_S are $1S \subseteq 2S \subseteq 3S \subseteq \ldots$. Hence, K_S is finitely Rees cogenerated. We claim that K_S is not finitely cogenerated. For each $n \in K_S$, define $f_n : K_S \longrightarrow K_S$ by $f_n(m) = m * n$. It can be easily checked that $\bigcap_{n\in\mathbb{N}} \ker f_n = \Delta_K$. But, for each finite subset J of K, $\bigcap_{n\in J} \ker f_n \neq \Delta_K$. Therefore, K is not finitely cogenerated.

Using the Unique Decomposition Theorem (see [9, Theorem 1.5.10]), we shall obtain the structure of finitely (Rees) cogenerated S-acts.

Proposition 3.5. *Every finitely* (*Rees*) *cogenerated S-act is a finite coproduct of indecomposable S-acts.*

Proof. Suppose that A_S is finitely cogenerated. As we know, A_S has a unique decomposition into indecomposable subacts $\{A_i \mid i \in I\}$, that is, $A = \coprod_{i\in I} A_i$. Let $B_i = \coprod_{j\neq i} A_j$ for each $i \in I$. Then, B_i is a proper subact of A and $\bigcap_{i\in I} \rho_{B_i} = \Delta_A$. Now, since A_S is finitely cogenerated, there exists a finite subset J of I such that $\bigcap_{j\in J} \rho_{B_j} = \Delta_A$. If $J \neq I$ and $i \in I \setminus J$, since $A_i \subseteq B_j$ for each $j \in J$, then $A_i \times A_i \subseteq \bigcap_{j\in J} \rho_{B_j}$, which is a contradiction. Therefore, $A = \coprod_{j\in J} A_j$ and we are done. □

Recall from [9, Definition 2.5.31] that an S-act is called *completely reducible* if it is a coproduct of simple subacts. Now, the previous proposition allows us to deduce the following corollary.

Corollary 3.6. *Every finitely* (*Rees*) *cogenerated, completely reducible S-act is finitely generated.*

Proposition 3.7. *Every non-zero finitely cogenerated S-act contains a minimal congruence. In particular, every non-zero finitely Rees cogenerated S-act contains a minimal subact.*

Proof. Let A_S be a finitely cogenerated S-act, and $\mathfrak{A}$ be the set of all non-diagonal congruences on A_S. Then, $\nabla_A = (A \times A) \in \mathfrak{A}$ and $\supseteq$ makes $\mathfrak{A}$ into a poset. Let $\{\rho_i \mid i \in I\}$ be a chain in $\mathfrak{A}$. If $\bigcap_{i\in I} \rho_i = \Delta_A$, then by Proposition 3.2, $\bigcap_{j\in J} \rho_j = \Delta_A$ for some finite subset J of I. Since $\{\rho_i \mid i \in I\}$ is a chain, $\rho_k = \Delta_A$ for some $k \in I$, which is a contradiction. Thus, by Zorn's Lemma, $\mathfrak{A}$ contains a minimal element.

Replacing congruence with Rees congruence in this proof, we obtain the second part. □

Proposition 3.8. *Let $f : A \longrightarrow B$ be an essential monomorphism. If A_S is finitely cogenerated, then so is B. In particular, every essential extension (injective envelope) of a finitely cogenerated S-act is again finitely cogenerated.*

Proof. Let A_S be finitely cogenerated, and $f : A \longrightarrow B$ an essential monomorphism. Suppose that $g : B \longrightarrow \prod_{i\in I} A_i$ is a monomorphism. Then, $gf : A \longrightarrow \prod_{i\in I} A_i$ is a monomorphism, and since A_S is finitely cogenerated, $A \xrightarrow{gf} \prod_{i\in I} A_i \xrightarrow{\pi} \prod_{j\in J} A_j$ is a monomorphism for some finite subset J of I. Now, we find that f is an essential monomorphism and $\pi g f$ is a monomorphism, implying that πg is a monomorphism. Thus, B is finitely cogenerated. □

Now, let's consider finitely (Rees) cogenerated factor S-acts.

Proposition 3.9. *Let A_S be an S-act, and θ be a congruence on A_S. Then, A/θ is finitely ((Rees) cogenerated if and only if for any family of (Rees) congruences $\{\rho_i \mid i \in I\}$ on A_S, if $\bigcap_{i\in I} \rho_i = \theta$, then $\bigcap_{j\in J} \rho_j = \theta$ for some finite subset J of I.*

Proof. Necessity. Let θ be a congruence on an S-act A_S such that A/θ is finitely cogenerated. Let $\bigcap_{i\in I} \rho_i = \theta$, where $\rho_i \in Con(A)$ for each $i \in I$, define

$$\overline{\rho_i} = \{([a]_\theta, [b]_\theta) \mid (a, b) \in \rho_i\}.$$

It can be easily checked that $\overline{\rho_i} \in Con(A/\theta)$ and $\bigcap_{i\in I} \overline{\rho_i} = \Delta_{A/\theta}$. By our assumption, $\bigcap_{j\in J} \overline{\rho_j} = \Delta_{A/\theta}$ for some finite subset J of I. Thus, $\bigcap_{j\in j} \rho_j = \theta$.

Sufficiency. To show that A/θ is finitely cogenerated, suppose that $\bigcap_{i\in I}\sigma_i = \Delta_{A/\theta}$, where $\sigma_i \in Con(A/\theta)$ for each $i \in I$. Define

$$\rho_i = \{(a,b) \mid ([a]_\theta, [b]_\theta) \in \sigma_i\}.$$

It can be easily checked that $\rho_i \in Con(A)$ and $\bigcap_{i\in I}\rho_i = \theta$. By our assumption, $\bigcap_{j\in J}\rho_j = \theta$ for some finite subset J of I. Therefore, $\bigcap_{j\in j}\sigma_j = \Delta_{A/\theta}$, and the result follows.

For the case of finitely Rees cogenerated, it is sufficient to restrict congruences to Rees congruences. □

Recall from [11] that $\mathrm{Rad}(A)$ is the intersection of all maximal subacts of A_S. If A_S contains no maximal subacts, we let $\mathrm{Rad}(A) = A$. If $\mathrm{Rad}(A) \neq \emptyset$, then $\mathrm{Rad}(A)$ is a subact of A_S. Now we further consider the factor act $A/\mathrm{Rad}(A)$.

Proposition 3.10. *If $A/\mathrm{Rad}(A)$ is finitely Rees cogenerated, then it is cogenerated by finitely many θ-simple S-acts. Moreover, A_S has only finitely many maximal subacts.*

Proof. If A_S contains no maximal subacts, since $\mathrm{Rad}(A) = A$, the result follows. Otherwise, suppose that $A/\mathrm{Rad}(A)$ is finitely Rees cogenerated. Let

$$\mathrm{Rad}(A) = \bigcap_{i\in I}\{M_i \mid M_i \text{ is a maximal subact of} A\}\,.$$

Define $f : A \longrightarrow \prod_{i\in I} A/M_i$ by $f(a) = ([a_i]_{\rho_{M_i}})$. Then, f is an epimorphism such that $\ker f = \rho_{\mathrm{Rad}(A)} = \bigcap_{i\in I}\rho_{M_i}$. Using the homomorphism theorem, $\overline{f} : A/\mathrm{Rad}(A) \cong \prod_{i\in I} A/M_i$. Since $A/\mathrm{Rad}(A)$ is finitely Rees cogenerated, we find that $A/\mathrm{Rad}(A) \cong \prod_{j\in J} A/M_j$ for a finite subset J of I. Moreover, since M_i is maximal, A/M_i is θ-simple and the result follows. To show the second part, let $B = \bigcap_{j\in J} M_j$, and so $A/\mathrm{Rad}(A) \cong A/B$. Then $\mathrm{Rad}(A) = B = \bigcap_{j\in J} M_j$, and the set of maximal subacts of A_S is finite. □

Proposition 3.7 together with the fact that $S(A) \subseteq \mathrm{Soc}(A)$ yield that if A_S is finitely cogenerated, then $\mathrm{Soc}(A) \neq \emptyset$. Now, we use the concepts of essentiality and socle to characterize finitely cogenerated S-acts.

Theorem 3.11. *An S-act A_S is finitely cogenerated if and only if $Soc(A)$ is a finitely cogenerated large subact of A_S.*

Proof. If A_S is finitely cogenerated, then the same is also true for each of its subacts, and in particular for $\mathrm{Soc}(A)$. To prove $\mathrm{Soc}(A) \subseteq' A$, suppose that $\theta \in Con(A)$ satisfies $\theta \cap \rho_{\mathrm{Soc}(A)} \neq \Delta_A$. It is clear that $\rho_{\mathrm{Soc}(A)} = \bigcap_{L \subseteq' A} \rho_L$. So, $(\bigcap_{L \subseteq' A} \rho_L) \cap \theta = \Delta_A$. Since A_S is finitely cogenerated, there exist $L_1, \ldots, L_n \subseteq' A$ such that $(\bigcap_{i=1}^{n} \rho_{L_i}) \cap \theta = \Delta_A$. Then, the fact that each L_i is large implies $\theta = \Delta_A$, and the result follows.

On the other hand, every essential extension of a finitely cogenerated S-act is again finitely cogenerated. $\square$

Recall that an S-act A_S is said to be a *subdirect product* of the family $\{A/\rho_i \mid i \in I\}$ if $\bigcap_{i \in I} \rho_i = \Delta_A$. This means that the natural epimorphisms $\pi_i : A_S \longrightarrow A/\rho_i$ form a monomorphic family. An S-act A_S is called *subdirectly irreducible* if every set of congruences $\{\rho_i \mid i \in I\}$ on A_S with $\bigcap_{i \in I} \rho_i = \Delta_A$ contains Δ_A. Also, an S-act A_S is called *irreducible* if any intersection of a finite number of non-diagonal congruences is non-diagonal. It is clear that every subdirectly irreducible S-act is finitely cogenerated. The following result can be deduced from the definition of being subdirectly irreducible.

Proposition 3.12. *For any S-act A_S, the following are equivalent.*

(i) *A_S is subdirectly irreducible.*

(ii) *There exist distinct elements a and a' of A_S such that every morphism $f : A \longrightarrow B$ with $(a, a') \notin \ker f$ is a monomorphism.*

(iii) *There exist distinct elements a and a' of A_S such that $\rho(a, a')$ is the minimum proper congruence of A_S.*

(iii) *If $f : A \longrightarrow \prod_{i \in I} A_i$ is a monomorphism, then $\pi f : A \longrightarrow A_j$ is already a monomorphism for some $j \in I$.*

(iv) *Every subact of A_S is subdirectly irreducible.*

(v) *A_S is a finitely cogenerated irreducible S-act.*

By Birkhoff's theorem for acts, [7], any non-trivial S-act is a subdirect product of subdirectly irreducible S-acts. Now, using part (iii) of Proposition 3.2, we obtain the following result.

Corollary 3.13. *If A_S is a finitely cogenerated S-act, then it is isomorphic to a subdirect product of finitely many subdirectly irreducible S-acts.*

4 Characterization of the S-acts cogenerating S

From [10, Proposition 2.6] it follows that S_S cogenerates an S-act A_S if and only if A_S is torsionless, such acts are characterized in [10]. Let us turn to the question of when an act cogenerates S. This section concerns with the properties of S-acts which (finitely) cogenerate S_S, or generate the injective envelope $E(S)$ of S_S. We introduce the classes of (strongly) cofaithful S-acts and give characterizations of monoids S such that all faithful acts are cofaithful.

Let A_S be an S-act and $a \in A_S$. Then, $\lambda_a : S_S \longrightarrow A_S$ is defined by $\lambda_a(s) = as$ for every $s \in S$. The kernel congruence $\ker \lambda_a$ on S_S is called the *annihilator congruence* of $a \in A_S$. Recall from [1] that the *right annihilator* of A_S is defined by

$$R_S(A) = \{(s,t) \in S \times S \mid as = at, \text{ for all } a \in A\},$$

which is a two-sided congruence on S. We call A_S a *faithful* S-act if for $s, t \in S$, the equality $as = at$ for all $a \in A$ implies $s = t$. Clearly, $R_S(A) = \bigcap_{a \in A} \ker \lambda_a$ and A_S is faithful in case $R_S(A) = \Delta_S$. On the other hand, for each S-act A_S,

$$cotr_S(A_S) = \bigcap_{g \in \mathrm{Hom}(S_S, A_S)} \ker g = \bigcap_{a \in A} \ker \lambda_a = R_S(A).$$

The next theorem will be a useful description of faithful S-acts.

Theorem 4.1. *For each S-act A_S, the following are equivalent.*

(i) *A_S is faithful.*

(ii) *A_S cogenerates S.*

(iii) *A_S cogenerates every projective S-act.*

(iv) *A_S cogenerates every free S-act.*

(v) *A_S cogenerates a generator S-act.*

Proof. Since $cotr_S(A_S) = R_S(A)$, clearly (i) and (ii) are equivalent. It suffices to show (ii) $\Rightarrow$ (iii). Let $S \hookrightarrow A^J$. Suppose that $P = \coprod_{i \in I} e_i S$ is a projective S-act. Since $e_i S$ is a retract of S and $S \in Cog(A)$, we deduce that $e_i S \in Cog(A)$. Using Proposition 2.8, since $\mathrm{Hom}(A, A) \neq \emptyset$, $\coprod_{i \in I} A$

is a cogenerator for $Cog(A)$. If $f_i : e_iS \to A^J$, then $\coprod_{i\in I} f_i : \coprod_{i\in I} e_iS \hookrightarrow \coprod_{i\in I} A^J \hookrightarrow (\coprod_{i\in I} A)^J$. Therefore, $P \in Cog(\coprod_{i\in I} A) = Cog(A)$, as desired. □

From the previous theorem, we know that faithful S-acts can be characterized as those S-acts cogenerate S_S or, equivalently, cogenerate every projective S-act. In the category of modules, the concept of co-faithful is the dual notion of faithful as the modules which generate every injective module, which is equivalent to modules finitely cogenerate R, such modules are also called subgenerators of Mod-R. Unlike the case for modules, these properties are no longer valid for S-acts. This description allows us to define the following dual notions:

Definition 4.2. Let A_S be an S-act.

(i) A_S is called *cofaithful* in case A_S finitely cogenerates S_S, i.e., there exists a positive integer n such that S_S can be embedded to A^n.

(ii) A_S is called *subgenerator* in case it generates every injective S-act.

Lemma 4.3. *Let A_S be an S-act. The following are equivalent:*

(i) *A_S is cofaithful.*

(ii) *There exists a finite subset B of elements of A_S such that $R_S(B) = \Delta_S$.*

Proof. (i) $\Rightarrow$ (ii). Let $f : S \to A^n$ be a monomorphism. Suppose $f(1) = (a_1, ..., a_n)$, and set $B = \{a_1, ..., a_n\}$. If $(s,t) \in R_S(B)$, then $a_is = a_it$ for each $1 \leq i \leq n$. Clearly, $f(s) = f(t)$, and thus $s = t$.

(ii) $\Rightarrow$ (i). If $R_S(\{a_1, ..., a_n\}) = \Delta_S$, then $\lambda_{(a_1,...,a_n)} : S \to A^n$ is a monomorphism. □

Corollary 4.4. *A cofaithful S-act contains a finitely generated faithful subact. Moreover, if S is a commutative monoid, the converse is valid.*

Proof. Let A_S be cofaithful. Then there exists a finite subset $\{a_1, ..., a_n\}$ of A_S with $R_S(\{a_1, ..., a_n\}) = \Delta_S$. Set $B = \bigcup_{i=1}^n a_iS$. Clearly, $R_S(B) \subseteq R_S(\{a_1, ..., a_n\}) = \Delta_S$, and B is faithful. To show the second part, suppose that a finitely generated subact $B = \bigcup_{i=1}^n a_iS$ of A_S is faithful. Since S is commutative, $R_S(\{a_1, ..., a_n\}) = R_S(B) = \Delta_S$, and thus A_S is cofaithful. □

As we know, the notion of A_S finitely cogenerates S_S means there exists a positive integer n such that S_S can be embedded to A^n. If $n = 1$, i.e., $S \hookrightarrow A$, we say that A_S *cyclically cogenerates* S_S.

Proposition 4.5. *Let A_S be an S-act. The following are equivalent:*

(i) *A_S is a subgenerator.*

(ii) *A_S generates $E(S)$.*

(iii) *A_S cyclically cogenerates S_S.*

(iv) *There exists an element $a \in A$ such that $R_S(\{a\}) = \ker \lambda_a = \Delta_S$.*

(v) *A_S contains a cyclic generator subact.*

Proof. By an argument similar to that of Lemma 4.3, one can prove (iii) $\Leftrightarrow$ (iv). The implications (iv) $\Leftrightarrow$ (v) and (i) $\Rightarrow$ (ii) are clear.

(ii) $\Rightarrow$ (iv). Let $f : \coprod_I A \to E(S)$ be an epimorphism and $\iota : S \hookrightarrow E(S)$. For $\iota(1) \in E(S)$, $f(a) = \iota(1)$ for some $a \in A$. It is easy to see that $R_S(\{a\}) = \Delta_S$.

(iii) $\Rightarrow$ (i). Let E be an injective S-act and $f : S \hookrightarrow A$ be an monomorphism. For each $b \in E$, there exists a homomorphism $g_b : A \to E$ such that $g_b f = \lambda_b$. So we have the homomorphism $g = \coprod_{b\in E} g_b : \coprod_{b\in E} A \to \coprod_{b\in E} E$. Hence $\coprod_{b\in E} A \to \coprod_{b\in E} E \to E$ is an epimorphism, as desired. $\square$

It is easily checked that the following implications are valid,

generator $\Longrightarrow$ subgenerator $\Longrightarrow$ cofaithful $\Longrightarrow$ faithful.

The following example shows that these implications are strict.

Example 4.6. (i) The implication cofaithful $\Longrightarrow$ faithful is strict:
Let $S = (\mathbb{N}, \min)\dot{\cup}\{\varepsilon\}$, where ε denotes the externally adjoint identity, and $A_S = S \setminus \{\varepsilon\}$. Obviously, A_S is faithful but not cofaithful.

(ii) The implication subgenerator $\Longrightarrow$ cofaithful is strict:
Let $S = \{1, 0, e, f\}$ be the semilattice where $ef = fe = 0$, and take $A_S = \{e, f, 0\}$. Clearly, A_S is not a subgenerator. But $R_S(\{e, f\}) = \Delta_S$, and so A_S is cofaithful.

(iii) The implication generator $\Longrightarrow$ subgenerator is strict:
Let $S = (\mathbb{N}, .)$, and $A_S = \mathbb{N} \coprod^{2^{\mathbb{N}}} \mathbb{N}$. By [15, Example 2.2], A_S is not a generator. But $S \hookrightarrow A$, and so A_S is a subgenerator.

Concluding this section, we pointed out the conditions on a monoid S under which the converses of implications are true. Recall that a monoid S is said to be *right self-injective* if S_S is injective.

Theorem 4.7. *For a monoid S the following are equivalent:*

(i) *Every subgenerator S-act is a generator.*

(ii) *$E(S)$ is a generator.*

(iii) *S_S is right self-injective.*

Proof. (i)$\Rightarrow$ (ii) is clear.

(ii)$\Rightarrow$(iii): Since $E(S)$ is a generator, S is a retract of $E(S)$, and so S_S is injective.

(iii)$\Rightarrow$(i): Let A_S be a subgenerator S-act. Then there exists an S-morphism $f : S \hookrightarrow A$, and injectivity of S_S implies that S_S is a retract of A_S. Thus A_S is a generator. $\square$

By an argument similar to that of Corollary 4.4, It is not difficult to obtain the following result.

Proposition 4.8. *If S_S is irreducible, then every cofaithful S-act is a subgenerator. Moreover, if S is a commutative monoid, the converse is true.*

Using Theorem 4.1, the following result can be obtained.

Proposition 4.9. *S_S is finitely cogenerated if and only if every faithful S-act is cofaithful.*

References

[1] Abbas, M.S. and Baanoon, H.R., *On fully stable acts*, Ital. J. Pure Appl. Math. 34 (2015), 389-396.

[2] Anderson, F., Fuller, K., "Rings and Categories of Modules", Springer-Verlag, 1992.

[3] Barzegar, H., *Essentiality in the category of S-acts*, Eur. J. Pure Appl. Math. 9(1) (2016), 19-26.

[4] Davvaz and B., Nazemian, Z., *Chain conditions on commutative monoids*, Semigroup Forum 100 (2020), 732–742.

[5] Estaji, A.A and Shabani, M., *A note on multiplication S-acts*, Far East J. Math. Sci. 36(2) (2010), 133-150.

[6] Feller, E.H. and Gantos, R.L., *Indecomposable and injective S-acts with zero*, Math. Nachr. 41 (1969), 37-48.

[7] Grätzer, G., "Universal Algebra", D. Van Nostrand, Princeton et al., 1968.

[8] Herrlich, H. and Strecker, G., "Category Theory", Allyn and Bacon, Boston, 1973.

[9] Kilp, M., Knauer, U., and Mikhalev, A., "Monoids, Acts and Categories", W. de Gruyter, 2000.

[10] Kilp, M. and Knauer, U., *On torsionless and dense acts*, Semigroup Forum 63 (2001), 396-414.

[11] Khosravi, R. and Roueentan, M., *Co-uniform and hollow S-acts over monoids*, Commun. Korean Math. Soc. 37(2) (2022), 347–358.

[12] Knauer, U. and Normak, P., *Morita duality for monoids* , Semigroup Forum 40 (1973), 39-57.

[13] Moghaddasi, G.H., *Cogenerator and subdirectly ireducible in the category of S-posets*, J. Algebr. Sys. 2(2) (2014), 137-146.

[14] Normak, P., *Cogenerator acts*, In: A. Cherubini, C. Bonzini, and C. Tibiletti, "Semigroups, Algebraic Theory and Applications to Formal Languages and Codes", World Scientific, 1993, 191-198.

[15] Sedaghatjoo, M., *A correction on the paper: Characterization of monoids by properties of generators*, Comm. Algebra 41 (2013), 4662-4666.

[16] Wisbauer, R., "Foundations of Module and Ring Theory", Gordon and Breach Science Publishers, Reading, 1991.

Roghaieh Khosravi *Department of Mathematics, Faculty of Sciences, Fasa University, Fasa, Iran.*

Email: khosravi@fasau.ac.ir

Xingliang Liang *Department of Mathematics, Shaanxi University of Science and Technology, Xi′an, Shaanxi, P.R. China.*

Email: lxl_119@126.com

Mohammad Roueentan *College of Engineering, Lamerd Higher Education Center, Shiraz University of Technology, Lamerd, Iran.*
Email: m.rooeintan@yahoo.com