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Primitive hyperideals and hyperstructure spaces of hyperrings

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The author dedicates this article to Themba Dube in celebration of his 65th birthday

Abstract. We introduce primitive hyperideals of a hyperring R and show how they are related to R itself, and to maximal and prime hyperideals of R. We endow a Jacobson topology on the set of primitive hyperideals of R and study the topological properties of the corresponding hyperstructure space.

1 Introduction

The notion of *multi-valued* algebraic structures was first considered in [19], where *hypergroups* were introduced. A hypergroup is a generalization of a group created by allowing the binary operation to be multi-valued. Later, in [17], the concept of a *hyperring* was introduced. Since their inception, (mostly commutative) hyperrings have been extensively studied in algebraic and geometric contexts. In [7] (see also [6]), a comprehensive account of var-

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ious algebraic properties of hyperrings (as well as their generalizations) can be found. For applications of hyperrings in geometry, we refer the reader to [3–5, 16]. There have been extensive studies conducted on hypermodules (over commutative hyperrings) and their topological aspects. For example, [21] studied topological properties of second subhypermodules over commutative hyperrings. For a study of free and cyclic hypermodules, we refer to [20]. The role of supplements in Krasner hypermodules is examined in [25] (see also [1]) and related to normal π-projectivity. For other aspects of hypermodules, see [2, 9].

It is well known (see [11] and [12]) that for a noncommutative ring, the notion of primitive ideals plays a crucial role in determining its structure. Furthermore, in [12], a hull-kernel-type topology was endowed on the set of all primitive ideals of a ring, and representations of biregular rings were studied. Primitive ideals have also proven to be immensely important in understanding the structural aspects of modules [12, 23], Lie algebras [18], enveloping algebras [8, 14], PI-algebras [13], quantum groups [15], skew polynomial rings [10], and others.

The aim of this paper is to introduce primitive hyperideals of a (Krasner) hyperring and study some of their properties. We show the relations between prime, maximal, and primitive hyperideals of a hyperring and also characterize simple hypermodules. Similar to [12], we impose a Jacobson topology on the set of primitive hyperideals of a hyperring and investigate the topological properties of the corresponding hyperstructure space. We characterize irreducible closed subsets of a hyperstructure space and prove that every irreducible closed subset of a hyperstructure space has a unique generic point. We give a sufficient condition for the space to be Noetherian and study continuous maps between such spaces.

2 Preliminaries

Suppose R is a nonempty set and $\mathscr{P}^*(R)$ is the set of all nonempty subsets of R. A *Krasner hyperring* is a system $(R, +, \cdot, -, 0)$ such that

(I) $(R, +, 0)$ is a *canonical hypergroup*, that is, $+\colon R \times R \to \mathscr{P}^*(R)$ is a hyperoperation on R satisfying the following properties for all $a, b, c \in R$:

(i) $a + b = b + a$;

(ii) $a + (b + c) = (a + b) + c$;

(iii) there exists $0 \in A$ such that $a + 0 = \{a\}$;

(iv) for every a, there exists a unique $-a \in A$ such that $0 \in a - a$;

(v) if $a \in b + c$, then $c \in -b + a$ and $a \in c - b$,

(II) $(R, \cdot)$ is a semigroup,
(III) $a \cdot 0 = 0 \cdot a = 0$, and
(IV) $a \cdot (b + c) = a \cdot b + a \cdot c$,
(V) $(a + b) \cdot c = a \cdot c + b \cdot c$,
for all $a, b, c \in R$.

A hyperring R is called *unital* if R has a multiplicative identity, that is, there exists $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$. For simplicity, we shall write $a \cdot b$ as ab. We will restrict our focus to Krasner hyperrings in this paper, so if we refer to a hyperring, it will be a Krasner hyperring.

A nonempty subset S of a hyperring R is said to be a *subhyperring* of R if $(S, +, \cdot)$ is itself a hyperring. A subhypergroup $\mathfrak{a}$ of a hyperring R is called a *left (right) hyperideal* of R if $r \cdot a \in \mathfrak{a}\,(a \cdot r \in \mathfrak{a})$ for all $r \in R, a \in \mathfrak{a}$. If $\mathfrak{a}$ is both a left and right hyperideal then $\mathfrak{a}$ is called a *two-sided hyperideal* or simply a *hyperideal*. Unless otherwise stated, we assume all hyperideals are two-sided. If $\mathfrak{a}$ is a hyperideal of R, then we can form the *quotient* hyperring $R/\mathfrak{a} = \{\mathfrak{a} + r \mid r \in R\}$ with the following two operations:

$$(\mathfrak{a} + r_1) + (\mathfrak{a} + r_2) = \{\mathfrak{a} + r \mid r \in r_1 + r_2\};$$
$$(\mathfrak{a} + r_1)(\mathfrak{a} + r_2) = \mathfrak{a} + r_1 r_2.$$

The following result is known, but the proof is included for completeness.

Proposition 2.1. *If $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ is a nonempty family of hyperideals of a hyperring R, then the following are also hyperideals of R.*

(i) $\bigcap_{\lambda \in \Lambda} \mathfrak{a}_\lambda$

(ii) $\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda = \{x \mid x \in \sum_{\lambda \in \Lambda} a_\lambda, a_\lambda \in \mathfrak{a}_\lambda\}$

Proof. (i) Suppose that $x, y \in \bigcap_{\lambda \in \Lambda} \mathfrak{a}_\lambda$. Then $x, y \in \mathfrak{a}_\lambda$ for all $\lambda \in \Lambda$. Since each $\mathfrak{a}_\lambda$ is a hyperideal, it follows that $x - y \in \mathfrak{a}_\lambda$ for all $\lambda \in \Lambda$. This implies that $x - y \subseteq \bigcap_{\lambda \in \Lambda} \mathfrak{a}_\lambda$. Now let $r \in R$. For each $\lambda \in \Lambda$, since $\mathfrak{a}_\lambda$ is a hyperideal of R, it follows that $rx \in \mathfrak{a}_\lambda$ and $xr \in \mathfrak{a}_\lambda$, and hence we conclude that $rx \in \bigcap_{\lambda\in\Lambda} \mathfrak{a}_\lambda$ and $xr \in \bigcap_{\lambda\in\Lambda} \mathfrak{a}_\lambda$.

(ii) Suppose that $x, y \in \sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda$. Then $x \in \sum_{\lambda\in\Lambda} a_\lambda$ for some $a_\lambda \in \mathfrak{a}_\lambda$ and $y \in \sum_{\lambda\in\Lambda} b_\lambda$ for some $b_\lambda \in \mathfrak{a}_\lambda$. Since each $\mathfrak{a}_\lambda$ is a hyperideal, it follows that $a_\lambda - b_\lambda \subseteq \mathfrak{a}_\lambda$ for $\lambda \in \Lambda$. This implies that $x - y \subseteq \sum_{\lambda\in\Lambda}(a_\lambda - b_\lambda)$, where $a_\lambda - b_\lambda \subseteq \mathfrak{a}_\lambda$, so $x - y \subseteq \sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda$. Now let $r \in R$. For each $\lambda \in \Lambda$, since $\mathfrak{a}_\lambda$ is a hyperideal of R, it follows that $ra_\lambda \in \mathfrak{a}_\lambda$ and $a_\lambda r \in \mathfrak{a}_\lambda$, for each $\lambda \in \Lambda$ and hence we conclude that $rx \in \sum_{\lambda\in\Lambda} ra_\lambda$ and $xr \in \sum_{\lambda\in\Lambda} a_\lambda r$. □

Recall that if $\mathfrak{a}$ and $\mathfrak{b}$ are nonempty subsets of a hyperring R, then the product $\mathfrak{ab}$ is defined by

$$\mathfrak{ab} = \left\{ x \mid x \in \sum_{i=1}^{n} a_i b_i, a_i \in \mathfrak{a}, b_i \in \mathfrak{b}, n \in \mathbb{Z}^+ \right\}.$$

Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ are hyperideals, $\mathfrak{ab}$ is also a hyperideal of R (see [7, p. 87]). Let X be a subset of a hyperring R. Let $\{\mathfrak{a}_i \mid i \in I\}$ be the family of all hyperideals in R which contain X. Then $\bigcap_{i\in I} \mathfrak{a}_i$, is called the hyperideal *generated by* X and we denoted it by $\langle X \rangle$. A proper hyperideal $\mathfrak{m}$ of a hyperring R is called *maximal* if the only hyperideals of R that contain $\mathfrak{m}$ are $\mathfrak{m}$ itself and R. A proper hyperideal $\mathfrak{p}$ of a hyperring R is called *prime* if for every pair of hyperideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, $\mathfrak{ab} \subseteq \mathfrak{p}$ implies either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

Lemma 2.2. *Every proper right hyperideal $\mathfrak{a}$ of a unital hyperring R is contained in a right maximal hyperideal of R.*

Proof. Suppose $\mathcal{U} = \{\mathfrak{u} \mid \mathfrak{u} \supseteq \mathfrak{a}, \mathfrak{u}$ is a proper hyperideal of $R\}$. Since $\mathfrak{a} \in \mathcal{U}$, the set $\mathcal{U}$ is nonempty. Consider a chain $\{\mathfrak{c}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{U}$. Then $\mathfrak{c} = \bigcup_{\lambda\in\Lambda} \mathfrak{c}_\lambda$ is a proper hyperideal of R which is an upper bound of the chain $\{\mathfrak{c}_\lambda\}_{\lambda\in\Lambda}$. Moreover, $\mathfrak{c} \neq R$ because $1 \notin \mathfrak{c}$. Hence by Zorn's lemma $\mathcal{U}$ contains a maximal element $\mathfrak{m}$, which is a maximal hyperideal of R containing $\mathfrak{a}$. □

3 Primitive hyperideals

As for rings, in order to define primitive hyperideals of a hyperring, we require the notion of simple hypermodules. In the next subsection we first study simple hypermodules and their annihilators.

3.1 Simple hypermodules Recall from [20] that a (*right*) *Krasner* R-*hypermodule* M is a canonical hypergroup M endowed with an external composition $M \times R \to M$ (defined by $(m, r) \mapsto mr$) satisfying the conditions:

(i) $(m + m')r = mr + m'r$;

(ii) $m(r + r') = mr + mr'$;

(iii) $m(rr') = (mr)r'$;

(iv) $m0 = 0$;

for all $m, m' \in M$ and $r, r' \in R$. If, moreover, R has a multiplicative identity 1 and $m1 = m$ for all $m \in M$, then M is called *unital*. We shall only consider right Krasner R-hypermodules and hence from now on we drop the adjective "right Krasner" and simply say R-hypermodule.

If an R-hypermodule M is generated by a single element m of M, then M is called *cyclic*, and we denote it by $\langle m \rangle$ or Rm. The proof of the following property of an R-hypermodule can be found in [24].

Lemma 3.1. *If M is an R*-hypermodule *then $(-m)r = -(mr) = m(-r)$ for all $r \in R$ and $m \in M$.*

A *subhypermodule* S of a hypermodule M is a subcanonical hypergroup of M such that $sr \subseteq S$, for all $r \in R$ and for all $s \in S$. If M, N are R-hypermodules, then a (*strong*) R-*hypermodule homomorphism* from M into N is a map $\mu \colon M \to N$ such that $\mu(m + m') = \mu(m) + \mu(m')$ and $\mu(mr) = \mu(m)r$ for all $r \in R$ and for all $m, m' \in M$. A hypermodule homomorphism μ is called an *isomorphism* if μ is also a bijection on the underlying sets.

If M is a R-hypermodule and K is a subhypermodule of M, then the set $M/K = \{K + a \mid a \in M\}$ endowed with a hyperoperation $+ : M/K \times M/K \to \mathscr{P}^*(M/K)$ and an R-action $\cdot : M/K \times R \to M/K$ respectively defined as:

$$(K + a) + (K + a') = \{K + b \mid b \in a + a'\};$$
$$(K + a) \cdot r = \{K + b \mid b \in ar\},$$

for every $a, a', b \in M$ and $r \in R$, is called the *quotient hypermodule* of M. It is easy to show (see [24, Corollary 2.2.8]) that $\mathtt{ker}(\mu)$ is a subhypermodule of M and $\mathtt{im}(\mu)$ is a subhypermodule of N. As for modules over rings, we also have the fundamental theorem of homomorphisms for hypermodules.

Proposition 3.2. [24, Theorem 2.2.14] *If $\mu\colon M \to M'$ is a hypermodule homomorphism, then $M/\mathtt{ker}(\mu)$ is isomorphic to $\mathtt{im}(\mu)$.*

An R-hypermodule M is called *simple* if $RM \neq 0$ and M has no subhypermodules other than 0 and M. The following proposition characterizes a simple hypermodule as a cyclic hypermodule generated by a nonzero element.

Proposition 3.3. *A nonzero R-hypermodule M is simple if and only if $M = mR$ for every nonzero $m \in M$.*

Proof. If M is simple, there exists a $0 \neq m \in M$ such that mR is a nonzero subhypermodule of M and we have that $mR = M$. Conversely, if $N \neq 0$ is a subhypermodule of M, then N must contain a nonzero element, say m of M. Then we have that $M = mR \subseteq N$, showing that $N = M$. □

The following example of subhypermodule is going to play an important role in studying properties of primitive hyperideals.

Lemma 3.4. *If M is a R-hypermodule and $\mathfrak{a}$ a hyperideal of R, then*

$$M\mathfrak{a} = \left\{ \sum_{i=1}^{k} m_i a_i \mid m_i \in M, a_i \in \mathfrak{a}, k \in \mathbb{Z}^+ \right\}$$

is a subhypermodule of M.

Proof. Let $\sum_{i=1}^{k} m_i a_i$ and $\sum_{j=1}^{l} m_j a_j$ be two elements of $M\mathfrak{a}$. Then

$$\sum_{i=1}^{k} m_i a_i - \sum_{j=i}^{l} m_j a_j = \sum_{i=1}^{k} m_i a_i + \sum_{j=1}^{l} (-m_j) a_j$$

where $-m_j \in M$ since $(M, +)$ is a canonical hypergroup. Hence, $\sum_{i=1}^{k} m_i a_i - \sum_{j=1}^{l} m_j a_j \subseteq M\mathfrak{a}$. Now let $r \in R$. Then

$$\left(\sum_{i=1}^{k} m_i a_i \right) r = \sum_{i=1}^{k} m_i (a_i r)$$

where $a_i r \in R$ since $\mathfrak{a}$ is a hyperideal of R. Thus $\left(\sum_{i=1}^{k} m_i a_i\right) r \in M\mathfrak{a}$. □

If M is a R-hypermodule then the additive subhypergroup Mr of M generated by the elements of the form $\{mr \mid m \in M, r \in R\}$ is a subhypermodule of M. The (*right*) *annihilator* of a R-hypermodule M is defined by

$$\mathtt{Ann}_R(M) = \{r \in R \mid mr = 0 \text{ for all } \ m \in M\}.$$

When $M = \{m\}$, we write $\mathtt{Ann}_R(m)$ for $\mathtt{Ann}_R(\{m\})$. If $\mathtt{Ann}_R(M) = \{0\}$ then M is said to be a *faithful* R-hypermodule. Like in rings, we have the following.

Lemma 3.5. *An annihilator $\mathtt{Ann}_R(M)$ is a hyperideal of R.*

Proof. Let $x, x' \in \mathtt{Ann}_R(M)$, $r \in R$, and $m \in M$. Then

$$m(x - x') = mx + m(-x') = mx - mx' = 0 + 0 = 0,$$

where the second equality follows from Lemma 3.1. Furthermore, $m(xr) = (mx)r = 0r = 0$ and $m(rx) = (mr)x = 0$. Thus, $\mathtt{Ann}_R(M)$ is a hyperideal of R. □

3.2 Primitivity A proper hyperideal of a hyperring R is called *primitive* if it is the annihilator of a simple R-hypermodule. We shall denote the set of all primitive hyperideals of R by $\mathtt{Prim}(R)$. A hyperring R is said to be *primitive* if $\{0\}$ is a primitive hyperideal of R. The next two propositions show some implications between maximal, prime, and primitive hyperideals.

Proposition 3.6. *Every primitive hyperideal is a prime hyperideal.*

Proof. Suppose that $\mathfrak{p} = \mathtt{Ann}_R(M)$ for some simple R-hypermodule M, and that $\mathfrak{b}$ is a hyperideal of R such that $M\mathfrak{b} \neq 0$, that is, $\mathfrak{b} \not\subseteq \mathfrak{p}$. Since M is simple, we must have that $M\mathfrak{b} = M$. If $\mathfrak{a}$ is a nonzero hyperideal of R, then

$$M(\mathfrak{b}\mathfrak{a}) = (M\mathfrak{b})\mathfrak{a} = M\mathfrak{a} = M, \tag{3.1}$$

which implies that $M\mathfrak{a} \neq 0$, that is, $\mathfrak{a} \not\subseteq \mathfrak{p}$. Therefore, from (3.1) it follows that $\mathfrak{b}\mathfrak{a} \not\subseteq \mathfrak{p}$. □

Proposition 3.7. *Every maximal hyperideal of a unital hyperring is a primitive hyperideal.*

Proof. Suppose $\mathfrak{a}$ is maximal hyperideal of a hyperring R. Then by Lemma 2.2, $\mathfrak{a}$ is contained in a maximal right hyperideal $\mathfrak{b}$ of R and $\mathfrak{a} \subseteq \mathtt{Ann}(R/\mathfrak{b})$. Since $\mathfrak{a}$ is a maximal hyperideal of R, we must have that $\mathfrak{a} = \mathtt{Ann}(R/\mathfrak{b})$, and thus $\mathfrak{a}$ is the annihilator of a simple R-hypermodule $R/\mathfrak{b}$. □

Example 3.8. Let $R = \{a, b, c, d, e, f\}$ be a set with the hyperoperation $\oplus$ and the multiplication $\odot$ defined as follows:

$\oplus$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	$\{a, b\}$	d	$\{c, d\}$	f	$\{e, f\}$
c	c	d	c	d	$\{a, c, e\}$	$\{b, d, f\}$
d	d	$\{c, d\}$	d	$\{c, d\}$	$\{b, d, f\}$	R
e	e	f	$\{a, c, e\}$	$\{b, d, f\}$	e	f
f	f	$\{e, f\}$	$\{b, d, f\}$	R	f	$\{e, f\}$

and

$\odot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	b	a	b
c	a	a	c	c	e	e
d	a	b	c	d	e	f
e	a	a	e	e	c	c
f	a	b	e	f	c	d

Then, $(R, \oplus, \odot)$ is a Krasner hyperring []. Since

$$\begin{aligned} d\cdot a &= a, & a\cdot d &= a,\\ d\cdot b &= b, & b\cdot d &= b,\\ d\cdot c &= c, & c\cdot d &= c,\\ d\cdot d &= d. & &\\ d\cdot e &= e, & e\cdot d &= e,\\ d\cdot f &= f, & f\cdot d &= f, \end{aligned}$$

it follows that R is a unital hyperring. It is easy to check that $M_1 = \{a, b\}$ and $M_2 = \{a, c, e\}$ are maximal hyperideals of R. Hence, by Proposition 3.7, we conclude that M_1 and M_2 are primitive hyperideals.

From the definition at the start of this subsection, we have that a hyperring R is primitive if and only if the zero hyperideal of R is a primitive hyperideal. This equivalence can further be generalized for an arbitrary primitive hyperideal of R.

Proposition 3.9. *A hyperideal $\mathfrak{p}$ of a hyperring R is primitive if and only if $R/\mathfrak{p}$ is a primitive hyperring.*

Proof. Suppose $\mathfrak{p}$ is primitive hyperideal of R and let M be a simple R-hypermodule such that $\mathfrak{p} = \mathtt{Ann}(M)$. If we define $m(\mathfrak{p}+r) = mr$, for all $r \in R, m \in M$, then the additive canonical hypergroup of M is also a simple $R/\mathfrak{p}$-hypermodule. On the other hand, since $\mathtt{Ann}(M) \subseteq \mathfrak{p}$, we have that M is a faithful $R/\mathfrak{p}$-hypermodule. Conversely, suppose that N is a faithful simple $R/\mathfrak{p}$-hypermodule and for all $r \in R, n \in N$, define $nr = n(\mathfrak{p}+r)$. Then the additive canonical hypergroup of N becomes a simple R-hypermodule with $\mathtt{Ann}(N) = \mathfrak{p}$. □

Primitive hyperideals are also related to right maximal hyperideals, as we will see in the next proposition. We will need the following result.

Lemma 3.10. *Let R be a hyperring. An R-hypermodule M is simple if and only if M is isomorphic to $R/\mathfrak{m}$ for some maximal right hyperideal $\mathfrak{m}$ of R.*

Proof. Let M be a simple R-hypermodule. Choose $0 \neq m \in M$. Then $mR = M$ and hence $\psi : R \to M$, defined by $\psi(r) = mr$, is a surjective R-hypermodule homomorphism. Its kernel $\mathfrak{m}$ is a right hyperideal of R and by Proposition 3.2, we have $R/\mathfrak{m} \cong M$. To show that $\mathfrak{m}$ is maximal, let $\mathfrak{b}$ be a right hyperideal of R such that $\mathfrak{m} \subseteq \mathfrak{b} \subseteq R$. Then $\mathfrak{b}/\mathfrak{a}$ is a subhypermodule of $R/\mathfrak{m}$. Now since $R/\mathfrak{m}$ is isomorphic to M and M is simple, we must have either $\mathfrak{b}/\mathfrak{m} = 0$ or $\mathfrak{b}/\mathfrak{a} = R/\mathfrak{a}$, and thus, either $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b} = R$, which implies that $\mathfrak{m}$ is maximal. Conversely, let $\mathfrak{m}$ be a maximal hyperideal of R and consider a subhypermodule N of $R/\mathfrak{m}$. It is easy to see that $\mathfrak{b} = \{r \in R \mid \mathfrak{m} + r \in N\}$ is a right hyperideal of R containing $\mathfrak{m}$. Thus $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b} = R$, giving that $N = 0$ or $N = R/\mathfrak{m}$. Thus $R/\mathfrak{m}$ is a simple R-hypermodule. □

Proposition 3.11. *If $\mathfrak{p}$ is a primitive hyperideal of a hyperring R then there exists a maximal right hyperideal $\mathfrak{m}$ of R such that*

$$\mathfrak{p} = \{r \in R \mid Rr \subseteq \mathfrak{m}\}. \tag{3.2}$$

Conversely, if $\mathfrak{m}$ is a maximal right hyperideal of R and if $R^2 \nsubseteq \mathfrak{m}$, then the hyperideal $\mathfrak{p}$ defined in (3.2) is primitive.

Proof. If $\mathfrak{p} = \mathtt{Ann}_R(M)$, for some simple R-hypermodule M, then by Lemma 3.10, there exists a maximal right hyperideal $\mathfrak{m}$ of R such that $M \cong R/\mathfrak{m}$. This implies $\mathfrak{p} = \mathtt{Ann}_R(R/\mathfrak{m})$ and hence condition (3.2) is satisfied. Conversely, if we assume that $\mathfrak{m}$ is a maximal right hyperideal of R, then again by Lemma 3.10, $R/\mathfrak{m}$ is a simple R-hypermodule, and therefore, $\mathtt{Ann}_R(R/\mathfrak{m}) = \mathfrak{p}$, a primitive hyperideal of R. □

Corollary 3.12. *Every maximal right hyperideal of a unital hyperring contains a primitive hyperideal.*

4 Hyperstructure spaces

We shall introduce Jacobson topology in $\mathtt{Prim}(R)$, the set of primitive hyperideals of a hyperring R, by defining a closure operator for the subsets of $\mathtt{Prim}(R)$. Once we have a closure operator, closed sets are defined as sets which are invariant under this closure operator.

Suppose S is a subset of $\mathtt{Prim}(R)$. Set $\mathcal{K}_S = \bigcap_{\mathfrak{q}\in S} \mathfrak{q}$. We define the closure of the set S as

$$\mathtt{Cl}(S) = \{\mathfrak{p} \in \mathtt{Prim}(R) \mid \mathfrak{p} \supseteq \mathcal{K}_S\}. \tag{4.1}$$

If $S = \{\mathfrak{s}\}$, we will write $\mathtt{Cl}(\{\mathfrak{s}\})$ as $\mathtt{Cl}(\mathfrak{s})$. We wish to verify that the closure operation defined in (4.1) satisfies Kuratowski's closure conditions.

Proposition 4.1. *The sets $\{\mathtt{Cl}(S)\}_{S\subseteq \mathtt{Prim}(R)}$ satisfy the following conditions for all subsets S and T of the hyperstructure space $\mathtt{Prim}(R)$:*

(i) $\mathtt{Cl}(\emptyset) = \emptyset$;

(ii) $\mathtt{Cl}(S) \supseteq S$;

(iii) $\mathtt{Cl}(\mathtt{Cl}(S)) = \mathtt{Cl}(S)$;

(iv) $\mathtt{Cl}(S \cup T) = \mathtt{Cl}(S) \cup \mathtt{Cl}(T)$.

Proof. The proofs of ((i))-((iii)) are straightforward, whereas for ((iv)), it is easy to see that $\mathtt{Cl}(S \cup T) \supseteq \mathtt{Cl}(S) \cup \mathtt{Cl}(T)$. To obtain the other inclusion, let $\mathfrak{p} \in \mathtt{Cl}(S \cup T)$. Then

$$\mathfrak{p} \supseteq \mathcal{K}_{S\cup T} = \mathcal{K}_S \cap \mathcal{K}_T.$$

Since $\mathcal{K}_S$ and $\mathcal{K}_T$ are hyperideals of the hyperring R, it follows that

$$\mathcal{K}_S\mathcal{K}_T \subseteq \mathcal{K}_S \cap \mathcal{K}_T \subseteq \mathfrak{p}.$$

Since by Proposition 3.6, $\mathfrak{p}$ is prime, either $\mathcal{K}_S \subseteq \mathfrak{p}$ or $\mathcal{K}_T \subseteq \mathfrak{p}$. This means either $\mathfrak{p} \in \mathtt{Cl}(S)$ or $\mathfrak{p} \in \mathtt{Cl}(T)$. Thus $\mathtt{Cl}(S \cup T) \subseteq \mathtt{Cl}(S) \cup \mathtt{Cl}(T)$. □

The set $\mathtt{Prim}(R)$ of primitive hyperideals of a hyperring R topologized (the Jacobson topology) by the closure operator defined in (4.1) is called the *hyperstructure space* of the hyperring R. If S is a subset of a hyperring R, then

$$\mathcal{O}(S) = \{\mathfrak{p} \in \mathtt{Prim}(R) \mid \mathfrak{p} \not\supseteq \mathcal{K}_S\}$$

is a typical open subset of this topology. It is evident from (4.1) that if $\mathfrak{p} \neq \mathfrak{p}'$ for any two $\mathfrak{p}, \mathfrak{p}' \in \mathtt{Prim}(R)$, then $\mathtt{Cl}(\mathfrak{p}) \neq \mathtt{Cl}(\mathfrak{p}')$. Thus we have the following.

Proposition 4.2. *Every hyperstructure space* $\mathtt{Prim}(R)$ *is a* T_0*-space.*

Using the finite intersection property, we can obtain compactness of the hyperstructure space.

Theorem 4.3. *If R is a unital hyperring then the hyperstructure space* $\mathtt{Prim}(R)$ *is compact.*

Proof. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets of a hyperstructure space $\mathtt{Prim}(R)$ such that $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$. Then a primitive hyperideal $\mathfrak{p} \in \bigcap_{\lambda\in\Lambda} C_\lambda$ if and only if $\mathfrak{p} \supseteq \sum_{\lambda\in\Lambda} \mathcal{K}_{C_\lambda}$. Since $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$, we must have that $\sum_{\lambda\in\Lambda} \mathcal{K}_{C_\lambda} = R$. In particular, we obtain that $1 = \sum_{i=1}^{n} \mathcal{K}_{C_{\lambda_i}}$ for a suitable finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ. This in turn implies that $\bigcap_{i=1}^{n} C_{\lambda_i} = \emptyset$, and hence $\mathtt{Prim}(R)$ is compact. □

Recall that a nonempty closed subset C of a topological space X is *irreducible* if $C \neq C_1 \cup C_2$ for any two proper closed subsets C_1, C_2 of C. A maximal irreducible subset of a topological space X is called an *irreducible component* of X. A point x in a closed subset C is called a *generic point* of C if $C = \mathtt{Cl}(x)$.

Lemma 4.4. $\{\mathtt{Cl}(\mathfrak{p})\}_{\mathfrak{p}\in\mathtt{Prim}(R)}$ *are the only irreducible closed subsets of a hyperstructure space* $\mathtt{Prim}(R)$.

Proof. Since $\{\mathfrak{p}\}$ is irreducible, so is $\texttt{Cl}(\mathfrak{p})$. Suppose $\texttt{Cl}(\mathfrak{a})$ is an irreducible closed subset of $\texttt{Prim}(R)$ and $\mathfrak{a} \notin \texttt{Prim}(R)$. This implies there exist hyperideals $\mathfrak{b}$ and $\mathfrak{c}$ of R such that $\mathfrak{b} \nsubseteq \mathfrak{a}$ and $\mathfrak{c} \nsubseteq \mathfrak{a}$, but $\mathfrak{bc} \subseteq \mathfrak{a}$. Then

$$\texttt{Cl}(\langle \mathfrak{a}, \mathfrak{b} \rangle) \cup \texttt{Cl}(\langle \mathfrak{a}, \mathfrak{c} \rangle) = \texttt{Cl}(\langle \mathfrak{a}, \mathfrak{bc} \rangle) = \texttt{Cl}(\mathfrak{a}).$$

But $\texttt{Cl}(\langle \mathfrak{a}, \mathfrak{b} \rangle) \neq \texttt{Cl}(\mathfrak{a})$ and $\texttt{Cl}(\langle \mathfrak{a}, \mathfrak{c} \rangle) \neq \texttt{Cl}(\mathfrak{a})$, and hence $\texttt{Cl}(\mathfrak{a})$ is not irreducible. □

Proposition 4.5. *Every irreducible closed subset of* $\texttt{Prim}(R)$ *has a unique generic point.*

Proof. The existence of a generic point follows from Lemma 4.4, and the uniqueness of such a point follows from Proposition 4.2. □

The irreducible components of a hyperstructure space can be characterised in terms of minimal primitive hyperideals, as shown in the following result.

Proposition 4.6. *The irreducible components of a hyperstructure space* $\texttt{Prim}(R)$ *are the closed sets* $\texttt{Cl}(\mathfrak{p})$*, where* $\mathfrak{p}$ *is a minimal primitive hyperideal of* R*.*

Proof. If $\mathfrak{p}$ is a minimal primitive hyperideal, then by Lemma 4.4, $\texttt{Cl}(\mathfrak{p})$ is irreducible. If $\texttt{Cl}(\mathfrak{p})$ is not a maximal irreducible subset of $\texttt{Prim}(S)$, then there exists a maximal irreducible subset $\texttt{Cl}(\mathfrak{p}')$ with $\mathfrak{p}' \in \texttt{Prim}(S)$ such that $\texttt{Cl}(\mathfrak{p}) \subsetneq \texttt{Cl}(\mathfrak{p}')$. This implies that $\mathfrak{p} \in \texttt{Cl}(\mathfrak{p}')$ and hence $\mathfrak{p}' \subsetneq \mathfrak{p}$, contradicting the minimality property of $\mathfrak{p}$. □

Recall that a hyperring is called *Noetherian* if it satisfies the ascending chain condition, whereas a topological space X is called *Noetherian* if the descending chain condition holds for closed subsets of X. A relation between these two notions is shown in the following.

Proposition 4.7. *If a hyperring* R *is Noetherian, then* $\texttt{Prim}(R)$ *is a Noetherian hyperstructure space.*

Proof. It suffices to show that a collection of closed sets in $\texttt{Prim}(R)$ satisfy the descending chain condition. Let $\texttt{Cl}(\mathfrak{a}_1) \supseteq \texttt{Cl}(\mathfrak{a}_2) \supseteq \cdots$ be a descending chain of closed sets in $\texttt{Prim}(R)$. Then, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ is an ascending chain of hyperideals in R. Since the hyperring R is Noetherian, the chain stabilizes at some $n \in \mathbb{N}$. Hence, $\mathtt{Cl}(\mathfrak{a}_n) = \mathtt{Cl}(\mathfrak{a}_{n+k})$ for any k. Thus $\mathtt{Prim}(R)$ is Noetherian. □

Corollary 4.8. *The set of minimal primitive hyperideals in a Noetherian hyperring is finite.*

Proof. By Proposition 4.7, $\mathtt{Prim}(R)$ is Noetherian, thus $\mathtt{Prim}(R)$ has finitely many irreducible components. By Proposition 4.6, every irreducible closed subset of $\mathtt{Prim}(R)$ is of the form $\mathtt{Cl}(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal primitive hyperideal. Thus $\mathtt{Cl}(\mathfrak{p})$ is an irreducible component if and only if $\mathfrak{p}$ is a minimal primitive hyperideal. Hence, R has only finitely many minimal primitive hyperideals. □

In general, a hyperstructure space is not T_1. However, with an added restriction we can characterize such spaces.

Theorem 4.9. *An hyperstructure space* $\mathtt{Prim}(R)$ *is a* T_1*-hyperstructure space if and only if* $\mathtt{Prim}(R)$ *coincides with the set* $\mathtt{Max}(R)$ *of maximal hyperideals of* R.

Proof. By Proposition 3.7, $\mathtt{Max}(R) \subseteq \mathtt{Prim}(R)$. So, it is sufficient to prove the result for the other inclusion. Let $\mathfrak{a} \in \mathtt{Prim}(R)$. Then $\mathfrak{a} \in \mathtt{Cl}(\mathfrak{a})$. Let $\mathfrak{m}$ be a maximal hyperideal with $\mathfrak{a} \subseteq \mathfrak{m}$. Then

$$\mathfrak{m} \in \mathtt{Cl}(\mathfrak{a}) = \{\mathfrak{a}\},$$

where the equality follows from $\mathtt{Prim}(R)$ being a T_1-space. Therefore $\mathfrak{m} = \mathfrak{a}$, showing that $\mathtt{Prim}(R) \subseteq \mathtt{Max}(R)$. Conversely, in $\mathtt{Max}(R)$, $\mathtt{Cl}(\mathfrak{m}) = \{\mathfrak{m}\}$ for every maximal hyperideal $\mathfrak{m}$, so that $\mathfrak{m} \in \mathtt{Cl}(\mathfrak{m})$, showing that the hyperstructure space is T_1. □

A strong hyperring homomorphism induces a continuous map between corresponding hyperstructure spaces. We now study this continuity and homeomorphisms between such spaces.

Proposition 4.10. *Suppose* $\phi\colon R \to R'$ *is a strong hyperring homomorphism and define the map* $\phi_*\colon \mathtt{Prim}(R') \to \mathtt{Prim}(R)$ *by* $\phi_*(\mathfrak{p}) = \phi(^{-1}\mathfrak{p})$, *where* $\mathfrak{p} \in \mathtt{Prim}(R')$. *Then* ϕ_* *is a continuous map.*

Proof. To show ϕ_* is continuous, we first show that $\phi(^{-1}\mathfrak{p}) \in \mathtt{Prim}(R)$, whenever $\mathfrak{p} \in \mathtt{Prim}(R')$. Note that $\phi(^{-1}\mathfrak{p})$ is a hyperideal of R. Suppose $\mathfrak{p} = \mathtt{Ann}_{R'}(M)$ for some simple R'-hypermodule. Then by the "change of hyperrings" property of hypermodules, $\phi(^{-1}\mathfrak{p})$ is the annihilator of the simple R'-hypermodule M obtained by defining $sm = \phi(s)m$. Therefore $\phi(^{-1}\mathfrak{p}) \in \mathtt{Prim}(R)$. Now consider a closed subset $\mathtt{Cl}(\mathfrak{a})$ of $\mathtt{Prim}(R)$. Then for any $\mathfrak{q} \in \mathtt{Prim}(R')$, we have the following sequence of equivalent statements:

$$\mathfrak{q} \in \phi_*(^{-1}\mathtt{Cl}(\mathfrak{a})) \Leftrightarrow \phi(^{-1}\mathfrak{q}) \in \mathtt{Cl}(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \phi(^{-1}\mathfrak{q}) \Leftrightarrow \mathfrak{q} \in \mathtt{Cl}(\langle\phi(\mathfrak{a})\rangle).$$

These prove the desired continuity of ϕ_*. □

Proposition 4.11. *If* $\mathfrak{a}$ *is a hyperideal of the hyperring* R, *then* $\mathtt{Cl}(\mathfrak{a})$ *is homeomorphic to the hyperstructure space* $\mathtt{Prim}(R/\mathfrak{a})$.

Proof. We shall in fact prove more, i.e., if $\phi\colon R \to R'$ is a strong hyperring homomorphism and if ϕ is surjective, then the hyperstructure space $\mathtt{Prim}(R')$ is homeomorphic to the closed subset $\mathtt{Cl}(\mathtt{ker}(\phi))$ of the hyperstructure space $\mathtt{Prim}(R)$. The desired result will then follow by taking the quotient map $R \to R/\mathfrak{a}$.

Since $\mathfrak{o} \subseteq \mathfrak{b}$ for all $\mathfrak{b} \in \mathtt{Prim}(R')$, we have that $\mathtt{ker}(\phi) \subseteq \phi(^{-1}\mathfrak{b})$, or, in other words $f^*(\mathfrak{b}) \in \mathtt{Cl}(\mathtt{ker}(\phi))$. This implies that $\mathtt{im}(\phi^*) = \mathtt{Cl}(\mathtt{ker}(\phi))$. Since for all $\mathfrak{b} \in \mathtt{Prim}(R')$, $\phi(\phi^*(\mathfrak{b})) = \phi(\phi(^{-1}\mathfrak{b})) = \mathfrak{b}$, the map ϕ^* is injective. To show that ϕ^* is a closed map, first we observe that for any closed subset $\mathtt{Cl}(\mathfrak{a})$ of $\mathtt{Prim}(R')$, we have that:

$$\phi^*(\mathtt{Cl}(\mathfrak{a})) = \phi(^{-1}\mathtt{Cl}(\mathfrak{a})) = \phi\{^{-1}\mathfrak{i}' \in \mathtt{Prim}(R') \mid \mathfrak{a} \subseteq \mathfrak{i}'\} = \mathtt{Cl}(\phi(^{-1}\mathfrak{a})).$$

Now if C is a closed subset of $\mathtt{Prim}(R')$ and $C = \mathtt{C}l(\mathfrak{a})$, then $\phi^*(C) = \phi(^{-1}\mathtt{C}l(\mathfrak{a})) = \mathtt{Cl}(\phi(^{-1}\mathfrak{a}))$, a closed subset of $\mathtt{Prim}(R)$. Since by Proposition 4.10, ϕ^* is continuous, we have the desired claim. □

Corollary 4.12. *The hyperstructure spaces* $\mathtt{Prim}(R)$ *and* $\mathtt{Prim}(R)/\sqrt{\mathfrak{o}}$ *are homeomorphic, where* $\sqrt{\mathfrak{o}}$ *is the nil radical of* R.

Proposition 4.13. *Let* ϕ^* *be as in Proposition 4.10. Then* $\phi^*(\mathtt{Prim}(R'))$ *is dense in* $\mathtt{Prim}(R)$ *if and only if* $\mathtt{ker}(\phi) \subseteq \sqrt{\mathfrak{o}}$.

Proof. We first show that $\mathtt{Cl}(\phi^*(\mathtt{Cl}(\mathfrak{b}))) = \mathtt{Cl}(\phi(^{-1}\mathfrak{b}))$, for all hyperideals $\mathfrak{b}$ of R'. To this end, let $\mathfrak{s} \in \phi^*(\mathtt{Cl}(\mathfrak{b}))$. This implies that $\phi(\mathfrak{s}) \in \mathtt{Cl}(\mathfrak{b})$, which means $\mathfrak{b} \subseteq \phi(\mathfrak{s})$. In other words, $\mathfrak{s} \in \mathtt{Cl}(\phi(^{-1}\mathfrak{b}))$. The other inclusion follows from the fact that $\phi(^{-1}\mathtt{Cl}(\mathfrak{b})) = \mathtt{Cl}(\phi(^{-1}\mathfrak{b}))$. Since

$$\mathtt{Cl}(\phi^*(\mathtt{Prim}(R'))) = \phi^*(\mathtt{Cl}(\mathfrak{o})) = \mathtt{Cl}(\phi(^{-1}\mathfrak{o})) = \mathtt{Cl}(\mathtt{ker}(\phi)),$$

we see that $\mathtt{Cl}(\mathtt{ker}(\phi))$ is equal to $\mathtt{Prim}(R))$ if and only if $\mathtt{ker}(\phi) \subseteq \sqrt{\mathfrak{o}}$. □

5 Conclusion

This paper had two main aims. The first was to introduce the notion of primitive hyperideals of a (Krasner) hyperring and study their properties. The second was to impose a Jacobson topology on the set of primitive hyperideals of a hyperring and investigate the topological properties of the corresponding hyperstructure space.

As part of the first aim we showed the relation between prime, maximal, and primitive hyperideals of a hyperring and also characterized simple hypermodules. We showed how the hyperideal is related to R itself, and to maximal and prime hyperideals of R.

As part of the second aim, we investigated the topological properties of the corresponding hyperstructure space. We characterized irreducible closed subsets of a hyperstructure space and proved that every irreducible closed subset of a hyperstructure space has a unique generic point. Finally we close with a sufficient condition for the space to be Noetherian and looked at continuous maps between such spaces.

As a continuation of this work, one may consider the following. Using the primitive hyperideals of hyperrings that have been introduced here, it would be interesting to investigate a structure theory of hyperrings as developed in [12] for rings.

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