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On relatively connected sublocales and J-frames

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Dedicated to Prof. Themba Dube on the occasion of his 65th birthday

Abstract. In this paper, we present a study of relatively connected sublocales. The development of this study is subsequently utilized to characterize what we call C-normal frames. Normal frames are C-normal, the converse is not true. Some results concerning J-frames are presented; amongst other things, we prove that regular continuous frames are rim-compact and that the converse is true for J-frames. The latter is used to show that the least compactification of a regular continuous J-frame coincides with its Freudenthal compactification.

1 Introduction

In [8], Michael introduced the concept of a J-space as a Hausdorff space X such that every binary closed cover with a compact intersection has a compact member. In the same paper, he introduced the concept of rela-

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tively connected subsets, explored more in [12]. A localic view of relatively connected subsets and the pointfree analog of J-spaces are presented in this note. While relatively connected sublocales need not be connected, we show that dense and open sublocales do not distinguish between connectedness and relative connectedness. We introduce C-normal frames as a conservative extension of C-normal spaces and characterize them as frames with the property that the join of any two closed connected disjoint sublocales is not relatively connected. Furthermore, we show that the class of normal frames is properly contained in the class of C-normal frames and that C-normality is a hereditary property with respect to closed sublocales. A partition of a J-frame into complemented elements is always finite and has exactly one compact member. The proof of the latter invokes the axiom of countable dependent choice (CDC). We conclude the paper by characterizing non-compact regular continuous J-frames via the perfectness of their least compactification.

2 Preliminaries

We begin with a collection of the basic frame-theoretic definitions we shall use in the sequel.

2.1 Basic frame theory A complete lattice L is called:

(i) a *frame* if it satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\},$$

(ii) a *co-frame* if it satisfies the infinite distributive law:

$$x \vee \bigwedge S = \bigwedge \{x \vee s : s \in S\},$$

for every $x \in L$ and every $S \subseteq L$. The top element and the bottom element of L will be denoted by 1 and 0, respectively. A *frame homomorphism* is a map $h : L \longrightarrow M$, between the frames L and M, that preserves finite meets (including 1) and arbitrary joins (including 0). Associated with such a homomorphism is its *right adjoint* $h_* : M \longrightarrow L$ satisfying $h(x) \leq y$ if and only if $x \leq h_*(y)$ for all $x \in L$ and $y \in M$, where $h_*(y) = \bigvee\{x \in L : h(x) \leq y\}$. A complete lattice L is called *continuous* if for each $a \in L$,

$a = \bigvee\{x \in L : x \ll a\}$, where $x \ll a$ means that for any $S \subseteq L$ with $a \leq \bigvee S$, there exists a finite $F \subseteq S$ such that $x \leq \bigvee F$. If $x \ll a$ we say that x is *well below* a. For any $a, b \in L$, we say that a is *rather below* b, written $a \prec b$, if $a \wedge c = 0$ and $c \vee b = 1$, for some $c \in L$. Note that $a \prec b$ if and only if $a^* \vee b = 1$, where a^* is the *pseudocomplement* of a, that is, $a^* = \bigvee\{x \in L : x \wedge a = 0\}$. A frame L is called *regular* if $a = \bigvee\{x \in L : x \prec a\}$, for all $a \in L$. An element $a \in L$ is complemented if $a \vee a^* = 1$. An element $a \in L$ is *compact* if whenever $a \leq \bigvee S$ for some $S \subseteq L$, then $a \leq \bigvee S_0$ for some finite $S_0 \subseteq S$. A frame L is *compact* if the top element is compact.

A frame L is *normal* if for any $a, b \in L$ with $a \vee b = 1$, there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee v = 1 = b \vee u$. Equivalently, L is normal if for any $a, b \in L$ with $a \vee b = 1$, there exists $u \in L$ such that $a \vee u^* = 1 = b \vee u$. An element $c \in L$ is *connected* if whenever $c = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. A frame L is *connected* if its top element is connected. A *partition* of a frame L (complete lattice in general) is a collection $\{a_i\}_{i \in I} \subseteq L$ such that: $a_i \neq 0$ for each $i \in I$, $a_i \wedge a_j = 0$ for $i \neq j$, and $1 = \bigvee\limits_{i \in I} a_i$. For more details on the theory of frames, we refer the reader to [3], [7] and [14].

2.2 Notes on sublocales Let L be a frame. We have the binary operation $\to$ on L, called the *Heyting operation*, that satisfies the property that for all a, b, c in a frame L, $c \leq a \to b$ if and only if $c \wedge a \leq b$. A *sublocale* of a frame L is a subset S of L such that S is closed under arbitrary meets, and for each $x \in L$ and each $s \in S$, $x \to s \in S$. Sublocales are frames in their own right.

For any $a \in L$, the set $\mathfrak{c}_L(a) = \{x \in L : a \leq x\}$ is a sublocale of frame L that is called the *closed sublocale* associated with a. The set $\mathfrak{o}_L(a) = \{a \to x : x \in L\}$ is referred to as an *open sublocale* associated with a. When we think frame-theoretically, we shall write $\uparrow a$ for $\mathfrak{c}_L(a)$. We speak of the *trivial* (or *void*) sublocale, $\mathsf{O} := \{1\}$, and say that sublocales S and T are *disjoint* if $S \cap T = \mathsf{O}$. Note that $\mathsf{O} = \{1\} \subseteq S$, so $1 \in S$ for any sublocale S. A sublocale is compact if every cover by open sublocales of the ambient frame (equivalently, of itself) can be reduced to a finite subcover. It is easy to see that an element $a \in L$ is compact if and only if $\mathfrak{o}_L(a)$ is compact.

The lattice $\mathcal{S}(L)$ of all sublocales of a frame L is a co-frame under inclusion (see [14, Theorem III.3.2.1]). Here, meets are precisely the intersections, O is the bottom element, and L is the top element of $\mathcal{S}(L)$. The joins are defined by the formula:

$$\bigvee_{i\in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}$$

for any $\{S_i\}_{i\in I} \subseteq \mathcal{S}(L)$. In this paper, joins and meets of sublocales of L with be taken in the co-frame $\mathcal{S}(L)$. The *supplement* of a sublocale S of the frame L is

$$L \setminus S = \bigcap\{R \in \mathcal{S}(L) : R \vee S = L\} = \bigvee\{T \in \mathcal{S}(L) : T \cap S = \mathsf{O}\}.$$

Note that $S \vee (L \setminus S) = L$. We say that a sublocale S is *complemented* if $S \cap (L \setminus S) = \mathsf{O}$. In this case, $L \setminus S$ is the lattice theoretic complement of S in $\mathcal{S}(L)$.

We say that a sublocale S is *connected* if the top element of S (same as the top element of L) is connected in S. Equivalently, a non-void sublocale S of a frame L is connected if and only if whenever $a, b \in L$ and $S \subseteq \mathfrak{o}_L(a) \vee \mathfrak{o}_L(b)$ with $S \cap \mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathsf{O}$, either $S \cap \mathfrak{o}_L(a) = \mathsf{O}$ or $S \cap \mathfrak{o}_L(b) = \mathsf{O}$. This is true if and only if, whenever $S \subseteq \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ with $S \cap \mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$, either $S \cap \mathfrak{c}_L(b) = \mathsf{O}$ or $S \cap \mathfrak{c}_L(b) = \mathsf{O}$. It is not difficult to show that an element $a \in L$ is connected if and only if the associated open sublocale $\mathfrak{o}_L(a)$ is connected.

Consider the inclusion map $j : A \hookrightarrow X$ of the subset $A \subseteq X$ of a space X. One has the frame homomorphism $\mathfrak{O}j : \mathfrak{O}X \longrightarrow \mathfrak{O}A$, whose right adjoint $(\mathfrak{O}j)_* : \mathfrak{O}A \longrightarrow \mathfrak{O}X$ is given by:

$$(\mathfrak{O}j)_*(U) = X \setminus \mathrm{cl}_X(A \setminus U), \quad \text{for all} \quad U \in \mathfrak{O}A.$$

The sublocale $\widetilde{A}$ of the frame $\mathfrak{O}X$ *induced* by A is the frame given by

$$\widetilde{A} = (\mathfrak{O}j)_*(\mathfrak{O}A) = \{(\mathfrak{O}j)_*(U) : U \in \mathfrak{O}A\}.$$

For general details on sublocales, see Chapters III and VI in [14] and Section 2 of Chapter II in [7].

3 Relatively connected sublocales

The notion of relatively connected subsets of topological spaces was first introduced by Michael [8]. A subset A of a topological space X is called *relatively connected* in X if no open $U \supseteq A$ in X has a disjoint open cover $\{U_1, U_2\}$ with $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$. In [12], Mthethwa and Taherifar provided more insight and characterized relatively connected subsets of space using various concepts. In particular, they showed that A is relatively connected in X if and only if for any open sets U_1, U_2 in X such that $A \subseteq U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$, either $A \subseteq U_1$ or $A \subseteq U_2$ (see [12, Lemma 2.1]). Using the latter, we lift the notion of relatively connected subsets from spaces to a localic setting as follows:

Definition 3.1. [11] A sublocale S of a frame L is *relatively connected* in L if whenever $S \subseteq U_1 \vee U_2$ where U_1, U_2 are open sublocales of L such that $U_1 \cap U_2 = \mathsf{0}$, either $S \cap U_1 = \mathsf{0}$ or $S \cap U_2 = \mathsf{0}$.

Remark 3.2. Of course, in the conclusion part of the definition above, we could equivalently use $S \subseteq U_1$ or $S \subseteq U_2$. An equivalent formulation of relative connectedness in terms of elements may be formulated in the following manner:

(1) A sublocale S of a frame L is relatively connected in L if and only if whenever $x, y \in L$ and $S \subseteq \mathfrak{o}(x) \vee \mathfrak{o}(y)$, with $x \wedge y = 0$, then $S \cap \mathfrak{o}_L(x) = \mathsf{0}$ or $S \cap \mathfrak{o}_L(y) = \mathsf{0}$.

(2) A sublocale S of a frame L is relatively connected in L if and only if whenever $x, y \in L$ and $S \cap \mathfrak{c}_L(x) \cap \mathfrak{c}_L(y) = \mathsf{0}$, with $x \wedge y = 0$, then $S \cap \mathfrak{c}_L(x) = \mathsf{0}$ or $S \cap \mathfrak{c}_L(y) = \mathsf{0}$.

Recall that a T_D*-space* is a topological space X with the property that, for each $x \in X$, there is an open $U \ni x$ such that $U \setminus \{x\}$ is also open. Let us recall from [14] that $A \subseteq B \Longrightarrow \widetilde{A} \subseteq \widetilde{B}$, and in a T_D-space, $A \subseteq B \Longleftrightarrow \widetilde{A} \subseteq \widetilde{B}$. If K is a closed subset and U is an open in a space X, it is easy to check that the induced sublocales $\widetilde{K}$ and $\widetilde{U}$ are given by: $\widetilde{K} = \mathfrak{c}_{\mathfrak{O}X}(X \setminus K)$ and $\widetilde{U} = \mathfrak{o}_{\mathfrak{O}X}(U)$.

Proposition 3.3. *Let X be a T_D-space, and $A \subseteq X$. Then A is relatively connected in X if and only if $\widetilde{A}$ is relatively connected in $\mathfrak{O}X$.*

Proof. ($\Longrightarrow$) Suppose that $A \subseteq X$ is relatively connected in X. Let $\widetilde{A} \subseteq \mathfrak{o}_{\mathfrak{O}X}(U_1) \cup \mathfrak{o}_{\mathfrak{O}X}(U_2)$ with $\mathfrak{o}_{\mathfrak{O}X}(U_1) \cap \mathfrak{o}_{\mathfrak{O}X}(U_2) = \mathsf{0}$, where $\widetilde{U_1, U_2} \in \mathfrak{O}X$. From $\widetilde{A} \subseteq \mathfrak{o}_{\mathfrak{O}X}(U_1) \cup \mathfrak{o}_{\mathfrak{O}X}(U_2)$, we get $\widetilde{A} \subseteq \mathfrak{o}(U_1 \cup U_2) = \widetilde{U_1 \cup U_2}$. Since X is a T_D-space, $A \subseteq U_1 \cup U_2$. Now, observe that $\mathfrak{o}_{\mathfrak{O}X}(U_1) \cap \mathfrak{o}_{\mathfrak{O}X}(U_2) = \mathsf{0} \Longleftrightarrow \mathfrak{o}_{\mathfrak{O}X}(U_1 \cap U_2) = \mathsf{0} \Longleftrightarrow \widetilde{U_1 \cap U_2} = \widetilde{\emptyset} \Longleftrightarrow U_1 \cap U_2 = \emptyset$, where the last equivalence follows by the fact that X is a T_D-space. Now, $A \subseteq U_1 \cup U_2$, and $U_1 \cap U_2 = \emptyset$. Since A is relatively connected in X, then $A \subseteq U_1$ or $A \subseteq U_2$. Thus, $\widetilde{A} \subseteq \widetilde{U_1}$ or $\widetilde{A} \subseteq \widetilde{U_2}$. That is, $\widetilde{A} \subseteq \mathfrak{o}_{\mathfrak{O}X}(U_1)$ or $\widetilde{A} \subseteq \mathfrak{o}_{\mathfrak{O}X}(U_2)$. It follows that $\widetilde{A} \cap \mathfrak{o}_{\mathfrak{O}X}(U_1) = \mathsf{0}$ or $\widetilde{A} \cap \mathfrak{o}_{\mathfrak{O}X}(U_2) = \mathsf{0}$. Therefore, $\widetilde{A}$ is relatively connected in $\mathfrak{O}X$.
($\Longleftarrow$) Follows by an argument similar to the one provided for the forward direction. □

It follows by the proposition above and [8, Example 9.8] that relatively connected sublocales need not be connected. Let $0_S = \bigwedge S$, where S is a sublocale of L. Recall that the *closure*, denoted by $\overline{S}$, of S in a frame L is given by

$$\overline{S} = \mathfrak{c}_L(0_S) = \{x \in L : x \geq \bigwedge S\} = \bigcap\{\mathfrak{c}_L(a) : S \subseteq \mathfrak{c}_L(a), a \in L\}.$$

The following statements are proved in [11, Proposition 3.8]:

Proposition 3.4. *Let $S \in \mathcal{S}(L)$. The following statements are true:*

1. *If S is connected, then S is relatively connected in L.*
2. *If S is relatively connected, then $\overline{S}$ is relatively connected in L.*

Next, we record some results concerning conditions under which relatively connected sublocales are connected. Before we do so, note that, just like in spaces, we have:

Lemma 3.5. *A sublocale S is dense in a frame L if and only if $S \cap U \neq \mathsf{0}$ for any non-void open $U \in \mathcal{S}(L)$.*

Proof. ($\Longrightarrow$) Let S be dense and U open with $U \neq \mathsf{0}$. Then $L \setminus U$ is closed and if $S \cap U = \mathsf{0}$, then $S \subseteq L \setminus U$. Hence, $L = \overline{S} \subseteq \overline{L \setminus U} \subseteq L \setminus U$. This implies that $U = \mathsf{0}$, which is a contradiction. Therefore $S \cap U \neq \mathsf{0}$.

($\Longleftarrow$) Let $S \in \mathcal{S}(L)$ be such that $S \cap U \neq \mathsf{0}$ for any non-void open $U \in \mathcal{S}(L)$. Note that $\overline{S}$ is closed, so $L \setminus \overline{S}$ is open and $\overline{S} \cap (L \setminus \overline{S}) = \mathsf{0}$. We have $S \subseteq \overline{S}$, therefore $S \cap (L \setminus \overline{S}) = \mathsf{0}$. Hence, $L \setminus \overline{S} = \mathsf{0}$, by the hypothesis. Thus, $L = \overline{S} \vee (L \setminus \overline{S}) = \overline{S}$. That is, S is dense. □

Proposition 3.6. *A dense sublocale is relatively connected in a frame L if and only if it is connected.*

Proof. ($\Longrightarrow$): Suppose that S is dense and relative connected in L. We want to show that S is connected. Suppose $S \subseteq U_1 \vee U_2$, where $S \cap U_1 \cap U_2 = \mathsf{0}$. Since S is dense and $U_1 \cap U_2$ is open, then $U_1 \cap U_2 = \mathsf{0}$, by the lemma above. Since S is relatively connected in L, then $S \cap U_1 = \mathsf{0}$ or $S \cap U_2 = \mathsf{0}$. Therefore, S is connected.
($\Longleftarrow$): This follows by Proposition 3.4(1). □

Proposition 3.7. *An open sublocale is relatively connected in a frame L if and only if it is connected.*

Proof. ($\Longleftarrow$): This follows by Proposition 3.4(1).

($\Longrightarrow$): Let $U \in \mathcal{S}(L)$ be open and assume that U is relatively connected in L. Suppose $U \subseteq U_1 \vee U_2$, where $U \cap U_1 \cap U_2 = \mathsf{0}$. From $U \subseteq U_1 \vee U_2$, we get:

$$U = U \cap (U_1 \vee U_2) = (U \cap U_1) \vee (U \cap U_2).$$

Now, $U \cap U_1$ and $U \cap U_2$ are disjoint open sublocales of L; indeed, $(U \cap U_1) \cap (U \cap U_2) = U \cap U_1 \cap U_2 = \mathsf{0}$. Since U is relatively connected in L, then $U \cap U_1 = \mathsf{0}$ or $U \cap U_2 = \mathsf{0}$. Thus, U is connected. □

Proposition 3.8. *If a frame L has a relatively connected dense sublocale, then L is connected.*

Proof. Let S be a relatively connected dense sublocale in L. We have that $\overline{S}$ is relatively connected in L by Proposition 3.4(2). Therefore, $\overline{S}$ is relatively connected and dense in L. Hence, $\overline{S}$ is connected by Proposition 3.6. But $\overline{S} = L$, so L is connected. □

Before we close this section, let us remark that in a normal frame L, any relatively connected closed sublocale of L is connected; the proof of this is provided in [11, Proposition 3.10].

4 C-normal frames

C-normal spaces were introduced in [12] as spaces X such that whenever A and B are two disjoint closed connected subsets in X, there are two disjoint open sets U, V in X with $A \subseteq U$ and $B \subseteq V$. We give a pointfree extension of this notion in terms of elements below:

Definition 4.1. A frame L is *C-normal* if whenever $a, b \in L$ and $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are connected with $a \vee b = 1$, then there exist $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$.

From a localic viewpoint, C-normality can be formulated as follows (this is equivalent to the definition presented in terms of elements above):

> *A frame L is C-normal if and only if whenever $S, T \in \mathcal{S}(L)$ are connected and closed with $S \cap T = \mathsf{O}$, then there exist opens $U, V \in \mathcal{S}(L)$ such that $S \subseteq U$ and $T \subseteq V$, with $U \cap V = \mathsf{O}$.*

The frame theoretic definition of C-normality is a conservative pointfree extension of C-normality of spaces. The proof of this uses the fact that a space is connected if and only if the frame of its open sets is connected:

Proposition 4.2. *A topological space X is C-normal if and only if $\mathfrak{O}X$ is a C-normal frame.*

Proof. ($\Longrightarrow$): Suppose X is C-normal. We prove that $\mathfrak{O}X$ is a C-normal frame. Take connected sublocales $\mathfrak{c}_{\mathfrak{O}X}(A) = {\uparrow}A$ and $\mathfrak{c}_{\mathfrak{O}X}(A) = {\uparrow}B$ with $A \cup B = 1_{\mathfrak{O}X}$, where $A, B \in \mathfrak{O}X$. Since ${\uparrow}A \cong \mathfrak{O}(X \setminus A)$ and ${\uparrow}B \cong \mathfrak{O}(X \setminus B)$, then $\mathfrak{O}(X \setminus A)$ and $\mathfrak{O}(X \setminus B)$ are connected frames, by conservativeness of connectedness. Thus, $X \setminus A$ and $X \setminus B$ are closed connected subsets of X. We also have that $(X \setminus A) \cap (X \setminus B) = \emptyset$, since $A \cup B = X$. Now, X is C-normal, so $(X \setminus A) \subseteq U$ and $(X \setminus B) \subseteq V$, for some open sets $U, V \in X$, with $U \cap V = \emptyset$. Note that $X = (X \setminus A) \cup A \subseteq U \cup A = X$. Similarly, $B \cup V = X$. Therefore, $\mathfrak{O}X$ is a C-normal frame.

($\Longleftarrow$): Suppose that $\mathfrak{O}X$ is a C-normal frame. We want to show that X is C-normal. Let A and B be closed connected subsets of X such that $A \cap B = \emptyset$. Therefore $\mathfrak{O}(A)$ is connected. But $\mathfrak{O}(A) = \mathfrak{O}(X \setminus (X \setminus A)) \cong {\uparrow}(X \setminus A)$. So ${\uparrow}(X \setminus A)$ is connected. Similarly, ${\uparrow}(X \setminus B)$ is connected. From $A \cap B = \emptyset$, we have that $(X \setminus A) \cup (X \setminus B) = X = 1_{\mathfrak{O}X}$. Since $\mathfrak{O}X$ is

C-normal, there exist $U, V \in \mathfrak{O}X$ such that $(X \setminus A) \cup U = X = (X \setminus B) \cup V$ and $U \cap V = \emptyset$. Finally, $(X \setminus A) \cup U = X$ implies that $A \subseteq U$. Similarly, $B \subseteq V$. Hence X is C-normal. □

Remark 4.3. It is clear from the definition that every normal frame is C-normal. The converse is not true. It is observed in [12] that $E \times E$ is a C-normal space, where E is the Sorgenfrey line. It is well-known that the Sorgenfrey plane $E \times E$ is not a normal space, even though E is normal. Since C-normality and normality in frames are conservative extensions of the corresponding concepts in spaces, then $\mathfrak{O}(E \times E)$ is a C-normal frame that is not a normal frame.

We do not know whether C-normality is hereditary in general, but closed sublocales do inherit this property. To prove this, recall that for any sublocale S of a frame L, we have the onto frame homomorphism $\nu_S : L \longrightarrow S$ defined by $\nu_S(a) = \bigwedge\{s \in S : a \leq s\}$. In particular, $\nu_{\mathfrak{c}_L(x)}(a) = a \vee x$ for any $a, x \in L$. Denote the finite joins in S by $\sqcup_S$ and let $\mathfrak{c}_S(a) = \{x \in S : x \geq a\}$ denote the closed sublocale of S associated $a \in S$.

Proposition 4.4. *A closed sublocale of a C-normal frame is C-normal.*

Proof. Let L be a C-normal frame and $x \in L$. We must show that $\mathfrak{c}_L(x)$ is C-normal when viewed as a frame. Let $a, b \in \mathfrak{c}_L(x)$ and suppose that $\mathfrak{c}_{\mathfrak{c}_L(x)}(a)$ and $\mathfrak{c}_{\mathfrak{c}_L(x)}(b)$ are closed connected sublocales of $\mathfrak{c}_L(x)$ such that $a \sqcup_{\mathfrak{c}_L(x)} b = 1$. We need to find $u, v \in \mathfrak{c}_L(x)$ such that $a \sqcup_{\mathfrak{c}_L(x)} u = 1 = b \sqcup_{\mathfrak{c}_L(x)} v$ and $u \wedge v = x$. Now, observe that $\mathfrak{c}_{\mathfrak{c}_L(x)}(a) = \mathfrak{c}_L(a) \cap \mathfrak{c}_L(x) = \mathfrak{c}_L(x \vee a)$, and $\mathfrak{c}_{\mathfrak{c}_L(x)}(b) = \mathfrak{c}_L(x \vee b)$. Notice that $1 = a \sqcup_{\mathfrak{c}_L(x)} b = \nu_{\mathfrak{c}_L(x)}(a \vee b) = a \vee b \vee x = (a \vee x) \vee (b \vee x)$. Since $\mathfrak{c}_L(x \vee a)$ and $\mathfrak{c}_L(x \vee b)$ are closed connected sublocales of L and L is C-normal, then there exist $c, d \in L$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$. Let $u = c \vee x$ and $v = d \vee x$. Then $u, v \in \mathfrak{c}_L(x)$. On the one hand, $a \sqcup_{\mathfrak{c}_L(x)} u = \nu_{\mathfrak{c}_L(x)}(a \vee u) = a \vee u \vee x = a \vee c \vee x = 1$, and similarly, $b \sqcup_{\mathfrak{c}_L(x)} v = 1$. On the other hand, $u \wedge v = (c \vee x) \wedge (d \vee x) = (c \wedge d) \vee x = 0 \vee x = x$. Therefore, $\mathfrak{c}_L(x)$ is C-normal. □

We end this section by a characterization of C-normality via relatively connected closed sublocales, the proof of which uses the fact that for $a, b \in L$, $a \vee b = 1 \Longleftrightarrow \mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(b)$:

Proposition 4.5. *A frame L is C-normal if and only if the join of any two nontrivial closed connected disjoint sublocales is not relatively connected in L.*

Proof. ($\Longrightarrow$): Suppose that L is C-normal. Let $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ be nontrivial connected with $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$. We need to show that $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ is not relatively connected in L. From $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$, we get $a \vee b = 1$. Since L is C-normal, then there exist $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. From $a \vee u = 1$, we get $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(u)$, and from $b \vee u = 1$, we get $\mathfrak{c}_L(b) \subseteq \mathfrak{o}_L(v)$. Since $u \wedge v = 0$, then $\mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}$. We now have, $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) \subseteq \mathfrak{o}_L(u) \vee \mathfrak{o}_L(v)$. However, $(\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) \cap \mathfrak{o}_L(u) = \mathfrak{c}_L(a) \vee (\mathfrak{c}_L(b) \cap \mathfrak{o}_L(u)) \neq \mathsf{O}$. Similarly, $(\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) \cap \mathfrak{o}_L(v) \neq \mathsf{O}$. Hence $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ is not relatively connected in L.
($\Longleftarrow$): Suppose that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are nontrivial connected sublocales of L and $a \vee b = 1$. Since $a \vee b = 1$, then $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$. By the hypothesis, $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ is not relatively connected in L. Hence, there exist $\mathfrak{o}_L(u), \mathfrak{o}_L(v)$ such that $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) \subseteq \mathfrak{o}_L(u) \vee \mathfrak{o}_L(v)$ and $\mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}$ with $(\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) \cap \mathfrak{o}_L(u) \neq \mathsf{O}$ and $(\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) \cap \mathfrak{o}_L(v) \neq \mathsf{O}$. We now have $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(u) \vee \mathfrak{o}_L(v)$ and $\mathfrak{c}_L(a) \cap \mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}$. Since $\mathfrak{c}_L(a)$ is connected, then $\mathfrak{c}_L(a) \cap \mathfrak{o}_L(u) = \mathsf{O}$ or $\mathfrak{c}_L(a) \cap \mathfrak{o}_L(v) = \mathsf{O}$. Thus, $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(u)$ or $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(v)$, i.e., $a \vee u = 1$ or $a \vee v = 1$. Similarly since $\mathfrak{c}_L(b)$ is connected, either $b \vee u = 1$ or $b \vee v = 1$. We now have: $a \vee u = 1$ and $b \vee v = 1$, or $a \vee v = 1$ and $b \vee u = 1$. Thus, $a \vee u = 1 = b \vee v$, and $u \wedge v = 0$, the latter follows from $\mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}$. □

5 More on J-frames

Let us begin this section by recalling the definition of the pointfree analog of the notion of a J-space:

Definition 5.1. [11] We say that a frame L is a *J-frame* if for any $a, b \in L$ such that $a \wedge b = 0$ and $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)$ is compact, either $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(b)$ is compact.

Remark 5.2. Using the fact that for any space X and an open U in X, $\uparrow U \cong \mathfrak{O}(X \setminus U)$, it is not difficult to see that:

A Hausdorff space X is a J-space if and only if $\mathfrak{O}X$ is a J-frame.

Here is a neat characterization of J-frames:

Proposition 5.3. *The following conditions are equivalent for a frame L:*

(1) *L is a J-frame.*

(2) *Whenever $a \in L$ and $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(a^*)$ is compact, then either $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(a^*)$ is compact.*

Proof. (1) $\Longrightarrow$ (2) This is clear from the definition of J-frame.
(2) $\Longrightarrow$ (1) Take $a, b \in L$ such that $a \wedge b = 0$ and $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)$ is compact. We need to show that $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(b)$ is compact. Now, $b \leq a^*$ implies that $\mathfrak{c}_L(a^*) \subseteq \mathfrak{c}_L(b)$, therefore $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(a^*) \subseteq \mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)$, and so $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(a^*)$ is compact. Thus, $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(a^*)$ is compact, by the hypothesis. If $\mathfrak{c}_L(a)$ is compact, then we are done. If $\mathfrak{c}_L(a^*)$ is compact, then $\mathfrak{c}_L(a^*) \vee (\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b))$ is compact. But $\mathfrak{c}_L(a^*) \vee (\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)) = \mathfrak{c}_L(a^*) \vee \mathfrak{c}_L(b)$. Hence $\mathfrak{c}_L(b)$ is compact. □

The result below, whose proof requires the CDC principle, says that a J-frame can only be partitioned by finitely many complemented elements; and in such a partition, precisely one element is not compact. Since this result is trivial for compact frames (indeed, all compact frames are J-frames by [11, Proposition 4.7]), we exclude this case. Before we present the proof, let us recall from [14, Proposition XIII.1.1] that: elements a and b are mutual complements in L if and only if the closed sublocales $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are mutual complements in $\mathcal{S}(L)$, and this is true if and only if the open sublocales $\mathfrak{o}_L(a)$ and $\mathfrak{o}_L(b)$ are mutual complements in $\mathcal{S}(L)$. If a is complemented and we write a^c for the complement of a, then the latter implies that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(a^c)$ are mutual complements in $\mathcal{S}(L)$, and $\mathfrak{o}_L(a)$ and $\mathfrak{o}_L(a^c)$ are mutual complements in $\mathcal{S}(L)$.

Proposition 5.4. *Let L be a non-compact J-frame and A be a partition of L by complemented elements. Then there exists exactly one $a_0 \in A$ such that a_0 is non-compact. Furthermore, A is finite.*

Proof. We start by showing that there exists an element $a_0 \in A$ such that a_0 is not compact. We proceed by contradiction, so to this end, assume that a_i is compact for every $a_i \in A$. Note that in this case, A cannot be finite: for if A has m elements for some $m \in \mathbb{N}$, then $1 = \bigvee_{i=1}^{m} a_i$. The latter implies that L is compact, which is a contradiction. So, assume A contains infinitely many distinct compact elements. Hence, under CDC, we may select a countable set $\{a_n\}_{n\in\mathbb{N}}$ containing distinct elements of A. Put $a = \bigvee\{a_{2n-1} : n \in \mathbb{N}\}$. Since A is a partition,

$$\bigvee A = a \vee \bigvee_{i\in I}\{a_i \in A : a_i \neq a_{2n-1}, n \in \mathbb{N}\} = 1,$$

and by repeated use of the frame law and by the disjointness of elements in A, we get:

$$\begin{aligned} a \wedge \bigvee_{i\in I}\{a_i \in A : a_i \neq a_{2n-1}, n \in \mathbb{N}\} =& \bigvee_{i\in I}\{a \wedge a_i : a_i \neq a_{2n-1}, n \in \mathbb{N}\} \\ =& 0. \end{aligned}$$

So, $a^c = \bigvee_{i\in I}\{a_i \in A : a_i \neq a_{2n-1}, n \in \mathbb{N}\}$. Thus, both a and a^c are not compact elements of L since each is an infinite join of elements in L. However, $a \vee a^c = 1$ implies that $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(a^c)$ is compact. We also have that $a \wedge a^c = 0$, and since L is a J-frame, then $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(a^c)$ is compact, by Proposition 5.3. Now, complementedness of a implies that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(a^c)$ are complements of each other in $\mathcal{S}(L)$. But $\mathcal{S}(L)$ is a distributive lattice, so complements are unique, thus:

$$\mathfrak{o}_L(a) = \mathfrak{c}_L(a^c) = \bigvee\{\mathfrak{o}_L(a_{2n-1}) : n \in \mathbb{N}\},$$

and similarly,

$$\mathfrak{c}_L(a) = \mathfrak{o}_L(a^c) = \bigvee_{i\in I}\{\mathfrak{o}_L(a_i) \in A : a_i \neq a_{2n-1}, n \in \mathbb{N}\}.$$

Hence, both $\mathfrak{c}_L(a^c)$ and $\mathfrak{c}_L(a)$ are not compact, as they are both infinite joins of open sublocales. This is a contradiction. Therefore, there exists an element in A which is non-compact. We now show that only one element in A has this property. To this end, fix $a_0 \in A$ such that a_0 is a non-compact element. We will show that any element in A different from a_0 is compact. Start by recalling that $a_0 \vee a_0^c = 1$, therefore $\mathfrak{c}_L(a_0) \cap \mathfrak{c}_L(a_0^c) = \mathsf{0}$ is compact. Since $a_0 \wedge a_0^c = 0$, and L is a J-frame, then $\mathfrak{c}_L(a_0)$ or $\mathfrak{c}_L(a_0^c)$ is compact. However, since $\mathfrak{c}_L(a_0^c) = \mathfrak{o}_L(a_0)$ and a_0 is non-compact, then $\mathfrak{c}_L(a_0^c)$ is non-compact. Thus, $\mathfrak{c}_L(a_0)$ must be compact. Now, take any $a_k \in A$ such that $a_k \neq a_0$. Then $a_k \wedge a_0 = 0$, and this implies that $a_0 \leq a_k^c$, whence $\mathfrak{c}_L(a_k^c) \subseteq \mathfrak{c}_L(a_0)$. Therefore $\mathfrak{c}_L(a_k^c)$ is compact. But $\mathfrak{c}_L(a_k^c) = \mathfrak{o}_L(a_k)$, so a_k is compact. To see that A is finite, notice that a_0^c is compact and $a_0^c \leq \bigvee A$, so $a_0^c \leq \bigvee_{i=1}^{m} \{a_i : a_i \in A\}$, for some $m \in \mathbb{N}$. Hence $1 = a_0 \vee \bigvee_{i=1}^{m} \{a_i : a_i \in A\}$, and so $A = \{a_0, a_1, \ldots, a_m\}$. □

Definition 5.5. A frame L is rim-compact if it is regular and the set $\{x \in L : \mathfrak{c}_L(x) \cap \mathfrak{c}_L(x^*)$ is compact$\}$ is a basis for L.

Lemma 5.6. *Any regular continuous frame is rim-compact.*

Proof. Suppose that L is a regular continuous frame and let $a \in L$. Then $a = \bigvee\{x \in L : x \ll a\}$. Now, $x \ll a$ and $a \leq 1$ implies that $x \ll 1$. So $\mathfrak{c}_L(x^*)$ is compact, by [2, Proposition 3.3]. Hence $\mathfrak{c}_L(x) \cap \mathfrak{c}_L(x^*)$ is compact, and so $a = \bigvee\{x \in L : \mathfrak{c}_L(x) \cap \mathfrak{c}_L(x^*)$ is compact$\}$. Thus L is rim-compact. □

For J-frames, the converse of Lemma 5.6 holds true:

Proposition 5.7. *A rim-compact J-frame is regular continuous.*

Proof. Suppose that L is rim-compact and let $a \in L$ such that $a \leq \bigvee S$ for some $S \subseteq L$. Using regularity of L and the fact that $B_L = \{x \in L : \mathfrak{c}_L(x) \cap \mathfrak{c}_L(x^*)$ is compact$\}$ is a basis for L (see [1, Remark 4.4]) we may write $a = \bigvee\{b : b \prec a,\ b \in B_L\}$. Now, $b \prec a$ if and only if $b^* \vee a = 1$. It follows that $b^* \vee \bigvee S = 1$. If L is compact, then $b^* \vee \bigvee S_0 = 1$ for some finite $S_0 \subseteq S$. Hence $b \leq \bigvee S_0$, and so $b \ll a$. So, suppose that L is a non-compact frame. Note that $b \in B_L$ implies that $b^* \in B_L$, so $\mathfrak{c}_L(b^*) \cap \mathfrak{c}_L(b^{**})$ is compact. Thus, $\mathfrak{c}_L(b^*)$ or $\mathfrak{c}_L(b^{**})$ is compact, by Proposition 5.3. If $\mathfrak{c}_L(b^*)$ is compact, then $b^* \vee \bigvee S = 1$ implies that $b^* \vee \bigvee S_1 = 1$ for some finite $S_1 \subseteq S$, hence $b \leq \bigvee S_1$ and we are done. Now, suppose $\mathfrak{c}_L(b^{**})$ is compact. Since $b^* \vee a = 1$ implies that $b^{**} \leq a$, then $\mathfrak{c}_L(a)$ is compact. The latter implies that $\mathfrak{c}_L(\bigvee S)$ is compact. Since $b^* \vee \bigvee S = \bigvee\{b^* \vee s : s \in S\} = 1$, it follows that $\bigvee_{i=1}^{n} \{b^* \vee s_i : s_i \in S\} = 1$, for some $s_i \in S$. Thus $b \leq \bigvee_{i=1}^{n} \{s_i : s_i \in S\}$, so $b \ll a$. Hence, L is regular continuous. □

6 Some compactifications of J-frames

The main goal of this section is to show that the least compactification is identical to the Freudenthal compactification precisely when L is a J-frame. Before we do this, let us recall some nomenclature. A *compactification* of a frame L is a dense onto frame homomorphism $h : M \longrightarrow L$, where M is compact regular. We say that a compactification $h : M \longrightarrow L$ is *perfect* if its right adjoint preserves disjoint binary joins. A *strong inclusion* on a frame L is a binary relation $\lhd$ on L such that: (1) if $x \leq a \lhd b \leq y$, then, $x \lhd y$, (2) $\lhd$ is a sublattice of $L \times L$, (3) $a \lhd b$ implies $a \prec b$, (4) $a \lhd b$ implies $a \lhd c \lhd b$, for some $c \in L$ (the interpolation property), (5) $a \lhd b$ implies $b^* \lhd a^*$, and, (6) $a = \bigvee\{x \in L : x \lhd a\}$, for all $a \in L$. Compactifications of a frame L are in a one-to-one correspondence with strong inclusions on L; see [4] for details. If two strong inclusions are equal, so are the corresponding compactifications, and conversely. For more on the theory of compactifications and perfect compactifications of frames, we refer the reader to papers by Baboolal ([1] and [2]), Banaschewski [4], Mthethwa ([9] and [10]).

Recall from Baboolal [1] that a regular frame L is *rim-compact* if it has a basis B such that $\mathfrak{c}_L(b) \cap \mathfrak{c}_L(b^*)$ is compact for every $b \in B$. It is shown in [1] that a rim-compact frame possesses a compactification $\bigvee : \gamma L \longrightarrow L$ whose remainder is zero-dimensional (also see Ferreria et. al [5]) and its corresponding strong inclusion $\lhd_{B_L}$ is defined by: $a \lhd_{B_L} b$ if and only if $a \prec c \prec b$ for some $c \in B_L$, where B_L is a totality of elements $u \in L$ such that $\mathfrak{c}_L(u) \cap \mathfrak{c}_L(u^*)$ is compact. In [1], Baboolal called this compactification the *Freudenthal compactification*. To further justify this terminology, Mthethwa [10] proved that, just like in the classical case, the following statements are true: $\bigvee : \gamma L \longrightarrow L$ is a maximal compactification whose remainder is zero-dimensional, and that $\bigvee : \gamma L \longrightarrow L$ is a minimal perfect compactification for the class of rim-compact frames.

Banaschewski [4] defined the strong inclusion $\lhd$ on a regular continuous frame L as follows: $a \lhd b$ if and only if $a \prec b$ and either $\mathfrak{c}_L(a^*)$ or $\mathfrak{c}_L(b)$ is compact. It is shown in the proof of [4, Proposition 4] that $\lhd$ is the smallest strong inclusion on L and therefore the compactification, $\bigvee : \mathfrak{J}_\lhd L \longrightarrow L$, corresponding to $\lhd$ is the least compactification for L. Here, $\mathfrak{J}_\lhd L$ is the set of all strongly regular ideals with respect to $\lhd$. Baboolal established some conditions under which this compactification is perfect in [2].

Now that we have seen that regular continuous frames are rim-compact, one can speak of the Freudenthal compactification for such frames. We end our paper with the following result:

Proposition 6.1. *A non-compact regular continuous frame L is a J-frame if and only if $\gamma L \cong \mathfrak{J}_{\lhd} L$.*

Proof. ($\Longrightarrow$) Let L be a non-compact regular continuous frame. Suppose L is a J-frame. We always have $\lhd \subseteq \lhd_{B_L}$, by [4, Proposition 4]. It remains to show that $\lhd_{B_L} \subseteq \lhd$. Suppose $a \lhd_{B_L} b$, that is, suppose $a \prec c \prec b$ for some $c \in B_L$. Clearly $a \prec b$, and since $c \in B_L$, $\mathfrak{c}_L(c) \cap \mathfrak{c}_L(c^*)$ is compact. Since L is a J-frame, $\mathfrak{c}_L(c)$ or $\mathfrak{c}_L(c^*)$ is compact, by Proposition 5.3. If $\mathfrak{c}_L(c)$ is compact, then $\mathfrak{c}_L(b)$ is compact since $c \leq b$. If $\mathfrak{c}_L(c^*)$ is compact, then $\mathfrak{c}_L(a^*)$ is compact since $a \leq c$. Thus, $a \lhd b$.

($\Longleftarrow$) Suppose $\gamma L = \mathfrak{J}_{\lhd} L$, where L is a non-compact regular continuous frame. This implies that $\bigvee : \mathfrak{J}_{\lhd} L \longrightarrow L$ is perfect. Therefore, by (1)$\Longrightarrow$(3) of [2, Theorem 4.2], whenever $\mathfrak{c}_L(u) \cap \mathfrak{c}_L(v)$ is compact, $u, v \in L, u \wedge v = 0$, then either $\mathfrak{c}_L(u)$ or $\mathfrak{c}_L(v)$ is compact. That is, L is a J-frame. $\square$

Since the Freudenthal compactification is perfect (see [1, Proposition 4.10]), the result above asserts that a non-compact regular continuous frame L is a J-frame precisely when its least compactification is perfect. This contributes to the known conditions (provided in [2]) under which the least compactification is perfect.

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