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A note on idempotent semirings

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Dedicated to Themba Dube on the occasion of his 65th birthday

Abstract. For a commutative semiring S, by an S-algebra we mean a commutative semiring A equipped with a homomorphism $S \to A$. We show that the subvariety of S-algebras determined by the identities $1+2x=1$ and $x^2=x$ is closed under non-empty colimits. The (known) closedness of the category of Boolean rings and of the category of distributive lattices under non-empty colimits in the category of commutative semirings both follow from this general statement.

1 Introduction

Let us recall:

1.1. A *commutative semiring* is an algebraic structure of the form $S = (S, 0, +, 1, \cdot)$ in which $(S, 0, +)$ and $(S, 1, \cdot)$ are commutative monoids with

$$x0 = 0, \quad x(y+z) = xy + xz$$

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for all $x, y, z \in S$. Here and below we use standard notational agreements: e.g. $xy + xz$ means $(x \cdot y) + (x \cdot z)$. The category of commutative semirings will be denoted by CSR; as expected, its morphisms are semiring homomorphisms, that is, maps $f : S \to S'$ of (commutative) semirings with

$$f(0) = 0, \ \ f(s + t) = f(s) + f(t), \ \ f(1) = 1, \ \ f(st) = f(s)f(t)$$

for all $s, t \in S$.

1.2. For a (commutative) semiring S, an S-*module* (or S-*semimodule*) is an algebraic structure of the form $A = (A, 0, +, h)$ in which $(A, 0, +)$ is a commutative monoid and $h : S \times A \to A$ is a map written as $(s, a) \mapsto sa$ that has

$$\begin{aligned} &1a = a, \quad s(ta) = (st)a, \quad s0 = 0, \quad s(a + b) = sa + sb, \\ &0a = 0, \quad (s + t)a = sa + ta \end{aligned}$$

for all $s, t \in S$ and $a, b \in A$. The category of S-modules will be denoted by S-mod.

1.3. For S above, a *commutative* S-*algebra* is a pair (A, f) in which A is a commutative semiring, and $f : S \to A$ is a semiring homomorphism. Accordingly, the category of commutative S-algebras is just the comma category $(S \downarrow \mathsf{CSR})$. Equivalently, a commutative S-algebra can be defined as an algebraic structure of the form $A = (A, 0, +, h, 1, \cdot)$ in which $A = (A, 0, +, h)$ is an S-module and $A = (A, 0, +, 1, \cdot)$ is a commutative semiring with $s(ab) = (sa)b$ for all $s \in S$ and $a, b \in A$; we will then have $sa = f(s)a$.

1.4. If S is a commutative ring, then $(S \downarrow \mathsf{CSR})$ is the ordinary category of commutative (unital) S-algebras. In particular: if $S = \mathbb{Z}$ is the ring of integers, then $(S \downarrow \mathsf{CSR})$ is the category CRings of commutative rings; if $S = \{0, 1\}$ with $1+1 = 0$, then $(S \downarrow \mathsf{CSR})$ is the category CRings_2 of commutative rings of characteristic 2 (=the category of commutative algebras over the two-element field). The category CRings_2 contains the category BRings of Boolean rings (=commutative rings satisfying the identity $x^2 = x$).

1.5. If $S = \{0, 1\}$ with $1 + 1 = 1$, then $(S \downarrow \mathsf{CSR})$ is the category AICSR of additively idempotent commutative semirings (=commutative semirings satisfying the identity $2 = 1$, or, equivalently, the identity $2x = x$). This category contains the category DLat of distributive lattices.

In this paper we present a general result (Corollary 2.3 in the next section) on S-semialgebras, which implies (known) closedness of the categories of Boolean rings and of distributive lattices under non-empty colimits (or, equivalently, just under binary coproducts) in the category of commutative semirings.

2 The general case

Let us first mention an obvious general fact:

Lemma 1. *Let $\mathcal{V}$ be a variety of universal algebras, $\mathcal{W}$ a subvariety of $\mathcal{V}$, and I the initial object of $\mathcal{W}$ (=the free algebra in $\mathcal{W}$ on the empty set). If $\mathcal{W} \approx (I \downarrow \mathcal{W})$ is coreflective in $(I \downarrow \mathcal{V})$, then $\mathcal{W}$ is closed under non-empty colimits in $\mathcal{V}$.* □

Then we take:

- $\mathcal{V} = (S \downarrow \mathsf{CSR})$;
- $\mathcal{W} = (S \downarrow \mathsf{CSR})^*$ to be the subvariety of $(S \downarrow \mathsf{CSR})$ consisting of all S-algebras satisfying the identities $1 + 2x = 1$ and $x^2 = x$. This makes $I \approx S/E$, where E is the smallest congruence on S containing $(1 + 2s, 1)$ and (s^2, s) for each $s \in S$.

Theorem 2.1. *Let $I \approx S/E$ be as above. The variety*

$$(S \downarrow \mathsf{CSR})^* \approx (I \downarrow (S \downarrow \mathsf{CSR})^*)$$

of commutative S-algebras satisfying the identities $1 + 2x = 1$ and $x^2 = x$ is a coreflective subcategory of $(I \downarrow \mathsf{CSR}) \approx (I \downarrow (S \downarrow \mathsf{CSR}))$.

Proof. It suffices to show that, for each $A \in (I \downarrow \mathcal{V})$, the set

$$A' = \{a \in A \mid 1 + 2a = 1 \ \&\ a^2 = a\}$$

is a subalgebra of A, that is, to show the following:

(i) $a, b \in A' \Rightarrow a + b \in A'$;

(ii) for each $s \in S$, $a \in A' \Rightarrow sa \in A'$;

(iii) $1 \in A'$;

(iv) $a, b \in A' \Rightarrow ab \in A'$;

Indeed, (i): Suppose a and b are in A'. Then $1+2(a+b)=1+2a+2b=1+2b=1$ and $(a+b)^2=a^2+2ab+b^2=a+2ab+b=a(1+2b)+b=a+b$.

(iii): $1+2\cdot 1=1$ since this equality holds in I.

(iv): Suppose a and b are in A'. Then

$$1+2ab=1+2a+2ab=1+a+a(1+2b)=1+a+a=1+2a=1$$

and $(ab)^2=a^2b^2=ab$.

(ii) follows from (iv) since $sa=(s1)a$ and $(s1)$ is in A' (since $s1$ is the image of class of s under the homomorphism $I\to A$). □

From Lemma 3.1 and Theorem 3.2 we obtain:

Corollary 2.2. *The variety $(S\downarrow \mathsf{CSR})^*$ of commutative S-algebras satisfying the identities $1+2x=1$ and $x^2=x$ is closed under non-empty colimits in the variety $(S\downarrow \mathsf{CSR})$ of all commutative S-algebras.* □

Taking S to be the ring of natural numbers, we obtain the following special cases of Theorem 2.2 and Corollary 2.3:

Corollary 2.3. *The variety CSR^* of commutative semirings satisfying the identities $1+2x=1$ and $x^2=x$ is coreflective in the variety $(\{0,1,2\}\downarrow \mathsf{CSR})$, where $1+2=1$ in $\{0,1,2\}$.* □

Corollary 2.4. *The variety CSR^* above is closed under non-empty colimits in CSR.* □

3 Boolean rings and distributive lattices

If an object A of $(\{0,1,2\}\downarrow \mathsf{CSR})$ belongs to $(\{0,1\}\downarrow \mathsf{CSR})$ with $1+1=0$ in $\{0,1\}$ making $(\{0,1\}\downarrow \mathsf{CSR})=\mathsf{CRings}_2$, then

$$\{a\in A\mid 1+2a=1\ \&\ a^2=a\}=\{a\in A\mid 2a=0\ \&\ a^2=a\}.$$

But if it is the case with $1+1=1$ making $(\{0,1\}\downarrow \mathsf{CSR})=\mathsf{AICSR}$, then

$$\{a\in A\mid 1+2a=1\ \&\ a^2=a\}=\{a\in A\mid 1+a=1\ \&\ a^2=a\}.$$

Therefore we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathsf{CRings}_2 & \longrightarrow & (\{0,1,2\} \downarrow \mathsf{CSR}) & \longleftarrow & \mathsf{AICSR} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{BRings} & \longrightarrow & \mathsf{CSR}^* & \longleftarrow & \mathsf{DLat}
\end{array}$$

where the horizontal arrows are inclusion functors while the left-hand and right-hand vertical arrows are the coreflections induced by the coreflection of Corollary 2.4 represented by the middle vertical arrow. Since CRings_2 and AICSR both being of the form $(\{0,1\} \downarrow \mathsf{CSR})$ (with different $1+1$ in $\{0,1\}$) are closed in CSR under non-empty colimits, we conclude that both BRings and DLat are also closed in CSR under non-empty colimits. That is, as promised in our Introduction, these two known results follow from what we have done in general (in Section 2).

4 Two additional remarks

4.1. The Reader might ask, what is special about $(S \downarrow \mathsf{CSR})$? The answer consists of the following observations:

- $(S \downarrow \mathsf{CSR})$ is the category of commutative monoids in the monoidal category S-mod having therefore 'good' colimits; indeed, its binary coproducts are given by tensor products.
- The monoidal category structure of S-mod is determined by the fact that it is a commutative variety of universal algebras.
- A commutative variety of universal algebras is semi-additive if and only if it is of the form S-mod for some commutative semiring S. This immediately follows from the equivalence $1. \Leftrightarrow 5.$ in Theorem 2.1 of [2], which refers to [1] for the proof.

4.2. The coreflectivity of DLat in AICSR is a 'finitary copy' of the coreflectivity of the category of frames in the category of quantales, see Section C1.1 of [3]: in fact A_f on Page 479 there is the same as our $\{a \in A \mid 1 + a = 1 \ \& \ a^2 = a\}$.

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