Categories and
General Algebraic Structures
with Applications
Volume 22, Number 1, January 2025, 113-137.
https://doi.org/10.48308/cgasa.20.1.113

Towards free localic algebras

Partha Pratim Ghosh, Inderasan Naidoo, and Nathan Mulaja Tshakatumba*

Abstract. The purpose of this paper is to establish that the underlying object functor from the category of models of a Lawvere theory to the base category creates limits which exist in the base category and creates coequalisers of all parallel pairs of homomorphisms whose underlying pairs admit a split coequaliser in the base category. Furthermore, we show that for a small-complete category with a *well-behaved* proper factorisation structure, the underlying object functor admits a left adjoint, and hence this underlying object functor is monadic in the sense of Beck's theorem. In particular, this establishes the existence of free localic algebras for any Lawvere theory, generalising the known results for the existence of free localic groups.

1 Introduction

Algebraic theories were first introduced in [22], Lawvere's PhD Thesis and later as a TAC preprint where the one-sorted case was introduced; these

* Corresponding author
Keywords: Beck's Theorem, Freyd's Adjoint Functor Theorem, co-well-powered category, Lawvere theory, monadicity, nearly PTT.
Mathematics Subject Classification [2010]: 06D10, 18A40 (Primary), 06D15 (Secondary), 18D99 (Tertiary).
Received: 2 December 2023, Accepted: 8 April 2024.
ISSN: Print 2345-5853, Online 2345-5861.

correspond to the equational theories of Birkhoff [6]. Multi-sorted algebras were first considered by Higgins in [19], and then popularised by Birkhoff and Lipson in [7]. In a review of Higgin's paper [19] (see MR0163940), Heller suggested a connection with Lawvere's approach which was first done by Bénabou in [5]. The definition of algebraic theory adopted in the sequel is provided without any reference to sorting; this sort-free approach corresponds to the *general theory of sketches* (see Ehresmann [15]; and Bastiani and Ehresmann [4]). A recent account of algebraic theory appears in Adámek and Rosický [2].

The quest in this paper is to establish when models of a Lawvere theory in a category with finite products are monadic over the base category. This result will then be applied to the specific case Loc of locales, which are considered to be *generalised topological spaces.* For a category $\mathbb{A}$ with finite products and given a *Lawvere* theory $\mathbb{T}$, the *underlying object* functor from the category of $\mathbb{A}$-models of $\mathbb{T}$ (also known as $\mathbb{T}$-algebras) is obtained to be *nearly PTT* (see Section 4). Hence, it remains to be understood when these models admit a free algebra functor, making the underlying object functor monadic as per Beck's theorem (see Mac Lane [23, Theorem VI.7]). It turns out that any small-complete category with a well-behaved proper (E, M) factorisation system and a small set of E-quotients (up to equivalence) does admit a free algebra functor (see Theorem 5.4). Thus, every locale admits a free algebra for any Lawvere theory and the category of localic algebras is monadic over the category of locales - this generalises the existence of free localic groups by Isebll *et. al.* in [20]. The main intent of whether the category of localic algebras is homological or semi-abelian is investigated in subsequent papers by the authors.

The paper is organised as follows: §2 collects all the necessary known results, provides the essential background material and terminologies and establishes the notations that we will use in the paper.

In §3 we provide the requisite notion of a *Lawvere theory.* We also consider factorisation systems and the notion of a *categorical context.* We show that a context $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ in which the E-morphisms are closed under finite products together with a Lawvere theory produces a proper factorisation system on the category of $\mathbb{T}$-algebras (Theorem 3.10).

In §4 we establish that the underlying object functor from the category of models of a Lawvere theory in a category with finite products is *nearly PTT.*

In §5 we establish sufficient conditions for the existence of a left adjoint to the underlying object functor (see Theorem 5.4). §6 considers an application to locales and concludes the paper with an open question to be investigated in forthcoming papers by the authors.

2 Preliminaries

The preliminaries for this paper are kept to a minimum, to the basic notions of category theory, as can either be obtained from [23] or [8]. Note that in this paper we follow the von Neumann-Bernays-Gödel (NBG) form of set theory (see [24] for details); a set X is said to be a *small set* if there exists a set Y such that $X \in Y$ and 2^Y denotes the set of all small subsets of the set Y.

We recall that an adjunction $\begin{tikzcd} \mathbb{B} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift right, "F"'] \end{tikzcd}$ with unit $1_{\mathbb{A}} \xrightarrow{\eta} U \circ F$ and co-unit $F \circ U \xrightarrow{\epsilon} 1_{\mathbb{B}}$ induces the monad $\mathcal{T} = (U \circ F, \eta, U\epsilon F)$ on $\mathbb{A}$, the Eilenberg-Moore category $\mathbb{A}^{\mathcal{T}}$ of $\mathcal{T}$-algebras and hence the comparison functor $\mathbb{B} \xrightarrow{K} \mathbb{A}^{\mathcal{T}}$ as shown in the following diagram of categories and functors

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[dr, "F^{\mathcal{T}}"'] & \mathbb{B} \arrow[r, "U"] \arrow[d, "K"] & \mathbb{A} \\ & \mathbb{A}^{\mathcal{T}} \arrow[ur, "U^{\mathcal{T}}"'] & \end{tikzcd}$$

where $F^{\mathcal{T}}$ is the induced free $\mathcal{T}$-algebra functor, $U^{\mathcal{T}}$ is the forgetful functor inducing the adjunction $F^{\mathcal{T}} \dashv U^{\mathcal{T}}$.

The functor U is said to be monadic if K is an isomorphism of categories (see [23, Chapter VI], for details).

Theorem 2.1. (Theorem VI.7.1 [23]) *In the setup described above, the following statements are equivalent:*

1. *The functor $\mathbb{B} \xrightarrow{K} \mathbb{A}^{T}$ is an isomorphism of categories;*
2. *The functor $\mathbb{B} \xrightarrow{U} \mathbb{A}$ creates coequalisers for those parallel $\mathbb{B}$-pairs whose U-image has an absolute coequaliser in $\mathbb{A}$.*
3. *The functor $\mathbb{B} \xrightarrow{U} \mathbb{A}$ creates coequalisers for those parallel $\mathbb{B}$-pairs whose U-image has a split coequaliser in $\mathbb{A}$.*

Remark 2.2. Given a functor $\mathbb{B} \xrightarrow{U} \mathbb{A}$, let $\mathcal{S}_U$ be the set of all parallel pairs in $\mathbb{B}$ whose U-image have split coequalisers in $\mathbb{A}$. The functor U is said to be PTT if U has a left adjoint, $\mathbb{B}$ admits coequalisers of all parallel pairs in $\mathcal{S}_U$ and U preserves and reflects coequalisers for all parallel pairs in $\mathcal{S}_U$. If U is PTT, then the comparison functor $\mathbb{B} \xrightarrow{K} \mathbb{A}^{\mathcal{T}}$ is an equivalence of categories (see [23, Exercises 5,6 of §VI.7] and ensuing discussions).

A major thrust in category theory is towards finding conditions ensuring the existence of left adjoints. One of the main tools in this direction is:

Theorem 2.3. (Freyd's Adjoint Functor Theorem [23, Theorem V.6.2]) *Given a functor* $\mathbb{B} \xrightarrow{\quad U \quad} \mathbb{A}$ *with* $\mathbb{B}$ *locally small and small-complete,* U *has a left adjoint if and only if it preserves all (small) limits and satisfies the following* Solution Set Condition*.*

Solution Set Condition: for each object $\mathbb{A}$*-object* A*, there is a small set* $\mathscr{F}_A$ *of* $\mathbb{A}$*-morphisms with domain* A *and codomain of the form* $U(B)$ *for some* $\mathbb{B}$*-object* B *such that for every* $\mathbb{A}$*-morphism* $A \xrightarrow{f} U(X)$ *there exists a morphism* $A \xrightarrow{g} U(B)$ *in* $\mathscr{F}_A$ *and a* $\mathbb{B}$*-morphism* $B \xrightarrow{h} X$ *such that the diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & U(B) \arrow[d, "U(h)"] \\ & U(X) \end{tikzcd}$$

commutes.

Remark 2.4. A small set $\mathscr{A}$ of objects of a category is said to be *weakly initial* if for each object X of the category, there exists a morphism $A \xrightarrow{f} X$, with $A \in \mathscr{A}$. Theorem 2.3 then states that a functor U defined on a locally small and small-complete category has a left adjoint if and only if U preserves all (small) limits and for each object A of the codomain, the slice category $(A \downarrow U)^1$ has a weakly initial object.

In practice, to ensure the existence of a left adjoint to $\mathbb{B} \xrightarrow{U} \mathbb{A}$, it is necessary to ensure that U preserves limits which exists in $\mathbb{B}$, and to somehow

1. The objects are morphisms of the form $\mathbb{A} \xrightarrow{f} U(B)$ and morphisms are $f \xrightarrow{u} f'$, where $f' = U(u) \circ f$, with appropriately defined composition and identities.

ensure the existence of a weakly initial set in each slice category $(A \downarrow U)$ (with A an $\mathbb{A}$-object). In most *algebraic* situations, this is ensured by the existence of *generated subalgebras*. For instance, for the forgetful functor $\texttt{Grp} \xrightarrow{U} \texttt{Set}$, while preservation of limits is evident, the functions $X \xrightarrow{f} U(G)$ are completely determined by subgroups of G generated by subsets of $U(G)$ of cardinality not exceeding the cardinality of X. This leads to the notion of *spanning*:

Definition 2.5. Given any functor $\mathbb{B} \xrightarrow{U} \mathbb{A}$, an $\mathbb{A}$-morphism $A \xrightarrow{f} U(B)$ is said to *span* B if $f = (Um) \circ f'$ for some monomorphism $M \xrightarrow{m} B$ in $\mathbb{B}$, then m is an isomorphism.

For instance, given the forgetful functor $\texttt{Grp} \xrightarrow{U} \texttt{Set}$, $A \xrightarrow{f} U(B)$ spans B if and only if $\{f(a) : a \in A\}$ generates B. A similar situation occurs for several cases of *algebraic structures* over $\texttt{Set}$ and other constructs as well. Thus, the notion of *spanning* exactly replaces the notion of generating and shall be used in this paper to prove the existence of left adjoints.

3 Algebraic Theories

The original concept of an algebraic theory was introduced by Lawvere in [22] emanating from his doctoral thesis wherein he presented a categorical approach to universal algebra. Adámek, Rosický and Vitale in [3] presented a more general definition of an algebraic theory and its algebras. In their sense [[3], Definition 1.1], an algebraic theory is a small category $\mathbb{T}$ with finite products. An *algebra* for the theory $\mathbb{T}$ is a functor $\mathbb{T} \xrightarrow{F} \texttt{Set}$ preserving finite products. $Alg\mathbb{T}$ denotes the category of algebras of $\mathbb{T}$. Morphisms, called *homomorphisms*, are the natural transformations. A category is *algebraic* if it is equivalent to $Alg\mathbb{T}$ for some algebraic theory $\mathbb{T}$. In this paper, we will adopt the following definition from Borceux and Bourn ([9, Definition A.1.9]), which defines a special type of an algebraic theory known as a *Lawvere Theory*.

Definition 3.1. A *Lawvere theory* is a category $\mathbb{T}$ with a denumerable set of objects written $T^0, T^1, \ldots, T^n, \ldots$ $(n \in \mathbb{N})$ such that T^n is the n-th *power of* T.

An $\mathbb{A}$*-model of an algebraic theory* $\mathbb{T}$ or a $\mathbb{T}$-algebra is a finite-product-preserving functor $\mathbb{T} \xrightarrow{A} \mathbb{A}$. Given two $\mathbb{T}$-algebras A and B, a natural transformation $\begin{tikzcd} \mathbb{T} \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbb{A} \end{tikzcd}$ is called a $\mathbb{T}$-algebra *homomorphism* and $[\mathbb{T}, \mathbb{A}]$ is the full subcategory of $\mathbb{A}^{\mathbb{T}}$ consisting of $\mathbb{T}$-algebras and $\mathbb{T}$-algebra homomorphisms.

If $\mathbb{T}$ is a Lawvere theory then the *underlying object functor*:

$$U: \qquad \left.\begin{array}{ccc} [\mathbb{T}, \mathbb{A}] & \longrightarrow & \mathbb{A} \\ & & \\ A & \longmapsto & A(T) \\ {\scriptstyle \alpha}\Big\downarrow & \longmapsto & \Big\downarrow{\scriptstyle \alpha_T} \\ B & \longmapsto & B(T) \end{array}\right\}$$

returns the underlying object $A(T)$ for each $\mathbb{T}$-algebra A and the *underlying morphism* α_T for any $\mathbb{T}$-algebra homomorphism α.

Remark 3.2. Given a Lawvere theory with the specified object T, the product projections are $T^n \xrightarrow{p_i^n} T$, $1 \leq i \leq n \in \mathbb{N}$; T^0 is the terminal object 1, and for each object K of $\mathbb{T}$, the unique morphism from K to 1 is $K \xrightarrow{t_K} 1$.

The following principle shall be utilised quite often in the paper.

Proposition 3.3. *Given a Lawvere theory* $\mathbb{T}$ *and* $\mathbb{T}$*-algebras* $\mathbb{T} \underset{B}{\overset{A}{\rightrightarrows}} \mathbb{A}$*, an* $\mathbb{A}$*-morphism* $A(T) \xrightarrow{f} B(T)$ *is the underlying morphism of a unique* $\mathbb{T}$*-algebra homomorphism* $A \xrightarrow{\phi} B$ *if and only if for each* $n \geq 0$ *and each* n*-ary operation* $T^n \xrightarrow{\omega} T$*, the diagram:*

$$\begin{tikzcd} A(T)^n \arrow[r, "f^n"] \arrow[d, "A\omega"'] & B(T)^n \arrow[d, "B\omega"] \\ A(T) \arrow[r, "f"'] & B(T) \end{tikzcd}$$

commutes, where f^n is the n-fold product of the morphism f.

Proof. Since $\mathbb{T}$-algebras are finite-product preserving, for each $n \geq 1$, $A(T^n)$ is the n-fold product of $A(T)$, with $A(T^n) \xrightarrow{Ap_i^n} A(T)$ $(1 \leq i \leq n)$ as product projections. Hence for any $\mathbb{T}$-algebra homomorphism $A \xrightarrow{\phi} B$, the diagram

$$\begin{tikzcd} A(T^n) \arrow[r, "\phi_{T^n}"] \arrow[d, "Ap_i^n"'] & B(T^n) \arrow[d, "Bp_i^n"] \\ A(T) \arrow[r, "\phi_T"'] & B(T) \end{tikzcd}$$

commutes, implying $\phi_{T^n} = (\phi_T)^n$. Consequently the underlying functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ is faithful.
The *only if* part of the statement is trivial; for the *if* part, faithfulness guarantees the uniqueness, so that it only remains to show the existence of a $\mathbb{T}$-algebra homomorphism $A \xrightarrow{\phi} B$ with $\phi_T = f$. If $T^n \xrightarrow{(\omega_1,\omega_2,\ldots,\omega_m)} T^m$ is a morphism of $\mathbb{T}$ then

$$\begin{aligned} f^m \circ (A\omega_1, A\omega_2, \ldots, A\omega_m) &= (f \circ A\omega_1, f \circ A\omega_2, \ldots, f \circ A\omega_m) \\ &= (B\omega_1 \circ f^n, B\omega_2 \circ f^n, \ldots, B\omega_m \circ f^n) \\ &\qquad\qquad \text{(hypothesis of } \textit{if} \text{ part)} \\ &= (B\omega_1, B\omega_2, \ldots, B\omega_m) \circ f^n \end{aligned}$$

proving $A \xrightarrow{\phi} B$ where $\phi_T = f$. □

In particular, for any Lawvere theory $\mathbb{T}$, a $\mathbb{T}$-algebra homomorphism is a monomorphism (respectively, epimorphism) in $[\mathbb{T}, \mathbb{A}]$ if the underlying morphism is a monomorphism (respectively, epimorphism) in $\mathbb{T}$.

3.1 Factorisation systems An important structure on a category $\mathbb{A}$ is a factorisation structure, initiated by Isbell in [21] as *bicategorical structures* and developed by Freyd and Kelly in [16]. For an excellent account of the latter see Carboni *et. al.* [13].

In line with the development in [13], given the outer commutative square

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, dashed, "w"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

the morphism e is said to be *orthogonal* to the morphism m, written $e \downarrow m$, if there exists a unique morphism w such that $u = w \circ e, v = m \circ w$. In [13], the morphism w is often referred as the *diagonal fill-in* that makes the diagram commutative.

As usual the relation $\downarrow$ on morphisms induces a Galois connection

$$\begin{tikzcd} 2^{\mathbb{A}_1} \arrow[r, shift left, "-^{\uparrow}"] \arrow[r, phantom, "\perp"] & (2^{\mathbb{A}_1})^{op} \arrow[l, shift left, "-^{\downarrow}"] \end{tikzcd} ,$$

where for $\mathcal{H} \subseteq 2^{\mathbb{A}_1}$,

$$\mathcal{H}^{\uparrow} = \{e \in \mathbb{A}_1 : m \in \mathcal{H} \Rightarrow e \downarrow m\} ,$$

and

$$\mathcal{H}^{\downarrow} = \{m \in \mathbb{A}_1 : e \in \mathcal{H} \Rightarrow e \downarrow m\} .$$

A *prefactorisation* system is a pair (E, M) of subsets of morphisms such that $\mathsf{E}^{\downarrow} = \mathsf{M}$ and $\mathsf{M}^{\uparrow} = \mathsf{E}$.

The next result is found in Carboni *et. al.* ([13, Proposition 2.2]).

Proposition 3.4. *Given any prefactorisation system* (E, M) *on a category* $\mathbb{A}$ *the following statements are true:*

a) M *contains isomorphisms and is closed under compositions.*

b) M *is stable under pullbacks.*

c) $g \circ f \in \mathsf{M} \Rightarrow f \in \mathsf{M}$, *if* $g \in \mathsf{M}$ *or* g *is a monomorphism.*

d) *If* $\mathbb{B} \underset{G}{\overset{F}{\underset{\to}{\overset{\to}{\Downarrow \alpha}}}} \mathbb{A}$ *is a natural transformation with components* $\alpha_B \in \mathsf{M}$ $(B \in \mathbb{B}_0)$*, and* $\varprojlim F$ *and* $\varprojlim G$ *exist, then* $\varprojlim F \xrightarrow{\varprojlim \alpha} \varprojlim G$ *is in* M*.*

Definition 3.5. A prefactorisation system (E, M) on a category $\mathbb{A}$ is a *factorisation system* if for each morphism $A \xrightarrow{f} B$ there exists a factorisation:

$$\begin{tikzcd} A \arrow[r, two heads, "f^{\mathsf{E}}"'] \arrow[rr, bend left, "f"] & I_f \arrow[r, tail, "f^{\mathsf{M}}"'] & B \end{tikzcd}$$

with $f^{\mathsf{E}} \in \mathsf{E}$ and $f^{\mathsf{M}} \in \mathsf{M}$.

Furthermore, (E, M) is a *proper factorisation structure* if E is a set of epimorphisms and M is a set of monomorphisms.

Remark 3.6. In context of Definition 3.5: evidently from orthogonality, the factorisation $f = f^{\mathsf{M}} \circ f^{\mathsf{E}}$ is unique, up to a unique isomorphism.

Remark 3.7. An E-morphism e (respectively, an M-morphism m) shall henceforth be denoted as $\bullet \overset{e}{\twoheadrightarrow} \bullet$ (respectively, $\bullet \overset{m}{\rightarrowtail} \bullet$).

Remark 3.8. In case when (E, M) is a proper factorisation system, an *admissible subobject* of an object $X \in \mathbb{A}_0$ is an M-morphism $M \overset{m}{\rightarrowtail} X$ with codomain X; two equivalent admissible subobjects are identified making the set $\mathtt{Sub}_{\mathsf{M}}(X)$ of admissible subobjects of X a partially ordered set with 1_X as its largest element. In presence of pullbacks, $\mathtt{Sub}_{\mathsf{M}}(X)$ is a meet semi-lattice, the smallest element being written as $\emptyset_X \overset{\sigma_X}{\rightarrowtail} X$. Moreover, given a morphism $X \xrightarrow{f} Y$, admissible subobjects $m \in \mathtt{Sub}_{\mathsf{M}}(X), n \in \mathtt{Sub}_{\mathsf{M}}(Y)$, the (E, M)-factorisation of $f \circ m$ and the pullback of n along f, as shown in the left and right hand diagrams below:

$$\begin{tikzcd} \mathsf{dom} m \arrow[r, two heads, "(f|_m)"] \arrow[d, tail, "m"] & I_{f \circ m} \arrow[d, tail, "\exists_f m"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} f^{-1}\mathsf{dom} n \arrow[r, "f_n"] \arrow[d, tail, "f^{-1}n"'] & \mathsf{dom} n \arrow[d, tail, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

induce a Galois connection $\mathtt{Sub}_{\mathsf{M}}(X) \underset{f^{-1}}{\overset{\exists_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathtt{Sub}_{\mathsf{M}}(Y)$, where the subobject $\exists_f m$ is the *image of* m *under* f, $f^{-1}n$ is the *preimage of* n *under* f, the morphism f_n is the *corestriction of* f *along* n while $(f|_m)$ is the *restriction of* f *along* m.

3.2 Categorical contexts The notion of a *categorical context* (or *context* for short) was investigated by Ghosh in [18] as a consequence of his study in [17]. In the latter it is shown that the notion of a neighbourhood system on an object in a category can be provided with minimal assumptions on the category.

Definition 3.9. A *context* is a triple $(\mathbb{A}, \mathsf{E}, \mathsf{M})$, where $\mathbb{A}$ is a small-complete category with finite coproducts, (E, M) is a proper factorisation system such that for each object X the set $\mathtt{Sub}_{\mathsf{M}}(X)$ of admissible subobjects of X is a complete lattice.

Every small-complete and small-co-complete category possesses a canonical proper factorisation system (see Adámek, Herrlich and Strecker [1, Proposition 14.11]). Thus, examples of contexts abound (see [17] for other examples).

Theorem 3.10. *Assume* $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ *is a context in which* E *is closed under non-empty finite products (i.e.,* $e, e' \in \mathsf{E} \Rightarrow e \times e' \in \mathsf{E}$*),* $\mathbb{T}$ *is a Lawvere theory and*

$$\mathsf{E}^{\mathbb{T}} = \big\{\xi \in [\mathbb{T}, \mathbb{A}]_1 : \xi_T \in \mathsf{E}\big\},$$

and

$$\mathsf{M}^{\mathbb{T}} = \big\{\nu \in [\mathbb{T}, \mathbb{A}]_1 : \nu_T \in \mathsf{M}\big\}.$$

Then $(\mathsf{E}^{\mathbb{T}}, \mathsf{M}^{\mathbb{T}})$ *is a proper factorisation structure on* $[\mathbb{T}, \mathbb{A}]$.

Proof. Firstly, consider the diagram:

$$\begin{tikzcd} A \arrow[r, "\xi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\nu"'] & D \end{tikzcd} \qquad (\star)$$

of $\mathbb{T}$-algebras and homomorphisms.

For $\xi \in \mathsf{E}^{\mathbb{T}}$ and $\nu \in \mathsf{M}^{\mathbb{T}}$, since $\xi_T \downarrow \nu_T$ there exists a unique morphism $B(T) \xrightarrow{g} C(T)$ making the top rectangle in the diagram below commutative:

$$\begin{tikzcd}
A(T) \arrow[r, two heads, "\xi_T"] \arrow[d, "\alpha_T"'] & B(T) \arrow[d, "\beta_T"] \arrow[dl, dotted, "!\, g"'] & \\
C(T) \arrow[r, tail, "\nu_T"'] & D(T) & \\
& D(T^n) \arrow[u, "D\omega"] & B(T^n) \arrow[l, "\beta_T^n"'] \arrow[uul, "B\omega"'] \arrow[dl, "g^n"] \\
& C(T^n) \arrow[u, "\nu_T^n"] \arrow[uul, "C\omega"] &
\end{tikzcd}$$

If $T^n \xrightarrow{\omega} T$ is a morphism of $\mathbb{T}$, then:

$$\begin{aligned}
\nu_T \circ g \circ B\omega &= \beta_T \circ B\omega \\
&= D\omega \circ \beta_T^n && \text{(naturality of } \beta) \\
&= D\omega \circ \nu_T^n \circ g^n \\
&= \nu_T \circ C\omega \circ g^n && \text{(naturality of } \nu),
\end{aligned}$$

implying $g \circ B\omega = C\omega \circ g^n$ since ν_T is a monomorphism, in particular. Hence there exists a unique $\mathbb{T}$-algebra homomorphism $B \xrightarrow{\gamma} C$ with $\gamma_T = g$ (Proposition 3.3) such that $\beta = \nu \circ \gamma, \alpha = \gamma \circ \xi$. Thus: $\xi \in \mathsf{E}^{\mathbb{T}}, \nu \in \mathsf{M}^{\mathbb{T}} \Rightarrow \xi \downarrow \nu$.

Given any $\mathbb{T}$-algebra homomorphism $A \xrightarrow{\alpha} C$, there exists an (E, M)-factorisation of α_T as $\alpha_T = \alpha_T^{\mathsf{M}} \circ \alpha_T^{\mathsf{E}}$. Choose and fix $T^n \xrightarrow{\omega} T$ of $\mathbb{T}$. If $n \geq 1$, then since E is closed under binary products $(\alpha_T^{\mathsf{E}})^n \in \mathsf{E}$ (using the Principle of Mathematical Induction) and since M is closed under limits (Proposition 3.4) $(\alpha_T^{\mathsf{M}})^n \in \mathsf{M}$. Consider the left-hand diagram below:

$$
\begin{tikzcd}
A(T^n) \arrow[rr, "\alpha_T^n"] \arrow[dr, two heads, "(\alpha_T^{\mathsf{E}})^n"'] \arrow[ddd, "A\omega"'] & & C(T^n) \arrow[ddd, "C\omega"] \\
& I_{\alpha_T}^n \arrow[ur, tail, "(\alpha_T^{\mathsf{M}})^n"'] \arrow[d, dashed, "!\hat{\omega}"] & \\
& I_{\alpha_T} \arrow[dr, tail, "\alpha_T^{\mathsf{M}}"] & \\
A(T) \arrow[ur, two heads, "\alpha_T^{\mathsf{E}}"] \arrow[rr, "\alpha_T"'] & & C(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, equal] \arrow[dr, equal] \arrow[ddd, "A\omega"'] & & 1 \arrow[ddd, "C\omega"] \\
& 1 \arrow[ur, equal] \arrow[d, dashed, "!\hat{\omega}"] & \\
& I_{\alpha_T} \arrow[dr, tail, "\alpha_T^{\mathsf{M}}"] & \\
A(T) \arrow[ur, two heads, "\alpha_T^{\mathsf{E}}"'] \arrow[rr, "\alpha_T"'] & & C(T)
\end{tikzcd}
$$

where the outer square commutes due to naturality of α; since

$$C\omega\circ(\alpha_T^{\mathsf{M}})^n\circ(\alpha_T^{\mathsf{E}})^n = C\omega\circ\alpha_T^n = \alpha_T\circ A\omega = \alpha_T^{\mathsf{M}}\circ\alpha_T^{\mathsf{E}}\circ A\omega,$$

there exists a unique morphism $\hat{\omega}$ making the whole diagram commute. If $n = 0$, then from the right-hand diagram $\hat{\omega} = \alpha_T^{\mathsf{E}}\circ A\omega$ is the unique morphism making the diagram commute.

More generally, if $T^n \xrightarrow{(\omega_1,\omega_2,\ldots,\omega_m)} T^m$ is a $\mathbb{T}$-morphism, then there are morphisms $I_{\alpha_T}^n \xrightarrow{\hat{\omega}_i} I_{\alpha_T}$ such that $\alpha_T^{\mathsf{M}}\circ\hat{\omega}_i = C\omega_i\circ(\alpha_T^{\mathsf{M}})^n$ implying

$$
\begin{aligned}
(C\omega_1, C\omega_2, \ldots, C\omega_m)\circ(\alpha_T^{\mathsf{M}})^n &= (C\omega_1\circ(\alpha_T^{\mathsf{M}})^n, C\omega_2\circ(\alpha_T^{\mathsf{M}})^n, \ldots, C\omega_m\circ(\alpha_T^{\mathsf{M}})^n) \\
&= (\alpha_T^{\mathsf{M}}\circ\hat{\omega}_1, \alpha_T^{\mathsf{M}}\circ\hat{\omega}_2, \ldots, \alpha_T^{\mathsf{M}}\circ\hat{\omega}_m) \\
&= (\alpha_T^{\mathsf{M}})^m\circ(\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m)
\end{aligned}
$$

This proves that $\mathbb{T} \xrightarrow{I_\alpha} \mathbb{A}$ defined by

$$\big(T^n \xrightarrow{(\omega_1,\omega_2,\ldots,\omega_m)} T^m\big) \overset{I_\alpha}{\longmapsto} \big(I_{\alpha_T}^n \xrightarrow{(\hat{\omega_1},\hat{\omega_2},\ldots,\hat{\omega_m})} I_{\alpha_T}^m\big)$$

is a $[\mathbb{T}, \mathbb{A}]$-algebra and we obtain the diagram
$$
\begin{tikzcd}
A \arrow[r, "\alpha^{\mathsf{E}^{\mathbb{T}}}"] \arrow[rr, bend right, "\alpha"'] & I_\alpha \arrow[r, "\alpha^{\mathsf{M}^{\mathbb{T}}}"] & C
\end{tikzcd}
$$
in $[\mathbb{T}, \mathbb{A}]$

where $\alpha_T^{\mathsf{E}^{\mathbb{T}}} = \alpha_T^{\mathsf{E}}, \alpha_T^{\mathsf{M}^{\mathbb{T}}} = \alpha_T^{\mathsf{M}}$. Thus, every $\mathbb{T}$-algebra homomorphism factors as an $\mathsf{E}^{\mathbb{T}}$-morphism followed by an $\mathsf{M}^{\mathbb{T}}$-morphism.

Finally, if $A \xrightarrow{\xi} B$ is a $\mathbb{T}$-algebra homomorphism with $\xi \in \left(\mathsf{M}^{\mathbb{T}}\right)^{\uparrow}$, then there exists a factorisation
$$\begin{tikzcd} A \arrow[r, "\xi^{\mathsf{E}^{\mathbb{T}}}"'] \arrow[rr, bend left, "\xi"] & I_\xi \arrow[r, "\xi^{\mathsf{M}^{\mathbb{T}}}"'] & B \end{tikzcd}$$
in $[\mathbb{T}, \mathbb{A}]$ with $\xi^{\mathsf{E}^{\mathbb{T}}} \in \mathsf{E}^{\mathbb{T}}, \xi^{\mathsf{M}^{\mathbb{T}}} \in \mathsf{M}^{\mathbb{T}}$. Hence there exists a diagram like $(\star)$ wherein $\beta = \mathbf{1}_B$, $\nu = \xi^{\mathsf{M}^{\mathbb{T}}}$ and $\alpha = \xi^{\mathsf{E}^{\mathbb{T}}}$; since from choice $\xi \downarrow \xi^{\mathsf{M}^{\mathbb{T}}}$, there exists a unique $\mathbb{T}$-algebra homomorphism $B \xrightarrow{\gamma} I_\xi$ such that $\xi^{\mathsf{M}^{\mathbb{T}}} \circ \gamma = \mathbf{1}_B$. In particular, $\xi_T^{\mathsf{M}^{\mathbb{T}}}$ is an isomorphism (since $\xi_T^{\mathsf{M}^{\mathbb{T}}}$ is monic); hence $\xi^{\mathsf{M}^{\mathbb{T}}}$ is an isomorphism, proving $\xi \in \mathsf{E}^{\mathbb{T}}$. This shows $\left(\mathsf{M}^{\mathbb{T}}\right)^{\uparrow} \subseteq \mathsf{E}^{\mathbb{T}}$; similarly $(\mathsf{E}^{\mathbb{T}})^{\downarrow} \subseteq \mathsf{M}^{\mathbb{T}}$. Hence $\left(\mathsf{E}^{\mathbb{T}}, \mathsf{M}^{\mathbb{T}}\right)$ is a proper factorisation structure on $[\mathbb{T}, \mathbb{A}]$. $\square$

4 Nearly PTT forgetful functor

Recall the notion of *PTT* for a functor $\mathbb{B} \xrightarrow{U} \mathbb{A}$ (see Remark 2.2). This section shows that the *underlying object* functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ creates limits and is *nearly PTT*.

Theorem 4.1. *If $\mathbb{A}$ is a category with finite products and $\mathbb{T}$ is a Lawvere theory, then the underlying functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ creates limits which exist in $\mathbb{A}$.*

Proof. Assume $\mathbb{D} \xrightarrow{P} [\mathbb{T}, \mathbb{A}]$ is a functor with $\delta_A \xrightarrow{\alpha} U \circ P$ the limiting cone. If $T^n \xrightarrow{\omega} T$ is an n-ary operation in $\mathbb{T}$, then for any morphism $D \xrightarrow{d} D'$ of $\mathbb{D}$ we obtain the diagram:

$$\begin{tikzcd} A^n \arrow[r, "\alpha_D^n"] & P(D)(T^n) \arrow[r, "(Pd)_T^n"] \arrow[d, "P(D)\omega"'] & P(D')(T^n) \arrow[d, "P(D')\omega"] \\ & P(D)(T) \arrow[r, "(Pd)_T"'] & P(D')(T) \end{tikzcd}$$

where the square commutes from naturality of $P(D) \xrightarrow{Pd} P(D')$, and since α is natural, the morphisms $\alpha_D^{(\omega)} = (P(D)\omega) \circ \alpha_D^n$ yield a natural transformation $\delta_{A^n} \xrightarrow{\alpha^{(\omega)}} U \circ P$. Since α is a limiting cone, there exists a unique morphism $A^n \xrightarrow{\hat{\omega}} A$ such that $\alpha_D^{(\omega)} = \alpha_D \circ \hat{\omega}$, i.e., for each $D \in \mathbb{D}_0$ the diagram:

$$\begin{tikzcd} A^n \arrow[r, "\alpha_D^n"] \arrow[d, dotted, "!\,\hat{\omega}"'] & P(D)(T^n) \arrow[d, "P(D)\omega"] \\ A \arrow[r, "\alpha_D"'] & P(D)(T) \end{tikzcd}$$

commutes. In particular, since $P(D)$ is a $\mathbb{T}$-algebra, $\widehat{p_i^n} = p_i^{A,n}$ for each $i = 1, 2, \ldots, n$, where $A^n \xrightarrow{p_i^{A,n}} A$ are the product projections for each $i = 1, 2, \ldots, n$.

Consequently, for any morphism $T^n \xrightarrow{(\omega_1, \omega_2, \ldots, \omega_m)} T^m$ of $\mathbb{T}$ the computation:

$$\begin{aligned} \alpha_D^m \circ (\hat{\omega_1}, \hat{\omega_2}, \ldots, \hat{\omega_m}) &= (\alpha_D \circ \hat{\omega_1}, \alpha_D \circ \hat{\omega_2}, \ldots, \alpha_D \circ \hat{\omega_m}) \\ &= (P(D)\omega_1 \circ \alpha_D^n, P(D)\omega_2 \circ \alpha_D^n, \ldots, P(D)\omega_m \circ \alpha_D^n) \\ &= (P(D)\omega_1, P(D)\omega_2, \ldots, P(D)\omega_m) \circ \alpha_D^n \end{aligned}$$

yields the unique $\mathbb{T}$-algebra $\mathbb{T} \xrightarrow{\hat{A}} \mathbb{A}$ defined by:

$$\left(T^n \xrightarrow{(\omega_1, \omega_2, \ldots, \omega_m)} T^m \right) \overset{\hat{A}}{\longmapsto} \left(A^n \xrightarrow{(\hat{\omega_1}, \hat{\omega_2}, \ldots, \hat{\omega_m})} A^m \right),$$

and a $\mathbb{T}$-algebra homomorphism $\hat{A} \xrightarrow{\hat{\alpha}_D} P(D)$ defined by $(\hat{\alpha}_D)_T = \alpha_D$ for each $D \in \mathbb{D}_0$. Hence there exists a natural transformation $\delta_{\hat{A}} \xrightarrow{\hat{\alpha}} P$ such that $U\hat{\alpha} = \alpha$. If $\delta_B \xrightarrow{\beta} P$ is another cone over P with vertex the $\mathbb{T}$-algebra B, then since α is a limiting cone, there exists a unique morphism $B(T) \xrightarrow{f} \hat{A}(T)$ such that $U\beta_D = \alpha_D \circ f$ for each $D \in \mathbb{D}_0$.

Since for any n-ary operation $T^n \xrightarrow{\omega} T$:

$$\begin{aligned} \alpha_D \circ \hat{\omega} \circ f^n &= (P(D)\omega) \circ \alpha_D^n \circ f^n \\ &= (P(D)\omega) \circ (\alpha_D \circ f)^n \\ &= (P(D)\omega) \circ (\beta_D)_T^n \\ &= (\beta_D)_T \circ B\omega && \text{(naturality of } \beta\text{)} \\ &= \alpha_D \circ f \circ (B\omega), \end{aligned}$$

for each $D \in \mathbb{D}_0$. Hence, since α is a limiting cone, $f \circ (B\omega) = \hat{\omega} \circ f^n$. Hence from Proposition 3.3 there exists a unique $\mathbb{T}$-algebra homomorphism $B \xrightarrow{\phi} \hat{A}$ such that $\phi_T = f$. Furthermore, ϕ is unique such that $\beta = \hat{\alpha} \circ \phi$, proving $\hat{A} = \varprojlim P$. Hence U creates whichever limits exist in $\mathbb{A}$, and this lift is unique. $\square$

Theorem 4.2. *If $\mathbb{A}$ is a category with finite products and $\mathbb{T}$ is a Lawvere theory, then the underlying functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ creates coequalisers for those parallel pairs in $[\mathbb{T}, \mathbb{A}]$ whose U-image has a split coequaliser in $\mathbb{A}$.*

Proof. Assume $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$ is a pair of $\mathbb{T}$-algebra homomorphisms such that there exists a split coequaliser in $\mathbb{A}$ as shown in the bottom row of the diagram:

$$
\begin{tikzcd}[column sep=large, row sep=large]
A(T^n) \arrow[r, shift left, "\alpha_T^n"] \arrow[r, shift right, "\beta_T^n"'] \arrow[d, "{(A\omega_1, A\omega_2, \ldots, A\omega_m)}"'] & B(T^n) \arrow[l, bend right=40, "t^n"'] \arrow[r, two heads, "h^n"] \arrow[d, "{(B\omega_1, B\omega_2, \ldots, B\omega_m)}"'] & C^n \arrow[l, bend right=40, "s^n"'] \arrow[d, dotted, "!\tilde{\omega}"] \\
A(T^m) \arrow[r, shift left=2, "\alpha_T^m"] \arrow[r, shift right=2, "\beta_T^m"'] \arrow[d, "Ap_j^m"'] & B(T^m) \arrow[l, "t^m" description] \arrow[r, two heads, shift left, "h^m"] \arrow[d, "Bp_j^m"'] & C^m \arrow[l, shift left, "s^m"] \arrow[d, "p_j^{C,m}"] \\
A(T) \arrow[r, shift left, "\alpha_T"] \arrow[r, shift right, "\beta_T"'] & B(T) \arrow[l, bend left=40, "t"] \arrow[r, two heads, "h"] \arrow[dr, bend right, "\xi_T"'] & C \arrow[l, bend left=40, "s"] \arrow[d, dashed, "!f"] \\
 & & D(T)
\end{tikzcd}
$$

i.e., $h \circ \alpha_T = h \circ \beta_T, h \circ s = \mathbf{1}_C, \alpha_T \circ t = \mathbf{1}_{B(T)}, \beta_T \circ t = s \circ h$. Choose and fix a morphism $T^n \xrightarrow{(\omega_1, \omega_2, \ldots, \omega_m)} T^m$ of $\mathbb{T}$. Since split coequalisers are absolute, the middle and the top rows are split coequalisers.

Since

$$h^m \circ (B\omega_1, B\omega_2, \ldots, B\omega_m) \circ \alpha_T^n = h^m \circ (B\omega_1, B\omega_2, \ldots, B\omega_m) \circ \beta_T^n$$

the top row coequaliser forces the existence of a unique morphism $\tilde{\omega}$ making the whole diagram to commute since:

$$\begin{aligned} p_j^{C,m} \circ \tilde{\omega} \circ h^n &= p_j^{C,m} \circ h^m \circ (B\omega_1, B\omega_2, \ldots, B\omega_m) \\ &= h \circ B p_j^m \circ (B\omega_1, B\omega_2, \ldots, B\omega_m) \\ &= h \circ B\omega_j \\ &= \tilde{\omega}_j \circ h^n, \end{aligned}$$

where taking $m = 1$, $C^n \xrightarrow{\tilde{\omega}_j} C$ is the unique function such that $h \circ B\omega_j = \tilde{\omega}_j \circ h^n$ for each $1 \leq j \leq m$. Hence $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m)$; furthermore, this shows $\mathbb{T} \xrightarrow{\tilde{C}} \mathbb{A}$ is a $\mathbb{T}$-algebra defined by:

$$\left(T^n \xrightarrow{(\omega_1, \omega_2, \ldots, \omega_m)} T^m \right) \overset{\tilde{C}}{\mapsto} \left(C^n \xrightarrow{(\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m)} C^m \right),$$

.

Proposition 3.3 ensures the existence of a unique $\mathbb{T}$-algebra homomorphism $B \xrightarrow{\eta} C$ such that $\eta_T = h$. Evidently, $\eta \circ \alpha = \eta \circ \beta$; if $B \xrightarrow{\xi} D$ is a $\mathbb{T}$-algebra homomorphism such that $\xi \circ \alpha = \xi \circ \beta$ then from the coequaliser h there exists a unique morphism $C \xrightarrow{f} D$ such that $\xi_T = f \circ h$. If $T^m \xrightarrow{\omega} T$ is an m-ary operation from $\mathbb{T}$ then since:

$$\begin{aligned} f \circ \tilde{C}\omega \circ h^m &= f \circ \tilde{\omega} \circ h^m \\ &= f \circ h \circ B\omega && \text{(naturality of } \eta) \\ &= \xi_T \circ B\omega \\ &= D\omega \circ \xi_T^m && \text{(naturality of } \xi) \\ &= D\omega \circ f^m \circ h^m \end{aligned}$$

and h^m is a coequaliser, $f \circ \tilde{C}\omega = D\omega \circ f^m$. Hence from Proposition 3.3, there exists a unique $\mathbb{T}$-algebra homomorphism $\tilde{C} \xrightarrow{\phi} D$ such that $\phi_T = f$ and $\xi = \phi \circ \eta$. This proves $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \xrightarrow{\eta} \tilde{C}$ is the unique coequaliser in $[\mathbb{T}, \mathbb{A}]$ such that $U\eta = h$, completing the proof. □

5 Existence of a left adjoint of the underlying functor

In this section let $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context and $\mathbb{T}$ a Lawvere theory. For any $\mathbb{T}$-algebra $\mathbb{T} \xrightarrow{A} \mathbb{A}$ let $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$ be the set of all $\nu \in \mathsf{M}^{\mathbb{T}}$ with codomain A,

and let
$$\begin{tikzcd} \mathbb{T} \arrow[r, bend left=60, "N"{name=N}] \arrow[r, "A"{name=A}] \arrow[r, bend right=60, "N'"'{name=NP}] & \mathbb{A} \arrow[from=N, to=A, Rightarrow, "\nu"] \arrow[from=NP, to=A, Rightarrow, "\nu'"'] \end{tikzcd}$$
be any such two, then ν and ν' shall be considered *equal*

in $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$ if and only if there exists a natural isomorphism $N \xrightarrow{\pi} N'$ with $\nu = \nu' \circ \pi$. In other words, since $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ is faithful, this means ν_T and ν'_T denote equivalent admissible subobjects of $A(T)$ and hence $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$ is the set of equivalence classes of $\mathsf{M}^{\mathbb{T}}$-morphisms with codomain A under the equivalence relation described above. Hence the set $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$ denotes the set of all M-subalgebras of A. Indeed, the complete lattices of admissible subobjects are not asserted to be small sets in the definition of a context (see Definition 3.9), even though for most of the contexts this turns out to be the case.

In the special case when $\mathbb{A} = \mathtt{Set}$, since it is a context with a $(\mathtt{Surjection}, \mathtt{Injection})$ factorisation structure, the $\mathtt{Injection}$-subalgebras are precisely the subalgebras.

Lemma 5.1. *For each $\mathbb{T}$-algebra A, the set $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}}(A)$ of M-subalgebras of A is a complete lattice.*

Proof. Surely, for $\nu,' \nu' \in \mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$, if $\nu \leq \nu'$ as monomorphisms in $[\mathbb{T}, \mathbb{A}]$ then $\nu_T \leq \nu'_T$. Conversely, if $N \xrightarrow{\nu} A \xleftarrow{\nu'} N'$ with $\nu, \nu' \in \mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} A$ such that $\nu_T \leq \nu'_T$ then there exists a morphism p making the lower triangle in the diagram below commute.

$$
\begin{tikzcd}
N(T^n) \arrow[rr, "p^n"] \arrow[dr, "\nu_T^n"'] \arrow[dd, "N\omega"'] & & N'(T^n) \arrow[dl, "(\nu'_T)^n"] \arrow[dd, "N'\omega"] \\
& A(T^n) \arrow[dd, "A\omega"] & \\
N(T) \arrow[rr, "p"] \arrow[dr, tail, "\nu_T"'] & & N'(T) \arrow[dl, tail, "\nu'_T"] \\
& A(T) &
\end{tikzcd}
$$

If $T^n \xrightarrow{\omega} T$ is an n-ary operation, then using naturality of ν and ν', and the fact that $\nu'_T \in \mathsf{M}$, then the back vertical rectangle in the diagram commutes. Hence there exists a unique $\mathbb{T}$-algebra homomorphism $N \xrightarrow{\pi} N'$ with $\pi_T = p$ such that $\nu = \nu' \circ \pi$. Hence the underlying functor reflects the partial order structure on subobjects of $A(T)$ to $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}}A$.

Let $\left(N_i \xrightarrow{\nu_i} A\right)_{i \in I}$ be a family of M-subalgebras of A with intersection in $\mathtt{Sub}_{\mathsf{M}}(A(T))$ given by $\begin{tikzcd} M \arrow[r, tail, "m_i"'] \arrow[rr, tail, bend left, "m"] & N_i(T) \arrow[r, tail, "(\nu_i)_T"'] & A(T) \end{tikzcd}$. Theorem 4.1 ensures the existence of a unique intersection $\begin{tikzcd} \hat{M} \arrow[r, tail, "\mu_i"'] \arrow[rr, tail, bend left, "\mu"] & N_i \arrow[r, tail, "\nu_i"'] & A \end{tikzcd}$ in $[\mathbb{T}, \mathbb{A}]$ such that $U(\tilde{M}) = M, \mu_T = m$ and $(\mu_i)_T = m_i$ $(i \in I)$. Thus the set of M-subalgebras of A is closed under meets (and hence has a largest subalgebra $\mathbf{1}_A$). Since $\mathtt{Sub}_{\mathsf{M}}(A(T))$ is a complete lattice, $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}}A$ has all meets and hence is a complete lattice. $\square$

5.1 Spanning morphisms The definition of spanning morphisms (Definition 2.5) is modified to this set up:

Definition 5.2. Given the underlying object functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$, an $\mathbb{A}$-object A, a $\mathbb{T}$-algebra B and an $\mathbb{A}$-morphism $A \xrightarrow{f} U(B)$, f *spans* B if $f = \nu_T \circ f'$ for any $\nu \in \mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}}B$, then $\nu = \mathbf{1}_B$.

The following is an adaptation of Mac Lane [23, Lemma V.7, Page 127] to the case in this paper.

Lemma 5.3. *For every object $A \in \mathbb{A}_0$, every object $A \xrightarrow{f} U(B)$ of $(A \downarrow U)$ factors through an object $A \xrightarrow{g} U(S)$ of $(A \downarrow U)$ where g spans S and S is an M-subalgebra of B.*

Proof. Let $\mathscr{S}$ be the set of all the M-subalgebras $M \xrightarrow{\nu} B$ of B such that f factors through ν, i.e., there exists a $A \xrightarrow{f_\nu} U(M)$ such that $f = U\nu \circ f_\nu$. Surely $\mathbf{1}_B \in \mathscr{S}$; since $\mathtt{Sub}_{\mathsf{M}^{\mathbb{T}}} B$ is a complete lattice (Lemma 5.1), the meet $s = \bigwedge_{\nu \in \mathscr{S}} \nu$ exists and let it be given by
$$\begin{tikzcd} S \arrow[r, tail, "s_\nu"] \arrow[rr, tail, bend right, "s"'] & M \arrow[r, tail, "\nu"] & B \end{tikzcd}.$$
Since $Us = \bigwedge_{\nu \in \mathscr{S}} U\nu$ (Theorem 4.1), there exists a unique morphism $A \xrightarrow{g} U(S)$ such that $f = Us \circ g$, i.e., f factors through s. Evidently from the construction of s, g spans S, completing the proof. $\square$

5.2 A Sufficient Condition for the Existence of a Left Adjoint

Since the underlying object functor $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ is known to create limits (Theorem 4.1), the key to finding a left adjoint would be in finding *natural conditions* on $\mathbb{A}$ which ensure that each $(A \downarrow U)$ (A being an $\mathbb{A}$-object) has a weakly initial set (see Remark 2.4). Taking a clue from the special case $\mathbb{A} = \mathtt{Set}$, it is enough to ensure conditions which guarantee a weakly initial set of spanning morphisms in each $(A \downarrow U)$.

Towards this goal, first consider the diagram:

$$
\begin{tikzcd}
 & & & A \arrow[dl, two heads, "f^{\mathsf{E}}"'] \arrow[dr, two heads, "g^{\mathsf{E}}"] \arrow[dlll, bend right, "f"'] \arrow[drrr, bend left, "g"] & & & \\
 & & I_f \arrow[rr, dashed, "!k"] \arrow[dl, tail, "i_f"'] \arrow[dll, bend left, "f^{\mathsf{M}}"] & & I_g \arrow[dr, tail, "i_g"] \arrow[drr, bend right, "g^{\mathsf{M}}"'] & & \\
 & U[I_f] \arrow[dl, tail, "U\alpha_f"'] & & & & U[I_g] \arrow[dr, tail, "U\alpha_g"] & \\
U(B) \arrow[rrrrrr, bend right, "U\beta"'] & & & & & & U(C)
\end{tikzcd}
$$

where $f \xrightarrow{\beta} g$ is a morphism of $(A \downarrow U)$, the (E, M)-factorisation of f, g further yielding the smallest M-subalgebras $[I_f], [I_g]$ of B, C respectively. Since $g = g^{\mathsf{M}} \circ g^{\mathsf{E}} = U\beta \circ f = U\beta \circ f^{\mathsf{M}} \circ f^{\mathsf{E}}$ and $f^{\mathsf{E}} \downarrow g^{\mathsf{M}}$, there exists the unique morphism k (which is evidently an E-morphism) making the whole diagram commute. Evidently, $i_f \circ f^{\mathsf{E}}$ spans $[I_f]$ and $i_g \circ g^{\mathsf{E}}$ spans $[I_g]$.

Let $\mathcal{E}(A)$ be the set of all morphisms $A \overset{e}{\twoheadrightarrow} E$ such that there exists an $f \in (A \downarrow U)_0$ which factors through e, i.e.,
$$
\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[dr, "f"'] & E \arrow[d, dotted] \\
 & U(B)
\end{tikzcd}
$$
. Further, any two equivalent E-morphisms e as above are identified. Also, let $\mathcal{S}(A)$ be the set of all objects $A \xrightarrow{p} U(P)$ of $(A \downarrow U)$ such that p spans P.

Choose and fix an $e \in \mathcal{E}(A)$; hence there exists a $A \xrightarrow{f} U(B)$ in $(A \downarrow U)$ and an $\mathbb{A}$-morphism k such that $f = k \circ e$. Using Lemma 5.3, there exists a $A \xrightarrow{g} U(B)$ such that g spans G (G being an $M^{\mathbb{T}}$-subalgebra of B) and hence a monomorphism $G \overset{h}{\rightarrowtail} B$ such that $f = U(h) \circ g$. Hence $U(h)$ is an admissible monomorphism, proving the existence of the unique morphism w making the diagram

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "g"'] & E \arrow[d, "k"] \arrow[dl, dashed, "!w"] \\ U(G) \arrow[r, tail, "U(h)"'] & U(B) \end{tikzcd}$$

commute, which in turn implies the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[dr, "g"] \arrow[ddr, "f"'] & E \arrow[d, "w"] \arrow[dd, bend left, "k"] \\ & U(G) \arrow[d, "U(h)"] \\ & U(B) \end{tikzcd}$$

Hence each $e \in \mathcal{E}(A)$ factors through some $g \in \mathcal{S}(A)$, proving an assignment

$$\left(A \overset{e}{\twoheadrightarrow} E \ \in \mathcal{E}(A)\right) \longmapsto \left(A \overset{e}{\twoheadrightarrow} U(G) \ \in \mathcal{S}(A)\right).$$

On the other hand, given any $A \xrightarrow{g} U(G)$ with g spanning G, surely since from above $A \overset{g^{\mathsf{E}}}{\twoheadrightarrow} I_g \overset{i_g}{\rightarrowtail} U([i_g]) \overset{U(\alpha_g)}{\rightarrowtail} U(G)$ and $i_g \circ g^{\mathsf{E}}$ spans $[I_g]$, the assignment is surjective.

Hence, conditions which ensure smallness of the set $\mathcal{E}(A)$ would ensure, by the Axiom of Replacement in NBG (see [24]), the smallness of the set $\mathcal{S}(A)$.

This yields:

Theorem 5.4. *If* $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ *is a context in which* $\mathbb{A}$ *is* E*-co-well powered and* $\mathbb{T}$ *is a Lawvere theory, then the underlying functor* $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ *has a left adjoint.*

Immediately from Theorem (5.4) and Theorem 2.1 we get:

Corollary 5.5. *If* $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ *is an* E*-co-well powered context, then for any Lawvere theory* $\mathbb{T}$*, the functor* $[\mathbb{T}, \mathbb{A}] \xrightarrow{U} \mathbb{A}$ *is monadic.*

6 Application to locales

$(\mathtt{Loc}, \mathtt{Epi}(\mathtt{Loc}), \mathtt{RegMon}(\mathtt{Loc}))$ is a regular-mono well-powered and co-well-powered context. This is known from Ghosh [17] and from Picado and Pultr [25]. Hence for any Lawvere theory $\mathbb{T}$, the category $[\mathbb{T}, \mathtt{Loc}]$ of localic $\mathbb{T}$-algebras has a localic free $\mathbb{T}$-algebra functor $\mathtt{Loc} \xrightarrow{F} [\mathbb{T}, \mathtt{Loc}]$ (Theorem 5.4). This generalises the existence result for free localic groups in Isbell *et. al.* [20].

Open Problem:
Recall from [9] that a category $\mathbb{A}$ is said to be :

1. *Protomodular* if it is finitely complete and the base change functor for fibrations of points reflects isomorphisms.
2. *Homological* if it is pointed, Barr-regular and protomodular.
3. *Semi-abelian* if it admits finite coproducts, is Barr-exact and homological.

The characterisation of varieties of Universal Algebra for protomodularity and semi-abelianness has been provided by Janelidze and Bourn [[12], Theorem 1.1], from which the following two theorems emanate:

Theorem 6.1. *A variety $\mathbb{V}$ of universal algebras is protomodular if and only if it has nullary terms $e_1, e_2, \ldots, e_n$, binary terms $\alpha_1, \alpha_2, \ldots, \alpha_n$ and an $(n+1)$-ary term θ satisfying the identities :*

$$\alpha_i(x, x) = e_i \quad \textit{for each } i = 1, 2, \ldots, n$$

and

$$\theta\big(x, t_1(x, y), t_2(x, y), \ldots, t_n(x, y)\big) = y$$

Theorem 6.2. *A variety $\mathbb{V}$ of universal algebras is semi-abelian if and only if it has one nullary term e, binary terms $\alpha_1, \alpha_2, \ldots, \alpha_n$ and an $(n+1)$-ary term θ satisfying the identities*

$$\alpha_i(x, x) = e$$

and

$$\theta\big(x, \alpha_1(x, y), \alpha_2(x, y), \ldots, \alpha_n(x, y)\big) = y$$

Subsequently, the protomodular, homological and semi-abelian varieties of topological spaces have been described by Borceux and Clementino in [10, 11], and a later account appears in the monograph [14] by Clementino. The discussion of this paper now leads to the following question, which is to be dealt in subsequent papers :

Given a semi-abelian Lawvere theory $\mathbb{T}$, describe the subcategories $\mathbb{A}$ of $\mathtt{Loc}$ such that $[\mathbb{T}, \mathbb{A}]$ is homological or semi-abelian.

Acknowledgement

This work forms part of the doctoral thesis by Tshakatumba, jointly supervised by Ghosh and Naidoo. Tshakatumba dearly acknowledges the support and mentorship of his supervisors. The first author, Ghosh, acknowledges support received in the form of a grant from the National Institute for Theoretical and Computational Sciences (NITheCS) of South Africa on *Space-like mathematical structures and related topics in algebra, logic and computation.* The second author, Naidoo, acknowledges the grant (Grant No :151034) received from the South African National Research Foundation (NRF). The third author, Tshakatumba, acknowledges the postgraduate bursary support received from the University of South Africa (UNISA). The authors gratefully thank Charles Msipha at the Tshwane University of Technology for discussions and suggestions during the course of this study.

The authors are indebted to the anonymous referee for the careful and thorough reading of the manuscript and for his/her helpful comments and constructive suggestions that have been incorporated into the final version of this paper.

References

[1] Adámek, J., Herrlich, H., and Strecker, G.E., "Abstract and Concrete Categories: the Joy of Cats", Repr. Theory Appl. Categ. 17, 1-507, 2006.

[2] Adámek, J. and Rosický, J., "Locally Presentable and Accessible Categories", Lond. Math. Soc. Lect. Note Ser. 189, Cambridge University Press, 1994.

[3] Adámek, J., Rosický, J., and Vitale, E.M., "Algebraic Theories: A Categorical Introduction to General Algebra with a foreword by F.W. Lawvere", Camb. Tracts Math. 184, Cambridge University Press, 2011.

[4] Bastiani, A. and Ehresmann, C., *Categories of sketched structures*, Cah. Topologie Géom. Différ. Catégoriques 13 (1972), 105-214.

[5] Bénabou, J., *Structures algébriques dans les catégories*, Cah. Topologie Géom. Différ. Catégoriques 10 (1968), 1-126.

[6] Birkhoff, G., *On the structure of abstract algebras*, Proc. Camb. Philos. Soc. 31 (1935), 433-454.

[7] Birkhoff, G. and Lipson, J.D., *Heterogeneous algebras*, J. Comb. Theory 8 (1970), 115-133.

[8] Borceux, F., "Handbook of Categorical Algebra, Volume 1: Basic Category Theory", Encycl. Math. Appl. 50, Cambridge University Press, 2008.

[9] Borceux, F. and Bourn, D., "Mal'cev, Protomodular, Homological and Semi-Abelian Categories", Mathematics and its Applications 566, Springer, 2004.

[10] Borceux, F., and Clementino, M.M., *Topological semi-abelian algebras*, Adv. Math. 190(2) (2005), 425-453.

[11] Borceux, F., and Clementino, M.M., *Topological protomodular algebras*, Topology Appl. 153(16) (2006), 3085-3100.

[12] Bourn, D., and Janelidze, G., *Characterization of protomodular varieties of universal algebras*, Theory Appl. Categ. 11 (2003), 143-147.

[13] Carboni, A., Janelidze, G., Kelly, G.M., and Paré, R., *On localization and stabilization for factorization systems*, Appl. Categ. Struct. 5(1) (1997), 1-58.

[14] Clementino, M.M., "An Invitation to Topological Semi-Abelian Algebras", in: New Perspectives in Algebra, Topology and Categories, Coimbra Mathematical Texts 1, Springer, 2021.

[15] Ehresmann, C., *Sur les structures algébriques*, C.R. Acad. Sci., Paris, Sér. A 264 (1967), 840-843.

[16] Freyd, P.J. and Kelly, G.M., *Categories of continuous functors. I*, J. Pure Appl. Algebra 2 (1972), 169-191.

[17] Ghosh, P.P., *Internal neighbourhood structures*, Algebra Univers. 81(12) (2020), `https://doi.org/10.1007/s00012-020-0640-2`.

[18] Ghosh, P.P., *Internal neighbourhood structures. II: Closure and closed morphisms*, Categ. Gen. Algebr. Struct. Appl. 18(1) (2023), 155-223.

[19] Higgins, Ph.J., *Algebras with a scheme of operators*, Math. Nachr. 27 (1963), 115-132.

[20] Isbell, J., Kříž, I., Pultr, A., and Rosický, J., *Remarks on localic groups*, in: Categorical Algebra and its Applications, Proc. 1st Conf., Louvain-la-Neuve/Belg. 1987, Lect. Notes Math. 1348 (1988), 154-172.

[21] Isbell, J.R., "Subobjects, Adequacy, Completeness and Categories of Algebras", Diss. Math. 36, Instytut Matematyczny Polskiej Akademi Nauk, 1964.

[22] Lawvere, F.W., *Functorial semantics of algebraic theories*, Proc. Natl. Acad. Sci. USA 50 (1963), 869-872.

[23] Mac Lane, S., "Categories for the Working Mathematician", Grad. Texts Math. 5, Springer, 1998.

[24] Monk, J.D., "Introduction to Set Theory", McGraw-Hill Book Co., 1969.

[25] Picado, J. and Pultr, A., "Frames and Locales: Topology without Points", Front. Math., Springer, 2012.

Partha Pratim Ghosh *Department of Mathematical Sciences, University of South Africa, P.O.Box 392, 0003 UNISA, South Africa.*

National Institute for Theoretical and Computational Sciences (NITheCS), South Africa.

Email: ghoshpp@unisa.ac.za

Inderasan Naidoo *Department of Mathematical Sciences, University of South Africa, P.O.Box 392, 0003 UNISA, South Africa.*

National Institute for Theoretical and Computational Sciences (NITheCS), South Africa.

Email: naidoi@unisa.ac.za

Nathan Mulaja Tshakatumba *Department of Mathematical Sciences, University of South Africa, P. O. Box 392, 0003 UNISA, South Africa.*

National Institute for Theoretical and Computational Sciences (NITheCS), South Africa.

Email: 46716866@mylife.unisa.ac.za