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Classification of Boolean algebras through von Neumann regular $\mathcal{C}^\infty-$rings

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Abstract. In this paper, we introduce the concept of a "von Neumann regular $\mathcal{C}^\infty$-ring", which is a model for a specific equational theory. We delve into the characteristics of these rings and demonstrate that each Boolean space can be effectively represented as the image of a von Neumann regular $\mathcal{C}^\infty$-ring through a specific functor. Additionally, we establish that every homomorphism between Boolean algebras can be expressed through a $\mathcal{C}^\infty$-ring homomorphism between von Neumann regular $\mathcal{C}^\infty$-rings.

1 Introduction

As per I. Moerdijk and G. Reyes, $\mathcal{C}^\infty$-rings were primarily introduced for their applications in Singularity Theory, as Weil algebras and jet spaces, and for constructing topos-models for Synthetic Differential Geometry. A remarkable topos is the one of sheaves over the small site comprising the category of all germ-determined $\mathcal{C}^\infty$-rings, along with an appropriate Grothendieck topology (for detailed information, refer to [15]).

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In recent years, the authors have been focused on exploring $\mathcal{C}^\infty$-rings and their applications, cf. [3], [5], [6], [10], [7], [8], [9], [4] and [2]. In this study, we present a valuable concept of a von Neumann regular $\mathcal{C}^\infty$-ring, showcasing pertinent results and detailing both their properties and those of the category they form.

As an application, we establish that every Boolean algebra can be canonically represented through a von Neumann regular $\mathcal{C}^\infty$-ring. Additionally, we demonstrate that each homomorphism between Boolean algebras can be depicted by a specific $\mathcal{C}^\infty$-ring homomorphism between von Neumann regular $\mathcal{C}^\infty$-rings. Specifically, the investigation of von Neumann regular $\mathcal{C}^\infty$-rings extends to the classification of Boolean spaces (and/or Boolean algebras) and their morphisms. A critical tool in developing these findings is the alignment between the concepts of Boolean spaces and "profinite spaces". Profinite spaces, along with their morphisms, constitute a category denoted as **Profinite**, forming the basis for the notion of a "condensed set." A condensed set is characterized as a functor (a specific sheaf) denoted by $C : \mathbf{Profinite}^{\mathrm{op}} \to \mathbf{Set}$. These condensed sets, in turn, represent a fundamental concept that plays a pivotal role in the recent theory of "Condensed Mathematics" developed by Dustin Clausen and Peter Scholze.

Overview of the Paper: In **Section 2**, we outline the fundamental preliminary concepts of $\mathcal{C}^\infty$-rings, delving into their universal algebra and their 'smooth commutative algebra.'

Moving on to **Section 3**, we introduce the concept of a von Neumann regular $\mathcal{C}^\infty$-ring, essentially a $\mathcal{C}^\infty$-ring with an underlying commutative unital ring being a von Neumann regular ring. We then investigate the main properties and characterizations associated with this concept.

In **Section 4**, we explore their practical applications. Firstly, we establish that each continuous map between profinite spaces can be accurately portrayed by a continuous map between boolean spaces, which are canonically associated with a $\mathcal{C}^\infty$-ring homomorphism between von Neumann regular $\mathcal{C}^\infty$-rings (see **Theorem 4.6**). Subsequently, we deduce that every homomorphism between Boolean algebras can be distinctly represented through a $\mathcal{C}^\infty$-ring homomorphism between von Neumann regular $\mathcal{C}^\infty$-rings (refer to **Theorem 4.7**).

2 Preliminaries on $\mathcal{C}^\infty$-rings

In this section we provide the main preliminary notions on $\mathcal{C}^\infty-$rings, with respect to their universal algebra (cf. [9]) and to their "smooth commutative algebra" (cf. [10]).

In order to formulate and study the concept of $\mathcal{C}^\infty-$ring, we use a first order language, $\mathcal{L}$, with a denumerable set of variables ($\mathrm{Var}(\mathcal{L}) = \{x_1, x_2, \cdots, x_n, \cdots\}$), whose nonlogical symbols are the symbols of $\mathcal{C}^\infty-$functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, with $m, n \in \mathbb{N}$, *i.e.*, the non-logical symbols consist only of function symbols: for each $n \in \mathbb{N}$, the $n-$ary **function symbols** of the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, *i.e.*, $\mathcal{F}_{(n)} = \{f^{(n)} \mid f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\}$. Thus, the set of function symbols of our language is given by:

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{(n)} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n).$$

Note that our set of constants is identified with the set of all $0-$ary function symbols, *i.e.*, $\mathcal{C} = \mathcal{F}_{(0)} = \mathcal{C}^\infty(\mathbb{R}^0) \cong \mathcal{C}^\infty(\{*\})$.

The terms of this language are defined, in the usual way, as the smallest set which comprises the individual variables, constant symbols and $n-$ary function symbols followed by n terms ($n \in \mathbb{N}$).

Functorially, a (set-theoretic) $\mathcal{C}^\infty-$ring is a finite product preserving functor from the category $\mathcal{C}^\infty$, whose objects are of the form $\mathbb{R}^n$, $n \in \mathbb{N}$, and whose morphisms are the smooth functions between them, *i.e.*, a finite product preserving functor:

$$A : \mathcal{C}^\infty \to \mathbf{Set}$$

Apart from the functorial definition and the "first-order language" definition we just gave, there are many equivalent descriptions. We focus, first, on the universal-algebraic description of a $\mathcal{C}^\infty-$ring in **Set**, given in the following:

Definition 2.1. A $\mathcal{C}^\infty-$**structure** is a pair $\mathfrak{A} = (A, \Phi)$, in which A is a non-empty set and:

$$\begin{array}{rccc} \Phi : & \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n \in \mathbb{N}} \mathrm{Func}\,(A^n; A) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \Phi(f) := (f^A : A^n \to A) \end{array},$$

is a function, that is, Φ interprets the **symbols**[1] of all smooth real functions of n variables as $n-$ary function symbols on A.

We call a $\mathcal{C}^\infty-$struture, $\mathfrak{A} = (A, \Phi)$, a $\mathcal{C}^\infty-$**ring**, whenever it preserves projections and all equations between smooth functions. More precisely, we have the following:

Definition 2.2. Let $\mathfrak{A} = (A, \Phi)$ be a $\mathcal{C}^\infty-$structure. We say that $\mathfrak{A}$ (or, when there is no danger of confusion, A) is a $\mathcal{C}^\infty-$**ring** if the following is true:

- Given any $n, k \in \mathbb{N}$ and any projection $\pi_k : \mathbb{R}^n \to \mathbb{R}$, we have:

$$(\forall x_1) \cdots (\forall x_n)(\Phi(\pi_k)(x_1, \cdots, x_n) = x_k).$$

- For every $f, g_1, \cdots g_n \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ with $m, n \in \mathbb{N}$, and every $h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = h \circ (g_1, \cdots, g_n)$, one has:

$$\Phi(f)(\vec{x}) = \Phi(h)(\Phi(g_1)(\vec{x}), \cdots, \Phi(g_n)(\vec{x}))$$

Definition 2.3. Let (A, Φ) and (B, Ψ) be two $\mathcal{C}^\infty-$rings. A function $\varphi : A \to B$ is called a **morphism of $\mathcal{C}^\infty-$rings** or **$\mathcal{C}^\infty$-homomorphism** if for any $n \in \mathbb{N}$ and any $f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}$, one has $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$, where $\varphi^{(n)} = (\varphi, \cdots, \varphi) : A^n \to B^n$.

Remark 2.4. (on universal algebraic constructions) It is not difficult to see that $\mathcal{C}^\infty-$structures, together with their morphisms (which we call $\mathcal{C}^\infty-$morphisms) compose a category, that we denote by $\mathcal{C}^\infty$**Str**, and that $\mathcal{C}^\infty-$rings, together with all the $\mathcal{C}^\infty-$morphisms between $\mathcal{C}^\infty-$rings (which we call $\mathcal{C}^\infty-$homomorphisms) compose a full subcategory, $\mathcal{C}^\infty$**Ring**. In particular, since $\mathcal{C}^\infty$**Ring** is a "variety of algebras", *i.e.* it is a class of $\mathcal{C}^\infty-$structures which satisfies a given set of equations, (or equivalently, by **Birkhoff's HSP Theorem**) it is closed under substructures, homomorphic images and products. Moreover:
• $\mathcal{C}^\infty$**Ring** is a concrete category and the forgetful functor, $U : \mathcal{C}^\infty$**Ring** $\to$ **Set**, creates directed inductive colimits;

[1]here considered simply as syntactic symbols rather than functions.

• Each set X freely generates a $\mathcal{C}^\infty$-ring, $\mathcal{C}^\infty(\mathbb{R}^X)$. In particular, the free $\mathcal{C}^\infty$-ring on n generators is (isomorphic to) $\mathcal{C}^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$. Moreover, $\mathcal{C}^\infty(\mathbb{R}^X) \cong \varinjlim_{X' \subset_{\text{fin}} X} \mathcal{C}^\infty(\mathbb{R}^{X'})$;
• Every $\mathcal{C}^\infty$−ring is the homomorphic image of some free $\mathcal{C}^\infty$−ring determined by some set, being isomorphic to the quotient of a free $\mathcal{C}^\infty$−ring by some congruence;
• The congruences of $\mathcal{C}^\infty$−rings are classified by their "ring-theoretical" ideals - which are the ideals of a $\mathcal{C}^\infty$−ring, or the "$\mathcal{C}^\infty$−ideals";
• In $\mathcal{C}^\infty$**Ring** one defines "the $\mathcal{C}^\infty$−coproduct" between two $\mathcal{C}^\infty$−rings $\mathfrak{A} = (A, \Phi)$ and $\mathfrak{B} = (B, \Psi)$, denoted by $A \otimes_\infty B$;
• Using free $\mathcal{C}^\infty$−rings and the $\mathcal{C}^\infty$−coproduct, one gets the "$\mathcal{C}^\infty$−ring of polynomials" on any set S of variables with coefficients in A, given by $A\{x_s \mid s \in S\} = A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S)$.

Remark 2.5. (**on smooth commutative algebra**). Every $\mathcal{C}^\infty$−ring has an underlying commutative unital ring, so there is a naturally defined forgetful functor $\widetilde{U} : \mathcal{C}^\infty\mathbf{Ring} \to \mathbf{CRing}$. Using such forgetful functor, one defines a $\mathcal{C}^\infty$−field (resp. $\mathcal{C}^\infty$−domain, local $\mathcal{C}^\infty$−ring) as a $\mathcal{C}^\infty$−ring $\mathfrak{A} = (A, \Phi)$ such that $\widetilde{U}(\mathfrak{A})$ is a field (resp. domain, local ring), when regarded as a commutative unital ring;
• In $\mathcal{C}^\infty$**Ring** one has the $\mathcal{C}^\infty$−ring of fractions of a $\mathcal{C}^\infty$−ring A with respect to any subset S of A, denoted by $A \stackrel{\eta_S}{\to} A\{S^{-1}\}$, in the same sense one has the ring of fractions with respect to a subset of a commutative unital ring, defined by the following two properties: (i) given any $a \in S$, $\eta_S(a) \in A^\times$ and (ii) given any $\mathcal{C}^\infty$−ring B and any $\mathcal{C}^\infty$−homomorphism $f : A \to B$ such that $(\forall a \in S)(f(a) \in B^\times)$ there is a unique $\mathcal{C}^\infty$−homomorphism $\widetilde{f} : A\{S^{-1}\} \to B$ such that $\widetilde{f} \circ \eta_S = f$.
• The $\mathcal{C}^\infty$−ring of fractions can be constructed using universal algebraic tools, and it is given by the quotient

$$A\{S^{-1}\} \cong A\{x_s \mid s \in S\}/\langle\{x_s \cdot s - 1 \mid s \in S\}\rangle.$$

• I. Moerdijk and G. Reyes introduced the notion of the $\mathcal{C}^\infty$−radical of an ideal I of a $\mathcal{C}^\infty$−ring $\mathfrak{A} = (A, \Phi)$ (thus, a ring-theoretical ideal) as the set:

$$\sqrt[\infty]{I} = \{a \in A \mid (A/I)\{a + I^{-1}\} \cong \{0\}\}$$

• The $\mathcal{C}^\infty$−spectrum of a $\mathcal{C}^\infty$−ring A is the topological space whose underlying set is $X = \{\mathfrak{p} \subseteq A \mid (\mathfrak{p} \text{ is a prime ideal})\&(\sqrt[\infty]{\mathfrak{p}} = \mathfrak{p})\}$ and whose topology

is generated by $\mathcal{B} = \{D^{\infty}(a) \mid a \in A\}$, where $D^{\infty}(a) = \{\mathfrak{p} \in X \mid a \notin \mathfrak{p}\}$. Moreover, $\mathcal{B}$ is closed under finite intersections and arbitrary reunions. We denote this topological space by $\mathrm{Spec}^{\infty}(A)$. Sometimes, when there is no danger of confusion, we write $\mathrm{Spec}^{\infty}(A)$ to denote the underlying set to this topological space, instead of X;
• The $\mathcal{C}^{\infty}-$radical of a $\mathcal{C}^{\infty}-$ideal I of a $\mathcal{C}^{\infty}-$ring A is characterised by:

$$\sqrt[\infty]{I} = \bigcap \{\mathfrak{p} \in \mathrm{Spec}^{\infty}(A) \mid I \subseteq \mathfrak{p}\}$$

• There is an *ad hoc* definition of saturation for $\mathcal{C}^{\infty}-$rings, the smooth saturation of a subset S of a $\mathcal{C}^{\infty}-$ring A, given by $S^{\infty-\mathrm{sat}} = \{a \in A \mid \eta_S(a) \in A\{S^{-1}\}^{\times}\}$. The smooth saturation is related to the $\mathcal{C}^{\infty}-$radical of an ideal $I \subseteq A$ by $\sqrt[\infty]{I} = \{a \in A \mid I \cap \{a\}^{\infty-\mathrm{sat}} \neq \varnothing\}$;
• Along with the notion of a $\mathcal{C}^{\infty}-$radical ideal we have the concept of a reduced $\mathcal{C}^{\infty}-$ring, which is a $\mathcal{C}^{\infty}-$ring $\mathfrak{A} = (A, \Phi)$ such that $\sqrt[\infty]{(0_A)} = (0_A)$.
• A $\mathcal{C}^{\infty}-$ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X : \mathrm{Open}\,(X)^{\mathrm{op}} \to \mathcal{C}^{\infty}\mathbf{Ring}$ is a sheaf. A morphism of $\mathcal{C}^{\infty}-$ringed spaces is a pair $(f, f^{\sharp}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $\mathcal{C}^{\infty}-$ringed spaces is a continuous map $f : X \to Y$ and a morphism of sheaves $f^{\sharp} : f^{\dashv}[\mathcal{O}_Y] \to \mathcal{O}_X$, where $f^{\dashv}[\mathcal{O}_Y]$ is described in **Definition 4.5** of [12].

3 On von Neumann regular $\mathcal{C}^{\infty}-$rings

We begin by registering a fact that is valid for any $\mathcal{C}^{\infty}-$ring regarding to its idempotents and localizations:

Lemma 3.1. *Let A be any $\mathcal{C}^{\infty}-$ring and $e \in A$ an idempotent element. There are unique isomorphisms:*

$$A\{e^{-1}\} \cong A/(1-e) \cong A \cdot e := \{a \cdot e \mid a \in A\}$$

Proof. It is straightforward. A detailed proof is given in Lemma 1 of [10]. □

Next we give a precise definition of a von Neumann regular $\mathcal{C}^{\infty}-$ring. Loosely speaking, it is a $\mathcal{C}^{\infty}-$ring (A, Φ) such that $\widetilde{U}(A, \Phi)$ is a von Neumann regular commutative unital ring.

Definition 3.2. Let $\mathfrak{A} = (A, \Phi)$ be a $\mathcal{C}^\infty-$ring. We say that $\mathfrak{A}$ is a **von Neumann regular $\mathcal{C}^\infty-$ring** if one (and thus all) of the following equivalent[2], conditions is satisfied:

(i) $(\forall a \in A)(\exists x \in A)(a = a^2x)$;

(ii) Every principal ideal of A is generated by an idempotent element, *i.e.*,

$$(\forall a \in A)(\exists e \in A)(\exists y \in A)(\exists z \in A)((e^2 = e)\&(ey = a)\&(az = e))$$

(iii) $(\forall a \in A)(\exists! b \in A)((a = a^2b)\&(b = b^2a))$

Example 3.3. Consider the set $\mathbb{R}^m = \mathcal{C}^\infty(\{*\}) \times \cdots \times \mathcal{C}^\infty(\{*\})$, together with the function:

$$\begin{array}{rccc} \Phi^{(m)}: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,((\mathbb{R}^m)^n, (\mathbb{R}^m)) \\ & \mathbb{R}^n \stackrel{f}{\to} \mathbb{R} & \mapsto & \mathbb{R}^m \times \cdots \times \mathbb{R}^m \stackrel{\Phi^{(m)}(f)}{\to} \mathbb{R}^m \end{array}$$

with:

$$\begin{array}{rccc} \Phi^{(m)}(f): & (\mathbb{R}^m)^n & \to & \mathbb{R}^m \\ & ((x_j^1)_{j=1}^m, \cdots, (x_j^n)_{j=1}^m) & \mapsto & (f((x_1^i)_{i=1}^n), \cdots, f((x_m^i)_{i=1}^n)) \end{array}$$

Therefore $\mathfrak{A} = (\mathbb{R}^m, \Phi^{(m)})$ is the product $\mathcal{C}^\infty-$ring. In this $\mathcal{C}^\infty-$ring we have, in particular, the following binary operation:

$$\begin{array}{rccc} \Phi^{(m)}(\cdot): & \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \\ & ((x_j)_{1\leq j\leq m}, (y_j)_{1\leq j\leq m}) & \mapsto & (x_1 \cdot y_1, \cdots, x_m \cdot y_m) \end{array}$$

so we write $(x_1, \cdots, x_m) \cdot (y_1, \cdots, y_m) = (x_1 \cdot y_1, \cdots, x_m \cdot y_m)$.

We claim that $\mathfrak{A} = (\mathbb{R}^m, \Phi^{(m)})$ is a von Neumann-regular $\mathcal{C}^\infty-$ring. In fact, given any $(a_1, \cdots, a_m) \in \mathbb{R}^m$, then for each $i \in \{1, \cdots, m\}$ such that $a_i \neq 0$, we take $x_i = a_i^{-1}$, and for each i such that $a_i = 0$, we take $x_i = 0$. The element $x = (x_i)_{i=1}^m \in \mathbb{R}^m$ is such that:

$$(a_1, \cdots, a_m) = (a_1^2, \cdots, a_m^2) \cdot (x_1, \cdots, x_m).$$

Thus, $\mathfrak{A} = (\mathbb{R}^m, \Phi^{(m)})$ is a von Neumann-regular $\mathcal{C}^\infty-$ring.

[2]See, Proposition 3.9, given later.

Remark 3.4. Observe that the construction given in the example above can be replicated by replacing $\mathbb{R}$ by any $\mathcal{C}^\infty-$field.

Remark 3.5. Let $\mathbb{T}^{'}$ be the theory of Von Neumann regular $\mathcal{C}^\infty$-rings in the language $\mathcal{L}' = \mathcal{L} \cup \{*\}$, where $*$ is an 1-ary function symbol, which contains:

- the (equational) $\mathcal{L}$-axioms for of $\mathcal{C}^\infty-$rings;
- the (equational) $\mathcal{L}'$-axiom

$$\sigma := (\forall x)((x \cdot x^* \cdot x = x)\&(x^* \cdot x \cdot x^* = x^*))$$

that is, $\mathbb{T}^{'} := \mathbb{T} \cup \{\sigma\}$. By the **Theorem of Extension by Definition** , we know that $\mathbb{T}^{'}$ is a conservative extension of $\mathbb{T}$.

An homomorphism of von Neumann $\mathcal{C}^\infty-$rings, A and B is simply a $\mathcal{C}^\infty-$homomorphism between these $\mathcal{C}^\infty-$rings. We have the following:

Definition 3.6. We denote by $\mathcal{C}^\infty$**vNRing** the category whose objects are von Neumann-regular $\mathcal{C}^\infty-$rings and whose morphisms are the $\mathcal{C}^\infty-$homomorphisms between them. Thus, $\mathcal{C}^\infty$**vNRing** is a full subcategory of $\mathcal{C}^\infty$**Ring**.

The following lemma tells us that, in $\mathcal{C}^\infty$**vNRing**, taking localizations and taking the ring of fractions with respect to a special element yields, up to isomorphisms, the same object.

Lemma 3.7. *If A is a von Neumann regular $\mathcal{C}^\infty-$ring, then given any $a \in A$ there is some idempotent element $e \in A$ such that $A\{a^{-1}\} \cong A\{e^{-1}\} \cong A/(1-e)$.*

Proof. Since $a \cdot y = e$ and $e \cdot x = a$, then $e \in \{a\}^{\infty-\mathrm{sat}}$ and $a \in \{e\}^{\infty-\mathrm{sat}}$. Thus, $\{a\}^{\infty-\mathrm{sat}} = \{e\}^{\infty-\mathrm{sat}}$, so $A\{a^{-1}\} \cong A\{e^{-1}\}$. By Lemma 3.1, it follows that $A\{a^{-1}\} \cong A\{e^{-1}\} \cong A/(1-e)$. □

Lemma 3.8. *Let A be a von Neumann-regular $\mathcal{C}^\infty-$ring, $S \subseteq A$ and let $\widetilde{U} : \mathcal{C}^\infty$**Ring** $\to$ **CRing** be the forgetful functor. Then:*

$$\widetilde{U}\left(A\{S^{-1}\}\right) = \widetilde{U}(A)[S^{-1}]$$

Proof. We prove the result first in the case $S = \{a\}$ for some $a \in A$. Since A is a von Neumann-regular $\mathcal{C}^\infty-$ring, by Lemma 3.7, given $a \in A$ there is some idempotent element $e \in A$ such that $(a) = (e)$ and $A\{a^{-1}\} \cong A\{e^{-1}\} \cong A/(1-e)$. Now, $A/(1-e) \cong A[e^{-1}]$, and $A[e^{-1}] \cong \widetilde{U}(A)[e^{-1}]$, and since $\widetilde{U}(A)[e^{-1}] \cong \widetilde{U}(A)[a^{-1}]$, as ordinary commutative rings [3] the result follows.

Whenever S is finite, we have $A\{S^{-1}\} = A\{a^{-1}\}$, for $a = \prod S$, and we can use the proof we have just made. For a general $S \subseteq A$, we write $S = \cup_{S' \subseteq_{\rm fin} S} S'$ and use the fact that $\widetilde{U} : \mathcal{C}^\infty\mathbf{Ring} \to \mathbf{CRing}$ preserves directed colimits and that $A\{S^{-1}\} \cong \varinjlim_{S' \subseteq_{\rm fin} S} A\{S'^{-1}\}$. $\square$

As a corollary, we have the following:

Proposition 3.9. $\mathcal{C}^\infty\mathbf{vNRing} \subseteq \mathcal{C}^\infty\mathbf{Ring}$ *is closed under localizations.*

The following result is an adaptation of Proposition 1 of [1] for the $\mathcal{C}^\infty-$case.

Theorem 3.10. *If A is a von Neumann regular $\mathcal{C}^\infty-$ring then A is a reduced $\mathcal{C}^\infty-$ring.*

Proof. By the **Lemma 3.7**, $\sqrt[\infty]{(0)} = \{a \in A \mid A\{a^{-1}\} \cong \{0\}\}$. Now, let $a \in \sqrt[\infty]{(0)}$ and let $e \in A$ be an idempotent element such that $(a) = (e)$. Then $A/(1-e) \cong A\{e^{-1}\} \cong A\{a^{-1}\}$. Thus, $A/(1-e) \cong \{0\}$ yields $1 \in (1-e)$, so there must exist some $z \in A$ such that $1 = z \cdot (1-e)$, and $(1-e)$ is an invertible idempotent of A, so $1 - e = 1$ and $e = 0$. Thus, $a = 0$, so $\sqrt[\infty]{(0)} \subseteq \{0\}$. $\square$

The following result shows us that whenever A is a von Neumann regular $\mathcal{C}^\infty-$ring, the notions of $\mathcal{C}^\infty-$spectrum, Zariski spectrum, maximal spectrum and thus, the structure sheaf of its affine scheme coincide.

Theorem 3.11. *Let A be a von Neumann regular $\mathcal{C}^\infty-$ring. Then:*

1) $\sqrt[\infty]{(0)} = \sqrt{(0)} = (0)$*;*
2) ${\rm Spec}^\infty(A) = {\rm Specm}(\widetilde{U}(A)) = {\rm Spec}(\widetilde{U}(A))$*, as topological spaces;*

[3] note that $\widetilde{U}(A/(1-e)) = \widetilde{U}(A)/(1-e) = \widetilde{U}(A)[e^{-1}]$

3) *The structure sheaf of A in the category $\mathcal{C}^\infty$***Ring** *is equal to the structure sheaf of $U(A)$ in the category* **CRing**.

Proof. Ad 1): By Theorem 3.10, since A is a von Neumann regular $\mathcal{C}^\infty-$ring, $\sqrt[\infty]{(0)} = (0)$. Since we always have $(0) \subseteq \sqrt{(0)} \subseteq \sqrt[\infty]{(0)}$, it follows that $\sqrt{(0)} = (0)$.

Ad 2): Note that in a von Neumann regular $\mathcal{C}^\infty-$ring every prime ideal is a maximal ideal. In fact, let $\mathfrak{p}$ be a prime ideal in A. Given $a+\mathfrak{p} \neq \mathfrak{p}$ in $A/\mathfrak{p}$, then $a+\mathfrak{p} \in (A/\mathfrak{p})^\times$. Since A is a von Neumann regular ring, there exists some $b \in A$ such that $aba = a$, so $a+\mathfrak{p} = aba+\mathfrak{p}$, $a+\mathfrak{p} = (ab+\mathfrak{p})\cdot(a+\mathfrak{p})$, therefore $ab+\mathfrak{p} = 1+\mathfrak{p}$ and, thus, $ab = 1$ in $A/\mathfrak{p}$.

Hence, every non-zero element of $A/\mathfrak{p}$ is invertible, so $A/\mathfrak{p}$ is a field. Under those circumstances, it follows that $\mathfrak{p}$ is a maximal ideal, so $\mathrm{Spec}\,(A) = \mathrm{Specm}\,(A)$.

We always have $\mathrm{Specm}\,(A) \subseteq \mathrm{Spec}^\infty\,(A)$ and $\mathrm{Spec}^\infty\,(A) \subseteq \mathrm{Spec}\,(A)$, so:

$$\mathrm{Spec}\,(A) \subseteq \mathrm{Specm}\,(A) \subseteq \mathrm{Spec}^\infty\,(A) \subseteq \mathrm{Spec}\,(A)$$

and $\mathrm{Spec}^\infty\,(A) = \mathrm{Spec}\,(A)$.

Note, also, that both the topological spaces $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}^\infty\,(A)$ have the same basic open sets, $D^\infty(a) = \{\mathfrak{p} \in \mathrm{Spec}^\infty\,(A) \mid a \notin \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \mid a \notin \mathfrak{p}\} = D(a)$, hence $\mathrm{Spec}^\infty\,(A) = \mathrm{Spec}\,(A)$ also as topological spaces.

Ad 3). It is an immediate consequence of 2. □

Proposition 3.12. *Let A be a $\mathcal{C}^\infty-$ring. Then the following are equivalent:*

(i) *A is von Neumann-regular, i.e.,* $(\forall a \in A)(\exists x \in A)(a = a^2x)$.

(ii) *Every principal ideal of A is generated by an idempotent element, i.e.,*

$$(\forall a \in A)(\exists e \in A)(\exists y \in A)(\exists z \in A)((e^2 = e)\&(ey = a)\&(az = e))$$

(iii) $(\forall a \in A)(\exists b \in A)((a = a^2b)\&(b = b^2a))$

Moreover, when A is von Neumann-regular, then A is ($\mathcal{C}^\infty-$)reduced (i.e., $\sqrt[\infty]{(0)} = \sqrt{(0)} = (0)$) and for each $a \in A$ the idempotent element e satisfying (ii) and the element b satisfying (iii) are uniquely determined.

Proof. The implication (iii) $\to$ (i) is obvious, so we omit the proof.

Ad (i) $\to$ (ii): Let $I = (a)$ be a principal ideal of A. By (i), there is $x \in A$ such that $a = a^2x$, so we define $e := ax$, which is idempotent since $e^2 = (ax)^2 = a^2x^2 = (a^2x)x = ax = e$. By definition, $e = ax \in (a) = I$, so $(e) \subseteq I$, and since $a = a^2x = (ax)a = ea$ we also have $a \in (e)$, so $I = (a) \subseteq (e)$. Hence, $I = (e)$.

Ad (ii) $\to$ (i): Let $a \in A$ be any element. By (ii) there are $e \in A$, $y \in A$ and $z \in A$ such that $e^2 = e$, $a = ey$ and $e = az$. Define $x := z^2y$, and we have $a^2x = a^2z^2y = e^2y = ey = a$.

Ad (i) $\to$ (iii): Let $a \in A$ be any element. By (i), there is some $x \in A$ such that $a = a^2x$. There can be many $x \in A$ satisfying this role, but there is a "minimal" one: the element ax is idempotent and we can project any chosen x down with this idempotent, obtaining $b := ax^2$. Then $aba = aab^2a = (ax)(ax)a = axa = a$ and $bab = (ax^2)a(ax^2) = (ax)^3x = (ax)x = b$.

Now suppose that A is a von Neumann-regular $\mathcal{C}^\infty$−ring, and let $a \in A$ be such that $a \in \sqrt[\infty]{(0)}$. Then let e be an idempotent such that $ey = a$, $az = e$, for some $y, z \in A$. Then a is such that $A\{a^{-1}\} \cong \{0\}$, and by Lemma 3.7 there is some idempotent $e \in A$ such that $A\{a^{-1}\} \cong A/(1-e)$. Now, $A\{a^{-1}\} \cong \{0\}$ occurs if and only if, $A/(1-e) \cong \{0\}$, *i.e.*, if and only if, $(1-e) = A$. Since $(1-e) = A$, it follows that $1 - e \in A^\times$, and since $e \cdot (1-e) = 0$, it follows by cancellation that $e = 0$, hence $a = ey = 0y = 0$.

Let $e, e' \in A$ be idempotents of an arbitrary ring satisfying $(e) = (e')$. Select $r, r' \in A$ such that $er' = e'$ and $e'r = e$. Then $e' = er' = er'e = e'e = e're' = e'r = e$. Thus, if an ideal is generated by an idempotent element, this element is uniquely determined.

Finally, let A be a von Neumann-regular $\mathcal{C}^\infty$−ring. Select a member $a \in A$ and consider $b, b' \in A$ such that $a^2b' = a = a^2b$, $b = b^2a$, $b' = {b'}^2a$. Then $[(b-b')a]^2 = (b-b')^2a^2 = (b-b')(ba^2 - b'a^2) = (b-b')(a-a) = (b-b')\cdot 0 = 0$ and $[(b-b')\cdot a]^2 \in (0)$. Since A is $\mathcal{C}^\infty$−reduced, $[(b-b')\cdot a]^2 \in (0) = \sqrt[\infty]{(0)}$. By item (1) of Theorem 3.11, $\sqrt[\infty]{(0)} = \sqrt{(0)}$, so $[(b-b')\cdot a]^2 \in \sqrt[\infty]{(0)} = \sqrt{(0)}$ and $(b-b')\cdot a = 0$. Therefore $b - b' = b^2a - {b'}^2a = (b^2 - {b'}^2)a = (b+b')(b-b')a = (b+b')\cdot 0 = 0$. □

Remark 3.13. Let A be a von Neumann-regular $\mathcal{C}^\infty$−ring and $e \in A$ be any idempotent element. Then $A \cdot e$ is a von Neumann-regular $\mathcal{C}^\infty$−ring.

Indeed, we have $A \cdot e \cong A/(1-e)$ and the latter is an homomorphic image of a von Neumann-regular $\mathcal{C}^\infty-$ring, namely $A/(1-e) = q[A]$.

Lemma 3.14. *Let A be a local $\mathcal{C}^\infty-$ring. The only idempotent elements of A are 0 and 1.*

Proof. See Lemma 4 of [10]. $\square$

Proposition 3.15. *Let A be a von Neumann-regular $\mathcal{C}^\infty-$ring whose only idempotent elements are 0 and 1. Then the following assertions are equivalent:*

(i) *A is a $\mathcal{C}^\infty-$field;*

(ii) *A is a $\mathcal{C}^\infty-$domain;*

(iii) *A is a local $\mathcal{C}^\infty-$ring.*

Proof. The implications (i) $\to$ (ii), (i) $\to$ (iii) are immediate, so we omit their proofs.

Ad (iii) $\to$ (i): Suppose A is a local $\mathcal{C}^\infty-$ring. Since A is a von Neumann-regular $\mathcal{C}^\infty-$ring, given any $x \in A \setminus \{0\}$ there exists some idempotent element $e \in A$ such that $(x) = (e)$. However, the only idempotent elements of A are, by Lemma 3.14, 0 and 1. We claim that $(x) = (1)$, otherwise we would have $(x) = (0)$, so $x = 0$.

Now, $(x) = (1)$ implies $1 \in (x)$, so there is some $y \in A$ such that $1 = x \cdot y = y \cdot x$, and x is invertible. Thus A is a $\mathcal{C}^\infty-$field.

Ad (ii) $\to$ (i): Suppose A is a $\mathcal{C}^\infty-$domain. Given any $x \in A \setminus \{0\}$, we have $(\forall y \in A \setminus \{0\})(x \cdot y \neq 0)$, so $(x) \neq (0)$. Since A is a von Neumann-regular $\mathcal{C}^\infty-$ring, (x) is generated by some non-zero idempotent element, namely, 1. Hence $(x) = (1)$ and $x \in A^\times$. $\square$

Proposition 3.16. *The inclusion functor $\imath : \mathcal{C}^\infty\mathbf{vNRing} \hookrightarrow \mathcal{C}^\infty\mathbf{Ring}$ creates filtered colimits, i.e., $\mathcal{C}^\infty\mathbf{vNRing}$ is closed in $\mathcal{C}^\infty\mathbf{Ring}$ under filtered colimits.*

Proof. (Sketch) Filtered colimits in $\mathcal{C}^\infty\mathbf{Ring}$ are formed by taking the colimit of the underlying sets and defining the $n-$ary functional symbol $f^{(n)}$ of an $n-$tuple $(a_1, \cdots, a_n)$ into a common $\mathcal{C}^\infty-$ring occurring in the diagram and taking the element $f^{(n)}(a_1, \cdots, a_n)$ there. Given a filtered poset

$(I, \leq)$ and a a filtered family of $\mathcal{C}^\infty-$rings, for every element $\alpha \in \varinjlim A_i$, there is some $i \in I$ and $a_i \in A_i$ such that $\alpha = [(a_i, i)]$. Since A_i is a von Neumann-regular $\mathcal{C}^\infty-$ring, there must exist some idempotent $e_i \in A_i$ such that $(a_i) = (e_i)$. It suffices to take $\eta = [(e_i, i)] \in \varinjlim A_i$, which is an idempotent element of $\varinjlim A_i$ such that $(\alpha) = ([(a_i, i)]) = ([(e_i, i)]) = (\eta)$. □

We have the following important result, which relates von Neumann-regular $\mathcal{C}^\infty-$rings to the topology of its smooth Zariski spectrum:

Theorem 3.17. *Let A be a $\mathcal{C}^\infty-$ring. The following assertions are equivalent:*

i) *A is a von Neumann-regular $\mathcal{C}^\infty-$ring;*
ii) *A is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$ring (i.e., $\sqrt[\infty]{(0)} = (0)$) and* $\mathrm{Spec}^\infty(A)$ *is a Boolean space, i.e.,a compact, Hausdorff and totally disconnected space.*

Proof. Ad $(i) \to (ii)$: it follows from item (1) of Theorem 3.11.

Since $\mathrm{Spec}^\infty(A)$ is a spectral space, we only need to show that $\mathcal{B} = \{D^\infty(a) \mid a \in A\}$ is a clopen basis for its topology.

Given any $a \in A$, since A is a von Neumann regular $\mathcal{C}^\infty-$ring, there is some idempotent element $e \in A$ such that $(a) = (e)$, so $D^\infty(a) = D^\infty(e)$. We claim that $\mathrm{Spec}^\infty(A) \setminus D^\infty(e) = D^\infty(1-e)$, hence $D^\infty(a) = D^\infty(e)$ is a clopen set.

In fact, from item (iii) of Lemma 1.2 of [14], $D^\infty(e) \cap D^\infty(1-e) = D^\infty(e \cdot (1-e)) = D^\infty(0) = \{\mathfrak{p} \in \mathrm{Spec}^\infty(A) \mid 0 \notin \mathfrak{p}\} = \varnothing$. Moreover, for every prime ideal $\mathfrak{p}$ we have $e \notin \mathfrak{p}$ or $(1-e) \notin \mathfrak{p}$ (for if this was not the case, we would have a prime ideal $\mathfrak{p}_0$ such that $e \in \mathfrak{p}_0$ and $(1-e) \in \mathfrak{p}_0$, so $1 = (1-e) + e \in \mathfrak{p}_0$, which would not be a proper ideal). Thus $\mathrm{Spec}^\infty(A) = D^\infty(e) \cup D^\infty(1-e)$.

Ad $(ii) \to (i)$. Since $\mathrm{Spec}^\infty(A)$ is a Boolean space, it is a Hausdorff space and for every $a \in A$, $D^\infty(a)$ is compact, hence it is closed. Thus, we conclude that for every $a \in A$, $D^\infty(a)$ is a clopen set, so $\mathrm{Spec}^\infty(A) \setminus D^\infty(a)$ is a clopen subset of $\mathrm{Spec}^\infty(A)$.

Now, for every clopen C in $\mathrm{Spec}^\infty(A)$ there is some $b \in A$ such that $C = D^\infty(b)$. Since C is clopen in $\mathrm{Spec}^\infty(A)$, it is in particular an open set, and since $\{D^\infty(a) \mid a \in A\}$ is a basis for the topology of $\mathrm{Spec}^\infty(A)$, there is a family, $\{b_i \in A \mid i \in I\}$, of elements of A such that $C = \cup_{i \in I} D^\infty(b_i)$. Since

C is compact, there is a finite subset $I' \subseteq I$ such that $C = \cup_{i \in I'} D^\infty(b_i)$. Applying the item (iii) of Lemma 1.4 of [14], we conclude that there is some element $b \in A$ such that $\bigcup_{i \in I'} D^\infty(b_i) = D^\infty(b)$.

Since $\mathrm{Spec}^\infty(A) \setminus D^\infty(a)$ is clopen, there is some $d \in A$ such that $\mathrm{Spec}^\infty(A) \setminus D^\infty(a) = D^\infty(d)$. We have $\varnothing = D^\infty(a) \cap D^\infty(d) = D^\infty(a \cdot d) = \{\mathfrak{p} \in \mathrm{Spec}^\infty(A) \mid a \cdot d \notin \mathfrak{p}\}$, so $(\forall \mathfrak{p} \in \mathrm{Spec}^\infty(A))(a \cdot d \in \mathfrak{p})$, hence $a \cdot d \in \bigcap \mathrm{Spec}^\infty(A) = \sqrt[\infty]{(0)} = (0)$, where the last equality is due to the fact that A is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$ring.

We have, then, $a \cdot d = 0$. Also, we have $D^\infty(a^2 + d^2) = D^\infty(a) \cup D^\infty(d) = \mathrm{Spec}^\infty(A) = D^\infty(1)$. By item (i) of Lemma 1.4 of [14], $D^\infty(a^2 + d^2) \subseteq D^\infty(1)$ implies $a^2 + d^2 \in \{1\}^{\infty-\mathrm{sat}}$. Since $\{1\}^{\infty-\mathrm{sat}} = A^\times$, it follows that $a^2 + d^2 \in A^\times$, so there is some $y \in A$ such that $y \cdot (a^2 + d^2) = 1, ya^2 + yd^2 = 1$. Since $a \cdot d = 0$, we get $a(a^2 y) + a(b^2 y) = a \cdot 1 = aa^2(a \cdot y) = a$. □

The following proposition will be useful to characterize the von Neumann-regular $\mathcal{C}^\infty-$rings by means of the ring of global sections of the structure sheaf of its affine scheme.

Proposition 3.18. *If a $\mathcal{C}^\infty-$ring A is a von-Neumann-regular $\mathcal{C}^\infty-$ring and $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$, then $A/\mathfrak{p} \cong A\{A \setminus \mathfrak{p}^{-1}\}$ and both are $\mathcal{C}^\infty-$fields.*

Proof. We are going to show that the only maximal ideal of $A\{A \setminus \mathfrak{p}^{-1}\}$, $\mathfrak{m}_\mathfrak{p}$ is such that $\mathfrak{m}_\mathfrak{p} \cong \{0\}$.

Let $\eta_\mathfrak{p} : A \to A\{A \setminus \mathfrak{p}^{-1}\}$ be the localization morphism of A with respect to $A \setminus \mathfrak{p}$. We have $\mathfrak{m}_\mathfrak{p} = \langle \eta_\mathfrak{p}[A \setminus \mathfrak{p}] \rangle = \{\eta_\mathfrak{p}(a)/\eta_\mathfrak{p}(b) \mid (a \in \mathfrak{p}) \& (b \in A \setminus \mathfrak{p})\}$. We must show that for every $a \in \mathfrak{p}$, $\eta_\mathfrak{p}(a) = 0$, which is equivalent, by Theorem 1.4 of [14], to assert that for every $a \in \mathfrak{p}$ there is some $c \in (A \setminus \mathfrak{p})^{\infty-\mathrm{sat}} = A \setminus \mathfrak{p}$ such that $c \cdot a = 0$ in A.

Ab absurdo, suppose $\mathfrak{m}_\mathfrak{p} \neq \{0\}$, so there is $a \in \mathfrak{p}$ such that $\eta_\mathfrak{p}(a) \neq 0$, i.e., such that for every $c \in A \setminus \mathfrak{p}$, $c \cdot a \neq 0$. Since A is a von Neumann-regular $\mathcal{C}^\infty-$ring, for this a there is some idempotent $e \in \mathfrak{p}$ such that $(a) = (e)$.

Since $a \in (e)$, there is some $\lambda \in A$ such that $a = \lambda \cdot a$, hence:

$$0 \neq \eta_\mathfrak{p}(a) = \eta_\mathfrak{p}(\lambda \cdot e) = \eta_\mathfrak{p}(\lambda) \cdot \eta_\mathfrak{p}(e)$$

and $\eta_\mathfrak{p}(e) \neq 0$.

Since $\eta_\mathfrak{p}(e) \neq 0$,

$$(\forall d \in A \setminus \mathfrak{p})(d \cdot e \neq 0). \tag{3.1}$$

Since e is an idempotent element, $1-e \notin \mathfrak{p}$, for if $1-e \in \mathfrak{p}$ then $e+(1-e)=1 \in \mathfrak{p}$ and $\mathfrak{p}$ would not be a proper prime ideal.

We have also:

$$e \cdot (1-e) = 0, \tag{3.2}$$

The equation (3.2) contradicts (3.1), so $\mathfrak{m}_\mathfrak{p} \cong \{0\}$ and $A\{A \setminus \mathfrak{p}^{-1}\}$ is a $\mathcal{C}^\infty-$field. □

As a consequence, we register another proof of $(iii) \to (i)$ of Proposition 3.15.

Corollary 3.19. *Let $\mathfrak{A} = (A, \Phi)$ be a local von Neumann-regular $\mathcal{C}^\infty-$ring. Then $\mathfrak{A}$ is a $\mathcal{C}^\infty-$field.*

Proof. (Sketch)Let $\mathfrak{m} \subseteq A$ be the unique maximal ideal of A. By Proposition 3.18, since A is von Neumann-regular, $A_\mathfrak{m} \cong A/\mathfrak{m}$, which is a $\mathcal{C}^\infty-$field. Also, $A_\mathfrak{m} = A\{A \setminus \mathfrak{m}^{-1}\} = A\{A^{\times^{-1}}\} \cong A$, and since $A_\mathfrak{m}$ is isomorphic to a $\mathcal{C}^\infty-$field, it follows that A is a $\mathcal{C}^\infty-$field. □

Summarizing, we have the following result:

Theorem 3.20. *If A is a von Neumann-regular $\mathcal{C}^\infty-$ring, then the set* $\mathrm{Spec}^\infty(A)$ *with the smooth Zariski topology,* Zar^∞, *is a Boolean topological space, by Theorem 3.17. Moreover, by Proposition 3.18, for every $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$,*

$$A_\mathfrak{p} = \varinjlim_{a \notin \mathfrak{p}} A\{a^{-1}\} \cong A\{A \setminus \mathfrak{p}^{-1}\}$$

is a $\mathcal{C}^\infty-$field.

The above theorem suggests us that von Neumann-regular $\mathcal{C}^\infty-$rings behave much like ordinary von Neumann-regular commutative unital rings. In the next sections we are going to explore this result using sheaf theoretic machinery.

Proposition 3.21. *The limit in $\mathcal{C}^\infty$**Ring** of a diagram of von Neumann-regular $\mathcal{C}^\infty$rings is a von Neumann-regular $\mathcal{C}^\infty-$ring. In particular, $\mathcal{C}^\infty$**vNRing** is a complete category and the inclusion functor from the category of all von Neumann regular $\mathcal{C}^\infty-$rings, $\mathcal{C}^\infty$**vNRing**, to $\mathcal{C}^\infty$**Ring** preserves all limits.*

Proof. (Sketch) It is clear from the definition that the class $\mathcal{C}^\infty$**vNRing** of von Neumann-regular $\mathcal{C}^\infty-$rings is closed under arbitrary products in the class $\mathcal{C}^\infty$**Ring**, of all $\mathcal{C}^\infty-$rings. Thus it suffices to show that it is closed under equalizers.

So let A, B be von Neumann-regular rings and $f, g : A \to B$ be $\mathcal{C}^\infty-$homomorphisms. Their equalizer in $\mathcal{C}^\infty$**Ring** is given by the set $E = \{a \in A \mid f(a) = g(a)\}$, endowed with the restricted ring operations from A.

To see that E is von Neumann-regular, we need to show that for $a \in E$, the (unique) element b satisfying $ab^2 = b$ and $a^2b = a$ also belongs to E. But this is true since the quasi-inverse element is unique and is preserved under $\mathcal{C}^\infty-$homomorphisms. □

Proposition 3.22. *The category* $\mathcal{C}^\infty$**vNRing** *is the smallest subcategory of* $\mathcal{C}^\infty$**Ring** *closed under limits containing all* $\mathcal{C}^\infty-$*fields.*

Proof. Clearly all $\mathcal{C}^\infty-$fields are von Neumann-regular $\mathcal{C}^\infty-$rings, and by Proposition 3.21 so are limits of $\mathcal{C}^\infty-$fields. Thus $\mathcal{C}^\infty$**vNRing** contains all limits of $\mathcal{C}^\infty-$fields. On the other hand the ring of global sections of a sheaf can be expressed as a limit of a diagram of products and ultraproducts of the stalks (by Lemma 2.5 of [13]). All these occurring (ultra)products are von Neumann-regular $\mathcal{C}^\infty-$rings as well and hence so is their limit, by Proposition 3.21. □

4 Von Neumann-regular $\mathcal{C}^\infty-$Rings and Boolean Algebras

In this section we also apply von Neumann regular $\mathcal{C}^\infty$-ring to naturally represent Boolean Algebras in a strong sense: i.e., not only all Boolean algebras are isomorphic to the Boolean algebra of idempotents of a von Neumann regular $\mathcal{C}^\infty$-ring, as every homomorphism between such Boolean algebras of idempotents is (essentially) induced by a $\mathcal{C}^\infty$-homomorphism.

Remark 4.1. By Stone Duality, there is an anti-equivalence of categories between the category of Boolean algebras, **BA**, and the category of Boolean spaces, **BoolSp**.

Under this anti-equivalence, a Boolean space (X, τ) is mapped to the Boolean algebra of clopen subsets of (X, τ), Clopen(X):

$$\begin{array}{rccc} \text{Clopen}: & \mathbf{BoolSp} & \to & \mathbf{BA} \\ & (X,\tau) \xrightarrow{f} (Y,\sigma) & \mapsto & \text{Clopen}\,(Y) \xrightarrow{f^{-1}\restriction} \text{Clopen}\,(X) \end{array}$$

The quasi-inverse functor is given by the Stone space functor: a Boolean algebra B is mapped to the Stone space of B, $\text{Stone}(B) = (\{U \subseteq B : U \text{ is an ultrafilter in } B\}, \tau_B)$, where τ_B is the topology whose basis is given by the image of the map $t_B : B \to \mathscr{P}(\text{Stone}(B))$ (the set of all subsets of $\text{Stone}(B)$), $b \mapsto t_B(b) = S_B(b) = \{U \in \text{Stone}(B) : b \in U\}$.

$$\begin{array}{rccc} \text{Stone}: & \mathbf{BA} & \to & \mathbf{BoolSp} \\ & B \xrightarrow{h} B' & \mapsto & \text{Stone}(B') \xrightarrow{h^{-1}\restriction} \text{Stone}(B) \end{array}$$

Remark 4.2. Let $(A', +', \cdot', 0', 1')$ be any commutative unital ring, $B(A') = \{e \in A' \mid e^2 = e\}$ and denote by $\wedge', \vee', *', \leq', 0'$ and $1'$ its respective associated Boolean algebra operations, relations and constant symbols as constructed above. Note that for any commutative unital ring homomorphism $f : A \to A'$, the map $B(f) := f\restriction_{B(A)}: B(A) \to B(A')$ is such that:

(i) $B(f)[B(A)] \subseteq B(A')$;

(ii) $(\forall e_1 \in A)(\forall e_2 \in A)(B(f)(e_1 \wedge e_2) = f\restriction_{B(A)}(e_1 \cdot e_2) = (f\restriction_{B(A)}(e_1)) \cdot' (f\restriction_{B(A)}(e_2)) = B(f)(e_1) \wedge' B(f)(e_2))$

(iii) $(\forall e_1 \in A)(\forall e_2 \in A)(B(f)(e_1 \vee e_2) = f\restriction_{B(A)}(e_1 + e_2 - e_1 \cdot e_2) = (f\restriction_{B(A)}(e_1)) +' (f\restriction_{B(A)}(e_2)) - f\restriction_{B(A)}(e_1) \cdot f\restriction_{B(A)}(e_2) = B(f)(e_1) \vee' B(f)(e_2))$

(iv) $(\forall e \in B(A))(B(f)(e^*) = f\restriction_{B(A)}(1 - e) = f\restriction_{B(A)}(1) - f\restriction_{B(A)}(e) = 1' - f\restriction_{B(A)}(e) = B(f)(e)^*)$

hence a morphism of Boolean algebras.

We also have, for every ring A, $B(\text{id}_A) = \text{id}_{B(A)}$ and given any $f : A \to A'$ and $f' : A' \to A''$, $B(f' \circ f) = B(f') \circ B(f)$, since $B(f) = f\restriction_{B(A)}$, so:

$$\begin{array}{rccc} B: & \mathbf{CRing} & \to & \mathbf{BA} \\ & A & \mapsto & B(A) \\ & A \xrightarrow{f} A' & \mapsto & B(A) \xrightarrow{B(f)} B(A') \end{array}$$

is a (covariant) functor.

Since we can regard any $\mathcal{C}^\infty-$ring A as a commutative unital ring via the forgetful functor $\widetilde{U} : \mathcal{C}^\infty\mathbf{Ring} \to \mathbf{CRing}$, we have a (covariant) functor:

$$\begin{array}{rccc} \widetilde{B}: & \mathcal{C}^\infty\mathbf{Ring} & \to & \mathbf{BA} \\ & A & \mapsto & \widetilde{B}(A) := (B \circ U)(A) \\ & A \xrightarrow{f} A' & \mapsto & \widetilde{B}(A) \xrightarrow{\widetilde{B}(f)} \widetilde{B}(A') \end{array}$$

Now, if A is any $\mathcal{C}^\infty-$ring, we can define the following map:

$$\begin{array}{rccc} \jmath_A: & \widetilde{B}(A) & \to & \mathrm{Clopen}\,(\mathrm{Spec}^\infty(A)) \\ & e & \mapsto & D^\infty(e) = \{\mathfrak{p} \in \mathrm{Spec}^\infty\,(A) \mid e \notin \mathfrak{p}\} \end{array}$$

Claim 1: The map defined above is a Boolean algebra homomorphism.

Note that for any $e \in B(A)$, $D^\infty(e^*) = D^\infty(1-e) = \mathrm{Spec}^\infty\,(A) \setminus D^\infty(e) = D^\infty(e)^*$, since:

$$D^\infty(e) \cap D^\infty(1-e) = D^\infty\,(e \cdot (1-e)) = D^\infty(0) = \varnothing$$

and

$$\begin{aligned} D^\infty(e) \cup D^\infty(e*) = D^\infty(e) \cup D^\infty(1-e) = D^\infty(e^2+(1-e)^2) = D^\infty(e+(1-e)) = \\ = D^\infty(1) = \mathrm{Spec}^\infty\,(A). \end{aligned}$$

Hence $\jmath_A(e^*) = \jmath_A(1-e) = D^\infty(1-e) = \mathrm{Spec}^\infty\,(A) \setminus D^\infty\,(e) = D^\infty(e)^* = \jmath_A(e)^*$.

By the item (iii) of Lemma 1.4 of [14], $D^\infty(e \cdot e') = D^\infty(e) \cap D^\infty(e')$, so $\jmath_A(e \wedge e') = D^\infty(e \cdot e') = D^\infty(e) \cap D^\infty(e') = \jmath_A(e) \cap \jmath_A(e')$.

Last,

$$\begin{aligned} \jmath_A(e \vee e') = \jmath_A(e+e'-e \cdot e') = D^\infty(e+e'-e \cdot e') = D^\infty(e^2) \cup D^\infty(e'-e \cdot e') = \\ D^\infty(e) \cup D^\infty(e' \cdot (1-e)) = D^\infty(e) \cup [D^\infty(e') \cap D^\infty(1-e)] = \\ = [D^\infty(e) \cup D^\infty(e')] \cap [D^\infty(e) \cup D^\infty(1-e)] = [D^\infty(e) \cup D^\infty(e')] \cap \mathrm{Spec}^\infty\,(A) = \\ = D^\infty(e) \cup D^\infty(e') = \jmath_A(e) \cup \jmath_A(e'), \end{aligned}$$

and the claim is proved.

Claim 2: $\jmath_A : \widetilde{B}(A) \to \mathrm{Clopen}\,(\mathrm{Spec}^\infty(A))$ is an injective map.

In order to prove that $\jmath_A : \widetilde{B}(A) \to \mathrm{Clopen}\,(\mathrm{Spec}^\infty\,(A))$ is an injective map, it suffices to show that $\jmath_A^{\dashv}[\mathrm{Spec}^\infty\,(A)] = \{1\}$.

In order to prove it, we need the following:

Claim 2.1: $(\jmath_A(e) = \mathrm{Spec}^\infty\,(A)) \iff (e \in A^\times)$.

In fact, $a \in A^\times \Rightarrow D^\infty(a) = \mathrm{Spec}^\infty\,(A)$. On the other hand, if $a \notin A^\times$ then there is some maximal $\mathcal{C}^\infty$−radical prime ideal $\mathfrak{m} \in \mathrm{Spec}^\infty\,(A)$ such that $a \in \mathfrak{m}$. If $\mathfrak{m} \notin D^\infty(a)$ then $\jmath_A(a) = D^\infty(a) \neq \mathrm{Spec}^\infty\,(A)$. Hence $\jmath_A(a) = \mathrm{Spec}^\infty\,(A) \Rightarrow a \in A^\times$, and the claim is proved.

Let $e \in \widetilde{B}(A)$ be such that $\jmath_A(e) = D^\infty(e) = \mathrm{Spec}^\infty\,(A)$, so by Claim 2.1, it follows that $e \in A^\times$ and since e is idempotent, $e = 1$, that is $\jmath_A^{\dashv}[\mathrm{Spec}^\infty\,(A)] = \{1\}$.

The injective map $\jmath_A : \widetilde{B}(A) \to \mathrm{Spec}^\infty\,(A)$ suggests that the idempotent elements of the Boolean algebra $\widetilde{B}(A)$ associated with a $\mathcal{C}^\infty$−ring A hold a strong relationship with the canonical basis of the Zariski topology of $\mathrm{Spec}^\infty(A)$. We are going to show that these idempotent elements, in the case of the von Neumann-regular $\mathcal{C}^\infty$−rings, represent *all the Boolean algebras.*

Theorem 4.3. *Let A be a von Neumann regular $\mathcal{C}^\infty$−ring. The map:*

$$\begin{array}{rccc} \jmath_A : & \widetilde{B}(A) & \to & \mathrm{Clopen}\,(\mathrm{Spec}^\infty(A)) \\ & e & \mapsto & D^\infty(e) = \{\mathfrak{p} \in \mathrm{Spec}^\infty\,(A) \mid e \notin \mathfrak{p}\} \end{array}$$

is an isomorphism of Boolean algebras.

Proof. We already know that $\jmath_A$ is an injective Boolean algebras homomorphism, so it suffices to prove that it is also surjective. This is achieved noting that given any $a_1, a_2, \cdots, a_n \in A$, there is some $b \in A$ such that $D^\infty(a_1) \cup D^\infty(a_2) \cup \cdots \cup D^\infty(a_n) = D^\infty(b)$, which is proved by induction, using item (iii) of Lemma 1.4 of [14] for the case $n = 2$.

Since A is a von Neumann-ring, given this $b \in A$, there is an idempotent element, e, such that $(b) = (e)$, so $D^{\infty}(b) = D^{\infty}(e)$. Thus, $\jmath_A$ is surjective, as claimed. □

Theorem 4.4. *We have the following diagram of categories, functors and a natural isomorphism:*

$$\begin{array}{ccc} \mathcal{C}^{\infty}\mathbf{vNRing} & \xrightarrow{\operatorname{Spec}^{\infty}} & \mathbf{BoolSp} \\ & \overset{\jmath}{\Nearrow} & \downarrow \text{Clopen} \\ & \searrow_{\widetilde{B}} & \mathbf{BA} \end{array} \tag{4.1}$$

Proof. First note that since A is a von Neumann-regular $\mathcal{C}^{\infty}-$ring, the set of the compact open subsets of (the Boolean space) $\operatorname{Spec}^{\infty}(A)$ equals $\operatorname{Clopen}(\operatorname{Spec}^{\infty}(A))$.

On the one hand, given a von Neumann regular $\mathcal{C}^{\infty}-$ring A, we have $\operatorname{Clopen}(\operatorname{Spec}^{\infty}(A)) = \jmath_A[\widetilde{B}(A)] = \{D^{\infty}(e) \mid e \in \widetilde{B}(A)\}$.

For every von Neumann regular $\mathcal{C}^{\infty}-$ring A, by Theorem 4.3, we have the following isomorphism of Boolean algebras:

$$\begin{array}{rccc} \jmath_A : & \widetilde{B}(A) & \to & \operatorname{Clopen}(\operatorname{Spec}^{\infty}(A)) \\ & e & \mapsto & D^{\infty}(e) \end{array}$$

It is easy to see that for every $\mathcal{C}^{\infty}-$homomorphism $f : A \to A'$, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{B}(A) & \xrightarrow{\jmath_A} & \operatorname{Clopen}(\operatorname{Spec}^{\infty}(A)) \\ {\scriptstyle \widetilde{B}(f)}\downarrow & & \downarrow{\scriptstyle \operatorname{Clopen}(\operatorname{Spec}^{\infty}(f))} \\ \widetilde{B}(A') & \xrightarrow[\jmath_{A'}]{} & \operatorname{Clopen}(\operatorname{Spec}^{\infty}(A')) \end{array}$$

In fact, given $e \in \widetilde{B}(A)$, we have, on the one hand, $\jmath_A(e) = D^{\infty}_A(e)$ and $\operatorname{Clopen}(\operatorname{Spec}^{\infty}(f))(D^{\infty}_A(e)) = D^{\infty}_{A'}(f(e))$. On the other hand, $\jmath_{A'} \circ \widetilde{B}(f)(e) = \jmath_{A'}(f(e)) = D^{\infty}_{A'}(f(e))$, so the diagram (4.1) commutes.

Thus, $\jmath$ is a natural transformation and $\jmath : \widetilde{B} \Longrightarrow \text{Clopen} \circ \text{Spec}^\infty$ is a natural isomorphism and the diagram "commutes" (up to natural isomorphism). □

The following lemma is a well-known result, of which the authors could not find a proof anywhere in the current literature. The authors provide a proof in Lemma 6 of [10].

Lemma 4.5. *Let (X, τ) be a Boolean topological space, and let:*

$$\mathcal{R} = \{R \subseteq X \times X \mid (R \textit{ is an equivalence relation on } X)\& \\ \&((X/R) \textit{ is a discrete compact space})\}$$

which is partially ordered by inclusion. Whenever $R_i, R_j \in \mathcal{R}$ are such that $R_j \subseteq R_i$, we have the continuous surjective map:

$$\begin{array}{rccc} \mu_{R_jR_i}: & (X/R_j) & \twoheadrightarrow & (X/R_i) \\ & [x]_{R_j} & \mapsto & [x]_{R_j} \end{array}$$

so we have the inverse system $\{(X/R_i); \mu_{R_jR_i} : (X/R_j) \to (X/R_i)\}$. By definition (see, for instance, [11]),

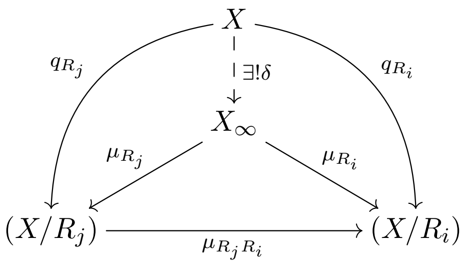
$$\varprojlim_{R \in \mathcal{R}} (X/R) = \{([x]_{R_i})_{R_i \in \mathcal{R}} \in \prod_{R \in \mathcal{R}} (X/R) \mid R_j \subseteq R_i \to ([x]_{R_i} = \mu_{R_jR_i}([x]_{R_j})\}$$

Let X_∞ denote $\varprojlim_{R \in \mathcal{R}} \frac{X}{R}$, so we have the following cone:

$$\begin{array}{ccc} & X_\infty & \\ {\scriptstyle \mu_{R_j}} \swarrow & & \searrow {\scriptstyle \mu_{R_i}} \\ (X/R_j) & \xrightarrow{\mu_{R_jR_i}} & (X/R_i) \end{array}$$

We consider X_∞ together with the induced subspace topology of $\prod_{R \in \mathcal{R}}(X/R)$.

By the universal property of X_∞, there is a unique continuous map map $\delta : X \to X_\infty$ such that the following diagram commutes:



We claim that such a $\delta : X \to X_\infty$ is a homeomorphism, so:

$$X \cong \varprojlim_{R \in \mathcal{R}} (X/R)$$

that is, X a profinite space.

Proof. See Lemma 6 of [10]. □

Let **BoolSp** be the category whose objects are all the Boolean spaces and whose morphisms are all the continuous functions between Boolean spaces. Given any Boolean space (X, τ), let

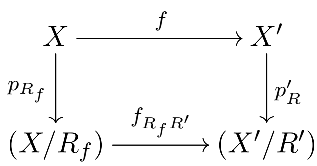
$$\mathcal{R}_X = \{R \subseteq X \times X \mid R \text{ is an equivalence relation on } X \text{ and } (X/R) \text{ is discrete and compact}\}$$

We are going to describe an equivalence functor between **BoolSp** and the category of profinite topological spaces: this is a known result, but we cannot found a reference containing a detailed description.

First we note that given any continuous function $f : X \to X'$ and any $R' \in \mathcal{R}_{X'}$,

$$R_f := (f \times f)^{\dashv}[R'] \subseteq X \times X$$

is an equivalence relation on X and the following diagram:



where $p_{R_f} : X \to (X/R_f)$ and $p_{R'} : X' \to (X'/R')$ are the canonical projections, commutes.

We know, by Theorem 4.3 of [11] that $f_{R_fR'} : (X/R_f) \to (X'/R')$ is a continuous map, and it is easy to see that $f_{R_fR'}$ is injective, as we are going to show.

Given any $[x]_{R_f}, [y]_{R_f} \in \dfrac{X}{R_f}$ such that $[x]_{R_f} \neq [y]_{R_f}$, *i.e.*, such that $(x,y) \notin R_f$, we have $(f(x), f(y)) \notin R'$, *i.e.*, $[f(x)]_{R'} \neq [f(y)]_{R'}$. Thus, since:

$$f_{R_fR'}([x]_{R_f}) = (f_{R_fR'} \circ p_{R_f})(x) = (p_{R'} \circ f)(x) = [f(x)]_{R'}$$

and

$$f_{R_fR'}([y]_{R_f}) = (f_{R_fR'} \circ p_{R_f})(y) = (p_{R'} \circ f)(y) = [f(y)]_{R'}$$

it follows that $f_{R_fR'}([x]_{R_f}) \neq f_{R_fR'}([y]_{R_f})$.

Since $f_{R_fR'} : \frac{X}{R_f} \to \frac{X'}{R'}$ is an injective continuous map and $\frac{X'}{R'}$ is discrete, it follows that given any $[x']_{R'} \in \frac{X'}{R'}$:

$$f^{\dashv}_{R_fR'}[\{[x']_{R'}\}] = \begin{cases} \varnothing, \text{ if } [x']_{R'} \notin f_{R_fR'}[X/R_f], \\ \{*\}, \text{ otherwise} \end{cases} ,$$

so every singleton of $\frac{X}{R_f}$ is an open subset of $\frac{X}{R_f}$, and $\frac{X}{R_f}$ is discrete. Also, since X is compact, $\frac{X}{R_f}$ is compact, and it follows that and $R_f \in \mathcal{R}_X$.

Now, if $R'_1, R'_2 \in \mathcal{R}_{X'}$ are such that $R'_1 \subseteq R'_2$, then $R'_{1f} \subseteq R'_{2f}$. In fact, given $(x,y) \in R'_{1f}$, we have $(f \times f)(x,y) \in R'_1$, and since $R'_1 \subseteq R'_2$, it follows that $(f \times f)(x,y) \in R'_2$, so $(x,y) \in R'_{2f}$.

Let X, Y, Z be Boolean spaces, $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be two Boolean spaces homomorphisms and $R \in \mathcal{R}_Z$. We easily see that $((g \circ f) \times (g \circ f))^{\dashv}[R] = (f \times f)^{\dashv}[(g \times g)^{\dashv}[R]]$.

Denoting $T := (f \times f)^{\dashv}[(g \times g)^{\dashv}[R]]$ and $S := (g \times g)^{\dashv}[R]$, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z \\ {\scriptstyle \mu_T}\downarrow & & \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \mu_R} \\ (X/T) & \xrightarrow{f_{TS}} & (Y/S) & \xrightarrow{g_{SR}} & (Z/R) \end{array}$$

$$(X/T) \xrightarrow{(g \circ f)_{TR}} (Z/R)$$

Given a continuous map between Boolean spaces, $f : X \to X'$, we can define a map $\check{f} : \varprojlim_{R \in \mathcal{R}_X} (X/R) \to \varprojlim_{R' \in \mathcal{R}_{X'}} (X'/R')$ in a functorial manner.

Let $R', S' \in \mathcal{R}_{X'}$ be any two equivalence relations such that $R' \subseteq S'$, so given $f : X \to X'$ the following diagram commutes:

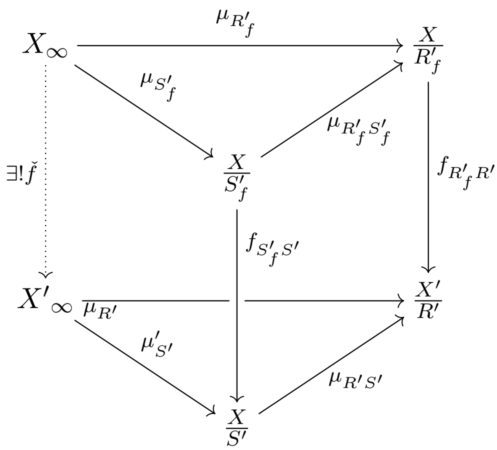
$$\begin{array}{ccc} (X/S'_f) & \xrightarrow{\mu_{R'_f S'_f}} & (X'/R'_f) \\ {\scriptstyle f_{S'_f S'}}\downarrow & & \downarrow{\scriptstyle f_{R'_f R'}} \\ (X'/S') & \xrightarrow{\mu_{R'S'}} & (X'/R') \end{array}$$

and since the diagram (4.2) commutes, the diagram (4.3) also commutes.

$$\begin{array}{ccc} & X_\infty & \\ {\scriptstyle \mu_{S'_f}}\swarrow & & \searrow{\scriptstyle \mu_{R'_f}} \\ (X/S'_f) & \xrightarrow[\mu_{R'_f S'_f}]{\qquad} & (X/R'_f) \end{array} \tag{4.2}$$

$$\begin{array}{ccc} & X_\infty & \\ {\scriptstyle f_{S'_f S'} \circ \mu_{S'_f}}\swarrow & & \searrow{\scriptstyle f_{R'_f R'} \mu_{R'_f}} \\ (X'/S') & \xrightarrow[\mu_{R'S'}]{\qquad} & (X'/R') \end{array} \tag{4.3}$$

By the universal property of X'_∞, there is a unique $\check{f} : X_\infty \to X'_\infty$ such that the following prism is commutative:



It can be proven that:

$$\begin{array}{cccc} \Delta : & \mathbf{BoolSp} & \to & \mathbf{ProfinSp} \\ & X & \mapsto & X_\infty = \varprojlim_{R \in \mathcal{R}_X} (X/R) \\ & X \xrightarrow{f} X' & \mapsto & X_\infty \xrightarrow{\check{f}} {X'}_\infty \end{array}$$

is a functor using the uniqueness properties regarding X_∞ and $\check{f}$ (for details, see [10]).

Theorem 4.6. *Let $\mathbb{K}$ be a $\mathcal{C}^\infty-$field. Following the notation of Lemma 4.5, define the contravariant functor:*

$$\begin{array}{cccc} \widehat{k} : & \mathbf{BoolSp} & \to & \mathcal{C}^\infty\mathbf{vNRing} \\ & (X, \tau) & \mapsto & A_X := \varinjlim_{R \in \mathcal{R}} \mathbb{K}^{U\left(\frac{X}{R}\right)} \end{array}$$

Then there is a natural isomorphism:

$$\epsilon : \mathrm{Id}_{\mathbf{BoolSp}} \overset{\cong}{\Rightarrow} \mathrm{Spec}^\infty \circ \widehat{k}$$

Therefore:

- *The functor $\widehat{k}$ is faithful;*
- *The functor* $\mathrm{Spec}^\infty : \mathcal{C}^\infty\mathbf{vNRing} \to \mathbf{BoolSp}$ *is "full up to conjugation";*

- *The functor* $\mathrm{Spec}^{\infty} : \mathcal{C}^{\infty}\mathbf{vNRing} \to \mathbf{BoolSp}$ *is isomorphism-dense. In particular: for each* (X, τ) *be a Boolean space, there is a von Neumann-regular* $\mathcal{C}^{\infty}-$*ring,* A_X*, such that* $\mathrm{Spec}^{\infty}(A_X) \approx X$.

Proof. By the Theorem 34, p. 118 of [6],

$$\mathrm{Spec}^{\infty}(A_X) \approx \varprojlim_{R \in \mathcal{R}} \mathrm{Spec}^{\infty}(\mathbb{K}^{U(X/R)}).$$

By the Theorem 33, p. 118 of [6],

$$\mathrm{Spec}^{\infty}(\mathbb{K}^{U(X/R)}) \approx X/R,$$

so

$$\varprojlim_{R \in \mathcal{R}} \mathrm{Spec}^{\infty}(\mathbb{K}^{U\left(\frac{X}{R}\right)}) \approx \varprojlim_{R \in \mathcal{R}} (X/R) \approx X$$

Since the homeomorphisms above are natural, just take $\epsilon_X : X \to \mathrm{Spec}^{\infty}(A_X)$ as the composition of these homeomorphisms.

In particular, $\mathrm{Spec}^{\infty}(A_X) \approx X$ and Spec^{∞} is an isomorphism-dense functor.

Let $\phi : X \to X'$ be a continuous function between Boolean spaces. Since ϵ is a natural isomorphism, we have $\phi = \epsilon_{X'}^{-1} \circ \mathrm{Spec}^{\infty}(\widehat{k}(\phi)) \circ \epsilon_X$. In particular, there exists a homomorphism of von Neumann regular $\mathcal{C}^{\infty}-$rings $f : A' \to A$ and homeomorphisms of Boolean spaces $\psi : X \to \mathrm{Spec}^{\infty}(A)$ and $\psi' : X' \to \mathrm{Spec}^{\infty}(A')$, such that

$$\phi = {\psi'}^{-1} \circ \mathrm{Spec}^{\infty}(f) \circ \psi,$$

thus Spec^{∞} is a full up to conjugatization functor.

Let $\phi, \psi : X \to X'$ be continuous functions between Boolean spaces such that $\widehat{k}(\phi) = \widehat{k}(\psi)$. Then

$$\phi = \epsilon_{X'}^{-1} \circ \mathrm{Spec}^{\infty}(\widehat{k}(\phi)) \circ \epsilon_X = \epsilon_{X'}^{-1} \circ \mathrm{Spec}^{\infty}(\widehat{k}(\psi)) \circ \epsilon_X = \psi,$$

thus $\widehat{k}$ is a faithful functor.

□

Theorem 4.7. *Let $\mathbb{K}$ be a $\mathcal{C}^\infty-$field. Defining the covariant functor (composition of contravariant functors):*

$$\check{K} = \widehat{k} \circ \text{Stone} : \mathbf{BA} \to \mathcal{C}^\infty\mathbf{vNRing}.$$

Then there is a natural isomorphism

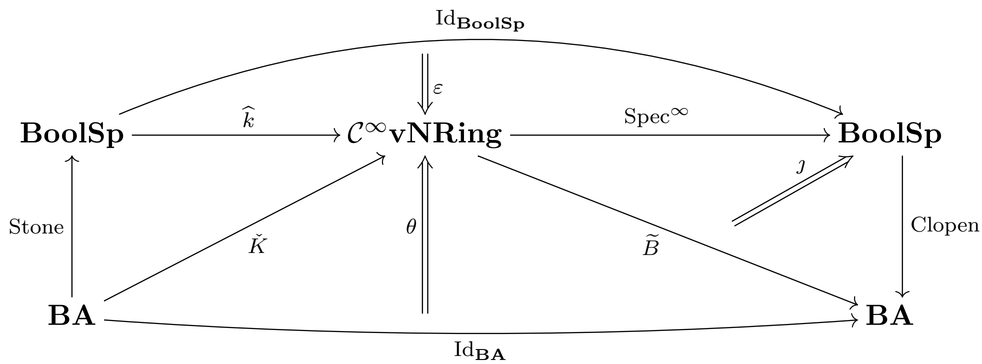
$$\theta : \text{Id}_{\mathbf{BA}} \overset{\cong}{\Rightarrow} \widetilde{B} \circ \check{K}.$$

Therefore:

- *The functor $\check{K}$ is faithful;*
- *The functor $\widetilde{B}$ is full up to conjugation;*
- *The functor $\widetilde{B} : \mathcal{C}^\infty - \mathbf{vNRing} \to \mathbf{BA}$ is isomorphism-dense. In particular: given any $\mathcal{C}^\infty-$field $\mathbb{K}$ and any Boolean algebra B, there is a von Neumann regular $\mathcal{C}^\infty-$ring which is a $\mathbb{K}-$algebra, $\check{K}(B)$, such that $\widetilde{B}(\check{K}(B)) \cong B$.*

Proof. This follows directly by a combination of the Theorem 4.6 above, Stone duality (Remark 4.1), Theorem 4.4 and Theorem 4.3. □

The diagram below summarizes the main functorial connections established in this section:



We finish this work with the following:

Remark 4.8. The theorem above leads us to the natural question(s):

- Is $\widetilde{B}$ an equivalence of categories, possibly with with $\check{K}$ being the quasi-inverse of $\widetilde{B}$, for some $\mathcal{C}^\infty-$field $\mathbb{K}$?

- For every $\mathcal{C}^\infty-$field $\mathbb{K}$ with $\mathrm{card}\,(\mathbb{K}) > \mathrm{card}\,(\mathbb{R})$, $\check{K}$ **can not be** a quasi-inverse of $\widetilde{B}$; in fact, since for every Boolean algebra A, $\check{K}(A)$ is in particular a $\mathbb{K}$-algebra, we have $\mathrm{card}(\check{K}(A)) \geq \mathrm{card}(\mathbb{K})$, whenever $A \ncong \{0\}$. Let V be a von Neumann-regular $\mathcal{C}^\infty-$ring; since the class of $\mathcal{C}^\infty-$fields is first-order axiomatizable in the language of $\mathcal{C}^\infty-$rings, and every $\mathcal{C}^\infty-$field is an infinite set (since it is, in particular, a non-trivial $\mathbb{R}$-algebra), then by the **Löwenheim-Skolem Theorem** (upward, which we denote by $\uparrow$ LS), there is a $\mathcal{C}^\infty-$field $\mathbb{K}$ such that $\mathrm{card}\,(\mathbb{K}) > \mathrm{card}\,(V)$. Therefore, $\mathrm{card}\,(\check{K}(\widetilde{B}\,(V))) \geq \mathrm{card}\,(\mathbb{K}) \overset{\uparrow \mathrm{LS}}{>} \mathrm{card}\,(V)$ and, thus, $\check{K}(\widetilde{B}\,(V)) \ncong V$. Since there exist non-trivial von Neumann-regular $\mathcal{C}^\infty-$rings V with $\mathrm{card}\,(V) = \mathrm{card}\,(\mathbb{R})$, this shows that there is no $\mathcal{C}^\infty-$field ,$\mathbb{K}$, with $\mathrm{card}\,(\mathbb{K}) > \mathrm{card}\,(\mathbb{R})$, such that $\check{K}$ is a quasi-inverse of $\widetilde{B}$.

- It is important to stress that the functor $\widetilde{B}$ **is not** an equivalence of categories. In fact, by $\uparrow$ LS, there are $\mathcal{C}^\infty-$fields, $\mathbb{K}$, with $\mathrm{card}\,(\mathbb{K}) > \mathrm{card}\,(\mathbb{R})$. We saw above that $\widetilde{B}$ is a "full up to conjugation functor", but if $\widetilde{B}$ were a full functor, then the Boolean algebra isomorphism $\widetilde{B}\,(\mathbb{K}) \cong \mathbf{2} \overset{\mathrm{id}_{\mathbf{2}}}{\to} \mathbf{2} \cong \widetilde{B}\,(\mathbb{R})$ should be the image of some $\mathcal{C}^\infty-$homomorphism $\mathbb{K} \to \mathbb{R}$: this is a contradiction since a $\mathcal{C}^\infty-$homomorphism between $\mathcal{C}^\infty-$fields must be injective. Therefore $\widetilde{B}$ is not a full functor, thus $\widetilde{B}$ is not an equivalence of categories.

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