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Inductive graded rings, hyperfields and quadratic forms

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Abstract. In [6] we developed a k-theory for the category of hyperbolic hyperfields (a category that contains a copy of the category of (pre)special groups): this construction extends, simultaneously, Milnor's k-theory ([20]) and Dickmann-Miraglia's k-theory ([13]). An abstract environment that encapsulate all them, and of course, provide an axiomatic approach to guide new extensions of the concept of K-theory in the context of the algebraic and abstract theories of quadratic forms is given by the concept of inductive graded rings a concept introduced in [9] in order to provide a solution of Marshall's signature conjecture in realm the algebraic theory of quadratic forms for Pythagorean fields. The goal of this work is twofold: (i) to provide a detailed analysis of some categories of inductive graded ring - a concept introduced in [9] in order to provide a solution of Marshall's signature conjecture in the algebraic theory of quadratic forms; (ii) apply this analysis to deepen the connections between the category of special hyperfields ([6]) - equivalent to the category of special groups ([10]) and the categories of inductive graded rings.

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Introduction

It can be said that the Algebraic Theory of Quadratic Forms (ATQF) was founded in 1937 by E. Witt, with the introduction of the concept of the Witt ring of a given field, constructed from the quadratic forms with coefficients in the field: given F, an arbitrary field of characteristic $\neq 2$, $W(F)$, the Witt ring of F, classifies the quadratic forms over F that are regular and anisotropic, being in one-to-one correspondence with them; thus the focus of the theory is the quadratic forms defined on the ground field where all their coefficients are invertible. In this way, the set of orders in F is in one-to-one correspondence with the set of minimal prime ideals of the Witt ring of F, and more, the set of orders in F provided with the Harrison's topology is a Boolean topological space that, by the bijection above, is identified with a subspace of the Zariski spectrum of the Witt ring of F.

Questions about the structure of Witt rings $W(F)$ could only be solved about three decades after Witt's original idea, through the introduction and analysis of concept of Pfister form. The Pfister forms of degree $n \in \mathbb{N}$, in turn, are generators of the power $I^n(F)$ of the fundamental ideal $I(F) \subseteq W(F)$ (the ideal determined by the anisotropic forms of even dimension).

Other finer questions about the powers of the fundamental ideal arose in the early 1970s: J. Milnor, in a seminal article from 1970 ([20]), determines a graduated ring $k_*(F)$ (from K-theory, reduced mod 2) associated with the field F, which interpolates, through graded ring morphisms

$$h_*(F) : k_*(F) \longrightarrow H^*(F) \text{ and } s_*(F) : k_*(F) \longrightarrow W_*(F),$$

the graded Witt ring

$$W_*(F) := \bigoplus_{n\in\mathbb{N}} I^n(F)/I^{n+1}(F)$$

and the graded cohomology ring

$$H^*(F) := \bigoplus_{n\in\mathbb{N}} H^n(Gal(F^s|F), \{\pm 1\}).$$

From Voevodski's proof of Milnor's conjectures, and the development of special groups theory (SG) – an abstract (and first-order) theory of Algebraic Theory of Quadratic Forms (ATFQ), introduced by M. Dickmann, and developed by him in partnership with F. Miraglia since the 1990s – it has been possible to demonstrate conjectures about signatures put by M. Marshall by T. Lam in the mid-1970s ([10], [9], [12]).

The SG theory, which faithfully codifies both the classical theory of quadratic forms over fields and the reduced theory of quadratic forms developed from the 1980s ([17]), allows us to naturally extend the construction of graded ring functors to all the special groups G: $W(G)$, $W_*(G)$, $k_*(G)$ ([10], [13]).

The key points in the demonstration of these conjectures for (pre-ordered) fields was a combination of methods: (i) the introduction of Boolean methods in the theory of quadratic forms through the SG theory -especially the Boolean hull functor ([10], [11]); (ii) the encoding of the original problems posed on signatures in questions on graded Witt rings; (iii) the use of Milnor's isomorphisms to transpose these questions to the graded ring of k-theory and the graded ring of cohomology; (iv) the use of Galois cohomology methods to finalize the resolution of the encoded problem.

In [6] we developed a k-theory for the category of hyperbolic hyperfields (a category that contains a copy of the category of (pre)special groups): this construction extends, simultaneously, Milnor's k-theory ([20]) and Dickmann-Miraglia's k-theory ([13]). An abstract environment that encapsulate all them, and of course, provide an axiomatic approach to guide new extensions of the concept of K-theory in the context of the algebraic and abstract theories of quadratic forms is given by the concept of inductive graded rings a concept introduced in [9] in order to provide a solution of Marshall's signature conjecture in realm the algebraic theory of quadratic forms for Pythagorean fields.

The goal of this work is twofold: (i) to provide a detailed analysis of some categories of inductive graded ring; (ii) apply this analysis to deepen the connections between the category of special hyperfields ([6]) - equivalent t groups ([10]) and the categories of inductive graded rings.

Outline of the work: Section 2 exposes the fundamental definitions and results that will be analysed in the present work: multirings/hyperfields, the K-theory of hyperfields and special groups. In Section 3 we introduce the concept of Inductive Graded Ring (IGR), that provides an axiomatic approach to guide new extensions of the concept of K-theory in the context of the algebraic and abstract theories of quadratic forms, and Section 4 establishes a more detailed analysis of the IGR category. In Section 5 we define subcategories of Igr that mimetize the following two central aspects of K-theories: i) the K-theory graded ring is "generated" by K_1; ii) the K-theory graded ring is defined by some convenient quotient of a graded tensor algebra. Section 6 provides some examples of functors of quadratic interest and Section 7 deals with the adjunction between PSG and Igr_h. We finish the work in Section 8 considering a general setting for "Marshall's Conjectures", that includes the previous case of the Igr's $W_*(F), k_*(F)$ for special hyperfields F.

We assume that the reader is familiar with some categorical results concerning adjunctions: mostly are based on [3], but the reader could also consult [18]. But for the benefit of the reader, we have included an appendix where we present some categorical results needed in this work.

1 Preliminaries: special groups, hyperbolic hyperfields and k-theory

1.1 Special Groups Firstly, we make a brief summary on special groups. Let A be a set and $\equiv$ a binary relation on $A \times A$. We extend $\equiv$ to a binary relation $\equiv_n$ on A^n, by induction on $n \geq 1$, as follows:

i) $\equiv_1$ is the diagonal relation $\Delta_A \subseteq A \times A$.

ii) $\equiv_2 = \equiv$.

iii) If $n \geq 3$,

$$\langle a_1, ..., a_n \rangle \equiv_n \langle b_1, ..., b_n \rangle \text{ iff there exists } x, y, z_3, ..., z_n \in A \text{ such that } \langle a_1, x \rangle \equiv \langle b_1, y \rangle,\ \langle a_2, ..., a_n \rangle \equiv_{n-1} \langle x, z_3, ..., z_n \rangle \text{ and } \langle b_2, ..., b_n \rangle \equiv_{n-1} \langle y, z_3, ..., z_n \rangle$$

Whenever clear from the context, we frequently abuse notation and indicate the afore-described extension $\equiv$ by the same symbol.

Definition 1.1 (Special Group, 1.2 of [10]). A **special group** is a tuple $(G, -1, \equiv)$, where G is a group of exponent 2, i.e, $g^2 = 1$ for all $g \in G$; -1 is a distinguished element of G, and $\equiv \subseteq G \times G \times G \times G$ is a relation (the special relation), satisfying the following axioms for all $a, b, c, d, x \in G$:

SG 0 $\equiv$ is an equivalence relation on G^2;

SG 1 $\langle a,b\rangle \equiv \langle b,a\rangle$;

SG 2 $\langle a,-a\rangle \equiv \langle 1,-1\rangle$;

SG 3 $\langle a,b\rangle \equiv \langle c,d\rangle \Rightarrow ab = cd$;

SG 4 $\langle a,b\rangle \equiv \langle c,d\rangle \Rightarrow \langle a,-c\rangle \equiv \langle -b,d\rangle$;

SG 5 $\langle a,b\rangle \equiv \langle c,d\rangle \Rightarrow \langle ga,gb\rangle \equiv \langle gc,gd\rangle$, for all $g \in G$.

SG 6 (3-transitivity) the extension of $\equiv$ for a binary relation on G^3 is a transitive relation.

A group of exponent 2, with a distinguished element -1, satisfying the axioms SG0-SG3 and SG5 is called a **proto special group**; a **pre special group** is a proto special group that also satisfy SG4. Thus a **special group** is a pre-special group that satisfies SG6 (or, equivalently, for each $n \geq 1$, $\equiv_n$ is an equivalence relation on G^n).

A n**-form** (or form of dimension $n \geq 1$) is an n-tuple of elements of a pre-SG G. An element $b \in G$ is **represented** on G by the form $\varphi = \langle a_1,...,a_n\rangle$, in symbols $b \in D_G(\varphi)$, if there exists $b_2,...,b_n \in G$ such that $\langle b,b_2,...,b_n\rangle \equiv \varphi$.

A pre-special group (or special group) $(G,-1,\equiv)$ is:
- **formally real** if $-1 \notin \bigcup_{n\in\mathbb{N}} D_G(n\langle 1\rangle)$[1] ;
- **reduced** if it is formally real and, for each $a \in G$, $a \in D_G(\langle 1,1\rangle)$ iff $a = 1$.

Definition 1.2 (1.1 of [10])**.** A map $(G,\equiv_G,-1) \xrightarrow{f} (H,\equiv_H,-1)$ between pre-special groups is a **morphism of pre-special groups or PSG-morphism** if $f: G \to H$ is a homomorphism of groups, $f(-1) = -1$ and for all $a,b,c,d \in G$

$$\langle a,b\rangle \equiv_G \langle c,d\rangle \Rightarrow \langle f(a),f(b)\rangle \equiv_H \langle f(c),f(d)\rangle$$

A **morphism of special groups or SG-morphism** is a pSG-morphism between the corresponding pre-special groups. The morphism f will be an isomorphism if is bijective and f,f^{-1} are PSG-morphisms.

[1]Here the notation $n\langle 1\rangle$ means the form $\langle a_1,...,a_n\rangle$ where $a_j = 1$ for all $j = 1,...,n$. In other words, $n\langle 1\rangle$ is the form $\langle 1,...,1\rangle$ with n entries equal to 1.

It can be verified that a special group G is formally real iff it admits some SG-morphism $f : G \to 2$. The category of special groups (respectively reduced special groups) and their morphisms will be denoted by SG (respectively RSG).

Definition 1.3 (2.4 [13])**.**

a) A reduced special group is [MC] if for all $n \geq 1$ and all forms φ over G,

$$\text{For all } \sigma \in X_G, \text{ if } \sigma(\varphi) \equiv 0 \bmod 2^n \text{ then } \varphi \in I^n G.$$

b) A reduced special group is [SMC] if for all $n \geq 1$, the multiplication by $\lambda(-1)$ is an injection of $k_n G$ in $k_{n+1} G$.

1.2 Multifields/Hyperfields

Roughly speaking, a multiring is a "ring" with a multivalued addition, a notion introduced in the 1950s by Krasner's works. The notion of multiring was joined to the quadratic forms tools by the hands of M. Marshall in the last decade ([19]). We gather the basic information about multirings/hyperfields and expand some details that we use in the context of K-theories. For more detailed calculations involving multirings/hyperfields and quadratic forms we indicate to the reader the reference [8] (or even [15] and [5]). Of course, multi-structures is an entire subject of research (which escapes from the "quadratic context"), and in this sense, we indicate the references [22], [23], [2].

Definition 1.4 (Adapted from Definition 2.1 in [19])**.** A multiring is a sextuple $(R, +, \cdot, -, 0, 1)$ where R is a non-empty set, $+ : R \times R \to \mathcal{P}(R) \setminus \{\emptyset\}$, $\cdot : R \times R \to R$ and $- : R \to R$ are functions, 0 and 1 are elements of R satisfying:

i) $(R, +, -, 0)$ is a commutative multigroup;

ii) $(R, \cdot, 1)$ is a monoid;

iii) $a0 = 0$ for all $a \in R$;

iv) If $c \in a + b$, then $cd \in ad + bd$ and $dc \in da + db$. Or equivalently, $(a + b)d \subseteq ab + bd$ and $d(a + b) \subseteq da + db$.

v) If the equalities holds, i.e, $(a + b)d = ab + bd$ and $d(a + b) = da + db$, we said that R is a **hyperring**.

A multiring is commutative if $(R, \cdot, 1)$ is a commutative monoid. A zero-divisor of a multiring R is a non-zero element $a \in R$ such that $ab = 0$ for another non-zero element $b \in R$. The multiring R is said to be a multidomain if do not have zero divisors, and R will be a multifield if $1 \neq 0$ and every non-zero element of R has multiplicative inverse.

Example 1.5.

a) Suppose that $(G, +, 0)$ is an abelian group. Defining $a + b = \{a + b\}$ and $r(g) = -g$, we have that $(G, +, r, 0)$ is an abelian multigroup. In this way, every ring, domain and field is a multiring, multidomain and hyperfield, respectively.

b) $Q_2 = \{-1, 0, 1\}$ is hyperfield with the usual product (in $\mathbb{Z}$) and the multivalued sum defined by relations

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1,\ (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

c) Let $K = \{0, 1\}$ with the usual product and the sum defined by relations $x + 0 = 0 + x = x$, $x \in K$ and $1 + 1 = \{0, 1\}$. This is a hyperfield called Krasner's hyperfield [16].

Now, another example that generalizes $Q_2 = \{-1, 0, 1\}$. Since this is a new one, we will provide the entire verification that it is a multiring:

Example 1.6 (Kaleidoscope, Example 2.7 in [8])**.** Let $n \in \mathbb{N}$ and define

$$X_n = \{-n, ..., 0, ..., n\} \subseteq \mathbb{Z}.$$

We define the n-**kaleidoscope multiring** by $(X_n, +, \cdot, -, 0, 1)$, where $- : X_n \to X_n$ is restriction of the opposite map in $\mathbb{Z}$, $+ : X_n \times X_n \to \mathcal{P}(X_n) \setminus \{\emptyset\}$ is given by the rules:

$$a + b = \begin{cases} \{a\}, & \text{if } b \neq -a \text{ and } |b| \leq |a| \\ \{b\}, & \text{if } b \neq -a \text{ and } |a| \leq |b| \\ \{-a, ..., 0, ..., a\} & \text{if } b = -a \end{cases},$$

and $\cdot : X_n \times X_n \to X_n$ is given by the rules:

$$a \cdot b = \begin{cases} \operatorname{sgn}(ab)\max\{|a|,|b|\} \text{ if } a,b \neq 0 \\ 0 \text{ if } a = 0 \text{ or } b = 0 \end{cases}.$$

With the above rules we have that $(X_n, +, \cdot, -, 0, 1)$ is a multiring.

Now, another example that generalizes $K = \{0, 1\}$.

Example 1.7 (H-hyperfield, Example 2.8 in [8])**.** Let $p \geq 1$ be a prime integer and $H_p := \{0, 1, ..., p-1\} \subseteq \mathbb{N}$. Now, define the binary multioperation and operation in H_p as follows:

$$a + b = \begin{cases} H_p \text{ if } a = b,\ a,b \neq 0 \\ \{a,b\} \text{ if } a \neq b,\ a,b \neq 0 \\ \{a\} \text{ if } b = 0 \\ \{b\} \text{ if } a = 0 \end{cases}$$

$$a \cdot b = k \text{ where } 0 \leq k < p \text{ and } k \equiv ab \bmod \text{p}.$$

$(H_p, +, \cdot, -, 0, 1)$ is a hyperfield such that for all $a \in H_p$, $-a = a$. In fact, these H_p are a kind of generalization of K, in the sense that $H_2 = K$.

There are many natural constructions on the category of multirings as: products, directed inductive limits, quotients by an ideal, localizations by multiplicative subsets and quotients by ideals. Now, we present some constructions that will be used further. For the first one, we need to restrict our category:

Definition 1.8 (Definition 3.1 of [5])**.** An **hyperbolic multiring** is a multiring R such that $1 - 1 = R$. The category of hyperbolic multirings and hyperbolic hyperfields will be denoted by HMR and HMF respectively.

Let F_1 and F_2 be two hyperbolic hyperfields. We define a new hyperbolic hyperfield $(F_1 \times_h F_2, +, -, \cdot, (0,0), (1,1))$ by the following: the underlying set of this structure is

$$F_1 \times_h F_2 := (\dot{F}_1 \times \dot{F}_2) \cup \{(0,0)\}.$$

For $(a, b), (c, d) \in F_1 \times_h F_2$ we define

$$\begin{aligned} -(a,b) &= (-a,-b), \\ (a,b)\cdot(c,d) &= (a\cdot c, b\cdot d), \\ (a,b)+(c,d) &= \{(e,f)\in F_1\times F_2 : e\in a+c \text{ and } f\in b+d\}\cap(F_1\times_h F_2). \end{aligned} \tag{1.1}$$

In other words, $(a, b) + (c, d)$ is defined in order to avoid elements of $F_1 \times F_2$ of type $(x, 0), (0, y)$, $x, y \neq 0$.

Theorem 1.9 (Product of Hyperbolic Hyperfields)**.** *Let F_1, F_2 be hyperbolic hyperfields and $F_1 \times_h F_2$ as above. Then $F_1 \times_h F_2$ is a hyperbolic hyperfield and satisfy the Universal Property of product for F_1 and F_2.*

In order to avoid confusion and mistakes, we denote the binary product in HMF by $F_1 \times_h F_2$. For hyperfields $\{F_i\}_{i\in I}$, we denote the product of this family by

$$\prod_{i\in I}^{h} F_i,$$

with underlying set defined by

$$\prod_{i\in I}^{h} F_i := \left(\prod_{i\in I} \dot{F_i}\right) \cup \{(0_i)_{i\in I}\}$$

and operations defined by rules similar to the ones defined in 1.1. If $I = \{1, ...n\}$, we denote

$$\prod_{i\in I}^{h} F_i = \prod_{\substack{i=1 \\ [h]}}^{n} F_i.$$

Example 1.10. Note that if F_1 (or F_2) is not hyperbolic, then $F_1 \times_h F_2$ is not a hyperfield in general. Let F_1 be a field (considered as a hyperfield), for example $F_1 = \mathbb{R}$ and F_2 be another hyperfield. Then if $a, b \in F_1$, we have $1 - 1 = \{0\}$, so $(1, a) + (-1, b) = \{0\} \times (a - b)$, and

$$[\{0\} \times (a - b)] \cap (F_1 \times_h F_2) = \emptyset.$$

Proposition 1.11 (3.13 of [8])**.** *Let $(G, \equiv, -1)$ be a special group and define $M(G) = G \cup \{0\}$ where $0 := \{G\}^2$. Then $(M(G), +, -, \cdot, 0, 1)$ is a hyperfield, where*

- $a \cdot b = \begin{cases} 0 \textit{ if } a = 0 \textit{ or } b = 0 \\ a \cdot b \textit{ otherwise} \end{cases}$
- $-(a) = (-1) \cdot a$
- $a + b = \begin{cases} \{b\} \textit{ if } a = 0 \\ \{a\} \textit{ if } b = 0 \\ M(G) \ \textit{ if } a = -b, \textit{ and } a \neq 0 \\ D_G(a, b) \textit{ otherwise} \end{cases}$

Corollary 1.12 (3.14-3.19 of [8])**.** *The correspondence $G \mapsto M(G)$ extends to an equivalence of categories $M : SG \to SMF$, from the category of special groups onto the category of* ***special multifields****.*

Definition 1.13 (Definition 3.2 of [5])**.** A **Dickmann-Miraglia multiring (or DM-multiring for short)** [3] is a pair (R, T) such that R is a multiring, $T \subseteq R$ is a multiplicative subset of $R \setminus \{0\}$, and (R, T) satisfies the following properties:

DM0 $R/_mT$ is hyperbolic.

DM1 If $\overline{a} \neq 0$ in $R/_mT$, then $\overline{a}^2 = \overline{1}$ in $R/_mT$. In other words, for all $a \in R \setminus \{0\}$, there are $r, s \in T$ such that $ar = s$.

DM2 For all $a \in R$, $(\overline{1} - \overline{a})(\overline{1} - \overline{a}) \subseteq (\overline{1} - \overline{a})$ in $R/_mT$

DM3 For all $a, b, x, y, z \in R \setminus \{0\}$, if

$$\begin{cases} \overline{a} \in \overline{x} + \overline{b} \\ \overline{b} \in \overline{y} + \overline{z} \end{cases} \quad \text{in } R/_mT,$$

then exist $\overline{v} \in \overline{x} + \overline{z}$ such that $\overline{a} \in \overline{y} + \overline{v}$ and $\overline{vb} \in \overline{xy} + \overline{az}$ in $R/_mT$.

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[2] Here, the choice of the zero element was ad hoc. Indeed, we can define $0 := \{x\}$ for any $x \notin G$.

[3] The name "Dickmann-Miraglia" is given in honor to professors Maximo Dickmann and Francisco Miraglia, the creators of the special group theory.

If R is a ring, we just say that (R,T) is a DM-ring, or R is a DM-ring. A Dickmann-Miraglia hyperfield (or DM-hyperfield) F is a hyperfield such that $(F,\{1\})$ is a DM-multiring (satisfies DM0-DM3). In other words, F is a DM-hyperfield if F is hyperbolic and for all $a,b,v,x,y,z\in F^*$,

i) $a^2=1$.

ii) $(1-a)(1-a)\subseteq(1-a)$.

iii) If $\begin{cases} a\in x+b \\ b\in y+z \end{cases}$ then there exists $v\in x+z$ such that $a\in y+v$ and $vb\in xy+az$.

Theorem 1.14 (Theorem 3.4 of [5])**.** *Let (R,T) be a DM-multiring and denote*

$$Sm(R,T)=(R/_mT).$$

Then $Sm(R)$ is a special hyperfield (thus $Sm(R,T)^\times$ is a special group).

Theorem 1.15 (Theorem 3.9 of [5])**.** *Let F be a hyperfield satisfying DM0-DM2. Then F satisfies DM3 if and only if satisfies SMF4. In other words, F is a DM-hyperfield if and only if it is a special hyperfield.*

In this sense, we define the following category:

Definition 1.16. A **pre-special hyperfield** is a hyperfield satisfying DM0, DM1 and DM2. In other words, a pre-special hyperfield is a hyperbolic hyperfield F such that for all $a\in\dot F$, $a^2=1$ and $(1-a)(1-a)\subseteq 1-a$.

The category of pre-special hyperfields will be denoted by $PSMF$.

Theorem 1.17. *Let G be a pre-special group and consider $(M(G),+,-,0,1)$, with operations defined by*

- $a\cdot b=\begin{cases} 0 \textit{ if } a=0 \textit{ or } b=0 \\ a\cdot b \textit{ otherwise}\end{cases}$
- $-(a)=(-1)\cdot a$
- $a+b=\begin{cases} \{b\} \textit{ if } a=0 \\ \{a\} \textit{ if } b=0 \\ M(G) \textit{ if } a=-b, \textit{ and } a\neq 0 \\ D_G(a,b) \textit{ otherwise}\end{cases}$

Then $M(G)$ is a pre-special multifield. Conversely, if F is a pre-special multifield then $(\dot F,\equiv_F,-1)$ is a pre-special group, where

$$\langle a,b\rangle\equiv_F\langle c,d\rangle \textit{ iff } ab=cd \textit{ and } a\in c+d.$$

We finish this section stating the following result established in [4]

Theorem 1.18 (Arason-Pfister Hauptsatz). *Let F be a special hyperfield, then it holds $AP_F(n)$, for all $n \geq 0$. In more details: for each $n \geq 0$ and For each $\varphi = \langle a_1, \cdots, a_k\rangle$, a non-empty ($k \geq 1$), regular ($a_i \in \dot{F}$) and anisotropic form, if $\varphi \in I^n(F)$, then $\dim(\varphi) \geq 2^n$ $\varphi \in I^n(F)$, if $\varphi \neq \emptyset$ is anisotropic, then $\dim_{W,F}(\varphi) \geq 2^n$.*

1.3 The K-theory for Hyperfields

In this section we describe the notion of K-theory of a hyperfield, introduced in [6] by essentially repeating the construction in [20] replacing the word "field" by "hyperfield" and explore some of this basic properties. Apart from the obvious resemblance, more technical aspects of this new theory can be developed (but with other proofs) in multistructure setting in parallel with classical K-theory.

Definition 1.19 (The K-theory of a Hyperfield (3.1 of [6])). For a hyperfield F, K_*F is the graded ring

$$K_*F = (K_0F, K_1F, K_2F, ...)$$

defined by the following rules: $K_0F := \mathbb{Z}$. K_1F is the multiplicative group $\dot{F}$ written additively. With this purpose, we fix the canonical "logarithm" isomorphism

$$\rho : \dot{F} \to K_1F,$$

where $\rho(ab) = \rho(a) + \rho(b)$. Then K_nF is defined to be the quotient of the tensor algebra

$$K_1F \otimes K_1F \otimes ... \otimes K_1F \; (n \text{ times})$$

by the (homogeneous) ideal generated by all $\rho(a) \otimes \rho(b)$, with $a, b \neq 0$ and $b \in 1 - a$.

In other words, for each $n \geq 2$,

$$K_nF := T^n(K_1F)/Q^n(K_1(F)),$$

where

$$T^n(K_1F) := K_1F \otimes_{\mathbb{Z}} K_1F \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} K_1F$$

and $Q^n(K_1(F))$ is the subgroup generated by all expressions of type $\rho(a_1) \otimes \rho(a_2) \otimes ... \otimes \rho(a_n)$ such that $a_{i+1} \in 1 - a_i$ for some i with $1 \leq i \leq n - 1$.

To avoid carrying the overline symbol, we will adopt all the conventions used in Dickmann-Miraglia's K-theory ([13]). Just as it happens with the previous K-theories, a generic element $\eta \in K_nF$ has the pattern

$$\eta = \rho(a_1) \otimes \rho(a_2) \otimes ... \otimes \rho(a_n)$$

for some $a_1, ..., a_n \in \dot{F}$, with $a_{i+1} \in 1 - a_i$ for some $1 \leq i < n$. Note that if F is a field, then "$b \in 1 - a$" just means $b = 1 - a$, and the hyperfield and Milnor's K-theory for F coincide.

Lemma 1.20 (Basic Properties I (3.2 of [6]))**.** *Let F be an hyperbolic hyperfield. Then*

a) $\rho(1) = 0$.

b) *For all $a \in \dot{F}$, $\rho(a)\rho(-a) = 0$ in K_2F.*

c) *For all $a, b \in \dot{F}$, $\rho(a)\rho(b) = -\rho(b)\rho(a)$ in K_2F.*

d) *For every $a_1, ..., a_n \in \dot{F}$ and every permutation $\sigma \in S_n$,*

$$\rho(a_{\sigma 1})...\rho(a_{\sigma i})...\rho(a_{\sigma n}) = sgn(\sigma)\rho(a_1)...\rho(a_n) \text{ in } K_nF.$$

e) *For every $\xi \in K_mF$ and $\eta \in K_nF$, $\eta\xi = (-1)^{mn}\xi\eta$ in $K_{m+n}F$.*

f) *For all $a \in \dot{F}$, $\rho(a)^2 = -\rho(a)\rho(-1)$.*

An element $a \in \dot{F}$ induces a morphism of graded rings

$$\omega^a = \{\omega^a_n\}_{n\geq 1} : K_*F \to K_*F$$

of degree 1, where $\omega^a_n : K_nF \to K_{n+1}F$ is the multiplication by $\rho(a)$. When $a = -1$, we write

$$\omega = \{\omega_n\}_{n\geq 1} = \{\omega^{-1}_n\}_{n\geq 1} = \omega^{-1}.$$

Proposition 1.21 (Adapted from 3.3 of [13])**.** *Let F, K be hyperbolic hyperfields and $\varphi : F \to L$ be a morphism. Then φ induces a morphism of graded rings*

$$\varphi_* = \{\varphi_n : n \geq 0\} : K_*F \to K_*L,$$

where $\varphi_0 = Id_{\mathbb{Z}}$ and for all $n \geq 1$, φ_n is given by the following rule on generators

$$\varphi_n(\rho(a_1)...\rho(a_n)) = \rho(\varphi(a_1))...\rho(\varphi(a_n)).$$

Moreover if φ is surjective then φ_* is also surjective, and if $\psi : L \to M$ is another morphism then

a) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ *and* $Id_* = Id$.

b) *For all* $a \in \dot{F}$ *the following diagram commute:*

$$\begin{array}{ccc} K_nF & \xrightarrow{\omega_n^a} & K_{n+1}F \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\ K_nL & \xrightarrow[\omega_n^{\varphi(a)}]{} & K_{n+1}L \end{array}$$

c) *For all* $n \geq 1$ *the following diagram commute:*

$$\begin{array}{ccc} K_nF & \xrightarrow{\omega_n^{-1}} & K_{n+1}F \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\ K_nL & \xrightarrow[\omega_n^{-1}]{} & K_{n+1}L \end{array}$$

In the hyperfield context we also have the reduced K-theory graded ring $k_*F = (k_0F, k_1F, ..., k_nF, ...)$, which is defined by the rule $k_nF := K_nF/2K_nF$ for all $n \geq 0$. Of course for all $n \geq 0$ we have an epimorphism $q : K_nF \to k_nF$ simply denoted by $q(a) := [a]$, $a \in K_nF$. It is immediate that k_nF is additively generated by $\{[\rho(a_1)]..[\rho(a_n)] : a_1, ..., a_n \in \dot{F}\}$. We simply denote such a generator by $\tilde{\rho}(a_1)...\tilde{\rho}(a_n)$ or even $\rho(a_1)...\rho(a_n)$ whenever the context allows it.

We also have some basic properties of the reduced K-theory, which proof is just a translation of 2.1 of [13]:

Lemma 1.22 (Adapted from 2.1 [13])**.** *Let* F *be a hyperbolic hyperfield,* $x, y, a_1, ..., a_n \in \dot{F}$ *and* σ *be a permutation on* n *elements.*

a) *In* k_2F, $\rho(a)^2 = \rho(a)\rho(-1)$. *Hence in* k_mF, $\rho(a)^m = \rho(a)\rho(-1)^{m-1}$, $m \geq 2$;

b) *In* k_2F, $\rho(a)\rho(b) = \rho(b)\rho(a)$;

c) *In* k_nF, $\rho(a_1)\rho(a_2)...\rho(a_n) = \rho(a_{\sigma 1})\rho(a_{\sigma 2})...\rho(a_{\sigma n})$;

d) *For* $n \geq 1$ *and* $\xi \in k_nF$, $\xi^2 = \rho(-1)^n\xi$;

e) *If* F *is a real reduced hyperfield, then* $x \in 1+y$ *and* $\rho(y)\rho(a_1)...\rho(a_n) = 0$ *implies*

$$\rho(x)\rho(a_1)\rho(a_2)...\rho(a_n) = 0.$$

Moreover the results in Proposition 1.21 continue to hold if we took $\varphi_* = \{\varphi_n : n \geq 0\} : k_*F \to k_*L$.

Proposition 1.23 (3.5 of [6]). *Let* F *be a (hyperbolic) hyperfield and* $T \subseteq F$ *be a multiplicative subset such that* $F^2 \subseteq T$. *Then, for each* $n \geq 1$

$$K_n(F/_mT^*) \cong k_n(F/_mT^*).$$

Theorem 1.24 (3.6 of [6]). *Let* F *be a hyperbolic hyperfield and* $T \subseteq F$ *be a multiplicative subset such that* $F^2 \subseteq T$. *Then there is an induced surjective morphism*

$$k(F) \to k(F/_mT^*).$$

Moreover, if $T = F^2$, *then*

$$k(F) \overset{\cong}{\to} k(F/_m\dot{F}^2).$$

2 Inductive graded rings: An abstract approach

After the three K-theories defined in the above sections, it is desirable (or, at least, suggestive) the rise of an abstract environment that encapsule all them, and of course, provide an axiomatic approach to guide new extensions of the concept of K-theory in the context of the algebraic and abstract theories of quadratic forms. The inductive graded rings fits this purpose. Here we will present two versions. The first one is:

Definition 2.1 (Inductive Graded Rings First Version (adapted from Definition 9.7 of [10])). An **inductive graded ring** (or **Igr** for short) is a structure $R = ((R_n)_{n\geq 0}, (h_n)_{n\geq 0}, *_{nm})$ where

i) $R_0 \cong \mathbb{F}_2$.

ii) R_n has a group structure $(R_n, +, 0, \top_n)$ of exponent 2 with a distinguished element $\top_n$.

iii) $h_n : R_n \to R_{n+1}$ is a group homomorphism such that $h_n(\top_n) = \top_{n+1}$.

iv) For all $n \geq 1$, $h_n = *_{1n}(\top_1, \_)$.

v) The binary operations $*_{nm} : R_n \times R_m \to R_{n+m}$, $n, m \in \mathbb{N}$ induces a commutative ring structure on the abelian group

$$R = \bigoplus_{n \geq 0} R_n$$

with $1 = \top_0$.

vi) For $0 \leq s \leq t$ define

$$h_s^t = \begin{cases} Id_{R_s} \text{ if } s = t \\ h_{t-1} \circ ... \circ h_{s+1} \circ h_s \text{ if } s < t. \end{cases}$$

Then if $p \geq n$ and $q \geq m$, for all $x \in R_n$ and $y \in R_m$,

$$h_n^p(x) * h_m^q(y) = h_{n+m}^{p+q}(x * y).$$

A **morphism** between Igr's R and S is a pair $f = (f, (f_n)_{n\geq 0})$ where $f_n : R_n \to S_n$ is a morphism of pointed groups and

$$f = \bigoplus_{n \geq 0} f_n : R \to S$$

is a morphism of commutative rings with unity. The category of inductive graded rings (in first version) and their morphisms will be denoted by Igr.

A first consequence of these definitions is that: if

$$f : ((R_n)_{n\geq 0}, (h_n)_{n\geq 0}, *_{nm}) \to ((S_n)_{n\geq 0}, (l_n)_{n\geq 0}, *_{nm})$$

is a morphism of Igr's then $f_{n+1} \circ h_n = l_n \circ f_n$.

$$\begin{array}{ccccccccccccc}
R_0 & \xrightarrow{h_0} & R_1 & \xrightarrow{h_1} & R_2 & \xrightarrow{h_2} & ... & \xrightarrow{h_{n-1}} & R_n & \xrightarrow{h_n} & R_{n+1} & \xrightarrow{h_{n+1}} & ... \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\
S_0 & \xrightarrow{l_0} & S_1 & \xrightarrow{l_1} & S_2 & \xrightarrow{l_2} & ... & \xrightarrow{l_{n-1}} & S_n & \xrightarrow{l_n} & S_{n+1} & \xrightarrow{l_{n+1}} & ...
\end{array}$$

In fact, since $R_0 \cong \mathbb{F}_2 \cong S_0$ and $f(1) = 1$, then $f_0 : R_0 \to S_0$ is the unique abelian group isomorphism and $f_1 \circ h_0 = l_0 \circ f_0$. If $n \geq 1$, for all $a_n \in R_n$ holds

$$\begin{aligned} f_{n+1} \circ h_n(a_n) &= f_{n+1} \circ (*_{1n}(\top_1, a_n)) = f_1(\top_1) *_{1n} f_n(a_n) \\ &= \top_1 *_{1n} f_n(a_n) = l_n(f_n(a_n)) = l_n \circ f_n(a_n). \end{aligned}$$

Example 2.2.

a) Let F be a field of characteristic not 2. The main actors here are WF, the Witt ring of F and IF, the fundamental ideal of WF. Is well know that I^nF, the n-th power of IF is additively generated by n-fold Pfister forms over F. Now, let $R_0 = WF/IF \cong \mathbb{F}_2$ and $R_n = I^nF/I^{n+1}F$. Finally, let $h_n = \_ \otimes \langle 1, 1\rangle$. With these prescriptions we have an inductive graded ring R associated to F.

b) The previous example still works if we change the Witt ring of a field F for the Witt ring of a (formally real) special group G.

Concerning k-theories, we register the followings:

Theorem 2.3.

a) *Let F be a field. Then k^{mil}_*F (the reduced Milnor K-theory) is an inductive graded ring.*

b) *Let G be a special group. Then k^{dm}_*G (the Dickmann-Miraglia K-theory of G) is an inductive graded ring.*

c) *Let F be a hyperbolic hyperfield. Then k^{mult}_*F (our reduced K-theory) is an inductive graded ring.*

Theorem 2.4 (Theorem 2.5 in [12])**.** *Let F be a field. The functor $G : Field_2 \to SG$ provides a functor $k'^{dm}_* : Field_2 \to Igr$ (the special group K-theory functor) given on the objects by $k'^{dm}_*(F) := k^{dm}_*(G(F))$ and on the morphisms $f : F \to K$ by $k'^{dm}_*(f) := G(f)_*$ (in the sense of Lemma 3.3 of [13]). Moreover, this functor commutes with the functors G and k, i.e, for all $F \in Field$, $k'^{dm}_*(F) = k^{dm}_*(G(F)) \cong k^{mil}_*(F)$.*

Theorem 2.5. *Let G be a special group. The equivalence of categories $M : SG \to SMF$ induces a functor $k'^{mult}_* : SG \to Igr$ given on the objects by*

$k'^{mult}_*(G) := k^{mult}_*(M(G))$ and on the morphisms $f : G \to H$ by $k'^{mult}_*(f) := k^{mult}_*(M(f))$. Moreover, this functor commutes with M and k^{dm}, i.e, for all $G \in SG$, $k'^{mult}_*(G) = k^{mult}_*(M(G)) \cong k^{dm}_*(G)$.

Theorem 2.6 (Interchanging K-theories Formulas)**.** *Let $F \in Field_2$. Then*

$$k^{mil}(F) \cong k^{dm}(G(F)) \cong k^{mult}(M(G(F))).$$

If F is formally real and T is a preordering of F, then

$$k^{dm}(G_T(F)) \cong k^{mult}(M(G_T(F))).$$

Moreover, since $M(G(F)) \cong F/_m\dot{F}^2$ and $M(G_T(F)) \cong F/_mT^*$, we get

$$k^{mil}(F) \cong k^{dm}(G(F)) \cong k^{mult}(F/_m\dot{F}^2) \text{ and}$$
$$k^{dm}(G_T(F)) \cong k^{mult}(F/_mT^*).$$

There is an alternative Definition for Igr with a first-order theoretic flavor. It is a technical framework that allows achieving some model-theoretic results.

Before define it, we need some preparation. First of all, we set up the language. Here, we will work with the poli-sorted framework (as established in Chapter 5 of [1]), which means the following:

Let S be a set (of sorts). For each $s \in S$ assume a countable set Var_s of **variables of sort** s (with the convention if $s \neq t$ then $\mathrm{Var}_s \cap \mathrm{Var}_t = \emptyset$). For each sort $s \in S$ an equality symbol $=_s$ (or just $=$); the connectives $\neg, \wedge, \vee, \to$ (not, and, or, implies); the quantifiers $\forall, \exists$ (for all, there exists).

A **finitary S-sorted language (or signature)** is a set $\mathcal{L} = (\mathcal{C}, \mathcal{F}, \mathcal{R})$ where:

i) $\mathcal{C}$ is the set of constant symbols. For each $c \in \mathcal{C}$ we assign an element $s \in S$, the sort of c;

ii) $\mathcal{F}$ is the set of functional symbols. For each $f \in \mathcal{F}$ we assign elements $s, s_1, ..., s_n \in S$, we say that f has arity $s_1 \times ... \times s_n$ and s is the value sort of f; and we use the notation $f : s_1 \times ... \times s_n \to s$.

iii) $\mathcal{R}$ is the set of relation symbols. $c \in \mathcal{C}$ we assign elements $s_1, ..., s_n \in S$, the arity of R; and we say that R has arity $s_1 \times ... \times s_n$.

A **$\mathcal{L}$-structure** $\mathcal{M}$ is, in this sense, prescribed by the following data:

i) The **domain or universe** of $\mathcal{M}$, which is an S-sorted set $|\mathcal{M}| := (M_s)_{s\in S}$.

ii) For each constant symbol $c \in \mathcal{C}$ of arity s, an element $c^{\mathcal{M}} \in M_s$.

iii) For each functional symbol $f \in \mathcal{F}$, $f : s_1 \times ... \times s_n \to s$, a function $f^{\mathcal{M}} : M_{s_1} \times ... \times M_{s_n} \to M_s$.

iv) For each relation symbol $R \in \mathcal{R}$ of arity $s_1 \times ... \times s_n$ a relation, i.e. a subset $R^{\mathcal{M}} \subseteq M_{s_1} \times ... \times M_{s_n}$.

A **$\mathcal{L}$-morphism** $\varphi : \mathcal{M} \to \mathcal{N}$ is a sequence of functions $\varphi = (\varphi_s)_s : |\mathcal{M}| \to |\mathcal{N}|$ such that

i) for all $c \in \mathcal{C}$ of arity s, $\varphi_s(c^{\mathcal{M}}) = c^{\mathcal{N}}$;

ii) for all $f : s_1 \times ... \times s_n \to s$, if $(a_1, ..., a_n) \in: M_{s_1} \times ... \times M_{s_n}$, then $\varphi_s(f^{\mathcal{M}}(a_1, ..., a_n)) = f^{\mathcal{N}}(\varphi_{s_1}(a_1), ..., \varphi_{s_n}(a_n))$;

iii) for all R of arity $s_1 \times ... \times s_n$, if $(a_1, ..., a_n) \in R^{\mathcal{M}}$ then $(\varphi(a_1), ..., \varphi(a_n)) \in R^{\mathcal{N}}$.

The category of $\mathcal{L}$-structures and $\mathcal{L}$-morphism in the poli-sorted language $\mathcal{L}$ will be denoted by $\mathrm{Str}_s(\mathcal{L})$.

The terms, formulas, occurrence and free variables definitions for the poli-sorted case are similar to the usual (single-sorted) first order ones. For example, the terms are defined as follows:

i) variables $x \in \mathrm{Var}_s$ and constants $c \in C_s$ are terms of value sort s;

ii) if $\vec{s} = \langle s_1, ..., s_n, s\rangle \in S^{n+1}$, $f \in \mathcal{F}$ with $f : s_1 \times ... \times s_n \to s$, and $\tau_1, ..., \tau_n$ are terms of value sorts $s_1, ..., s_n$ respectively, then $f(\tau_1, ..., \tau_n)$ is a term of sort s.

As usual, we may write $\tau : s$ to indicate that the term τ has value sort s.

For the formulas:

i) if $x, y \in \mathrm{Var}_s$ then $x = y$ is a formula; if $\vec{s} = \langle s_1, ..., s_n\rangle \in S^n$, $R \in \mathcal{R}$ of arity $s_1 \times ... \times s_n$ and $\tau_1, ..., \tau_n$ are terms of sort $s_1, ..., s_n$ respectively, then $R(\tau_1, ..., \tau_n)$ is a formula. These are the **atomic formulas**.

ii) If φ_1, φ_2 are formulas, then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ and $\varphi_1 \to \varphi_2$ are formulas.

iii) If φ is a formula and $x \in \mathrm{Var}_s$ $(s \in S)$, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

In our particular case, the set of sorts will be just $\mathbb{N}$. Then, for each $n, m \geq 0$, we set the following data:

i) $0_n, \top_n$ are constant symbols of arity n. We use $0_0 = 0$ and $\top_0 = 1$.

ii) $+_n : n \times n \to n$ is a binary operation symbol.

iii) $h_n : n \to (n+1)$ and $*_{n,m} : n \times m \to (n+m)$ are functional symbols.

The **(first order) language of inductive graded rings** $\mathcal{L}_{igr}$ is just the following language (in the poli-sorted sense):

$$\mathcal{L}_{igr} := \{0_n, \top_n, +_n, h_n, *_{nm} : n, m \geq 0\}.$$

The **(first order) theory of inductive graded rings** $T(\mathcal{L}_{igr})$ is the $\mathcal{L}_{igr}$-theory axiomatized by the following $\mathcal{L}_{igr}$-sentences, where we use $\cdot_n : 0 \times n \to n$ as an abbreviation for $*_{0n}$:

i) For $n \geq 0$, sentences saying that "$+_n, 0_n, \top_n$ induces a pointed left $\mathbb{F}_2$-module":

$$\begin{aligned}
&\forall x : n \forall y : n \forall z : n((x +_n y) +_n z = x +_n (y +_n z)) \\
&\forall x : n(x +_n 0_n = x) \\
&\forall x : n \forall y : n(x +_n y = y +_n x) \\
&\forall x : n(x +_n x = 0_n) \\
&\forall x : n(1 \cdot_n x = x) \\
&\forall x : n \forall y : n \forall a : 0(a \cdot_n (x +_n y) = a \cdot_n x +_n a \cdot_n y) \\
&\forall x : n \forall a : 0 \forall b : 0((a +_0 b) \cdot_n x = a \cdot_n x +_n b \cdot_n x)
\end{aligned}$$

ii) For $n \geq 0$, sentences saying that "h_n is a pointed $\mathbb{F}_2$-morphism":

$$\begin{aligned}
&\forall x : n \forall y : n(h_n(x +_n y) = h_n(x) +_{n+1} h_n(y)) \\
&\forall x : n \forall a : 0(h_n(a \cdot_n x) = a \cdot_n h_n(x)) \\
&h_n(\top_n) = \top_{n+1}
\end{aligned}$$

iii) Sentences saying that "$R_0 \cong \mathbb{F}_2$":

$$\begin{aligned}
&0_0 \neq \top_0 \\
&\forall x : 0(x = 0_0 \vee x = \top_0)
\end{aligned}$$

iv) Using the abbreviation $*_{n,m}(x,y) = x *_{n,m} y$, we write for $n,m \geq 0$ sentences saying that "$*_{n,m}$ is a biadditive function compatible with h_n":

$$\forall\, x : n \forall\, y : n \forall\, z : m(((x +_n y) *_{nm} z) = (x *_{mn} z +_{n+m} y *_{nm} z))$$
$$\forall\, x : n \forall\, y : m \forall\, z : m((x *_{mn} (y +_m z)) = (x *_{nm} y +_{n+m} x *_{nm} z))$$
$$\forall\, x : n \forall\, y : m(h_{n+m}(x *_{nm} y) = h_n(x) *_{nm} h_m(y))$$

v) Sentences describing "the induced ring with product induced by $*_{n,m}$, $n,m \geq 0$":

$$\forall\, x : n \forall\, y : m \forall\, z : p((x *_{n,m} y) *_{(m+n),p} z = x *_{n,(m+p)} (y *_{m,p} z))$$
$$\forall\, x : n \forall\, y : m(x *_{n,m} y = y *_{m,n} x)$$

vi) For $n \geq 1$, sentences saying that "$h_n = \top_1 *_{1n} \_$":

$$\forall\, x : n(h_n(x) = \top_1 *_{1n} x)$$

Now we are in a position to define another version of Igr:

Definition 2.7 (Inductive Graded Rings Second Version)**.** An **inductive graded ring** (or **(Igr)** for short) is a model for $T(\mathcal{L}_{igr})$, or in other words, a $\mathcal{L}_{igr}$-structure $\mathcal{R}$ such that $\mathcal{R} \models_{\mathcal{L}_{igr}} T(\mathcal{L}_{igr})$. We denote by Igr_2 the category of $\mathcal{L}_{igr}$-structures and $\mathcal{L}_{igr}$-morphisms.

Again, after some straightforward calculations we can check:

Theorem 2.8. *The categories Igr, Igr$_2$ are equivalent.*

Remark 2.9. Following a well-known procedure, it is possible to correspond theories on poly-sorted first-order languages with theories on traditional (single-sorted) first-order languages in such a way that the corresponding categories of models are equivalent. This allows a useful interchanging between model-theoretic results, in both directions. In particular, in the following, we will freely interchange the two notions of Igr indicated in this Section.

Theorem 2.6 gives a hint that the category of Igr is a good abstract environment for studying questions of "quadratic flavour". So a better understanding of categories of Igr's and its applications to quadratic forms theories is the main purpose of the next Sections in this work.

3 The first properties of Igr

In this section we discuss the theory of Igr's. Constructions like products, limits, colimits, ideals, quotients, kernel and image are not new and are obtained in a very straightforward manner (basically, putting those structures available for rings in a "coordinatewise" fashion), then in order to gain speed, we will present these facts leaving more detailed proofs to the reader.

Denote: $\mathrm{PMod}_{\mathbb{F}_2}$ the category of pointed $\mathbb{F}_2$-modules, Ring the category of commutative rings with unity and morphism that preserves these units and Ring_2 the full subcategory of the associative $\mathbb{F}_2$-algebras. We have a functorial correspondence $\mathrm{Ring}_2 \to \mathrm{Igr}$, given by the following diagram:

$$\begin{array}{ccccccccccccc}
A & & \mathbb{F}_2 & \xrightarrow{!} & A & \xrightarrow{id} & A & \xrightarrow{id} & \ldots & \xrightarrow{id} & A & \xrightarrow{id} & \ldots \\
{\scriptstyle f}\downarrow & \mapsto & {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & & & {\scriptstyle f}\downarrow & & \\
B & & \mathbb{F}_2 & \xrightarrow{!} & B & \xrightarrow{id} & B & \xrightarrow{id} & \ldots & \xrightarrow{id} & B & \xrightarrow{id} & \ldots
\end{array}$$

Here A is a $\mathrm{PMod}_{\mathbb{F}_2}$ where $\top_n = 1, n \geq 1$ and $\top_0 = 1 \in \mathbb{F}_2$.

Definition 3.1. The **trivial graded ring functor** $\mathbb{T} : \mathrm{Ring}_2 \to \mathrm{Igr}$ is the functor defined for $f : A \to B$ by $T(A)_0 := \mathbb{F}_2$, $T(f)_0 := id_{\mathbb{F}_2}$ and for all $n \geq 1$ we set $T(A)_n = A$ and $T(f)_n := f$.

Definition 3.2. We define the **associated $\mathbb{F}_2$-algebra functor** $\mathbb{A} : \mathrm{Igr} \to \mathrm{Ring}_2$ is the functor defined for $f : R \to S$ by

$$\mathbb{A}(R) := R_{\mathbb{A}} = \varinjlim_{n\geq 0} R_n \text{ and } \mathbb{A}(f) = f_{\mathbb{A}} := \varinjlim_{n\geq 0} f_n.$$

More explicitly, $\mathbb{A}(R) = (R_{\mathbb{A}}, 0, 1, +_{\mathbb{A}}, \cdot)$, where

i) $R_{\mathbb{A}} = \varinjlim_{n\geq 0} R_n$,

ii) $0 = [(0,0)]$ and $1 = [(1,0)]$,

iii) given $[(a_n, n)], [(b_m, m)] \in R_{\mathbb{A}}$ and setting $d \geq m, n$ we have

$$[(a_n, n)] + [(b_m, m)] = [(h_{nd}(a_n) + h_{md}(b_m), d)]$$

iv) given $[(a_n, n)], [(b_m, m)] \in R_{\mathbb{A}}$, we have

$$[(a_n, n)] \cdot [(b_m, m)] = [(a_n *_{nm} b_m, n + m)].$$

Proposition 3.3.

i) *The functor* $\mathbb{A}$ *is the left adjunct to* $\mathbb{T}$*.*

ii) *The functor* $\mathbb{T}$ *is full and faithful.*

iii) *The composite functor* $\mathbb{A} \circ \mathbb{T}$ *is naturally isomorphic to the functor* 1_{Ring_2}.

Proof. Let $R \in Igr$. We have

$$\mathbb{T}(\mathbb{A}(R)) = \mathbb{T}\left(\varinjlim_{m\geq 0} R_m\right).$$

In other words, for all $n \geq 1$

$$\mathbb{T}\left(\varinjlim_{m\geq 0} R_m\right)_n := \varinjlim_{m\geq 0} R_m.$$

Then, for all $n \geq 1$ we have a canonical embedding

$$\eta(R)_n : R_n \to \varinjlim_{m\geq 0} R_m = \mathbb{T}\left(\varinjlim_{m\geq 0} R_m\right)_n,$$

providing a morphism

$$\eta(R) : R \to \varinjlim_{m\geq 0} R_m = \mathbb{T}\left(\varinjlim_{m\geq 0} R_m\right).$$

For $f \in \mathrm{Igr}(R, S)$, taking $n \geq 1$ we have a commutative diagram

$$\begin{array}{ccc} R_n & \xrightarrow{\quad f_n \quad} & S_n \\ \Big\downarrow{\scriptstyle \eta(R)_n} & & \Big\downarrow{\scriptstyle \eta(S)_n} \\ \varinjlim_{m\geq 0} R_m & \xrightarrow[\varinjlim_{m\geq 0} f_m]{} & \varinjlim_{m\geq 0} S_m \end{array}$$

with the convention that $\eta(R)_0 = id_{\mathbb{F}_2}$. Then it is legitimate the definition of a natural transformation $\eta : 1_{\text{Igr}} \to \mathbb{T} \circ \mathbb{A}$ given by the rule $R \mapsto \eta(R)$.

Now let $A \in Ring_2$ and $g \in Ring_2(R, \mathbb{T}(A))$. Then for each $n \geq 0$, there is a morphism $g_n : R_n \to \mathbb{T}(A)_n = A$ and by the universal property of inductive limit we get a morphism

$$\varinjlim_{m\geq 0} g_n : \varinjlim_{m\geq 0} R_m \to A.$$

In fact, $\varinjlim_{m\geq 0} g_n = \mathbb{A}(g)$.

Now, using the fact that $\eta(R)_n$ is the morphism induced by the inductive limit we have for all $n \geq 0$ the following commutative diagram

$$\begin{array}{ccc} R_n & \xrightarrow{\eta(B)_n} & \varinjlim_{m\geq 0} R_m \\ & \searrow^{g_n} & \downarrow{\scriptstyle \varinjlim_{m\geq 0} g_n} \\ & & A \end{array}$$

In other words, $\eta(B)_n$ is the canonical morphism commuting the diagram

$$\begin{array}{ccc} R_n & \xrightarrow{\eta(B)_n} & \mathbb{T}(\mathbb{A}(R)) \\ & \searrow^{g_n} & \downarrow{\scriptstyle \mathbb{T}(\mathbb{A}(g_n))} \\ & & \mathbb{T}(A) \end{array}$$

and hence, $\mathbb{A}$ is the left adjoint of $\mathbb{T}$, proving item (i). By the very Definition of $\mathbb{A}$ and $\mathbb{T}$ we get item (iii), and using Proposition 7.13 we get item (ii). $\square$

Using Proposition 7.13 (and its dual version) we get the following Corollary.

Corollary 3.4.

i) $\mathbb{T} : Ring_2 \to Igr$ *preserves all projective limits.*

ii) *If I is such that Igr is I-inductively complete then for $\{A_i\}_{i\in I}$ in Igr we have*

$$\varinjlim_{i\in I} A_i \cong \mathbb{A}\left(\varinjlim_{i\in I} \mathbb{T}(A_i)\right).$$

iii) $\mathbb{F}_2 \in Ring_2$ *is the initial object in* $Ring_2$.

iv) $0 \in Ring_2$ *is the terminal object in* $Ring_2$.

v) $\mathbb{T}(\mathbb{F}_2)$ *is the initial object in Igr.*

vi) $\mathbb{T}(0)$ *is the terminal object in Igr.*

Now we discuss (essentially) the limits and colimits in Igr. Fix a non-empty set I and let $\{(R_i, \top_i, h_i)\}_{i\in I}$ be a family of Igr's. We start with the construction of the Igr-product

$$R = \prod_{i\in I} R_i.$$

For this, we define $R_0 \cong \mathbb{F}_2$ and for all $n \geq 1$, we define

$$R_n := \prod_{i\in I}(R_i)_n \text{ and } \top_n := \prod_{i\in I}(\top_i)_n.$$

In the sequel, we define $h_0 : \mathbb{F}_2 \to R_1$ as the only possible morphism and for $n \geq 1$, we define $h_n : R_n \to R_{n+1}$ by

$$h_n := \prod_{i\in I}(h_i)_n.$$

Definition 3.5.

i) The **space of orderings**, X_R, of the Igr R, is the set of Igr-morphisms $Igr(R, \mathbb{T}(\mathbb{F}_2)$. By the Proposition 3.3.(i), we have a natural bijection $Igr(R, \mathbb{T}(\mathbb{F}_2) \cong Ring_2(\mathbb{A}(R), \mathbb{F}_2)$, thus considering the discrete topologies on the $\mathbb{F}_2$-algebras $\mathbb{A}(R), \mathbb{F}_2$ and transporting the boolean topology in $Ring_2(\mathbb{A}(R), \mathbb{F}_2)$, we obtain a boolean topology on the space of orderings $X_R = Igr(R, \mathbb{T}(\mathbb{F}_2))$.

ii) The **boolean hull**, $B(R)$, of the Igr R, is the boolean ring canonically associated to the space of orderings of R by Stone duality: $B(R) := \mathcal{C}(X_R, \mathbb{F}_2)$.

iii) A Igr R is called **formally real** if $X_R \neq \emptyset$ (or, equivalently, if $B(R) \neq 0$).

Proposition 3.6. *Let I be a non-empty set and $\{(R_i, h_i)\}_{i\in I}$ be a family of Igr's. Then*

$$R = \prod_{i\in I} R_i$$

with the above rules is an Igr. Moreover it is the product in the category Igr.

Proof. Using Definition 2.1 is straightforward to verify that $(R, \top_n, h_n)$ is an Igr. Note that for each $i \in I$, we have an epimorphism $\pi_i : R \to R_i$ given by the following rules: for each $n \geq 0$ and each $(x_i)_{i\in I} \in R_n$, we define

$$(\pi_i)_n((x_i)_{i\in I}) := x_i.$$

Now, let $(Q, \{q_i\}_{i\in I})$ be another pair with Q being an Igr and $q_i : Q \to R_i$ being a morphism for each $i \in I$. Given $i \in I$ and $n \geq 0$, since $R_n := \prod_{i\in I}(R_i)_n$ is the product in the category of pointed $\mathbb{F}_2$-modules, we have an unique morphism $(q)_n : (Q)_n \to (R)_n$ such that $(\pi_i)_n \circ (q)_n = (q_i)_n$. Set $q_n := ((q_i)_{i\in I})_n$. By construction, q is the unique Igr-morphism such that $\pi_i \circ q = q_i$, completing the proof that R is in fact the product in the category Igr. □

Proposition 3.7.

i) *Let R be an Igr and let $X \subseteq R = \bigoplus_{n\in\mathbb{N}} R_n$. Then there exists the* **inductive graded subring generated by** X *(notation : $[X] \overset{i_X}{\hookrightarrow} R$): this is the least inductive graded subring of R such that $\forall n \in \mathbb{N}$, $X \cap R_n \subseteq [X]_n$.*

ii) *Let $\mathcal{I}$ be a small category and $\mathcal{R} : \mathcal{R} \to Igr$ be a diagram. Then there exists $\varprojlim_{i\in\mathcal{I}} \mathcal{R}_i$ in the category Igr.*

Proof.

i) It is enough consider S_X , the $\mathbb{F}_2$-subalgebra of $(\bigoplus_{n\in\mathbb{N}} R_n, *)$ generated by $X \cup \{\top_1\} \subseteq \bigoplus_{n\in\mathbb{N}} R_n$ and set $\forall n \in \mathbb{N}$, $[X]_n := s_x \cap R_n$.

ii) Just define $\varprojlim_{i\in\mathcal{I}} \mathcal{R}_i$ as the inductive graded subring of $\prod_{i\in obj(\mathcal{I})} \mathcal{R}_i$ generated by $X_D = \bigoplus_{n\in\mathbb{N}} X_n$ and $X_n := \varprojlim_{i\in\mathcal{I}}(\mathcal{R}_i)_n$ (projective limit of pointed $\mathbb{F}_2$-algebras).

□

Now we construct the Igr-tensor product of a finite family of Igr's, $\{R_i : i \in I\}$

$$R = \bigotimes_{i\in I} R_i.$$

For this, we define $R_0 \cong \mathbb{F}_2$ and for all $n \geq 1$, we define

$$R_n := \bigotimes_{i\in I} (R_i)_n,$$

$$(\otimes_{i\in I} a_i) *_{n,k} (\otimes_{i\in I} b_i) := \otimes_{i\in I}(a_i *^i_{n,k} b_i)$$

$$\text{and } \top_n := \otimes_{i\in I}(\top_i)_n.$$

In particular, if $I = \emptyset$, then $R_n = \{0\}, n \geq 1$. In the sequel, we define $h_0 : \mathbb{F}_2 \to R_1$ as the only possible morphism and for $n \geq 1$, we define $h_n : R_n \to R_{n+1}$ by

$$h_n := \bigotimes_{i\in I} (h_i)_n.$$

In other words, for a generator $\bigotimes_{i\in I} x_i \in R_n$, we have

$$h_n(\otimes_{i\in I} x_i) := \bigotimes_{i\in I} (h_i)_n(x_i).$$

Proposition 3.8. *Let I be a finite set and $\{(R_i, h_i)\}_{i\in I}$ be a family of Igr's. Then*

$$R = \bigotimes_{i\in I} R_i$$

with the above rules is an Igr. Moreover it is the coproduct in the category Igr.

Now suppose that $(I, \leq)$ is an upward directed poset and that $((R_i, h_i), \varphi_{ij})_{i\leq j\in I}$ is an inductive system of Igr's. We define the inductive limit

$$R = \varinjlim_{i\in I} R_i$$

by the following: for all $n \geq 0$ define

$$R_n := \varinjlim_{i\in I} (R_i)_n.$$

Note that

$$R_0 := \varinjlim_{i\in I} (R_i)_0 \cong \varinjlim_{i\in I} \mathbb{F}_2 \cong \mathbb{F}_2.$$

In the sequel, for $n \geq 1$ we define $h_n : R_n \to R_{n+1}$ by

$$h_n := \varinjlim_{i\in I} (h_i)_n.$$

Proposition 3.9. *Let $(I, \leq)$ is an upward directed poset and $((R_i, h_i), \varphi_{ij})_{i\in I}$ be a directed family of Igr's. Then*

$$R = \varinjlim_{i\in I} R_i$$

with the above rules is an Igr. Moreover, it is the inductive limit in the category Igr.

Proposition 3.10. *The general coproduct (general tensor product) of a family $\{R_i : i \in I\}$ in the category Igr is given by the combination of constructions:*

$$\bigotimes_{i\in I} R_i := \varinjlim_{I'\in P_{fin}(I)} \bigotimes_{i\in I'} R_i.$$

After discussing directed inductive colimits and coproducts, we will deal with ideals, quotients, and coequalizers.

Definition 3.11. Given $R \in$ Igr and $(J_n)_{n\geq 0}$ where $J_n \subseteq R_n$ for all $n \geq 0$. We say that J is a **graded ideal** of R where

$$J := \bigoplus_{n\geq 0} J_n \subseteq \bigoplus_{n\geq 0} R_n$$

is an ideal of $(R, *)$.

In particular, for all $n \geq 0$, $J_n \subseteq R_n$ is a graded $\mathbb{F}_2$-submodule of $(R_n, +_n, 0_n)$. For each $X \subseteq R$, there exists the ideal generated by X, denoted by $\langle X \rangle$. It is the smaller graded ideal of R such that for all $n \geq 0$, $(X \cap R_n) \subseteq [X]_n$. For this, just consider $\langle X \rangle$, the ideal of $(R, *)$ generated by $X \subseteq R$ and define $\langle X \rangle_n := \langle X \rangle \cap R_n$.

Definition 3.12. Let R, S be Igr's and $f : R \to S$ be a morphism. We define the **kernel** of f, notation $\mathrm{Ker}(f)$ by

$$\mathrm{Ker}(f)_n := \{x \in R_n : f_n(x) = 0\}$$

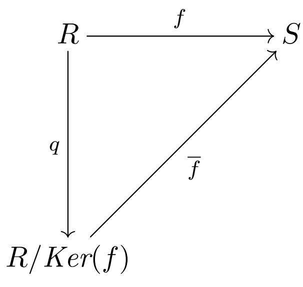
and **image** of f, notation $\mathrm{Im}(f)$ by

$$\mathrm{Im}(f)_n := \{f_n(x) \in S_n : x \in R_n\}.$$

Of course, $\mathrm{Ker}(f) \subseteq R$ is an ideal and $\mathrm{Im}(f) \subseteq S$ is an Igr.

Given $R \in \mathrm{Igr}$ and $J = (J_n)_{n \geq 0}$ a graded ideal of R, we define $R/J \in \mathrm{Igr}$, the **quotient inductive graded ring of** R **by** J: for all $n \geq 0$, $(R/J)_n := R_n/J_n$, where the distinguished element is $\top_n +_n J_n$. We have a canonical projection $q_J : R \to R/J$, "coordinatewise surjective" and therefore, an Igr-epimorphism.

Proposition 3.13 (Homomorphism Theorem)**.** *Let R, S be Igr's and $f : R \to S$ be a morphism. Then there exist an unique monomorphism $\overline{f} : R/Ker(f) \to S$ commuting the following diagram:*



where q is the canonical projection. In particular $R/Ker(f) \cong Im(f)$.

Proposition 3.14. *Let $R \overset{f}{\underset{g}{\rightrightarrows}} S$ be Igr-morphisms and consider $q_J : S \to S/J$ the quotient morphism where $J := \langle X \rangle$ is the graded ideal generated by $X_n := \{f_n(a) - g_n(a) : a \in R_n\}, n \in \mathbb{N}$. Then q_J is the coequalizer of f, g.*

Proposition 3.15. *Given $R, S \in Igr$ and $f \in Igr(R, S)$.*

i) *f is a Igr-monomorphism whenever for all $n \geq 0$ $f_n : R_n \to S_n$ is a monomorphism of pointed $\mathbb{F}_2$-modules iff for all $n \geq 0$, $f_n : R_n \to S_n$ is an injective homomorphism of pointed $\mathbb{F}_2$-modules.*

ii) *f is a Igr-epimorphism whenever for all $n \geq 0$ $f_n : R_n \to S_n$ is a epimorphism of pointed $\mathbb{F}_2$-modules iff for all $n \geq 0$, $f_n : R_n \to S_n$ is a surjective homomorphism of pointed $\mathbb{F}_2$-modules.*

iii) *f is a Igr-isomorphism iff for all $n \geq 0$ $f_n : R_n \to S_n$ is a isomorphism of pointed $\mathbb{F}_2$-modules iff for all $n \geq 0$, $f_n : R_n \to S_n$ is a bijective homomorphism of pointed $\mathbb{F}_2$-modules.*

Definition 3.16. We denote Igr_{fin} the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_{fin}) = \{R \in \mathrm{Obj}(\mathrm{Igr}) : |R_n| < \omega \text{ for all } n \geq 1\}.$$

Remark 3.17. Of course,

$$\left\{ R \in \mathrm{Obj}(\mathrm{Igr}) : \left| \bigoplus_{n \geq 1} R_n \right| < \omega \right\} \neq \mathrm{Obj}(\mathrm{Igr}_{fin}),$$

for example, in 2.6(a), if F is a Euclidian field (for instance, any real closed field), then $\bigoplus_{n \in \mathbb{N}} I^n F/I^{n+1}F \cong \mathbb{F}_2[x]$, thus the graded Witt ring of F (see definition 5.9) $W_*(F) \in \mathrm{Obj}(\mathrm{Igr}_{fin})$ but $\mathbb{F}_2[x]$ is not finite.

4 Relevant subcategories of Igr

The aim of this section is to define subcategories of Igr that mimetize the following two central aspects of K-theories:

1. The K-theory graded ring is "generated" by K_1;
2. The K-theory graded ring is defined by some convenient quotient of a graded tensor algebra.

Our desired category will be the intersection of two subcategories. The first one is obtained after we define the **graded subring generated by the level 1** functor

$$\mathbb{1} : \mathrm{Igr} \to \mathrm{Igr}.$$

We define it as follow: for an object $R = ((R_n)_{n\geq 0}, (h_n)_{n\geq 0}, *_{nm})$,

i) $\mathbb{1}(R)_0 := R_0 \cong \mathbb{F}_2$,

ii) $\mathbb{1}(R)_1 := R_1$,

iii) for $n \geq 2$,

$$\mathbb{1}(R)_n := \{x \in R_n : x = \sum_{j=1}^{r} a_{1j} *_{11} ... *_{11} a_{nj},$$
$$\text{with } a_{ij} \in R_1,\ 1 \leq i \leq n,\ 1 \leq j \leq r \text{ for some } r \geq 1\}.$$

Note that for all $n \geq 2$, R_n is generated by the expressions of type

$$d_1 *_{11} d_2 *_{11} ... *_{11} d_n,\ d_i \in R_1,\ i = 1, ..., n.$$

Of course, $\mathbb{1}(R)$ provides an inclusion $\iota_{\mathbb{1}(R)} : \mathbb{1}(R) \to R$ in the obvious way.

On the morphisms, for $f \in \mathrm{Igr}(R, S)$, we define $\mathbb{1}(f) \in \mathrm{Igr}(\mathbb{1}(R), \mathbb{1}(S))$ by the restriction $\mathbb{1}(f) = f \upharpoonright_{\mathbb{1}(R)}$. In other words, $\mathbb{1}(f)$ is the only Igr-morphisms that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{1}(R) & \xrightarrow{\iota_{\mathbb{1}(R)}} & R \\ {\scriptstyle \mathbb{1}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{1}(S) & \xrightarrow[\iota_{\mathbb{1}(S)}]{} & S \end{array}$$

Definition 4.1. We denote $\mathrm{Igr}_{\mathbb{1}}$ the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_{\mathbb{1}}) = \{R \in \mathrm{Igr} : \iota_{\mathbb{1}(R)} : \mathbb{1}(R) \to R \text{ is an isomorphism}\}.$$

Example 4.2.

i) If A is a $\mathbb{F}_2$-algebra, then $\mathbb{T}(A) \in obj(\mathrm{Igr}_{\mathbb{1}})$.

ii) If F is an hyperbolic hyperfield, then $k_*(F) \in obj(\mathrm{Igr}_{\mathbb{1}})$.

iii) If F is a special hyperfield (equivalently, $G = F \setminus \{0\}$ is a special group), then the graduate Witt ring of F (see Definition 5.9) $W_*(F) \in obj(\mathrm{Igr}_{\mathbb{1}})$.

iv) If F is a field with $char(F) \neq 2$, then, by a known result of Vladimir Voevodski,
$$\mathcal{H}^*(Gal(F^s|F), \{\pm 1\}) \in obj(\mathrm{Igr}_{\mathbb{1}}).$$

Proposition 4.3.

i) *For each $R \in Igr$ we have that $\iota_{\mathbb{1}(\mathbb{1}(R))} : \mathbb{1}(\mathbb{1}(R)) \to \mathbb{1}(R)$ is the identity arrow.*

ii) $\mathbb{1} \circ \mathbb{1} = \mathbb{1}$.

iii) *The functor $\mathbb{1} : Igr \to Igr_{\mathbb{1}}$ is the right adjoint of the inclusion functor $j_{\mathbb{1}} : Igr_{\mathbb{1}} \to Igr$.*

iv) *$j_{\mathbb{1}} : Igr_{\mathbb{1}} \to Igr$ creates inductive limits and to obtain the projective limits in $Igr_{\mathbb{1}}$ is sufficient restrict the projective limits obtained in Igr:*
$$\varprojlim_{i\in I} R_i \cong \left(\varprojlim_{i\in I} j_{\mathbb{1}}(R_i)\right)_{\mathbb{1}} \xrightarrow{\varprojlim_{i\in I} j_{\mathbb{1}}(R_i)} \varprojlim_{i\in I} j_{\mathbb{1}}(R_i).$$

Proof. Similar to Proposition 3.3. □

Now we define the second subcategory. We define the **quotient graded ring functor**
$$\mathcal{Q} : \mathrm{Igr} \to \mathrm{Igr}$$
as follow: for a object $R = ((R_n)_{n\geq 0}, (h_n)_{n\geq 0}, *_{nm})$, $\mathcal{Q}(R) := R/T$, where $T = (T_n)_{n\geq 0}$ is the ideal generated by $\{(\top_1 +_1 a) *_{11} a \in R_2 : a \in R_1\}$. More explicit,

i) $T_0 := \{0_0\} \subseteq R_0$,

ii) $T_1 := \{0_1\} \subseteq R_1$,

iii) for $n \geq 2$, $T_n \subseteq R_n$ is the pointed $\mathbb{F}_2$-submodule generated by

$$\{x \in R_n : x = y_l *_{l1} (\top_1 +_1 a_1) *_{11} a_1 *_{1r} z_r,$$
$$\text{with } a_1 \in R_1,\ y_l \in R_l,\ z_r \in R_r,\ l + r = n - 2\}.$$

Of course, $\mathcal{Q}(R)$ provides a projection $\pi_R : R \to \mathcal{Q}(R)$ in the obvious way.

On the morphisms, for $f \in \mathrm{Igr}(R, S)$, we define $\mathcal{Q}(f) \in \mathrm{Igr}(\mathcal{Q}(R), \mathcal{Q}(S))$ by the only Igr-morphisms that makes the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{\pi_R} & \mathcal{Q}(R) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}(f)} \\ S & \xrightarrow[\pi_S]{} & \mathcal{Q}(S) \end{array}$$

Definition 4.4. We denote Igr_h the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_h) = \{R \in \mathrm{Igr} : \pi_R : R \to \mathcal{Q}(R) \text{ is an isomorphism}\}.$$

Remark 4.5. Note that $R \in obj(Igr_h)$ iff for each $a \in R_1$, $a *_{11} \top_1 = a *_{11} a \in R_2$. Each R satisfying this condition is, in some sense, "hyperbolic" (see Proposition 6.2): this is the motivation of the index "h".

Example 4.6. i) Let A be a $\mathbb{F}_2$-algebra. Then $\mathbb{T}(A) \in obj(\mathrm{Igr}_h)$ iff A is a boolean ring (i.e., $\forall a \in A, a^2 = a$).

ii) If F is an hyperbolic hyperfield, then $k_*(F) \in obj(\mathrm{Igr}_h)$.

iii) If F is a special hyperfield (equivalently, $G = F \setminus \{0\}$ is a special group), then $W_*(F) \in obj(\mathrm{Igr}_h)$.

iv) If F is a field with $char(F) \neq 2$, then $\mathcal{H}^*(Gal(F^s|F), \{\pm 1\}) \in obj(\mathrm{Igr}_h)$.

Proposition 4.7.

i) *For each $R \in Igr$ we have that $\pi_{\mathcal{Q}(R)} : \mathcal{Q}(R) \to \mathcal{Q}(\mathcal{Q}(R))$ is an isomorphism.*

ii) $\mathcal{Q} \circ \mathcal{Q} = \mathcal{Q}$.

iii) *The functor* $\mathcal{Q} : Igr \to Igr_h$ *is the left adjoint of the inclusion functor* $j_q : Igr_{\mathcal{Q}} \to Igr$.

iv) $j_q : Igr_h \to Igr$ *creates projective limits and to obtain the inductive limits in* Igr_h *is sufficient restrict the inductive limits obtained in Igr:*

$$\varinjlim_{i\in I} j_q(R_i) \xrightarrow{\varinjlim_{i\in I} j_q(R_i)} \left(\varinjlim_{i\in I} j_q(R_i)\right)_{\mathcal{Q}} \cong \varinjlim_{i\in I} R_i.$$

Moreover, $j_q : Igr_h \to Igr$ *creates filtered inductive limits and quotients by graded ideals.*

Example 4.8. Are examples of inductive graded rings in Igr_+:

i) $\mathbb{T}(A)$, where A is a boolean ring;

ii) $k_*(F)$, where F is an hyperbolic hyperfield;

iii) $W_*(F)$, where F is an special hyperfield;

iv) $\mathcal{H}^*(Gal(F^s|F), \{\pm 1\})$, where F is a field with $char(F) \neq 2$.

Definition 4.9 (The Category Igr_+)**.** We denote by Igr_+ the full subcategory of Igr such that

$$\text{Obj}(\text{Igr}_+) = \text{Obj}(\text{Igr}_{\mathbb{1}}) \cap \text{Obj}(\text{Igr}_h).$$

We denote by $j_+ : \text{Igr}_+ \to \text{Igr}$ the inclusion functor.

Remark 4.10.

i) Note that the notion of an Igr, R, be in the subcategory Igr_h can be axiomatized by a first-order (finitary) sentence in L, the polysorted language for Igr's described earlier:

$$\forall a : 1, a *_{11} a = \top_1 *_{11} a$$

On the other hand, the concepts $R \in \text{Igr}_{\mathbb{1}}$ and $R \in Igr_+$ are axiomatized by $L_{\omega_1,\omega}$-sentences.

ii) Note that the subcategory $Igr_+ \hookrightarrow Igr$ is closed by filtered inductive limits.

In order to think of an object in Igr_+ as a graded ring of "K-theoretic type", we make the following convention.

Definition 4.11 (Exponential and Logarithm of an Igr)**.** Let $R \in \mathrm{Igr}_+$ and write R_1 **multiplicatively** by $(\Gamma(R), \cdot, 1, -1)$, i.e, fix an isomorphism $e_R : R_1 \to \Gamma(R)$ in order that $e_R(\top) = -1$ and $e_R(a+b) = a \cdot b$. Such isomorphism e_R is called **exponential** of R and $l_R = e_R^{-1}$ is called **logarithm** of R. In this sense, we can write $R_1 = \{l(a) : a \in \Gamma(R)\}$. We also denote $l(a) *_{11} l(b)$ simply by $l(a)l(b)$, $a, b \in \Gamma(R)$. We drop the superscript and write just e, l when the context allows it.

Using Definitions 4.9, 4.11 (and of course, Definitions 4.1 and 4.4 with an argument similar to the used in Lemma 1.20) we have the following properties.

Lemma 4.12 (First Properties)**.** *Let $R \in Igr_+$.*

i) $l(1) = 0$.

ii) *For all $n \geq 1$, $\eta \in R_n$ is generated by $l(a_1)...l(a_n)$ with $a_1, ..., a_n \in \Gamma(R)$.*

iii) $l(a)l(-a) = 0$ *and* $l(a)l(a) = l(-1)l(a)$ *for all* $a \in \Gamma(R)$.

iv) $l(a)l(b) = l(b)l(a)$ *for all* $a, b \in \Gamma(R)$.

v) *For every $a_1, ..., a_n \in \Gamma(R)$ and every permutation $\sigma \in S_n$,*

$$l(a_1)...l(a_i)...l(a_n) = sgn(\sigma)l(a_{\sigma 1})...l(a_{\sigma n}) \text{ in } R_n.$$

vi) *For all $\xi \in R_n$, $\eta \in R_n$,*

$$\xi\eta = \eta\xi.$$

vii) *For all $n \geq 1$,*

$$h_n(l(a_1)...l(a_n)) = l(-1)l(a_1)...l(a_n).$$

Proposition 4.13. *Let $R \in Igr_+$*

i) *For each $n \in \mathbb{N}$ and each $x \in R_n$, $x *_{n,n} x = \top_n *_{n,n} x \in R_{2n}$.*

ii) $\mathbb{A}(R) = \varinjlim_{n \in \mathbb{N}} R_n$ *is a boolean ring (or, equivalently, $\mathbb{T}(\mathbb{A}(R)) \in Igr_+$).*

Proof.

i) The property is clear if $n = 0$. If $n \geq 1$, then the property can be verified by induction on the number of generators $k \geq 1, x = \sum_{i=1}^{k} a_{1,i} *_{1,1} a_{2,i} *_{1,1} \cdots *_{1,1} a_{n,i} \in R_n$: if $k = 1$, then note that

$$\begin{aligned} x *_{n,n} x &= (a_1 * a_2 * \cdots * a_n) * (a_1 * a_2 * \cdots * a_n) \\ &= (a_1 * a_1) * (a_2 * a_2) * \cdots (a_n * a_n) \\ &= (\top_1 * a_1) * (\top_1 * a_2) * \cdots * (\top_1 * a_n) \\ &= (\top_n) * (a_1 * a_2 * \cdots * a_n); \end{aligned}$$

if $k > 1$, write $x = y + z$, where $y, z \in R_n$ are have $< k$ generator and then, by induction,

$$\begin{aligned} x *_{n,n} x &= (y + z) *_{n,n} (y + z) = y *_{n,n} y + y *_{n,n} z + z *_{n,n} y + z *_{n,n} z \\ &= y *_{n,n} y + z *_{n,n} z = \top_n *_{n,n} y + \top_n *_{n,n} z \\ &= \top_n *_{n,n} (y + z) = \top_n *_{n,n} x \end{aligned}$$

ii) This follows directly from item (i) and the definition of the ring structure in $\mathbb{A}(R) = \varinjlim_{n \in \mathbb{N}} R_n$.

□

By the previous Proposition and the universal property of the boolean hull of an Igr (Definition 3.5.(ii)), we obtain:

Corollary 4.14. *Let $R \in Igr_+$. Then:*

i) $X_{\mathbb{T}(\mathbb{A}(R))} \approx X_R$.

ii) $\mathbb{A}(R) \cong B(R)$.

Lemma 4.15.

i) *Given $R \in Igr_{\mathbb{1}}$, $S \in Igr$ and $f : S \to j_{\mathbb{1}}(R)$, we have: f is coordinatewise surjective iff $f_1 : S_1 \to R_1$ is a surjective morphism of pointed $\mathbb{F}_2$-modules.*

ii) *Given $R \in Igr_{\mathbb{1}}$, $S \in Igr$ and $f, h \in Igr(j_{\mathbb{1}}(R), S)$, we have $f = h$ if and only if $f_1 = h_1$.*

Let $R, S \in$ Igr. The inclusion function $\iota_R : \mathbb{1}(R) \to R$ and projection function $\pi_R : R \to \mathcal{Q}(R)$ induces respective natural transformations $\iota : \mathbb{1} \Rightarrow 1_{Igr}$ and $\pi : 1_{Igr} \Rightarrow \mathcal{Q}$. Moreover, we have a natural transformation $can : \mathcal{Q}\mathbb{1} \Rightarrow \mathbb{1}\mathcal{Q}$ given by the rule

$$can_n(l(a_1)...l(a_n)) := l(a_1)...l(a_n),\ n \geq 1.$$

In fact, can_n is well defined and is an isomorphism basically because both $\mathcal{Q}\mathbb{1}(R)$ and $\mathbb{1}\mathcal{Q}(R)$ are generated in level 1 by R_1 and both graded rings satisfies the relation $l(a)l(-a) = 0$. We have another immediate consequence of the previous results (and adjunctions):

Lemma 4.16.

i) *For all $R \in Igr_h$, $\mathbb{1}(R) \in Igr_+$ and can_R is an isomorphism.*

ii) *For all $R \in Igr_{\mathbb{1}}$, $\mathcal{Q}(R) \in Igr_+$ and can_R is an isomorphism.*

iii) *To get projective limits in Igr_+ is enough to restrict the projective limits obtained in Igr:*

$$\varprojlim_{i \in I} R_i \cong \mathbb{1}\left(\varprojlim_{i \in I} j_+(R_i)\right).$$

iv) *To get inductive limits in Igr_+ is enough to restrict the inductive limits obtained in Igr:*

$$\varinjlim_{i \in I} R_i \cong \mathcal{Q}\left(\varinjlim_{i \in I} j_+(R_i)\right).$$

5 Examples and constructions of quadratic interest

Definition 5.1. A **filtered ring** is a tuple $A = (A, (J_n)_{n\geq 0}, +, \cdot, 0, 1)$ where:

i) $(A, +, \cdot, 0, 1)$ is a commutative ring with unit.

ii) $J_0 = A$ and for all $n \geq 1$, $J_n \subseteq A$ is an ideal.

iii) For all $n, m \geq 0$, $n \leq m \Rightarrow J_n \supseteq J_m$.

iv) For all $n, m \geq 0$, $J_n \cdot J_m \subseteq J_{n+m}$.

v) $J_0/J_1 \cong \mathbb{F}_2$ (then $2 = 1 + 1 \in J_1$).

vi) For all $n \geq 0$, J_n/J_{n+1} is a group of exponent 2 (then $2 \cdot J_n \subseteq J_{n+1}$ and $2^n \in J_n$).

A **morphism** $f : A \to A'$ of filtered rings is a ring homomorphism such that $f(J_n) \subseteq J'_n$. The category of filtered rings will be denoted by FRing.

Definition 5.2. We define the **inductive graded ring associated functor**

$$Grad : \text{FRing} \to \text{Igr}$$

for $f \in \text{FRing}(A, B)$ as follow:

$$Grad(A) := ((Grad(A)_n)_{n\geq 0}, (t_n)_{n\geq 0}, *)$$

is the inductive graded ring where

i) For all $n \geq 0$, $Grad(A)_n := (J_n/J_{n+1}, +_n, 0_n, \top_n)$ is the exponent 2 group with distinguished element $\top_n := 2^n + J_{n+1}$.

ii) For all $n \geq 0$, $t_n : Grad(A)_n \to Grad(A)_{n+1}$ is defined by $t_n := 2 \cdot \_$, i.e, For all $a + J_{n+1} \in J_n/J_{n+1}$,,

$$t_n(a + J_{n+1}) := 2 \cdot a + J_{n+2} \in J_{n+1}/J_{n+2}.$$

Observe that $t_n(\top_n) = \top_{n+1}$, i.e, t_n is a morphism of pointed $\mathbb{F}_2$-modules.

iii) For all $n, m \geq 0$ the biadditive function $*_{nm} : Grad(A)_n \times Grad(A)_m \to Grad(A)_{n+m}$ is defined by the rule

$$(a_n + J_{n+1}) *_{mn} (b_m + J_{m+1}) = a_n \cdot b_m + J_{n+m+1} \in J_{n+m}/J_{n+m+1}.$$

The group $A_g := \bigoplus_{n\geq 0} Grad(A)_n$ of exponent 2 and the induced application $* : A_g \times A_g \to A_g$ are such that $(A_g, *)$ is a commutative ring with unit $\top_1 = (2 + J_2) \in J_1/J_2$.

iv) For all $n \geq 1$, $t_n = \top_1 *_{1n} \_$.

The morphism $Grad(f) \in \text{Igr}(Grad(A), Grad(A'))$ is defined by the following rules: for all $n \geq 0$, $f_n : Grad(A)_n \to Grad(A')_n$ is given by

$$f_n(a + J_{n+1}) := f_n(a) + J'_{n+1}.$$

Note that f_n a homomorphism of $\mathbb{F}_2$-pointed modules and $\bigoplus_{n\geq 0} f_n : (A_g, *) \to (A'_g, *)$ is a homomorphism of graded rings with unit.

Definition 5.3. The **functor of graded ring of continuous functions** over a space X

$$\mathcal{C}(X, \_) : \text{Igr} \to \text{Igr}$$

is the functor defined for $f : R \to S$ by

i) $\mathcal{C}(X, R)_0 := R_0 \cong \mathbb{F}_2$,

ii) for all $n \geq 1$, $\mathcal{C}(X, R)_n := \mathcal{C}(X, R_n)$ as a pointed $\mathbb{F}_2$-module,

iii) for all $n, m \geq 0$, $*^X_{nm} : \mathcal{C}(X, R_n) \times \mathcal{C}(X, R_m) \to \mathcal{C}(X, R_{n+m})$ is given by $(\alpha_n, \beta_m) \mapsto \alpha_n *^X_{nm} \beta_m$, where for $x \in X$,

$$\alpha_n *^X_{nm} \beta_m(x) = \alpha_n(x) *_{nm} \beta_m(x) \in R_{n+m}.$$

iv) $\mathcal{C}(X, f)_0 := f_0$ as an homomorphism of pointed $\mathbb{F}_2$-modules $R_0 \to S_0$.

v) for all $n \geq 1$,

$$\mathcal{C}(X, f)_n := \mathcal{C}(X, f_n) := f_n \circ \_ \in \text{PMod}_{\mathbb{F}_2}(\mathcal{C}(X, R_n), \mathcal{C}(X, S_n)).$$

Remark 5.4. Let X be a topological space and let $R \in \text{Igr}_{\mathbb{1}}$. Note that if X is compact or $R \in Igr_{fin}$, then $\mathcal{C}(X, R) \in \text{Igr}_{\mathbb{1}}$.

Definition 5.5. We define the **continuous function filtered ring functor**

$$\mathcal{C} : SG \to \text{FRing}$$

as follow: first, consider the functor $\mathcal{C}(X_{\_}, \mathbb{Z}) : SG \to \text{Ring}$, composition of the (contravariant) functors "associated ordering space" $X_{\_} : SG \to \text{Top}^{op}$ and "continuous functions in $\mathbb{Z}$ ring" $\mathcal{C}(\_, \mathbb{Z}) : \text{Top}^{op} \to \text{Ring}$ (here $\mathbb{Z}$ is endowed with the discrete topology).

Now we define the functor $\mathcal{C} : SG \to \text{FRing}$: given a special group $G \in SG$, we define

$$\mathcal{C}(G) := (R(G), (J_n(G))_{n \geq 0}, +, \cdot, 0, 1)$$

where

i) $(R(G), +, \cdot, 0, 1)$ is the subring of $\mathcal{C}(X_G, \mathbb{Z})$ of continuous functions of constant parity, i.e, $R(G) := J_0(G) \xrightarrow{i_0(G)} \mathcal{C}(X_G, \mathbb{Z})$ is the image of the monomorphism of rings with unit

$$j_0(G) : \mathcal{C}(X_G, 2\mathbb{Z}) \cup \mathcal{C}(X_G, 2\mathbb{Z} + 1) \to \mathcal{C}(X_G, \mathbb{Z}).$$

ii) For all $n \geq 1$, $J_n(G) \xrightarrow{i_n(G)} J_0(G)$ is the ideal of $R(G)$ (and also of $\mathcal{C}(X_G, \mathbb{Z})$) that is the image of the monomorphism of abelian groups

$$j_n(G) : \mathcal{C}(X_G, 2^n\mathbb{Z}) \to \mathcal{C}(X_G, 2\mathbb{Z}) \cup \mathcal{C}(X_G, 2\mathbb{Z}+1).$$

We also have $J_0(G)/J_1(G) \cong \mathbb{F}_2$ and for all $n, m \geq 0$:

a) If $n \geq m$ then $J_n(G) \supseteq J_m(G)$;

b) $J_n(G) \cdot J_m(G) \subseteq J_{n+m}(G)$;

c) $2J_n(G) = J_{n+1}(G) \Rightarrow J_n(G)/J_{n+1}(G)$ is an exponent 2 group.

On the morphisms, for $f \in SG(G, G')$, we define $\mathcal{C}(f) \in \text{FRing}(\mathcal{C}(G), \mathcal{C}(G'))$ by

$$\mathcal{C}(f)(h) = \mathcal{C}(X_f, \mathbb{Z})(h)$$

for $h \in \mathcal{C}(G)$. $\mathcal{C}(f)$ is well-defined because $\mathcal{C}(f) \in \text{Ring}(\mathcal{C}(G), \mathcal{C}(G'))$ and for all $n \geq 0$,

$$\mathcal{C}(f)(J_n(G)) \subseteq J_n(G').$$

Definition 5.6. We define the **continuous function graded ring functor** by

$$Grad \circ \mathcal{C} : SG \to \text{Igr}.$$

For convenience, we describe this functor now: given $G \in \text{SG}$,

$$Grad(\mathcal{C}(G)) := ((Grad(\mathcal{C}(G))_n)_{n\geq 0}, (t_n)_{n\geq 0}, \cdot)$$

where:

i) $Grad(\mathcal{C}(G))_n := (J_n(G)/J_{n+1}(G), \cdot, 0 \cdot J_{n+1}(G), 2^n J_{n+1}(G))$, where $2 \in \mathcal{C}(X_G, \mathbb{Z})$ is the constant function of value $2 \in 2\mathbb{Z} \subseteq \mathbb{Z}$.

ii) For all $n \geq 0$, $J_n(G)/J_{n+1}(G) \xrightarrow{t_2 = 2 \cdot \_} J_{n+1}(G)/J_{n+2}(G)$.

iii) For all $n, m \geq 0$,

$$*_{nm} : J_n(G)/J_{n+1}(G) \times J_m(G)/J_{m+1}(G) \to J_{n+m}(G)/J_{n+m+1}(G)$$

is given by

$$(h_n + J_{n+1}(G)) *_{nm} (k_m + J_{m+1}(G)) = h_n k_m + J_{n+m+1}(G).$$

On the morphisms, given $f \in SG(G, G')$, we have that

$$Grad(\mathcal{C}(f)) = (Grad(\mathcal{C}(f))_n)_{n\geq 0} \in \mathrm{Igr}(Grad(\mathcal{C}(G), Grad(\mathcal{C}(G')),$$

where for all $n \geq 0$, $Grad(\mathcal{C}(f))_n : Grad(\mathcal{C}(G))_n \to Grad(\mathcal{C}(G'))_n$ is such that

$$Grad(\mathcal{C}(f))_n(h + J_{n+1}(G)) = \mathcal{C}(f)(h) + J'_{n+1}(G').$$

Proposition 5.7.

a) *There is a natural isomorphism* $\theta : Grad \circ \mathcal{C} \xrightarrow{\cong} \mathbb{T} \circ \mathcal{C}(X_{\_}, \mathbb{F}_2)$. *In particular, for all* $G \in SG$, $Grad(\mathcal{C}(G)) \in Igr_+$.

b) *For all* $0 < n \leq m < \omega$, $2^{m-n} \cdot \_ : J_n(G)/J_{n+1}(G) \to J_m/J_{m+1}(G)$ *is an isomorphism of groups of exponent 2.*

c) *For all* $n \geq 1$, *there is an isomorphism of groups of exponent 2*

$$\theta_n(G) : J_n(G)/J_{n+1}(G) \xrightarrow{\cong} \mathcal{C}(X_G, \mathbb{F}_2),$$

given by the rule

$$\theta_n(h + J_n(G))(\sigma) := h_n(\sigma)/2^n \in \mathcal{C}(X_G, \mathbb{Z}/2\mathbb{Z}).$$

d) *For all* $0 < n \leq m < \omega$ *the following diagram commute:*

$$\begin{array}{ccc} J_n(G)/J_{n+1}(G) & \xrightarrow{2^{m-n}\cdot\_} & J_m(G)/J_{m+1}(G) \\ & {\scriptstyle\theta_n(G)} \searrow \quad \swarrow {\scriptstyle\theta_m(G)} & \\ & \mathcal{C}(X_G, \mathbb{F}_2) & \end{array}$$

Definition 5.8. We define the **filtered Witt ring functor**

$$W : SG \to \mathrm{FRing}$$

for $f \in \mathrm{SG}(G, H)$ as follow: given a special group $G \in SG$, we define

$$W(G) := (W(G), I^n(G)_{n\geq 0}, \oplus, \otimes, \langle\rangle, \langle 1\rangle)$$

where for all $n \geq 0$, $I^n(G)$ is the n-th power of the fundamental ideal

$$I(G) := \{\varphi \in W(G) : \dim_2(\varphi) = 0\}.$$

We define $W(f) \in \mathrm{FRing}(W(G), W(H))$ by the rule $W(f)(\varphi) := f \star \varphi$.

$W(G)$ is a filtered commutative ring with unit because:

i) $(W(G), \oplus, \otimes, \langle\rangle, \langle 1\rangle) \in \mathrm{Ring}$.

ii) For all $n \geq 0$, $I^n(G) \subseteq W(G)$ is an ideal.

iii) For all $n, m \geq 0$, $n \leq m \Rightarrow I^n(G) \supseteq I^m(G)$.

iv) For all $n, m \geq 0$, $I^n(G) \otimes I(G) \subseteq I^{n+m}(G)$.

v) $I^0(G) := W(G)$.

vi) $I^0(G)/I^1(G) \cong \mathbb{F}_2$.

vii) For all $n \geq 0$, $(I^n(G)/I^{n+1}(G), \oplus, \langle\rangle)$ is a group of exponent 2 with distinguished element $2^n + I^{n+1}(G)$, where $2^n = \otimes_{i<n}\langle 1, 1\rangle$.

Definition 5.9. We define the **graded Witt ring** functor

$$\mathrm{Grad} \circ W : SG \to \mathrm{Igr}.$$

We register, again, the following result:

Proposition 5.10. *For each $G \in SG$ we have Grad$(W(G)) \in Igr_+$.*

For each commutative ring with unit A, we have

$$t(A) = \{a \in A : \text{ exists } n \geq 0 \text{ with } n \cdot a = 0\} \subseteq A$$

is an ideal (the torsion ideal of A). The association $A \mapsto A/t(A)$ is the component on the objects of an endofunctor of Ring.

For each $G \in SG$ we have a ring homomorphism with unit $\mathrm{sgn}_G : W(G) \to \mathcal{C}(X_G, \mathbb{Z})$ given by the rule

$$\mathrm{sgn}_G(\langle a_0, ..., a_{n-1}\rangle)(\sigma) := \sum_{i=0}^{n-1} \sigma(a_i).$$

The Pfister's Local-Global principle says that sgn_G induces a monomorphism

$$\mathrm{rsgn}_G : W(G)/t(W(G)) \to \mathcal{C}(X_G, \mathbb{Z}).$$

For each $G \in SG$ we have $\mathrm{sgn}_G(W(G)) \subseteq \mathcal{C}(X_G, 2\mathbb{Z}) \cup \mathcal{C}(X_G, 2\mathbb{Z}+1)$ (since the signatures of classes of forms has the same parity of its dimension) and for all $n \geq 1$, $\mathrm{sgn}_G(I^n(G)) \subseteq \mathcal{C}(X_G, 2^n\mathbb{Z})$ (since $I^n(G)$ is the abelian subgroup of $W(G)$ generated by classes of Pfister forms of dimension 2^n).

$\mathrm{sgn} : W \to \mathcal{C}$ (respectively $\mathrm{rsgn} : W/t(W) \to \mathcal{C}$) is the natural transformation between functors

$$SG \overset{W}{\underset{\mathcal{C}}{\rightrightarrows}} \mathrm{FRing}$$

that provide natural transformations between functors $SG \rightrightarrows \mathrm{Igr}$:

$$\mathrm{Grad} \cdot \mathrm{sgn} : \mathrm{Grad} \circ W \to \mathrm{Grad} \circ \mathcal{C}, \text{ respectively}$$
$$\mathrm{Grad} \cdot \mathrm{rsgn} : \mathrm{Grad} \circ (W/t(W)) \to \mathrm{Grad} \circ \mathcal{C}.$$

Remember that [MC] ([LC]) and [WMC] ([WLC]) are conjectures about these natural transformations.

$\mathcal{C}$ is a particular case of W in the following sense: $\mathcal{C} : SG \to \mathrm{FRing}$ is naturally isomorphic to the composition of functors $SG \xrightarrow{\gamma\circ\beta} SG \xrightarrow{W} \mathrm{FRing}$.

6 The adjunction between PSG and $\mathbf{Igr}_h$

By the very Definition of the K-theory of hyperfields (with the notations in Theorem 1.21) we define the following functor[4].

Definition 6.1 (K-theories Functors)**.** With the notations of Theorem 1.21 we have a functors $k : HMF \to \mathrm{Igr}_+$, $k : PSMF \to \mathrm{Igr}_+$ induced by the reduced K-theory for hyperfields.

Now, let $R \in \mathrm{Igr}_h$. We define a hyperfield $(\Gamma(R), +, -\cdot, 0, 1)$ by the following: firstly, fix an exponential isomorphism $e_R : (R_1, +_1, 0_1, \top_1) \to$

[4]Here we will improve the adjunction presented in [6] with a more detailed proof, in order to motivate the k-stability (Definition 6.5), since it is central in the context of Galois group for special groups [7].

$(G(R), \cdot, 1, -1)$ (in agreement with Definition 4.11). This isomorphism makes, for example, an element $a *_{11} (\top_1 + b) \in R_2$, $a, b \in R_1$ take the form $(l_R(x)) *_{11} (l_R((-1) \cdot y)) \in R_2$, $x, y \in G(R)$. By an abuse of notation, we simply write $l_R(x)l_R(-y) \in R_2$, $x, y \in G(R)$. In this sense, an element in Q_2 has the form $l_R(x)l_R(-x)$, $x \in \Gamma(R)$, and we can extend this terminology for all Q_n, $n \geq 2$ (see Definition 4.4, and Lemma 4.12).

Now, let $\Gamma(R) := G(R) \cup \{0\}$ and for $a, b \in \Gamma(R)$ we define

$$\begin{aligned} -a &:= (-1) \cdot a, \\ a \cdot 0 = 0 \cdot a &:= 0, \\ a + 0 = 0 + a &= \{a\}, \\ a + (-a) &= \Gamma(R), \\ &\text{for } a, b \neq 0, a \neq -b \text{ define} \\ a + b &:= \{c \in \Gamma(R) : \text{there exist } d \in G(R) \text{ such that} \\ &\quad a \cdot b = c \cdot d \in G(R) \text{ and } l_R(a)l_R(b) = l_R(c)l_R(d) \in R_2\}. \end{aligned} \tag{6.1}$$

Proposition 6.2. *With the above rules,* $(\Gamma(R), +, -\cdot, 0, 1)$ *is a pre-special hyperfield.*

Proof. We will verify the conditions of Definition 1.4. Note that by the definition of multivalued sum once we proof that $\Gamma(R)$ is an hyperfield, it will be hyperbolic. In order to prove that $(\Gamma(R), +, -\cdot, 0, 1)$ is a multigroup we follow the steps below. Here we use freely the properties in Lemma 4.12.

i) Commutativity and $(a \in b + 0) \Leftrightarrow (a = b)$ are direct consequence of the definition of multivaluated sum and the fact that $l_R(a)l_R(b) = l_R(b)l_R(a)$.

ii) We will prove that if $c \in a + b$, then $a \in c - b$ and $b \in c - a$.

If $a = 0$ (or $b = 0$) or $a = -b$, then $c \in a + b$ means $c = a$ or $c \in a - a$. In both cases we get $a \in c - b$ and $b \in c - a$.

Now suppose $a, b \neq 0$ with $a \neq -b$. Let $c \in a + b$. Then $a \cdot b = c \cdot d$ and $l_R(a)l_R(b) = l_R(c)l_R(d) \in R_2$ for some $d \in G(R)$. Since $G(R)$ is a multiplicative group of exponent 2, we have $a \cdot d = b \cdot c$ (and hence $a \cdot (-d) = c \cdot (-b)$). Note that

$$l_R(a)l_R(-d) = l_R(a)l_R(-abc) = l_R(a)l_R(bc) = l_R(a)l_R(b) + l_R(a)l_R(c)$$

$$= l_R(c)l_R(d) + l_R(a)l_R(c) = l_R(c)l_R(d) + l_R(c)l_R(a)$$
$$= l_R(c)l_R(ad).$$

Similarly,

$$\begin{aligned} l_R(b)l_R(-c) &= l_R(b)l_R(-abd) = l_R(b)l_R(ad) = l_R(b)l_R(a) + l_R(b)l_R(d) \\ &= l_R(a)l_R(b) + l_R(b)l_R(d) = l_R(c)l_R(d) + l_R(b)l_R(d) \\ &= l_R(bc)l_R(d) = l_R(ad)l_R(d). \end{aligned}$$

Then

$$\begin{aligned} l_R(a)l_R(-d) - l_R(b)l_R(-c) &= l_R(c)l_R(ad) - l_R(ad)l_R(d) = \\ &= l_R(c)l_R(ad) - l_R(d)l_R(ad) = l_R(-cd)l_R(ad). \end{aligned}$$

But

$$\begin{aligned} l_R(-cd)l_R(ad) &= l_R(-cd)l_R(a) + l_R(-cd)l_R(d) = \\ &= l_R(-cd)l_R(a) + l_R(c)l_R(d) = l_R(a)l_R(-cd) + l_R(a)l_R(b) \\ &= l_R(a)l_R(-bcd) = l_R(a)l_R(-a) = 0. \end{aligned}$$

Then

$$l_R(a)l_R(-d) = l_R(b)l_R(-c),$$

proving that $a \in b - c$. Similarly we prove that $b \in -c + a$.

iii) Since $(G(R), \cdot, 1)$ is an abelian group, we conclude that $(\Gamma(R), \cdot, 1)$ is a commutative monoid. Beyond this, every nonzero element $a \in \Gamma(R)$ is such that $a^2 = 1$.

iv) $a \cdot 0 = 0$ for all $a \in \Gamma(R)$ is direct from definition.

v) For the distributive property, let $a, b, d \in \Gamma(R)$ and consider $x \in d(a+b)$. We need to prove that

$$x \in d \cdot a + d \cdot b. \qquad (*)$$

It is the case if $0 \in \{a, b, d\}$ or if $b = -a$. Now suppose $a, b, d \neq 0$ with $b \neq -a$. Then there exist $y \in G(R)$ such that $x = dy$ and $y \in a + b$. Moreover, there exist some $z \in G(R)$ such that $y \cdot z = a \cdot b$ and $l_R(y)l_R(z) = l_R(a)l_R(b)$.

If $0 \in \{a, b, d\}$ or if $b = -a$ there is nothing to prove. Now suppose $a, b, d \neq 0$ with $b \neq -a$. Therefore $(dy) \cdot (dz) = (da) \cdot (db)$ and

$$\begin{aligned} l_R(dy)l_R(dz) &= l_R(d)l_R(d) + l_R(d)l_R(z) + l_R(d)l_R(y) + l_R(y)l_R(z) \\ &= l_R(d)l_R(d) + l_R(d)[l_R(z) + l_R(y)] + l_R(y)l_R(z) \\ &= l_R(d)l_R(d) + l_R(d)l_R(yz) + l_R(y)l_R(z) \\ &= l_R(d)l_R(d) + l_R(d)l_R(ab) + l_R(a)l_R(b) \\ &= l_R(d)l_R(d) + l_R(d)l_R(a) + l_R(d)l_R(b) + l_R(a)l_R(b) \\ &= l_R(da)l_R(db), \end{aligned}$$

so $l_R(dy)l_R(dz) = l_R(da)l_R(db)$. Hence we have $x = dy \in d \cdot a + d \cdot b$.

vi) Using distributivity we have that for all $a, b, c, d \in \Gamma(R)$

$$d[(a + b) + c] = (da + db) + dc \text{ and } d[a + (b + c)] = da + (db + dc).$$

In fact, if $x \in (a + b) + c$, then $x \in y + c$ for $y \in a + b$. Hence

$$dx \in dy + dc \subseteq d(a + b) + dc = (da + db) + dc.$$

Conversely, if $z \in (da+db)+dc$, then $z = w+dc$, for some $w \in da+db = d(a + b)$. But in this case, $w = dt$ for some $t \in a + b$. Then

$$z \in dt + dc = d[t + c] \subseteq d[(a + b) + c].$$

Similarly we prove that $d[a + (b + c)] = da + (db + dc)$.

vii) Let $a \in \Gamma(R)$ and $x, y \in 1 - a$. If $a = 0$ or $a = 1$ then we automatically have $x \cdot y \in 1 - a$, so let $a \neq 0$ and $a \neq 1$. Then $x, y \in G(R)$ and there exist $p, q \in \Gamma(R)$ such that

$$\begin{aligned} x \cdot p = 1 \cdot a &\text{ and } l_R(x)l_R(p) = l_R(1)l_R(a) = 0 \\ y \cdot q = 1 \cdot a &\text{ and } l_R(y)l_R(q) = l_R(1)l_R(a) = 0. \end{aligned}$$

Then $(xy) \cdot (pqa) = 1 \cdot a$ and

$$\begin{aligned} l_R(xy)l_R(pqa) &= l_R(xy)l_R(p) + l_R(xy)l_R(q) + l_R(xy)l_R(a) \\ &= l_R(y)l_R(p) + l_R(x)l_R(q) + l_R(x)l_R(a) + l_R(y)l_R(a) \\ &= l_R(y)l_R(pa) + l_R(x)l_R(qa) \end{aligned}$$

$$= l_R(y)l_R(x) + l_R(x)l_R(y) = 0.$$

Then $xy \in 1 - a$, proving that $(1-a)(1-a) \subseteq (1-a)$. In particular, since $1 \in 1 - a$, we have $(1-a)(1-a) = (1-a)$.

viii) Finally, to prove associativity, we use Theorem 1.17. Let $\langle a, b\rangle \equiv \langle c, d\rangle$ the relation defined for $a, b, c, d \in \Gamma(R) \setminus \{0\}$ by

$$\langle a, b\rangle \equiv \langle c, d\rangle \text{ iff } ab = cd \text{ and } l_R(a)l_R(b) = l_R(c)l_R(d).$$

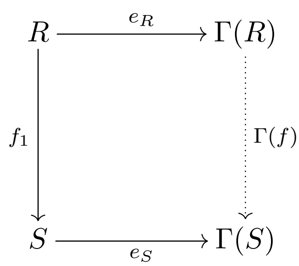
For $0 \notin \{a, b, c, d\}$, $a \neq -b$ and $ab = cd$, we have

$$a + b = c + d \text{ iff } \langle a, b\rangle \equiv \langle c, d\rangle.$$

Using items (i)-(vii) we get that $(\Gamma(R) \setminus \{0\}, \equiv, 1, -1)$ is a pre-special group. Then by Theorem 1.17 we have that $M(\Gamma(R) \setminus \{0\}) \cong \Gamma(R)$ is a pre-special hyperfield, and in particular, $\Gamma(R)$ is associative.

□

Definition 6.3. With the notations of Proposition 6.2 we have a functor $\Gamma : \text{Igr}_+ \to \text{PSMF}$ defined by the following rules: for $R \in \text{Igr}_+$, $\Gamma(R)$ is the special hyperfield obtained in Proposition 6.2 and for $f \in \text{Igr}_+(R, S)$, $\Gamma(f) : \Gamma(R) \to \Gamma(S)$ is the unique morphism such that the following diagram commute



In other words, for $x \in R$ we have

$$\Gamma(f)(x) = (e_S \circ f_1 \circ l_R)(x) = e_S(f_1(l_R(x))).$$

Theorem 6.4. *The functor $k : PSMF \to Igr_+$ is the left adjoint of $\Gamma : Igr_+ \to PSMF$. The unity of the adjoint is the natural transformation $\phi : 1_{PSMF} \to \Gamma \circ k$ defined for $F \in PSMF$ by $\phi_F = e_{k(F)} \circ \rho_F$.*

Proof. We show that for all $f \in \mathrm{PSMF}(F, \Gamma(R))$ there is an unique $f^\sharp$: $\mathrm{Igr}_+(k(F), R)$ such that $\Gamma(f^\sharp) \circ \phi_F = f$. Note that $\phi_F = e_{k(F)} \circ \rho_F$ is a group isomorphism (because $e_{k(F)}$ and ρ_F are group isomorphisms).

Let $f_0^\sharp : 1_{\mathbb{F}_2} : \mathbb{F}_2 \to \mathbb{F}_2$ and $f_1^\sharp := l_R \circ f \circ (\phi_F)^{-1} \circ e_{k(F)} : k_1(F) \to R_1$. For $n \geq 2$, define $h_n : \prod_{i=1}^n k_1(F) \to R_n$ by the rule

$$h_n(\rho(a_1), ..., \rho(a_n)) := l_R(f(a_1)) * ... * l_R(f(a_n)).$$

We have that h_n is multilinear and by the Universal Property of tensor products we have an induced morphism $\bigotimes_{i=1}^n k_n(F) \to R_n$ defined on the generators by

$$h_n(\rho(a_1) \otimes ... \otimes \rho(a_n)) := l_R(f(a_1)) * ... * l_R(f(a_n)).$$

Now let $\eta \in Q_n(F)$. Suppose without loss of generalities that $\eta = \rho(a_1) \otimes ... \otimes \rho(a_n)$ with $a_1 \in 1 - a_2$. Then $f(a_1) \in 1 - f(a_2)$ which imply $l_R(f(a_1)) \in 1 - l_R(f(a_2))$. Since $R_n \in \mathrm{Igr}_+$,

$$h_n(\eta) := h_n(\rho(a_1) \otimes ... \otimes \rho(a_n)) = l_R(f(a_1)) * ... * l_R(f(a_n)) = 0 \in R_n.$$

Then h_n factors through Q_n, and we have an induced morphism $\overline{h}_n$: $k_n(F) \to R_n$. We set $f_n^\sharp := \overline{h}_n$. In other words, $f_n^\sharp$ is defined on the generators by

$$f_n^\sharp(\rho(a_1)...\rho(a_n)) := l_R(f(a_1)) * ... * l_R(f(a_n)).$$

Finally, we have

$$\begin{aligned}
\Gamma(f^\sharp) \circ \phi_F &= [e_R \circ (f_1^\sharp) \circ e_{k(F)}^{-1}] \circ [e_{k(F)} \circ \rho_F] = e_R \circ (f_1^\sharp) \circ \rho_F \\
&= e_R \circ [l_R \circ f \circ (\phi_F)^{-1} \circ e_{k(F)}] \circ \rho_F \\
&= f \circ (\phi_F)^{-1} \circ [e_{k(F)} \circ \rho_F] \\
&= f \circ (\phi_F)^{-1} \circ \phi_F = f.
\end{aligned}$$

For the unicity, let $u, v \in \mathrm{Igr}_+(k(F), R)$ such that $\Gamma(u) \circ \phi_F = \Gamma(v) \circ \phi_F$. Since ϕ_F is an isomorphism we have $u_1 = v_1$ and since $k(F) \in \mathrm{Igr}_+$ we have $u = v$. □

As we have already seen in Theorem 6.4, there natural transformation $\phi_F : F \to \Gamma(k(F))$ is a group isomorphism. Now let $a, c, d \in F$ with $a \in c+d$. Then $\phi_F(a) \in \phi_F(c) + \phi_F(d)$, i.e, ϕ_F is a morphism of hyperfields. In fact, if $0 \in \{a, c, d\}$ there is nothing to prove. Let $0 \notin \{a, c, d\}$. To prove that $\phi_F(a) \in \phi_F(c) + \phi_F(d)$ we need to show that $\rho_F(a)\rho_F(acd) = \rho_F(c)\rho_F(d)$. In fact, from $a \in c + d$ we get $ac \in 1 + ad$, and then $\rho_F(ac)\rho_F(ad) = 0$. Moreover

$$\begin{aligned}
&\rho_F(a)\rho_F(acd) + \rho_F(c)\rho_F(d) = \\
&\rho_F(a)\rho_F(acd) + \rho_F(c)\rho_F(d) + \rho_F(ac)\rho_F(ad) = \\
&\rho_F(a)\rho_F(ac) + \rho_F(a)\rho_F(d) + \rho_F(c)\rho_F(d) + \rho_F(ac)\rho_F(ad) = \\
&[\rho_F(a)\rho_F(ac) + \rho_F(ac)\rho_F(ad)] + [\rho_F(a)\rho_F(d) + \rho_F(c)\rho_F(d)] = \\
&\rho_F(d)\rho_F(ac) + \rho_F(d)\rho_F(ac) = 0,
\end{aligned}$$

proving that $\phi_F(a) \in \phi_F(c) + \phi_F(d)$. Unfortunately we do not now if or where ϕ_F is a strong morphism. Then we propose the following definition.

Definition 6.5 (The k stability)**.** Let F be a pre-special hyperfield. We say that F is k**-stable** if $\phi_F : F \to \Gamma(k(F))$ is a strong morphism. Alternatively, F is k-stable if for all $a, b, c, d \in \dot{F}$, if $ab = cd$ then

$$\rho_F(a)\rho_f(b) = \rho_F(c)\rho_F(d) \text{ imply } ac \in 1 + cd.$$

We emphasize that if G is $AP(3)$ special group, then G is k-stable. In particular, every reduced special group is k-stable, and if F is a field of characteristic not 2, then $G(F)$ is also k-stable.

In the ongoing paper [4], it is established the Arason-Pfister Hauptsatz for **every special group** G, i.e., G satisfies $AP(n)$ for each $n \in \mathbb{N}$ (Theorem 1.18).

Proposition 6.6.

i) *For each special group G we have a PSG-isomorphism*

$$\Gamma(s_G) : \Gamma(k(G)) \to \Gamma(Grad(W(G))).$$

ii) *For each reduced special group G we have a PSG-isomorphism*

$$\kappa_G : G \to \Gamma(k(G)).$$

iii) *For each reduced special group G we have a PSG-isomorphism*

$$\omega_G : G \to \Gamma(Grad(W(G))).$$

Proposition 6.7. *Let G be a pre-special group. Then the following are equivalent:*

i) $G \in PSG_{fin}$.

ii) $k(G) \in Igr_{fin}$.

Proposition 6.8. *Let G be a special group. Then the following are equivalent:*

i) $G \in SG_{fin}$.

ii) $k(G) \in Igr_{fin}$.

iii) $(Grad \circ W)(G) \in Igr_{fin}$.

Proposition 6.9. *The canonical arrow*

$$can : \varinjlim_{i\in I} k(G_i) \to k\left(\varinjlim_{i\in I} G_i\right)$$

is an Igr_+-isomorphism as long as the I-colimits above exists.

Proposition 6.10. *The canonical arrow*

$$can : k\left(\varprojlim_{i\in I} G_i\right) \to \varprojlim_{i\in I} k(G_i)$$

is an Igr_+-morphism pointwise surjective, as long as the I-colimits above exists.

Remark 6.11. In [9] there is an interesting analysis identifying the boolean hull of a special group G (or special hyperfield $F = G \cup \{0\}$) with the boolean hull of the inductive graded rings $k_*(F), W_*(F) \in Igr_+$ (see the above Corollary 4.14). It could be interesting to compare the space of orderings of $R \in Igr_h$ and of $\Gamma(R) \in$ PSMF.

7 Igr and Marshall's conjecture

Using the Boolean hull functor, M. Dickmann and F. Miraglia provide an encoding of Marshall's signature conjecture ([MC]) for reduced special groups by the condition

$$\langle 1,1\rangle \otimes - : I^n(G)/I^{n+1}(G) \to I^{n+1}(G)/I^{n+2}(G)$$

to be injective, for each $n \in \mathbb{N}$. In fact they introduce the notion of a [SMC] reduced special group:

$$l(-1) \otimes - : k_n(G) \to k_{n+1}(G)$$

is injective, for each $n \in \mathbb{N}$. They establish that, [SMC] imply [MC], for every reduced special group G. Moreover (see 5.1 and 5.4 in [13]):

- The inductive limit of [SMC] groups is [SMC].
- The finite product of [SMC] groups is [SMC].
- $G(F)$ is [SMC], for every Pythagorean field F (with $(char(F) \neq 2)$.

Proposition 7.1.

i) *$s : k \to Grad \circ W$ is a "surjective" natural transformation, where for each $G \in SG$ and all $n \geq 1$, $s_n(G) : K_n(G) \to I^n(G)/I^{n+1}(G)$ is given by the rule*

$$s_n(G)\left(\sum_{i=0}^{s-1} l(g_{1,i}) \otimes ... \otimes l(g_{n,i}) + \mathcal{Q}_n(G)\right) := \overline{\bigotimes}_{i=0}^{s-1}[\langle 1, -g_{1,i}\rangle]\overline{\otimes}...\overline{\otimes}[\langle 1, -g_{n,i}\rangle]\overline{\otimes} I^{n+1}(G).$$

ii) *$r : Grad \circ W \to k$ is a natural transformation, where for each $G \in SG$ and all $n \geq 1$, $r^n_G : I^n(G)/I^{n+2}(G) \to k_{2n-1}(G)$ is given by the rule*

$$r_n(G)\left(\overline{\bigotimes}_{i=0}^{s-1}[\langle 1, -g_{1,i}\rangle]\overline{\otimes}...\overline{\otimes}[\langle 1, -g_{n,i}\rangle]\overline{\otimes} I^{n+1}(G)\right) := \sum_{i=0}^{s-1} l(-1)^{2^{n-1}-n} l(g_{1,i}) \otimes ... \otimes l(g_{n,i}) + \mathcal{Q}_{2n-1}(G)$$

iii) *For all* $n \geq 1$, $r_n(G) \circ s_n(G) = l(-1)^{2^{n-1}-n}\overline{\otimes}_-$.

iv) *We have an isomorphism of pointed* $\mathbb{F}_2$*-modules:*

$$s_G^1 : k_1(G) \xrightarrow{\cong} I^1(G)/I^2(G)$$
$$s_G^2 : k_2(G) \xrightarrow{\cong} I^2(G)/I^3(G).$$

v) *If* G *is [SMC] Then* $s_G : k(G) \to Grad \circ W(G)$ *is an isomorphism.*

We finish this work considering a general setting for "Marshall's Conjectures", that includes the previous case of the Igr's $W_*(F), k_*(F)$ for special hyperfields F.

Let $R \in Igr_+$. The ideal, $nil(R)$, in the ring $\bigoplus\limits_{n\in\mathbb{N}} R_n$, formed by all of its nilpotent elements, determines $N(R)$ an Igr-ideal of R, where for all $n \in \mathbb{N}$,

$$(N(R))_n := nil(R) \cap R_n.$$

Note that, by Proposition 4.13, for all $n \geq \mathbb{N}$,

$$(nil(R))_n = \{a \in R_n : \text{ exists } k \geq 1 \text{ with } \top_{kn} *_{kn,n} a = 0_{(k+1)n}\}.$$

Remark 7.2. Let $\rho : \mathbb{N} \to \mathbb{N}$ be an increasing function and for $n \in \mathbb{N}$ define

$$(N_\rho(R))_n = \{a \in R_n : \text{ exists } k \geq 1 \text{ with } \top_{\rho(n)} *_{\rho(n),n} a = 0_{\rho(n)+n}\}.$$

Then $(N_\rho(R))_n$ is a subgroup of R_n and, since $\rho(n+k) \geq \rho(n)$, we have $(N_\rho(R))_n *_{n,k} R_k \subseteq (N_\rho(R))_{n+k}$. Summing up, $((N_\rho(R))_n)_{n\in\mathbb{N}}$ is an Igr-ideal.

The following result is straightforward consequence of the Definitions and 3.3, 4.14.

Proposition 7.3. *For each* $R \in Igr_+$ *the following are equivalent:*

i) *For all* $n \leq m \in \mathbb{N}$, $ker(h_{nm}) = \{0_n\} \in R_n$.

ii) *The canonical morphism* $R \to \mathbb{T}(\mathbb{A}(R))$ *is pointwise injective.*

iii) *There exists a boolean ring* B *and a pointwise injective Igr-morphism* $R \to \mathbb{T}(B)$.

Moreover, if $R \in Igr_{fin}$*, these are equivalent to*

iv) $N(R) \cong \mathbb{T}(0) \in Igr$.

Motivated by item (i), we use the abbreviation MC(R) to say that R satisfies one (and hence all) of the above conditions.

In the following, we fix a category of L-structures $\mathcal{A}$ that is closed under directed inductive limits and a functor $F_* : \mathcal{A} \to Igr_+$ be a functor that preserves directed inductive limits. Examples of such kind of functors are $k_* : \mathrm{HMF} \to Igr_+$ and $W_* : \mathrm{HMF} \to Igr_+$, since such hyperfields can be conveniently described in the first-order relational language for multirings and it is closed under directed inductive limits. Related examples are the functors $k_* : SG \to Igr_+$ and $W_* : SG \to Igr_+$; note that SG is a full subcategory of $L_{SG} - Str$ that is closed under directed inductive limits **and** under arbitrary products.

Proposition 7.4. *Let $(I, \leq)$ be an upward directed poset and consider a functor $\Gamma : (I, \leq) \to \mathcal{A}$ such that $MC(F_*(\Gamma(i)))$, for all $i \in I$. Then*

$$MC\left(F_*\left(\varinjlim_{i\in I} \Gamma(i)\right)\right).$$

Proof. The hypothesis on F_* and the fact that the directed inductive limits in Igr_+ are pointwise, give us immediately that for each $n \in \mathbb{N}$, the mappings

$$h_n : F_n\left(\varinjlim_{i\in I} \Gamma(i)\right) \to F_{n+1}\left(\varinjlim_{i\in I} \Gamma(i)\right)$$

are isomorphic to the injective maps

$$\varinjlim_{i\in I} h_n^i : \varinjlim_{i\in I} F_n(\Gamma(i)) \to \varinjlim_{i\in I} F_{n+1}(\Gamma(i)).$$

Therefore it holds

$$MC\left(F_*\left(\varinjlim_{i\in I} \Gamma(i)\right)\right).$$

□

Corollary 7.5. *Let $F \subseteq P(I)$ be a filter and let $\{M_i : i \in I\}$ be a family of (non-empty) L-structures in $\mathcal{A}$. Suppose that $\mathcal{A}$ is closed under products and suppose that holds $MC(F_*(\prod_{i\in J} M_i))$, for each $J \in F$. Then holds $MC(F_*(\prod_{i\in J} M_i/F))$.*

Proof. This follows from the preceding result since, by a well-known model-theoretic result due to D. Ellerman ([14]), any reduced product of a family of (non-empty) L-structures, $\{M_i : i \in I\}$, module a filter $F \subseteq P(I)$, is canonically isomorphic to an upward directed inductive limit,

$$\varinjlim_{J\in F}\left(\prod_{i\in J} M_i\right) \cong \left(\prod_{i\in I} M_i\right)/F.$$

□

Proposition 7.6. *Let $F_* : \mathcal{A} \to Igr_+$ be a functor preserving pure embeddings. More precisely, we require that if $M, M' \in \mathcal{A}$ and $j : M \to M'$ is a pure L-embedding, then $F_*(j) : F_*(M) \to F_*(M')$ is a pure morphism of Igr's (described in the first-order polysorted language for Igr's).*

Proof. This follows from the well known characterization result:

Fact: Let L' be a first-order language and $f : A \to B$ be an L'-homomorphism. Then are equivalent

- $f : A \to B$ is a pure L'-embedding.
- There exists an elementary L'-embedding $e : A \to C$ and a L'-homomorphism $h : B \to C$, such that $e = h \circ f$.
- There exists an ultrapower A^I/U and a L'-homomorphism $g : B \to A^I/U$, such that $\delta_A^{(I,U)} = g \circ f$, where $\delta_A^{(I,U)} : A \to A^I/U$ is the diagonal (elementary) L'-embedding.

Since the morphism $j : M \to M'$ is a pure embedding, by the Fact there exists an ultrapower M^I/U and a L-homomorphism $g : M' \to M^I/U$, such that $\delta^M_{(I,U)} = g \circ j$, where $\delta_M^{(I,U)} : M \to M^I/U$ is the diagonal (elementary) L-embedding.

Since we have a canonical isomorphism

$$can : \varinjlim_{J\in U} M^J \overset{\cong}{\to} M^I/U,$$

applying the functor F_*, we obtain

$$F_*(M^I/U) \cong F_*\left(\varinjlim_{J\in U} M^J\right)$$

$$\cong \varinjlim_{J \in U} F^*(M^J) \to \varinjlim_{J \in U} (F^*(M))^J$$
$$\cong (F_*(M))^I/U.$$

Keeping track, we obtain that the above morphism

$$t : F_*(M^I/U) \to (F_*(M))^I/U$$

establishes a comparison between $F_*(\delta^M_{(I,U)}) : F_*(M) \to F_*(M^I/U)$ and $\delta^{F_*(M)}_{(I,U)}) : F_*(M) \to F_*(M)^I/U$

$$\delta^{F_*(M)}_{(I,U)}) = t \circ F_*(\delta^M_{(I,U)}).$$

Since $F_*(\delta^M_{(I,U)}) = F_*(g) \circ F_*(j)$, combining the equations we obtain

$$\delta^{F_*(M)}_{(I,U)}) = t \circ F_*(g) \circ F_*(j).$$

Applying again the Fact, we conclude that $F_*(j) : F_*(M) \to F_*(M')$ is a pure morphism of Igr's. □

Corollary 7.7. *For each $n \in \mathbb{N}$, the functor $F_n : \mathcal{A} \to PMod_{\mathbb{F}_2}$ preserves pure embeddings. More precisely, if $M, M' \in \mathcal{A}$ and $j : M \to M'$ is a pure L-embedding, then $F_n(j) : F_n(M) \to k_n(M')$ is a pure morphism of pointed $\mathbb{F}_2$-modules (described in the first-order single sorted language adequate). In particular $F_n(j) : F_n(M) \to F_n(M')$ is an injective morphism of pointed $\mathbb{F}_2$-modules.*

Corollary 7.8. *Let $M, M' \in \mathcal{A}$ and $j : M \to M'$ is a pure L-embedding. If $MC(F_*(M'))$, then $MC(F_*(M))$.*

Proof. This follows directly from the previous Corollary. Indeed, suppose that holds $MC(F_*(M'))$. Since $h'_n : F_n(M') \to F_{n+1}(M')$ and $F_n(j) : F_n(M) \to F_n(M')$ are injective morphisms, then, by a diagram chase, h_n :

$F_n(M) \to F_{n+1}(M)$ is an injective morphism too, thus holds $MC(F_*(M))$.

$$\begin{array}{ccc} F_nM & \xrightarrow{h_n} & F_{n+1}M \\ {\scriptstyle F_n(j)}\downarrow & & \downarrow{\scriptstyle F_{n+1}(j)} \\ F_n(M') & \xrightarrow[h'_n]{} & F_{n+1}(M') \end{array}$$

□

The Igr's functors W_*, k_* were extended by M. Dickmann and F.Miraglia from the category of fields of characteristic $\neq 2$ to the category of special groups (equivalently, the category of special hyperfields). Another relevant Igr functor, the graded cohomology ring, $H^*(Gal(F^s|F), \{\pm 1\})$ remains defined only on the field setting. The ongoing work [7] constitutes an attempt to provide an Igr functor associated to a (Galois) cohomology theory for special groups, based on the work of J. Minac and M. Spira [21]: we will define - by "generator and relations", $Gal(G)$, the *Galois Group of a special group* G, and provide some properties of this construction, as the encoding of the orderings on G.

Appendix: Some Categorical Facts

For the reader's convenience, we provide here some categorical results concerning adjunctions. Most of them are based on [3], but the reader could also consult [18].

Definition 7.9 (3.1.1 of [3])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and B an object of $\mathcal{B}$. A **reflection** of B along F is a pair (R_B, η_B) where

1) R_B is an object of A and $\eta_B : B \to F(R_B)$ is a morphism of $\mathcal{B}$.
2) If $A \in \mathcal{A}$ is another object and $b : B \to F(A)$ is a morphism of $\mathcal{B}$, there exists a unique morphism $a : R_B \to A$ in $\mathcal{A}$ such that $F(a) \circ \eta_B = b$.

Proposition 7.10 (3.1.2 of [3]). *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and B an object of $\mathcal{B}$. When the reflection of B along F exists, it is unique up to isomorphism.*

Definition 7.11 (3.1.4 of [3]). A functor $R : \mathcal{B} \to A$ is **left adjoint** to the functor $F : \mathcal{A} \to \mathcal{B}$ when there exists a natural transformation

$$\eta : 1_{\mathcal{B}} \Rightarrow F \circ R$$

such that for every $B \in \mathcal{B}$, $(R(B), \eta_B)$ is a reflection of B along F.

Theorem 7.12 (3.1.5 of [3]). *Consider two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. The following conditions are equivalent.*

1) *G is left adjoint of F.*
2) *There exist a natural transformation $\eta : 1_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon : G \to F \Rightarrow 1_{\mathcal{A}}$ such that*

$$(F * \varepsilon) \circ (\eta * F) = 1_F,\ (\varepsilon * G) \circ (G * \eta) = 1_G.$$

3) *There exist bijections*

$$\theta_{AB} : \mathcal{A}(G(B), A) \cong \mathcal{B}(B, F(A))$$

for every objects A and B, and those bijections are natural both in A and B.

4) *F is right adjoint of G.*

Proposition 7.13 (3.2.2 of [3]). *If the functor $F : \mathcal{A} \to \mathcal{B}$ has a left adjoint then F preserves all limits which turn out to exist in $\mathcal{A}$.*

Proposition 7.14 (3.4.1 of [3]). *Consider two functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ with G left adjoint to F with $\eta : 1_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon : G \circ F \Rightarrow 1_{\mathcal{A}}$ the two corresponding natural transformations. The following conditions are equivalent.*

1) *F is full and faithfull.*
2) *ε is an isomorphism.*

Under these conditions, $\eta * F$ and $G * \eta$ are isomorphisms as well.

Proposition 7.15 (3.4.3 of [3]). *Given a functor $F : \mathcal{A} \to \mathcal{B}$, the following conditions are equivalent:*

1) *F is full and faithfull and has a full and faithfull left adjoint G.*
2) *F has a left adjoint G and the two canonical natural transformations of the adjunction $\eta : 1_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon : G \to F \Rightarrow 1_{\mathcal{A}}$ are isomorphisms.*
3) *There exists a functor $G : \mathcal{B} \to \mathcal{A}$ and two arbitrary natural isomorphisms $1_{\mathcal{B}} \cong F \circ G$, $G \circ F \cong 1_{\mathcal{A}}$.*
4) *F is full and faitful and each object $B \in \mathcal{B}$ is isomorphic to an object of the form $F(A)$, for some $A \in \mathcal{A}$.*
5) *The dual condition of* (1).
6) *The dual condition of* (2).

Definition 7.16 (3.4.4 of [3]). The categories $\mathcal{A}, \mathcal{B}$ are **equivalent** if there exist a functor $F : \mathcal{A} \to \mathcal{B}$ satisfying the conditions of Proposition 7.15.

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