Categories and
General Algebraic Structures
with Applications

In press.

Baer criterion in locally presentable categories

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Abstract. In this paper, some Baer type criteria are considered for locally presentable categories. Recalling the notion of the classical Baer criterion for injectivity, it is shown that a locally presentable category which has enough injectives and coproduct injections, which are monomorphisms, satisfy such criterion if and only if the class of its injective objects is accessibly embedded in the category. Also, it is shown that this criterion is equivalent to the Baer type criterion that injectivity is equivalent to injectivity with respect to a subclass of monomorphisms.

It is also proved some Baer type criteria for locally λ-presentable categories for injectivity with respect to monomorphisms with λ-presentable domains and codomains, for a regular cardinal number λ. In particular, some Baer type criteria is found for varieties.

1 Introduction

Extending the domain of morphisms has been always an interesting and important problem in all areas of mathematics, and injectivity is one of the

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Keywords: Locally presentable categories, injectivity, Baer criterion..
Mathematics Subject Classification [2010]: 18C35, 18G05, 08B30, 18C05.
Received: 20 March 2024, Accepted: 24 June 2024.
ISSN: Print 2345-5853 Online 2345-5861.

most important concepts in algebra which is about extending the domain of homomorphisms. The Baer criterion for injectivity of R-modules for a ring R with unit is a well-known and interesting criterion which states that considering extensions of ideals to the ring R, is equivalent to considering the general extensions of R-modules. This criterion which was proved by Baer in [5], helped algebraists to characterize injective abelian groups as divisible abelian groups. Although the Baer criterion is not generally true even for rings and modules, mathematicians in the study of extensions are always seeking for a kind of Baer type criterion. A general Baer type criterion has been proved in [23, Theorem 3.7] for abelian categories, and another one was proved in [12] for Grothendieck categories. A Baer type criterion in a kind of non-additive categories was proved in [25, Theorem 1] (see also [24, Theorem 1.8]), where it is shown that in the category of right M-sets over a given monoid M, injectivity coincides with injectivity with respect to monomorphisms having cyclic codomains. Also, some other Baer type criteria have been proved in different categories (see for example, [21], [15], [16], and [8]).

In [27], we proved some Baer type criterion in abelian categories. In particular, it was shown there that injectivity with respect to monomorphisms with codomain F is equivalent to general injectivity, where F is the free object over the singleton set (Banaschewki and Ebrahimi proposed this problem for varieties). In that study, the notion of classical Baer criterion had also been introduced which meant injectivity is equivalent to the injectivity with respect to monomorphisms with a fixed codomain. Also, in [1], [2] , [3] and [22], injectivity classes as full subcategories of a kind of injective objects in locally presentable categories were studied.

In this paper, we study Baer type criterion and the classical Baer criterion in locally presentable categories. It is clear that if the classical Baer criterion holds in a category then we have a Baer type criterion in that category while it is seen that the converse need not be true. Also, it is shown that they are equivalent whenever the category has enough injectives and its coproduct injections are monic. Moreover, it is proved that under these conditions the classical Baer criterion is also equivalent to the condition that the class of all injective objects is accessibly embedded in the category. In particular, some results about the classical Baer criterion and a Baer type criterion is given for varieties.

We give two interesting Baer type criteria for injectivity with respect to some kind of classes of monomorphisms $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{M}' \subseteq \mathcal{M}$. Also, taking $\mathcal{M}_\lambda$ to be the set of all monomorphisms having λ-presentable domains and codomains, and taking a regular cardinal number μ with $\lambda < \mu$, it is shown that $\mathcal{M}_\lambda$-injectivity coincides with $\mathcal{M}_\mu$-injectivity under some conditions. We also give some examples of locally presentable categories which do or do not satisfy the classical Baer criterion.

2 Preliminaries

In this section, we recall the preliminaries which are needed in the sequel. First, some basic definitions concerning injectivity in a category $\mathcal{A}$, is given from [2].

Definition 2.1. (1) Let $\mathcal{A}$ be a category. An object B is said to be *injective with respect to* a morphism $m : S \to S^{'}$ provided that for each morphism $f : S \to B$ there exists a morphism $f^{'} : S^{'} \to B$ such that $f^{'} \circ m = f$. Let $\mathcal{M}$ be a class of morphisms in $\mathcal{A}$. Then, an object B is called $\mathcal{M}$-*injective*, if B is injective with respect to each morphism $m \in \mathcal{M}$. In the case that $\mathcal{M}$ is the class $\mathcal{M}ono$ of all monomorphisms, $\mathcal{M}$-injective objects are simply called *injective*.

(2) For each class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$, the full subcategory of all objects which are $\mathcal{M}$-injective is denoted by $\mathcal{M}$-Inj. If $\mathcal{M} = \mathcal{M}ono$, then $\mathcal{M}$-Inj is denoted by Inj.

(3) A full subcategory of $\mathcal{A}$ is called a (small-)*injectivity class* provided that it has the form $\mathcal{M}$-Inj for a (small) class $\mathcal{M}$ of morphisms in $\mathcal{A}$.

(4) We say that a category $\mathcal{A}$ has *enough injectives* if every object of $\mathcal{A}$ has an injective extension.

Example 2.2. (1) In the category **Ab** of all abelian groups and group homomorphisms, injective objects are exactly divisible groups (see [5]). Also, in this category if $\mathcal{M}$ is the class of ω-pure embeddings, then $\mathcal{M}$-injective groups are precisely the algebrically compact groups (see [19]).

(2) In the category **Pos** of all partially ordered sets and order-preserving functions, there does not exist any injective object. Although, if $\mathcal{M}$ is the class of all order-embeddings, then $\mathcal{M}$-injective objects are exactly complete posets (see [6]).

(3) In the category **Boo** of Boolean algebras and Boolean homomorphisms, injective objects are exactly complete Boolean algebras. The same is true for the category **DLatt** of distributive lattices (see [7]).

Remark 2.3. Injectivity classes in a small category $\mathcal{A}$ give rise to a functor $I : (\mathcal{P}(Mor\mathcal{A}), \subseteq) \to (\mathcal{P}(Obj\mathcal{A}), \supseteq)$ defined by $I(\mathcal{M}) = \mathcal{M}$-Inj, for any class $\mathcal{M}$ of morphisms. This functor has a right adjoint $T : (\mathcal{P}(Obj\mathcal{A}), \supseteq) \to (\mathcal{P}(Mor\mathcal{A}), \subseteq)$ which is defined by

$$T(\mathcal{O}) = \{f \in Mor\mathcal{A} \mid \forall A \in \mathcal{O},\ \text{A is \{f\}-injective}\}$$

for any class $\mathcal{O}$ of objects $\mathcal{A}$. This is because, a straightforward computation shows that $I(\mathcal{M}) \supseteq \mathcal{O}$ if and only if $\mathcal{M} \subseteq T(\mathcal{O})$ for each $\mathcal{M} \subseteq Mor\mathcal{A}$ and $\mathcal{O} \subseteq Obj\mathcal{A}$.

Now, we recall the classical Baer criterion from [27].

Definition 2.4. (1) Let $\mathcal{A}$ be a category and C be an object of $\mathcal{A}$. An object B in $\mathcal{A}$ is said to be *C-injective*, if B is $Sub(C)$-injective, where $Sub(C)$ is the class of all monomorphisms with codomain C (see [4] and [27]).

If there exists an object C such that $Sub(C)$-Inj=Inj, then it is said that the category $\mathcal{A}$ satisfies *the classical Baer criterion*.

(2) Suppose that $\mathcal{M}'$ and $\mathcal{M}$ are classes of morphisms in the category $\mathcal{A}$ such that $\mathcal{M}' \subseteq \mathcal{M}$. It is said that the *Baer criterion holds for* $\mathcal{M}'$ and $\mathcal{M}$ in $\mathcal{A}$, if $\mathcal{M}'$-Inj=$\mathcal{M}$-Inj.

If there exist classes $\mathcal{M}'$ and $\mathcal{M}$ with $\mathcal{M}' \subseteq \mathcal{M}$ such that the Baer criterion holds for $\mathcal{M}'$ and $\mathcal{M}$ in $\mathcal{A}$, we say that $\mathcal{A}$ *satisfies a Baer type criterion*.

Example 2.5. (1) Suppose that $\mathcal{A}$ is a Grothendieck category and G is a generator in $\mathcal{A}$. Then an object is injective if and only if it is $Sub(G)$-injective (see [27, Corollary 2.7]).

(2) Let M be the monoid $(\mathbb{N}^{\infty}, min)$. In [15], it is proved that in the category M-**Set** of M-sets and action-preserving (equivariant) maps between them, injectivity coincides with $Sub(M)$-injectivity, where M is the monoid with its operation as action. Notice that M as an M-set is in fact the free object of this category over a singleton set.

(3) Suppose that S is a cyclic semigroup and S^e is the monoid obtained by adjoining the identity e to it. In [16], it is shown that an object in S^e-**Set**

is injective if and only if it is $Sub(F)$-injective, where $F = S^e$ is the free object over a singleton set.

In the following, we recall the basic notions of locally presentable categories needed in the sequel from [1].

Definition 2.6. Given a regular cardinal number λ, an object K of a category is called *λ-presentable* provided that its hom-functor $hom\,(K, -)$ preserves λ-directed colimits.

An object is said to be *presentable* if it is λ-presentable, for some regular cardinal number λ.

Definition 2.7. A category is called *locally λ-presentable* provided that it is cocomplete, and has a set $\mathcal{K}$ of λ-presentable objects such that every object is a λ-directed colimit of some objects of $\mathcal{K}$.

A category is called *locally presentable*, if it is locally λ-presentable, for some regular cardinal number λ.

Notice that a locally λ-presentable category is also locally μ-presentable for all regular cardinal numbers μ with $\lambda \leq \mu$ (see [2, Remark, page 22]).

Notation 2.8. In a locally λ-presentable category $\mathcal{A}$, the skeleton of the subcategory of all λ-presentable objects is denoted by $Pres_\lambda \mathcal{A}$ (see [1]).

Example 2.9. Every variety of algebras is a locally presentable category, in fact varieties are locally λ-presentable for every regular cardinal number λ (see Corollary 3.7 and Remark page 22 of [2]).

Moreover, in every variety $\mathcal{V}$ of algebras of type $(n_\sigma)_{\sigma\in\Sigma}$ and for each regular cardinal number λ, λ-presentable objects are precisely the algebras presentable by less than λ generators and less than λ equations (see [2, Corollary 3.13]). Also, For $\lambda > card\ \Sigma$, they are precisely the algebras whose underlying sets have cardinality less than λ.

3 Baer type Criterion in Locally Presentable Categories

In this section, we seek for Baer type criteria in locally λ-presentable categories, for a regular cardinal number λ.

We first give two interesting Baer type criteria for injectivity with respect to some kind of classes $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{M}' \subseteq \mathcal{M}$.

Then, taking $\mathcal{M}_\lambda$ to be the set of all monomorphisms having λ-presentable domains and codomains, and taking a regular cardinal number μ with $\lambda < \mu$, it is shown that $\mathcal{M}_\lambda$-injectivity coincides with $\mathcal{M}_\mu$-injectivity under some conditions. In particular, as a consequence some Baer type criteria is found for varieties.

Recall from [3] that $\mathcal{M}_\lambda$-Inj is closed under λ-directed colimits. Here, we give another proof for this fact.

Lemma 3.1. *Let $\mathcal{A}$ be a locally λ-presentable category. Then, for every $\mathcal{M} \subseteq \mathcal{M}_\lambda$, the class $\mathcal{M}$-*Inj *is closed under λ-directed colimits.*

Proof. Suppose that $D : I \to \mathcal{A}$ is a λ-directed diagram such that for each $i \in I$, D_i is $\mathcal{M}$-injective. Let $c_i : D_i \to A$ be the colimit of the diagram D. We prove that A is $\mathcal{M}$-injective. Let $m : B \to C$ be an arbitrary morphism in $\mathcal{M}$. Since for each $i \in I$, D_i is $\mathcal{M}$-injective, the map $hom(-, D_i)(m) : hom(C, D_i) \to hom(B, D_i)$ is surjective. Thus, the natural transformation $hom(-, D-)(m) : hom(C, D-) \to hom(B, D-)$ is epic, and hence its colimit:

$$\varinjlim hom(-, D-)(m) : \varinjlim hom(C, D-) \to \varinjlim hom(B, D-)$$

is also epic (because the colimit functor is a left adjoint). But, B and C are λ-presentable objects, and so the hom-functors $hom(B, -)$ and $hom(C, -)$ preserve λ-directed colimits. Therefore, $hom(-, A)(m) : hom(C, A) \to hom(B, A)$ is surjective, and A is injective with respect to m. □

Now, we give a Baer type criterion with a straightforward proof.

Theorem 3.2. *Let $\mathcal{A}$ be a locally λ-presentable category, and coproduct injections in $\mathcal{A}$ be monic. If $C = \coprod_{S \in Pres_\lambda \mathcal{A}} S$ and $Sub(C) \subseteq \mathcal{M}_\lambda$, then $Sub(C)$-*Inj=*$\mathcal{M}_\lambda$-*Inj.

Proof. To prove the non clear part, let B be $Sub(C)$-injective. We show that B is $\mathcal{M}_\lambda$-injective. Let $f : A \to D$ be a monomorphism in $\mathcal{M}_\lambda$ and $g : A \to B$ be an arbitrary morphism. Assume that $h : D \to C$ is a coproduct morphism. Since B is $Sub(C)$-injective, there exists a morphism $k : C \to B$ such that $k \circ (h \circ f) = g$, and so $k \circ h : D \to B$ is such that $(k \circ h) \circ f = g$ and so B is $\mathcal{M}_\lambda$-injective. □

Recalling Example 2.9, as a consequence of the above theorem, we get the following Baer type criterion for varieties.

Corollary 3.3. *If $\mathcal{V}$ is a variety of type $(n_\sigma)_{\sigma\in\Sigma}$ and $\lambda > \Sigma$, then taking $\mathcal{M} = \{f : A \to B \mid A, B \in \mathcal{V},\ card\ A, B < \lambda\}$ and $C = \coprod\{A \mid card\ A < \lambda\}$, we have $Sub(C)$-Inj=$\mathcal{M}$-Inj.*

Another general Baer type criterion with an easy proof is as follows.

Theorem 3.4. *If $\mathcal{M}' \subseteq \mathcal{M}$ are classes of monomorphisms in a category $\mathcal{A}$ such that for every $m \in \mathcal{M}$, the domain of m has an $\mathcal{M}$-injective extension which belongs to $\mathcal{M}'$. Then $\mathcal{M}$-Inj=$\mathcal{M}'$-Inj.*

Proof. Since, $\mathcal{M}' \subseteq \mathcal{M}$, it is clear by the definition of $\mathcal{M}$-injectivity that $\mathcal{M}$-Inj $\subseteq \mathcal{M}'$-Inj. To prove the converse, let A be $\mathcal{M}'$-injective. Then A is also $\mathcal{M}$-injective, because if $m : B \to C$ belongs to $\mathcal{M}$ and $f : B \to A$ is an arbitrary morphism, then by assumption, B has an $\mathcal{M}$-injective extension, namely $h : B \to E(B)$ which belongs to $\mathcal{M}'$. Since $E(B)$ is $\mathcal{M}$-injective, there exists a morphism $k : C \to E(B)$ such that $k \circ m = h$, and since A is $\mathcal{M}'$-injective, there exists a morphism $k' : E(B) \to A$ such that $k' \circ h = f$. Therefore, $(k' \circ k) \circ m = f$, and A is $\mathcal{M}$-injective. □

Now, to prove a Baer type criterion for general locally presentable categories, we apply the above lemma and recall the following lemma.

Lemma 3.5. ([2], Exercise 1.0(3)) *Let $\mathcal{A}$ be a λ-filtered category and $\mathcal{S}$ be a full subcategory of the category $\mathcal{A}$. If for each object $A \in \mathcal{A}$ there exists a morphism $m : A \to S$ with $S \in \mathcal{S}$, then the inclusion functor $i : \mathcal{S} \to \mathcal{A}$ is cofinal. Moreover, the category $\mathcal{S}$ is λ-filtered.*

Theorem 3.6. *Let λ and μ be regular cardinal numbers such that $\lambda < \mu$ and $\mathcal{A}$ be a locally λ-presentable category. Suppose that for each $S \in Pres_\lambda\mathcal{A}$ there exists a monomorphism $r_S : S \to E(S)$ such that $E(S)$ is λ-presentable and $\mathcal{M}_\mu$-injective. If $\mathcal{M}_\mu$-Inj is closed under λ-filtered colimits, then $\{r_S \mid S \in Pres_\lambda\mathcal{A}\}$-Inj=$\mathcal{M}_\mu$-Inj.*

Proof. Since $\lambda < \mu$, we have $\mathcal{M}_\lambda \subseteq \mathcal{M}_\mu$ and so $\{r_S \mid S \in Pres_\lambda\mathcal{A}\} \subseteq \mathcal{M}_\mu$. This implies that every $\mathcal{M}_\mu$-injective object is $\{r_S \mid S \in Pres_\lambda\mathcal{A}\}$-injective. To prove the converse, suppose that B is an object in $\mathcal{A}$ such that B is r_S-injective, for all $S \in Pres_\lambda\mathcal{A}$. We show that B is $\mathcal{M}_\mu$-injective. First, we prove that the inclusion functor $i : \mathcal{S} \downarrow B \to Pres_\lambda \mathcal{A} \downarrow B$ is cofinal, where $\mathcal{S}$ is the full subcategory of $\mathcal{A}$ such that Obj $\mathcal{S} = \{E(S) \mid S \in Pres_\lambda \mathcal{A}\}$. Notice that, by Proposition 1.22 [2], the category $Pres_\lambda \mathcal{A} \downarrow B$ is λ-filtered. So in order to prove that the inclusion functor $i : \mathcal{S} \downarrow B \to Pres_\lambda \mathcal{A} \downarrow B$ is cofinal, by Lemma 3.5, it is sufficient to show that for each object $f : S \to B$ in $Pres_\lambda \mathcal{A} \downarrow B$ there exists a morphism m in $Pres_\lambda \mathcal{A} \downarrow B$ such that the codomain of m is an object in $\mathcal{S} \downarrow B$. In fact, for $f : S \to B$, the morphism m can be chosen to be $r_S : S \to E(S)$, since B is r_S-injective. Thus, the inclusion functor $i : \mathcal{S} \downarrow B \to Pres_\lambda \mathcal{A} \downarrow B$ is cofinal, so again by Lemma 3.5, the category $\mathcal{S} \downarrow B$ is λ-filtered. Notice that by Proposition 1.22 of [2], B is the colimit of the forgetful functor $U : Pres_\lambda \mathcal{A} \downarrow B \to \mathcal{A}$. From the fact that the inclusion functor i is cofinal, it follows that B is also a colimit for $U \circ i : \mathcal{S} \downarrow B \to \mathcal{A}$. Since $\mathcal{M}_\mu$-Inj is closed under λ-filtered colimits, we conclude that $B \in \mathcal{M}_\mu$-Inj. □

As a corollary of Theorems 3.4 and 3.6, we have the following proposition.

Proposition 3.7. *Let λ and μ be regular cardinal numbers such that $\lambda < \mu$ and $\mathcal{A}$ be a locally λ-presentable category. Suppose that for each $S \in Pres_\lambda \mathcal{A}$ there exists a monomorphism $r_S : S \to E(S)$ such that $E(S)$ is λ-presentable and $\mathcal{M}_\mu$-injective. If $\mathcal{M}_\mu$-*Inj *is closed under λ-filtered colimits, then $\mathcal{M}_\mu$-*Inj=*$\mathcal{M}_\lambda$-*Inj.

Proof. By hypotheses and applying Theorem 3.4, we get $\mathcal{M}_\lambda$-Inj=$\{r_S \mid S \in Pres_\lambda \mathcal{A}\}$-Inj. Also, by Theorem 3.6, we have $\{r_S \mid S \in Pres_\lambda \mathcal{A}\}$-Inj=$\mathcal{M}_\mu$-Inj. So, the result is concluded. □

Other interesting consequences of the above theorem are the following propositions about Baer criterion in varieties. Recall that varieties are locally presentable (see [2]).

Proposition 3.8. *Let $\mathcal{V}$ be a variety of type $(n_\sigma)_{\sigma \in \Sigma}$ such that $\mathcal{V}$ has enough injectives. Let λ, μ be regular cardinal numbers such that card $\Sigma < \lambda < \mu$. If for each $S \in Pres_\lambda \mathcal{V}$, there exists a monomorphism $r_s : S \to E(S)$ such that $E(S)$ is λ-presentable and $\mathcal{M}_\mu$-injective and $\mathcal{M}_\mu$-*Inj *is closed under λ-filtered colimits, then $\{r_S \mid S \in Pres_\lambda \mathcal{A}\}$-*Inj=*$\mathcal{M}_\mu$-*Inj.

The following example shows an instance of varieties fulfilling Proposition 3.8.

Example 3.9. The category **Set**-M of right M-sets satisfies a Baer type criterion. Let λ, μ be regular cardinal numbers such that $card\ M < \lambda < \mu$. Recall from [10] that for each M-set A, the monomorphism $r_A : A \to A^M$, $a \mapsto f_a$ ($f_a(s) = as$) is an injective extension of A, where $E(A) = A^M$ is the set of all functions from M to A with the obvious action which is in fact the cofree M-set on the set A. We prove that if S is a λ-presentable M-set, then so is $E(S)$. Notice that, by Example 2.9, S is λ-presentable if and only $card\ S < \lambda$. Thus, if S is λ-presentable, then $card\ E(S) < \lambda^{card\ M}$. Now, in the case that M is finite, $\lambda^{card\ M} = \lambda$ for any λ. Also, if M is infinite, we take $\lambda = 2^{card\ M}$, then $card\ E(S) < (2^{card\ M})^{card\ M} = 2^{card\ M} = \lambda$. Therefore in both cases, $card\ E(S) < \lambda$ and so $E(S)$ is λ-presentable. Finally, to show $\mathcal{M}_\mu$-Inj is closed under λ-filtered colimits, we observe that in this category $\mathcal{M}_\mu$-Inj=Inj=$\mathcal{M}_\lambda$-Inj, and apply Lemma 3.1. This is because, by Skornjakov-Baer critetion for M-sets, injectivity coincides with injectivity with respect to the class $\mathcal{M}$ of monomorphisms with cyclic codomains (quotients of the monoid M) which is a subset of $\mathcal{M}_\lambda \subseteq \mathcal{M}_\mu$. Thus by Proposition 3.8, $\{r_S \mid S \in Pres_\lambda \mathcal{A}\}$-Inj=$\mathcal{M}_\mu$-Inj.

4 The Classical Baer Criterion In Locally Presentable Categories

In this section, we consider the question that in which locally presentable categories does the classical Baer criterion hold. In Theorem 4.3 we give an equivalent condition for the classical Baer criterion to hold in locally presentable categories with enough injectives and monic coproduct injections.

Let $\mathcal{A}$ be a well-powered category with small coproducts such that the coproduct injections are monic. In the following proposition the necessary and sufficient condition such that the classical Baer criterion holds in the category $\mathcal{A}$ is given.

Proposition 4.1. *Suppose that $\mathcal{A}$ is a well-powered category with small coproducts such that the coproduct injections in it are monic. Then the classical Baer criterion holds in $\mathcal{A}$ if and only if there exists a set $\mathcal{M}$ of monomorphisms such that $\mathcal{M}$-*Inj*=*Inj*.*

Proof. To prove the non clear part, let $\mathcal{M}$ be a set of monomorphisms such that Inj=$\mathcal{M}$-Inj. Then taking $\mathcal{O}$ to be the set $\{codom(g) \mid g \in \mathcal{M}\}$ and

$C = \coprod_{A \in \mathcal{O}} A$, we get that $\mathcal{M}$-Inj=$Sub(C)$-Inj (one can also see the proof of Proposition 4.2 of [27]). □

Now, applying Proposition 4.1, we present some examples of locally presentable categories such that the classical Baer criterion holds in them.

Example 4.2. (1) If R is a ring with unity, then the category R-**Mod** of all left R-modules and module homomorphisms satisfies the classical Baer criterion. This is because by the famous criterion of Baer, in this category injectivity coincides with $Sub(R)$-injectivity (see [5]).

(2) Given a monoid M, the category **Set**-M of all right M-sets together with action-preserving maps, satisfies the classical Baer criterion. In more details, by the Skornajov-Baer criterion in **Set**-M, see Theorem 1 of [25] (or Theorem 1.8 of [24]), $\mathcal{M}$-Inj=Inj where $\mathcal{M}$ is the set of all monomorphisms with cyclic codomain.

(3) Every complete lattice as a category satisfies the classical Baer criterion. This is because in such a category every morphism is a monomorphism and it satisfies the hypotheses of Proposition 4.1.

Now we characterize locally presentable categories which have enough injectives and have monic coproduct injections such that the classical Baer criterion holds in them.

Theorem 4.3. *Suppose that $\mathcal{A}$ is a locally presentable category which has enough injectives and the coproduct injections in it are monic. Then the following are equivalent:*

(i) *The classical Baer criterion holds in $\mathcal{A}$.*

(ii) Inj *is accessibly embedded in $\mathcal{A}$, that is, there exists a regular cardinal λ such that* Inj *is closed under λ-directed colimits.*

Proof. (i) $\Rightarrow$ (ii) Suppose there exists an object $C \in \mathcal{A}$ such that $Sub(C)$-Inj=Inj. Since $Sub(C)$ is a set, there exists a regular cardinal λ such that all the morphisms in $Sub(C)$ have λ-presentable domain and codomain. So, by Lemma 3.1, Inj is closed under λ-directed colimits.

(ii) $\Rightarrow$ (i) From the facts that $\mathcal{A}$ has enough injectives and the classs Inj is accessibly embedded in $\mathcal{A}$, by Theorem 4.8 of [2], it follows that there exists a set $\mathcal{M}$ of morphisms in $\mathcal{A}$ such that $\mathcal{M}$-Inj=Inj. Thus, each injective object in $\mathcal{A}$ is injective with respect to each morphism in $\mathcal{M}$. Now, by Lemma 4.3 of [26], all morphisms in $\mathcal{M}$ are monic. So, the classical Baer criterion holds in $\mathcal{A}$, by Proposition 4.1. □

Example 4.4. (1) As stated in Example 4.2(2), the classical Baer criterion holds in **Set**-M. So the subcategory of injective acts is accessibly embedded in **Set**-M. In particular, if λ is a regular cardinal such that $\lambda > card\ M$, then by the proof of Theorem 4.3, the subcategory of injective acts is closed under λ-directed colimits.

(2) If $\mathcal{A}$ is a Grothendieck category, then by the above theorem, the subcategory of injective objects is accessibly embedded in $\mathcal{A}$ (for details, see [27, Corollary 4.7]).

Notice that in Theorem 4.3 the monicity of coproduct injections is used. A **question** that arises from Theorem 4.3 is that whether or not this condition is superfluous. Motivated by this question, we prove the following lemma concerning coproduct injections.

Lemma 4.5. *Suppose that $\mathcal{A}$ is a cocomplete category and T is the terminal object of $\mathcal{A}$. If $\mathcal{A}$ has enough injectives, then the following are equivalent:*

(i) *The coproduct injections for any family of objects are monic.*

(ii) *For each injective object E, there is a monorphism from T to E (that is, T is is a subobject of E).*

(iii) *$\mathcal{A}$ satisfies the common extension property, that is, for any given family $(A_i)_{i\in I}$ of objects, there is an object A such that for each $i \in I$, A_i is a subobject of A.*

Proof. (i) $\Leftrightarrow$ (iii): It is easily concluded from the (universal) property of coproducts.

(ii) $\Rightarrow$ (iii): In order to prove that $\mathcal{A}$ satisfies the common extesion property, it suffices to show that any family of injective objects has a common extension. Suppose that $(E_i)_{i\in I}$ is a family of injective objects. By hypothesis, for each $i \in I$ there exists a monomorphism $f_i : T \to E_i$. Suppose that $(g_i : E_i \to P)_{i\in I}$ is the multiple pushout of $(f_i : T \to E_i)_{i\in I}$. Take $i_0 \in I$ fixed. We see that $(h_i : E_i \to E_{i_0})_{i\in I}$ is a sink for $(f_i : T \to E_i)_{i\in I}$, where for $i \neq i_0$, $h_i : E_i \to E_{i_0}$ is the morphism which arises by injectivity of E_{i_0}, that is, $h_i \circ f_i = f_{i_0}$, and for $i = i_0$, $h_{i_0} = id_{E_{i_0}}$. Thus, by the universal property of multiple pushouts, there exist a unique morphism $g : P \to E_{i_0}$

such that $g \circ g_i = h_i$ for all $i \in I$. In particular, $g \circ g_{i_0} = id_{E_{i_0}}$, and so g_{i_0} is monic.

(i) $\Rightarrow$ (ii). Let the coproduct injections be monic for any family of objects and E be an arbitrary injective object. Suppose that $i_1 : E \to E \sqcup T$ and $i_2 : T \to E \sqcup T$ are the coproduct injection. Since E is injective, there exists a morphism $f : E \sqcup T \to E$ such that $f \circ i_1 = id_E$. We prove that $f \circ i_2 : T \to E$ is monic. If $g : E \to T$ is the unique morphism with the domain E and the codomain T, then $g \circ f \circ i_2 = id_T$, since T is the terminal object. Thus, $f \circ i_2$ is monic and T is a subobject of E. □

Suppose that $\mathcal{A}$ is a locally presentable category such that $\mathcal{A}$ has enough injectives. We give an equivalent condition for $\mathcal{A}$ in order to get the existence of a set of monomorphisms $\mathcal{M}$ such that $\mathcal{M}$-Inj=Inj.

Theorem 4.6. *Suppose that $\mathcal{A}$ is a locally presentable category which has enough injectives. Then the following conditions are equivalent:*

(i) *There exists a set $\mathcal{M}$ of monomorphisms such that $\mathcal{M}$-*Inj=Inj.

(ii) Inj *is accessibly embedded in $\mathcal{A}$.*

Proof. The proof is similar to the proof of Theorem 4.3. □

Now, as a consequence of Theorems 4.6 and 4.3, we get the following proposition.

Proposition 4.7. *Suppose that $\mathcal{A}$ is a locally presentable category such that $\mathcal{A}$ has enough injectives, and its coproduct injections are monic. Then, the following are equivalent:*

(i) *The classical Baer criterion holds in $\mathcal{A}$.*

(ii) *There exists a set $\mathcal{M}$ of monomorphisms such that $\mathcal{M}$-*Inj=Inj.

(iii) Inj *is accessibly embedded in $\mathcal{A}$.*

In the final part of this section, we give a lemma that helps us to prove the classical Baer criterion in some varieties.

Lemma 4.8. *Let $\mathcal{R}$ and $\mathcal{V}$ be categories, and $\mathcal{R}$ satisfies the classical Baer criterion. Let $F : \mathcal{R} \to \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{R}$ be an adjoint pair with $F \dashv G$, such that both preserve monomorphisms and G is full and faithful. Then, $\mathcal{V}$ satisfy the classical Baer criterion.*

Proof. On the contrary, let $\mathcal{V}$ do not satisfy the classical Baer criterion. Let C be an arbitrary object in the category $\mathcal{R}$. We show that there exists an object A in $\mathcal{R}$ such that A is $Sub(C)$-injective but it is not injective. Since the classical Baer criterion does not hold in $\mathcal{V}$, there exists an object B in $\mathcal{V}$ such that B is $Sub(F(C))$-injective which is not injective. We claim that $A := G(B)$ is $Sub(C)$-injective but it is not injective. From the fact that F preserves monomorphisms, it follows that $F(Sub(C)) \subseteq Sub(F(C))$. Thus, $G(B)$ is $Sub(C)$-injective. Now we prove that $G(B)$ is not injective. Since B is not injective in $\mathcal{V}$, there exist a monomorphism $i : P \to Q$ and a morphism $f : P \to B$ in $\mathcal{V}$ such that f does not have an extension through i. This implies that $G(f)$ does not have an extension through $G(i)$, because G is full and faithful. Thus, the object $G(B)$ of $\mathcal{R}$ is $Sub(C)$-injective but it is not injective. This contradicts the hypothesis that $\mathcal{R}$ satisfies the classical Baer criterion. □

Lemma 4.9. [2] *Let $\mathcal{V}$ be a variety, n be the maximum arity of $\mathcal{V}$. Let $\mathcal{A}$ be the full subcategory of $\mathcal{V}$ whose only object is the free object with n generators. Then there exists a right adjoint $G : \mathcal{V} \to \mathbf{Set}^{\mathcal{A}^{op}}$, which is both full and faithful.*

Proof. For each object B of $\mathcal{V}$, take $G(B)$ to be the hom-functor $hom(-, B)$ restricted to $\mathcal{A}^{op}$ and to each morphism $f : C \to D$ in $\mathcal{V}$, the restriction of $hom(-, f)$ (see Notation 1.25 of [2]). By Example 1.24(5) of [2], $\mathcal{A}$ is dense in $\mathcal{V}$, and so by Proposition 1.26 of [2], G is full and faithful. Moreover, by Proposition 1.27 of [2], G is a right adjoint. □

Theorem 4.10. *Let $\mathcal{V}$ be a variety and n be the maximum arity of $\mathcal{V}$. Let $\mathcal{A}$ be the full subcategory of $\mathcal{V}$ whose only object is the free object with n generators. Then $\mathcal{V}$ satisfies the classical Baer criterion, provided that the left adjoint F of the functor G given in Lemma* 4.9 *preserves monomorphisms.*

Proof. Take M to be the monoid $(hom_{\mathcal{A}}(F, F), \circ)$, where F is the free object with n generators. It is clear that $\mathcal{A}^{op}$ is isomorphic to the monoid M as a one object category. Therefore, $\mathbf{Set}^{\mathcal{A}^{op}}$ is isomorphic to $\mathbf{Set}^{M}$, and the latter is isomorphic to M-**Set**. But, by Example 4.2(2), M-**Set** satisfies the classical Baer criterion, so $\mathbf{Set}^{\mathcal{A}^{op}}$ satisfies the classical Baer criterion. Thus, by applying Lemma 4.8 to the adjunction given in Lemma 4.9, we conclude that $\mathcal{V}$ satisfies the classical Baer criterion. □

We close this section, by applying Lemma 4.8 to find some varieties which do **not satisfy** the classical Baer criterion.

Example 4.11. The category **Boo** does not satisfy the classical Baer criterion. This is because the class of injective objects in this category is exactly the class of complete Boolean algebras (see Example 2.2(3)) which is not a small injectivity class (see [1, Example 3.5(2)]).

Now, applying Lemma 4.8, we generalize the above example of **Boo**$=SP(\mathbf{2})$ to any equationally complete, congruence distributive variety of the form $\mathcal{A}=SP(T)$, where T is a finite ϕ-regular subdirectly irreducible algebra, and S, P are the subalgebra and direct product operators.

Theorem 4.12. *Let $\mathcal{R} = SP(T)$, where T is a ϕ-regular subdirectly irreducible algebra and $\mathcal{R}$ be an equationally complete, congruence distributive variety. Then $\mathcal{R}$ does not satisfy the classical Baer criterion.*

Proof. Recall from [13] that a ϕ-regular algebra is an algebra with at least one singleton subalgebra. Foster in [17] defined the functor $U : \mathbf{Boo} \to \mathcal{A}$ such that for $B \in \mathbf{Boo}$, $U(B) = T[B]$ is the Boolean extension of B (which is, by definition, a subset of B^T). Alan Day in [13] found a left adjoint $F : \mathcal{A} \to \mathbf{Boo}$ to the functor U which takes $A \in \mathcal{A}$ to the Boolean algebra of subsets of $Hom_{\mathcal{A}}(A,T)$ generated by the sets $X_A(a,M) = \{f \in Hom_{\mathcal{A}}(A,T) \mid f(a) \in M\}$, where $a \in A$ and $M \subseteq T$. It is shown in [13] that both F and U preserve monomorphisms and U is full and faithful. Now, since **Boo** does not satisfy the classical Baer criterion, by Lemma 4.8, we get that $\mathcal{R}$ does not satisfy the classical Baer criterion. □

Note 4.13. Notice that the proof of the above theorem would be constructive if one can find a Boolean algebra C which is $Sub(C)$-injective but it is not injective.

Acknowledgements

The authors thanks the referee for the very careful reading the paper and very useful comments that improved the paper. We also thanks Prof. M. Mehdi Ebrahimi for the useful comments he gave us through this reaserch work.

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