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Finitely presentable objects in $(Cb\text{-}\mathbf{Sets})_{\mathrm{fs}}$

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Abstract. Pitts generalized nominal sets to finitely supported Cb-sets by utilizing the monoid Cb of name substitutions instead of the monoid of finitary permutations over names. Finitely supported Cb-sets provide a framework for studying essential ideas of models of homotopy type theory at the level of convenient abstract categories.

Here, the interplay of two separate categories of finitely supported actions of a submonoid of $\mathrm{End}(\mathbb{D})$, for some countably infinite set $\mathbb{D}$, over sets is first investigated. In particular, we specify the structure of free objects. Then, in the category of finitely supported Cb-sets, we characterize the finitely presentable objects and provide a generator in this category.

1 Introduction

Given a countably infinite set $\mathbb{D}$, a permutation π on $\mathbb{D}$ is said to be *finitary* if it changes only a finite number of elements of $\mathbb{D}$. Consider the group $\mathrm{Perm}(\mathbb{D})$ of finitary permutations on $\mathbb{D}$ and take a set X equipped with an

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action of $\mathrm{Perm}(\mathbb{D})$ on it, that is, a $\mathrm{Perm}(\mathbb{D})$-set. An element $x \in X$ is said to have a *finite support* $S \subseteq \mathbb{D}$ if it is invariant (fixed) under the action of each element π of $\mathrm{Perm}(\mathbb{D})$ which fixes all the elements of S (that is, if $\pi s = s$, for all $s \in S$, then $\pi x = x$). A $\mathrm{Perm}(\mathbb{D})$-set X every element of which has a finite support is said to be a *nominal set*. Nominal sets are used to model the syntax of formal systems requiring variable binding operations, (see [8]). These sets have become a popular topic not only in semantics but also across various areas in mathematics [11, 15].

Gabbay [7] discusses the concept of nominal renaming sets, which are sets with a finitely supported atoms-renaming action. Pitts [14] then considers a special case of nominal renaming sets, known as finitely supported Cb-sets by adding two elements to $\mathbb{D}$, $\mathtt{0}$ and $\mathtt{1}$ and concentrating on the monoid Cb rather than the group $\mathrm{Perm}(\mathbb{D})$, where Cb is a submonoid of the monoid $\mathrm{End}(\mathbb{D})$ consisting of all maps on the countably infinite set $\mathbb{D}$. In these works by Gabbay and Pitts, finitely supported Cb-sets are utilized to analyze models of homotopy type theory.

When working in a category $\mathcal{C}$ one possible interesting thing would be to ask for the objects of $\mathcal{C}$ to be finite in some sense, since we are usually better at understanding finite things. A finite object in the category **Set** is just a finite set. However, the categorical way to characterize these objects is that: a set X is finite if and only if its homfunctor $(X, -) : \mathbf{Set} \to \mathbf{Set}$ preserves filtered colimits. In general algebraic categories an object whose homfunctor preserves filtered colimits is called *finitely presentable* [2]. Finitely presentable objects often play a significant role in categories, for instance, in the category of vector spaces over a field F, finitely presentable objects are precisely finite-dimensional ones, see also the other items of [2, Example 1.2], and it is always interesting to describe these objects in a category. Finitely presentable objects in the category of nominal sets have been characterized by Petrisan, see [12, Proposition 2.3.7]. Here we are going to describe these objects in the category of finitely supported Cb-sets.

In this paper, to put our work in context, we first review the necessary concepts. The construction of free Cb-sets over nominal sets is then shown in Section 3, where we also construct free finitely supported N-sets over a finitely supported M-set, in which M is a submonoid of N. The existence of a generator in a category provides useful information about that category. For instance, every object in a category containing all coproducts is a homomorphic image of a coproduct of generators, see [3, Proposition 6.3]. So in Section 4, in order to give additional valuable information about the category of finitely supported Cb-sets, we show that this category has a generator. Finally, finitely presentable finitely supported Cb-sets are characterized in Section 5.

2 Preliminaries

In this section, we give the necessary background on M-sets, finitely supported M-sets, and finitely supported Cb-sets. One can consult [6, 10, 14] for more information.

2.1 M-sets An (left) *M-set* for a monoid M with identity e is a set X equipped with a map $M \times X \to X, (m, x) \rightsquigarrow mx$, called an *action* of M on X, subject to $ex = x$ and $m(m'x) = (mm')x$, for all $x \in X$ and $m, m' \in M$.

By the category M-**Set** we mean the category of all M-sets and all *equivariant maps*, $f : X \to Y$ subject to $f(mx) = mf(x)$, for all $x \in X$ and $m \in M$, between them.

In the category M-**Set**, epimorphisms are exactly surjective equivariant maps (see [10, Proposition I.6.15]).

An element x of an M-set X is a *zero* (*fixed* or *equivariant*) element if $mx = x$, for all $m \in M$. We denote the set of all zero elements of an M-set X by $Z(X)$. An M-set X with *discrete action* is one in which all of its elements are zero.

A subset Y of an M-set X is an *M-subset* of Y if $my \in Y$, for all $m \in M$ and $y \in Y$. The subset $Z(X)$ of X is in fact an *M-subset* of X.

A *cyclic M-set* X is an M-set which is generated by only one element. In fact, that is of the form of $Mx = \{mx \mid m \in M\}$, for some $x \in X$.

An equivalence relation ρ on an M-set X is called a *congruence relation* on X if $x \,\rho\, x'$ implies $mx \,\rho\, mx'$, for $x, x' \in X$, $m \in M$. We denote the set of all congruences on X by $\mathrm{Con}(X)$.

Lemma 2.1. [10, Lemma I.4.37] *For $R \subseteq X \times X$, the smallest congruence on X containing R is denoted by $\rho(R)$. It is in fact, the congruence relation generated by R, and so $a \,\rho(R)\, b \Leftrightarrow a = b$ or $\exists m_1, \ldots, m_n \in M, p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in X$, where $i = 1, \ldots, n$, $(p_i, q_i) \in R$ or*

$(p_i, q_i) \in R$, such that there exists the following fence from a to b

$$
\begin{array}{ccccccccc}
a = m_1p_1 & & m_2p_2 & & m_3p_3 & & \cdots & & m_np_n \\
\updownarrow R & \diagup\!\!\diagup & \updownarrow R & \diagup\!\!\diagup & \updownarrow R & & & \diagup\!\!\diagup & \updownarrow R \\
m_1q_1 & & m_2q_2 & & \cdots & & m_{n-1}q_{n-1} & & m_nq_n = b
\end{array}
$$

2.2 Finitely supported M-sets In this subsection, we give some facts about finitely supported M-sets, where M is a submonoid of the monoid $\mathrm{End}(\mathbb{D})$ of maps on $\mathbb{D}$ together with composition and identity map.

Definition 2.2. Let X be an M-set and $x \in X$. Then,

(a) A subset $S \subseteq \mathbb{D}$ is a *support* of x, if

$$(\forall m, m' \in M)\,(m(s) = m'(s), (\forall s \in S)) \Rightarrow mx = m'x$$

If there exists a finite (possibly empty) support S of x, then we say that x is *finitely supported.*

(b) A *finitely supported M-set* is an M-set X all of whose elements have finite supports.

(c) A *nominal set* is a finitely supported $\mathrm{Perm}(\mathbb{D})$-set.

(d) An M-set X is called *uniformly* finitely supported if there exists a finite subset $S \subseteq \mathbb{D}$ such that S is a finite support of all elements of X.

Notation 2.3. We denote the full subcategory of M-**Set** consisting of all finitely supported M-sets by $(M\text{-}\mathbf{Set})_{\mathrm{fs}}$.

Proposition 2.4. *For each M-set X, the set*

$$X_{\mathrm{fs}} = \{x \in X \mid x \textit{ has a finite support in } X\}$$

is a finitely supported M-subset of X.

Proof. One can easily check that: if A is a finite support of x then $m(A)$ is a finite support of mx, for every $m \in M$ and $x \in X_{\mathrm{fs}}$. □

Example 2.5. (1) The set $\mathbb{D}$ is a finitely supported M-set, with the *canonical action* given by evaluation; that is, $\forall m \in M,\ d \in \mathbb{D},\ md = m(d)$. Also, for each $d \in \mathbb{D}$, the singleton $\{d\}$ is a finite support of d.

(2) Since $\mathcal{P}(\mathbb{D})$ together with the evaluation action $(\pi, A) \mapsto \pi A = \{\pi(a) | a \in A\}$, for every $\pi \in \mathrm{Perm}(\mathbb{D})$ and $A \in \mathcal{P}(\mathbb{D})$, is a $\mathrm{Perm}(\mathbb{D})$-set, by Proposition 2.4, $(\mathcal{P}(\mathbb{D}))_{\mathrm{fs}}$ is a finitely supported $\mathrm{Perm}(\mathbb{D})$-set.

(3) The set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$, consisting of all finite subsets of $\mathbb{D}$, together with the evaluation action is a finitely supported $\mathrm{Perm}(\mathbb{D})$-set.

(4) The sets $\mathbb{D}^n = \{(d_1, \cdots, d_n) \mid d_i \in \mathbb{D},\ i = 1, \cdots, n\}$ and $\mathbb{D}^{(n)} = \{(d_1, \cdots, d_n) \in \mathbb{D}^n \mid d_i \neq d_j, \text{ for every } i \neq j \in \{1, \cdots, n\}\}$ are finitely supported M-sets, with the action given by $m(d_1, \cdots, d_n) = (md_1, \cdots, md_n)$. For each $x = (d_1, \cdots, d_n)$, the finite set $\{d_1, \cdots, d_n\}$ is a finite support of x.

Theorem 2.6. *Let $f : X \to Y$ be an equivariant map between finitely supported M-sets and $x \in X$. Also, let S and S' be supports of x and $f(x)$, respectively. Then,*

(i) *S is also a support of $f(x)$.*

(ii) *If f is injective, then S' is a support of x.*

Proof. (i) Let $m, m' \in M$ and $m|_S = m'|_S$ in which S is a support of x. Then, we show that $mf(x) = m'f(x)$. Indeed, Definition 2.2 implies $mx = m'x$. So $mf(x) = f(mx) = f(m'x) = m'f(x)$, since f is equivariant.

(ii) Let S' be a support of $f(x)$, and $m|_{S'} = m'|_{S'}$, for some $m, m' \in M$. Then, we show that $mx = m'x$. First, because S' is a support of $f(x)$, we have $mf(x) = m'f(x)$. Notice that, f is equivariant, so $f(mx) = f(m'x)$. Now, since f is injective, we get that $mx = m'x$. □

As a result of Theorem 2.6(i) we have:

Corollary 2.7. *The category of finitely supported M-sets is a mono-coreflective subcategory of the category of M-sets.*

Definition 2.8. Let X be a finitely supported M-set and $x \in X$. Then, we say

(a) x has the *least finite support*, if the intersection of all finite supports of x is a support of x.

(b) *X admits the least support*, if each element of X has the least support.

We denote the least support of x by $\mathrm{supp}\, x$, for every $x \in X$.

Proposition 2.9. *Let X be a uniformly supported M-set which admits the least support. If* $\mathrm{Perm}(\mathbb{D}) \subseteq M$*, then X is discrete.*

Proof. Towards a contradiction, suppose $x \in X$ with $\mathrm{supp}\, x \neq \emptyset$. Since X is uniformly, there exists a finite subset $S \subseteq \mathbb{D}$ with $\mathrm{supp}\, x \subseteq S$, for all $x \in X$. Let $d_1 \notin S$ and $d \in \mathrm{supp}\, x$. Then $(d\ d_1)x$ is a non-zero element of X. So $d_1 = (d\ d_1)d \in (d\ d_1)\mathrm{supp}\, x = \mathrm{supp}\,(d\ d_1)x \subseteq S$ which is a contradiction. $\square$

Corollary 2.10. *Let X be a finite finitely supported M-set which admits the least support. If* $\mathrm{Perm}(\mathbb{D}) \subseteq M$*, then all elements of X are zero.*

Proof. Suppose $X = \{x_1, \ldots, x_k\}$ and $x_{i_1}, \ldots, x_{i_l}$ are all non-zero elements of X. Take $S = \bigcup_{j=1}^{l} \mathrm{supp}\, x_{i_j}$. Then, S is a finite support of x_i's and so X is uniformly supported M-set. Now, applying Proposition 2.9, the result holds. $\square$

Theorem 2.11. (**Presentation Theorem**) *Let X be a finitely supported M-set. Then, X is cyclic if and only if there exist a cyclic M-subset B of $\mathbb{D}^{(n)}$ and a congruence $\sim$ on B such that X is isomorphic to $B/\sim$.*

Proof. Notice that if X is singleton then $B = \mathbb{D}^{(n)}$ and $\sim = B \times B$. Suppose $X = Mx$ is a cyclic finitely supported M-set, for some non-zero element $x \in X$. Take $\{d_1, \ldots, d_n\}$ to be a support of x and $B = M(d_1, \ldots, d_n)$. Then, B is a cyclic M-subset of $\mathbb{D}^{(n)}$. Now, the assignment $\varphi : B \to Mx$ defined by $\varphi(m(d_1, \ldots, d_n)) = mx$, for every $m \in M$, is a surjective equivariant map. Indeed, if $m(d_1, \ldots, d_n) = m'(d_1, \ldots, d_n)$, for some $m, m' \in M$, then $md_i = m'd_i$, for each $1 \leq i \leq n$, and since $\mathrm{supp}\, x = \{d_1, \ldots, d_n\}$, by the definition of support, we have $mx = m'x$. Hence φ is well-defined. Obviously φ is surjective and equivariant. On the other hand, $\ker \varphi$ is a congruence relation on B. Therefore, $B/\ker \varphi$ is isomorphic to X.

To prove the converse, let B be a cyclic M-subset of $\mathbb{D}^{(n)}$ which satisfies the assumption. Then, $B = M(d_1, \ldots, d_n)$, where $(d_1, \ldots, d_n) \in \mathbb{D}^{(n)}$. We show that $B/\sim = M([(d_1, \ldots, d_k)]_\sim)$, and so, X is cyclic. Since $\sim$ is a congruence on B, we have $B/\sim$ is a finitely supported M-set. Thus, $M([(d_1, \ldots, d_k)]_\sim) \subseteq B/\sim$. Now, suppose $[\bar{b}]_\sim \in B/\sim$, for some $\bar{b} \in B$. Since $B = M(d_1, \ldots, d_n)$, we get $\bar{b} = m(d_1, \ldots, d_k)$, for some $m \in M$.

Hence
$[\bar{b}]_\sim = [m(d_1,\ldots,d_k)]_\sim = m([(d_1,\ldots,d_k)]_\sim) \in M([(d_1,\ldots,d_k)]_\sim)$. □

Lemma 2.12. [13, Homogeneity Lemma] *For any finite subsets S, S' of $\mathbb{D}$ and any bijection $f : S \to S'$, there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ that extends f to a bijection on the whole of $\mathbb{D}$ and that is the identity on the complement of $S \cup S'$:*

$$(\forall d \in S)\, \pi(d) = f(d) \wedge (\forall d \in \mathbb{D} \setminus (S \cup S'))\, \pi(d) = d.$$

2.3 Cb-sets

The following definition is given for $\mathsf{2} = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

Definition 2.13. [5, Definitions 2.1 and 2.2]

(a) An *injective finite substitution* is a map $\sigma : \mathbb{D} \to \mathbb{D} \cup \mathsf{2}$ for which $\mathbb{D}_\sigma = \{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite, and

$$(\forall d, d' \in \mathbb{D}),\ \ \sigma(d) = \sigma(d') \notin \mathsf{2} \Rightarrow d = d'.$$

(b) If $d \in \mathbb{D}$ and $b \in \mathsf{2}$, we write (b/d) for the finite substitution which maps d to b, and is the identity mapping on all the other elements of $\mathbb{D}$. Each (b/d) is called a *basic substitution.*

(c) If $d, d' \in \mathbb{D}$, then we write $(d\, d')$ for the finite substitution that transposes d and d', and keeps fixed all other elements. Each $(d\, d')$ is called a *transposition substitution.*

(d) Let Cb be the monoid whose elements are injective finite substitutions, with the monoid operation given by $\sigma \cdot \sigma' = \hat{\sigma}\sigma'$, where $\hat{\sigma} : \mathbb{D} \cup \mathsf{2} \to \mathbb{D} \cup \mathsf{2}$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined like σ. The identity element of Cb is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup \mathsf{2}$.

(e) The set $S \doteq \{\delta = (b_1/d_1) \cdots (b_k/d_k) \mid d_i \in \mathbb{D}, b_i \in \mathsf{2}\}$ is a subsemigroup of Cb. We denote the set $\{d_1, \ldots, d_k\}$ by $\mathbb{D}_\delta$, for every $\delta \in S$.

Remark 2.14. [5, Remark 2.3(ii)] For every $\delta = (b_1/d_1) \cdots (b_k/d_k) \in S$ and $\pi \in \mathrm{Perm}(\mathbb{D})$ we have $\pi\delta = \delta'\pi$, in which $\delta' = (b_1/\pi d_1) \cdots (b_k/\pi d_k)$, and $\delta\pi = \pi\delta''$, in which $\delta'' = (b_1/\pi^{-1} d_1) \cdots (b_k/\pi^{-1} d_k)$.

Theorem 2.15. [5, Theorem 2.6] *For the monoid Cb, we have $Cb =$ $\mathrm{Perm}(\mathbb{D}) \cup \mathrm{Perm}(\mathbb{D})S$.*

2.4 Finitely supported Cb-sets As noted previously, a finitely supported M-set is one in which every element has a finite support for the monoid M. We go over some facts concerning finitely supported Cb-sets in this section. See [5, 14] for further information.

Lemma 2.16. [14, Lemma 2.4] *Suppose X is a Cb-set, $x \in X$ and $b \in 2$. Also, let C be a finite subset of $\mathbb{D}$. Then, C is a support of x if and only if*

$$(\forall d \in \mathbb{D}) \quad d \notin C \Rightarrow (b/d)x = x.$$

Remark 2.17. [5, Remark 3.2 and Corollary 3.5] Let X be a Cb-set and $x \in X$.

(i) If X is finitely supported, then the set $\{d \in \mathbb{D} \mid (0/d)x \neq x\}$ is in fact the least finite support of x.

(ii) The element $x \in X$ is zero if and only if $\operatorname{supp} x = \emptyset$ if and only if $\delta\, x = x$, for all $\delta \in S$.

(iii) Every non-empty finitely supported Cb-set has a zero element.

Example 2.18. (1) The set $\mathbb{D} \cup 2$ is a finitely supported Cb-set, with the *canonical action* given by evaluation; that is,

$$\forall \sigma \in Cb,\ x \in \mathbb{D} \cup 2,\ \sigma x = \hat{\sigma}(x),$$

in which $\hat{\sigma}$ is defined as in Definition 2.13(d). Also, for each $d \in \mathbb{D}$, $\operatorname{supp} d = \{d\}$, and $\operatorname{supp} 0 = \operatorname{supp} 1 = \emptyset$.

(2) Let $X = \mathbb{D}^{(k)} \cup \{\mathsf{0}\}$, where k is a natural number, the set $\mathbb{D}^{(k)}$ is given in Example 2.5(4), and $\mathsf{0}$ is a zero element which is not included in $\mathbb{D}^{(k)}$. Then, X is a finitely supported Cb-set with the following action of Cb. Let $\sigma \in Cb$ and $x \in \mathbb{D}^{(k)}$. Then applying Theorem 2.15, $\sigma = \pi$ or $\sigma = \pi\delta$, where $\pi \in \operatorname{Perm}(\mathbb{D})$ and $\delta \in S$. For $\sigma = \pi$ or $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset$, define $\sigma x = \pi x$ and for $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$, define $\sigma x = \mathsf{0}$. Notice that, for each element $(d_1, \ldots, d_k)$, the set $\{d_1, \ldots, d_k\}$ is the support.

(3) The set $\mathcal{P}_{\mathrm{f}}(\mathbb{D} \cup 2) = \{Y \mid\ Y \text{ is a finite subset of } \mathbb{D} \cup 2\}$ is a finitely supported Cb-set with the natural Cb-action

$$* : Cb \times \mathcal{P}_{\mathrm{f}}(\mathbb{D} \cup 2) \to \mathcal{P}_{\mathrm{f}}(\mathbb{D} \cup 2),\ \sigma * Y = \sigma Y = \{\sigma y \mid\ y \in Y\}.$$

Notice that $\operatorname{supp} Y = Y \setminus 2$.

(4) All Cb-sets with the discrete action are clearly finitely supported Cb-sets, because of Remark 2.17(ii).

It is worth noting that, by Corollary 2.10, we have:

Corollary 2.19. *Finite nominal sets and finite finitely supported Cb-sets are discrete.*

Remark 2.20. [5, Notation and Remark 4.4] The sets $S_x \doteq \{\delta \in S \mid \delta x = x\}$ and $S'_x \doteq S \setminus S_x = \{\delta \in S \mid \delta x \neq x\}$ are two subsemigroups of S.

Lemma 2.21. [5, Lemma 3.4] *Let X be a non-empty finitely supported Cb-set and $x \in X$. Then,*

(i) *for $\delta \in S$, we have $\delta\, x = x$ if and only if $\mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset$.*

(ii) *for $\delta \in S$,* $\operatorname{supp} \delta x \subseteq \operatorname{supp} x \setminus \mathbb{D}_\delta$.

(iii) *for $\pi \in \mathrm{Perm}(\mathbb{D})$, we have $\operatorname{supp} \pi x = \pi \operatorname{supp} x$. In particular, $|\operatorname{supp} \pi x| = |\pi \operatorname{supp} x| = |\operatorname{supp} x|$.*

Remark 2.22. (i) If X is a non-empty finitely supported Cb-set, $x \in X$ and $\sigma \in Cb$, then by Theorem 2.15, $\sigma = \pi\delta$ where $\pi \in \mathrm{Perm}(\mathbb{D})$ and $\delta \in S$. Now, since $S = S_x \cup S'_x$, applying Remark 2.20, we obtain that $\delta \in S_x$ or $\delta \in S'_x$. Also, by Lemma 2.21, $\operatorname{supp} \sigma x = \operatorname{supp} \pi\delta x \subseteq \pi(\operatorname{supp} x \setminus \mathbb{D}_\delta)$.

(ii) We recall that a cyclic finitely supported Cb-set X is a finitely supported Cb-set generated by one element of X (i.e. $X = Cbx = \{\sigma x \mid \sigma \in Cb\}$, for some $x \in X$).

Lemma 2.23. [5, Lemma 4.5] *Let Cbx be a cyclic finitely supported Cb-set. Then,*

(i) *$Cbx = \mathrm{Perm}(\mathbb{D})S'_x x \cup \mathrm{Perm}(\mathbb{D})x$, and $\mathrm{Perm}(\mathbb{D})S'_x x \cap \mathrm{Perm}(\mathbb{D})x = \emptyset$.*

(ii) *the set $S'_x x$ is finite.*

Corollary 2.24. *Suppose X is a finitely supported Cb-set and $x \in X$. If $\delta \in S'_x$, then*

(i) *there exists $\delta_1 \in S'_x$ with $\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} x$ and $\delta x = \delta_1 x$.*

(ii) *$\delta|_{\operatorname{supp} x} = \delta_1|_{\operatorname{supp} x}$, for some $\delta_1 \in S'_x$.*

Proof. (i) Let $\delta \in S'_x$. Then, $\delta x \neq x$ and, by Lemma 2.21(i), $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$. Suppose $\delta = (b_1/d_1)\ldots(b_k/d_k)(b_{k+1}/d_{k+1})\cdots(b_n/d_n)$, wherein

$\{d_1, \ldots, d_k\} = \mathbb{D}_\delta \cap \operatorname{supp} x$. Then, for $i = k+1, \ldots, n$, we have $(b_i/d_i)x = x$ and thus $\delta x = (b_1/d_1) \ldots (b_k/d_k) \cdots (b_n/d_n)x = (b_1/d_1) \ldots (b_k/d_k)x$. Take $\delta_1 = (b_1/d_1) \cdots (b_k/d_k)$. So $\delta_1 \in S'_x$ and $\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} x$.

(ii) By (i), if $d \in \operatorname{supp} x$, then $(b_i/d_i)d = d$, for $i = k+1, \cdots, n$, and so $\delta(d) = \delta_1(d)$, as required. □

Corollary 2.25. *Every cyclic finitely supported Cb-set is a finite disjoint union of cyclic nominal sets.*

3 Interaction between finitely supported act categories

For a given monoid N, in order to study the category $(N\text{-}\mathbf{Set})_{\text{fs}}$ of finitely supported N-sets, it is crucial to find adjoint pairs between this category and other well-known categories such as **Set**, **Nom**, and $(Cb\text{-}\mathbf{Set})_{\text{fs}}$, especially the free functor, provided that there are any. We have divided this section into two subsections to do this. The free functor from finitely supported M-sets to finitely supported N-sets is found in the first subsection, where $M \leq N \leq \operatorname{End}(\mathbb{D})$. This is the composition of the forgetful functor $M\text{-}\mathbf{Set} \to \mathbf{Set}$ with the free functor $\mathbf{Set} \to N\text{-}\mathbf{Set}$ (left adjoint to the forgetful functor $N\text{-}\mathbf{Set} \to \mathbf{Set}$). Also the free functor from the category **Nom** to the category $(Cb\text{-}\mathbf{Set})_{\text{fs}}$ is given in the second subsection. In this section, we additionally construct a right adjoint for the forgetful functor $U : (N\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Set}$ and transfer certain important functors from the category **Nom** to the category $(Cb\text{-}\mathbf{Set})_{\text{fs}}$.

3.1 A free functor from $(M\text{-}\mathbf{Set})_{\text{fs}}$ to $(N\text{-}\mathbf{Set})_{\text{fs}}$

In this subsection, we consider N and M as two submonoids of $\operatorname{End}(\mathbb{D})$ with $M \leq N$, and recall that for any M-set X, the set $N \times X$ together with the action $(n, (n', x)) \mapsto (nn', x)$ is an N-set.

Definition 3.1. For any finitely supported M-set X we define the relation R_X over $N \times X$ by

$$(n, x)\, R_X\, (n', x') \Leftrightarrow \exists m \in M;\ mx = x' \text{ and } n'm|_S = n|_S,$$

in which S is a finite support of x. We also consider $\rho(R_X)$ to be the smallest congruence on $N \times X$ containing R_X.

Lemma 3.2. *Suppose $f : X \to Y$ is an equivariant map between finitely supported M-sets. Then $((n, f(x)), (n', f(x'))) \in R_Y$ and $nf(x) = n'f(x')$, if $((n, x), (n', x')) \in R_X$ or $((n', x'), (n, x)) \in R_X$, for every $n, n' \in N$ and $x, x' \in X$.*

Proof. To prove the statement, we assume $((n, x), (n', x')) \in R_X$. The other case is analogous. Since $((n, x), (n', x')) \in R_X$, by Definition 3.1, there exists $m \in M$ with $mx = x'$, and $n'm|_S = n|_S$, where S is a finite support of x. Since f is equivariant, $mf(x) = f(x')$ and also Theorem 2.6(i) implies S is a finite support of $f(x)$. Hence $((n, f(x)), (n', f(x'))) \in R_Y$, by Definition 3.1. Also since S is a finite support of $f(x)$, by Definition 2.2, we have $n'mf(x) = nf(x)$. Now, since f is equivariant, we get $nf(x) = n'mf(x) = n'f(mx) = n'f(x')$. □

Notation 3.3. We denote the N-set $(N \times X)/\rho(R_X)$ by $F(X)$, and the equivalence class $[(n, x)]_{\rho(R_X)}$ by x_n.

Remark 3.4. With this notation in mind and definition of the action of N over $F(X)$ one gets $n'x_n = x_{n'n}$, for every $n' \in N$ and $x_n \in F(X)$.

Lemma 3.5. *If S is a finite support of x, then $n(S)$ is a finite support of the equivalence class x_n.*

Proof. For every $n_1, n_2 \in N$ with $n_1|_{n(S)} = n_2|_{n(S)}$, we have $n_1n(d) = n_2n(d)$, for all $d \in S$. Thus, $n_1n|_S = n_2n|_S$. Since S is a finite support of x, and $idx = x$, we get that $((n_1n, x), (n_2n, x)) \in R_X$ and hence $((n_1n, x), (n_2n, x)) \in \rho(R_X)$. Now, by Remark 3.4, we have $n_1x_n = n_2x_n$. □

Corollary 3.6. *The N-set $F(X) = (N \times X)/\rho(R_X)$ is a finitely supported N-set.*

It is worth noting that since M is a submonoid of the monoid N, every finitely supported N-set can be considered as a finitely supported M-set. So one can consider the forgetful functor $U : (N\text{-}\mathbf{Set})_{\text{fs}} \to (M\text{-}\mathbf{Set})_{\text{fs}}$ which forgets the action of elements in $N \setminus M$ over each N-set and $U(f) = f$, for every equivariant map $f : X \to Y$ in $(N\text{-}\mathbf{Set})_{\text{fs}}$. It is also worth noting that, for every finitely supported N-set X with a finite support C for $x \in X$, C is a finite support for $x \in U(X)$.

Theorem 3.7. *The forgetful functor $U : (N\text{-}\mathbf{Set})_{\mathrm{fs}} \to (M\text{-}\mathbf{Set})_{\mathrm{fs}}$ has a left adjoint.*

Proof. Consider $F : (M\text{-}\mathbf{Set})_{\mathrm{fs}} \to (N\text{-}\mathbf{Set})_{\mathrm{fs}}$ mapping each $X \in (M\text{-}\mathbf{Set})_{\mathrm{fs}}$ to $F(X) = (N \times X)/\rho(R_X)$, and each equivariant map $f : X \to Y$ to $F(f) : F(X) \to F(Y)$, defined by $F(f)(x_n) = (f(x))_n$. The map $F(f)$ is well-defined, by Lemma 3.2, and obviously it is equivariant. It is a routine to check that F is a functor. Now, for every $X \in (M\text{-}\mathbf{Set})_{\mathrm{fs}}$, we define $\eta_X : X \to U(F(X)) = F(X)$ by $\eta_X(x) = x_{id}$, for every $x \in X$. The map η_X is equivariant, since $\eta_X(mx) = (mx)_{id} = x_{mid} = x_m = mx_{id} = m\eta_X(x)$, for every $x \in X$ and $m \in M$. To prove the universal property of η_X, suppose $f : X \to U(Y)$ is an equivariant map in $(M\text{-}\mathbf{Set})_{\mathrm{fs}}$, wherein Y is a finitely supported N-set. Then one can consider the diagram

$$\begin{array}{ccc} N \times X & \xrightarrow{\gamma_{N\times X}} & F(X) \\ {\scriptstyle id_N \times f}\big\downarrow & & \\ N \times Y & \xrightarrow[g]{} & Y, \end{array}$$

where $\gamma_{N\times X}$ is the canonical epimorphism mapping each $(n, x) \in N \times X$ to x_n and g is the action of N over Y. First we note that $\ker \gamma_{N\times X} \subseteq \ker(g(id_N \times f))$. Since if $x_n = x'_{n'}$, for some $x, x' \in X$ and $n, n' \in N$, then $nf(x) = n'f(x')$ follows from Lemma 3.2, and hence $((n, x), (n', x')) \in \ker(g(id_N \times f))$, as required. Now, by the Fundamental Theorem of Homomorphisms for N-sets, see [10, Theorem I.4.21], there exists a unique equivariant map $\hat{f} : F(X) \to Y$ in the category $(N\text{-}\mathbf{Set})_{\mathrm{fs}}$ such that $\hat{f}\gamma_{N\times X} = g(id_N \times f)$. So we have $U(\hat{f})\eta_X(x) = \hat{f}(x_{id}) = \hat{f}\gamma_{N\times X}(id, x) = g(id_N \times f)(id, x) = idf(x) = f(x)$. That is the following triangle is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & U(F(X)) \\ & {\scriptstyle f}\searrow & \vdots\, {\scriptstyle U(\hat{f})} \\ & & Y \end{array}$$

Obviously, $\hat{f}$ with this definition is unique. Also, for every equivariant map $f : X \to Y$ in M-**Set**, $Ff = \widehat{\eta_Y f}$, which makes the family $(\eta_X)_{X\in M\text{-}\mathbf{Set}}$ into a natural transformation. Indeed, for every equivariant map $f : X \to Y$ in

M-**Set** and every $x_n \in F(X)$ we have:

$$\begin{aligned}
\widehat{\eta_Y f}(x_n) &= n\eta_Y f(x) && \text{(by definition of } \widehat{\eta_Y f}) \\
&= n(f(x))_{id} && \text{(by definition of } \eta_Y) \\
&= f(x)_n \\
&= Ff(x).
\end{aligned}$$

□

Corollary 3.8. *If $X = Mx$ is a cyclic finitely supported M-set, then $F(X)$ is a cyclic finitely supported N-set.*

Proof. First, notice that $(mx)_{id} = mx_{id}$, for every $m \in M$. Thus, for every $n \in N$ and $m \in M$, we have $(mx)_n = n(mx)_{id} = (nm)x_{id}$, by Remark 3.4. Now, if $X = Mx$, then $F(X) = (N \times Mx)/\rho(R_X) = Nx_{id}$. □

Given a finitely supported N-set X and $M \leq N$, we define the relation $\sim$ over X as the following:

$$t \sim t' \iff \exists \pi \in \mathrm{Perm}(\mathbb{D}) \cap M, \quad \pi t = t',$$

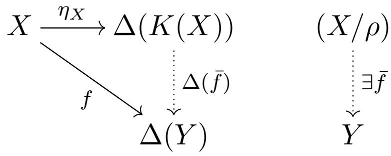
for every $t, t' \in X$. Let ρ be the least congruence generated by $\sim$. Then the set X/ρ of ρ-classes with the action $M \times (X/\rho) \to (X/\rho)$ defined by $m \cdot ([t]_\rho) = [(mt)]_\rho$ forms a finitely supported M-set. It is worth noting that if C is a finite support of t, then C is a finite support of $[t]_\rho$. Indeed, for every $m_1, m_2 \in M$ with $m_1|_C = m_2|_C$, we have $m_1 t = m_2 t$, and so, $m_1 \cdot ([t]_\rho) = [(m_1 t)]_\rho = [(m_2 t)]_\rho = m_2 \cdot ([t]_\rho)$.

We now consider the assignment $K : (N\text{-}\mathbf{Set})_{\text{fs}} \to (M\text{-}\mathbf{Set})_{\text{fs}}$, mapping each $X \in (N\text{-}\mathbf{Set})_{\text{fs}}$ to $K(X) = X/\rho$, and each equivariant map $f : X \to Y$ between finitely supported N-sets to $K(f) : X/\rho \to Y/\rho$ defined by $K(f)([x])_\rho = [f(x)]_\rho$, for every $[x]_\rho \in X/\rho$. The map $K(f)$ is well-defined, since if $[x_1]_\rho = [x_2]_\rho$, then there exists $\pi \in \mathrm{Perm}(\mathbb{D}) \cap M$ with $x_2 = \pi x_1$. Since f is equivariant, we get that $f(x_2) = f(\pi x_1) = \pi f(x_1)$. Therefore, $[f(x_1)]_\rho = [f(x_2)]_\rho$ and K is a functor.

We also consider the functor $\Delta : (M\text{-}\mathbf{Set})_{\text{fs}} \to (N\text{-}\mathbf{Set})_{\text{fs}}$ defined by $\Delta(X) = (X, \cdot)$, in which "$\cdot$" is the discrete action, for every $X \in (M\text{-}\mathbf{Set})_{\text{fs}}$, and $\Delta(f) = f$, for every equivariant map $f : X \to Y$. Since the action of ΔX is discrete, $\Delta f = f$ is equivariant.

Theorem 3.9. *The functor K is a left adjoint for Δ.*

Proof. For each finitely supported N-set X, consider $\eta_X : X \to \Delta KX$ mapping $x \mapsto [x]_\rho$. We show that η_X is an equivariant Δ-universal map. To do so, let $f : X \to \Delta Y$ be an equivariant map, for some $Y \in (M\text{-}\mathbf{Set})_{\mathrm{fs}}$. Then we define $\bar{f} : X/\rho \to Y$ by $\bar{f}([x]_\rho) = f(x)$, for every $[x]_\rho \in X/\rho$. This means that the following commutative triangle is completed by $\Delta(\bar{f})$.



Similar to the proof of well-definedness of $K(f)$, one can check that $\bar{f}$ is well-defined. Also we have $\Delta(\bar{f})\circ\eta_X(x) = \Delta\bar{f}([x]_\rho) = f(x)$. The uniqueness of $\bar{f}$ with $\Delta(\bar{f})\circ\eta_X = f$ follows from its definition. To prove that $(\eta_X)_{X\in(N\text{-}\mathbf{Set})_{\mathrm{fs}}}$ is a natural transformation, we note that, for every $f : X \to Y$ in $(N\text{-}\mathbf{Set})_{\mathrm{fs}}$ and every $[x]_\rho \in K(X)$,

$$\begin{aligned} \overline{\eta_Y f}([x]_\rho) &= \eta_Y f(x) && \text{(by definition of } \overline{\eta_Y f}) \\ &= [f(x)]_\rho && \text{(by definition of } \eta_Y) \\ &= K(f)([x]_\rho) && \text{(by definition of } K(f)). \end{aligned}$$

□

Now we define the functor $Z : (N\text{-}\mathbf{Set})_{\mathrm{fs}} \to (M\text{-}\mathbf{Set})_{\mathrm{fs}}$ by mapping each finitely supported N-set X to the set $Z(X)$, consists of all the zero elements of X, with the discrete action and $Z(f) = f$, for each $f : X \to Y$ in N-**Set**. It is worth noting that, since $\emptyset$ is a finitely supported M-set, $Z(X)$ can be empty for a given N-set X.

Theorem 3.10. *The functor Δ is a left adjoint for Z.*

Proof. It is straightforward to verify that $Z\Delta$ is the identity, $\eta_X = id_X$ is a universal map, for every $X \in (M\text{-}\mathbf{Set})_{\mathrm{fs}}$, and $(\eta_X)_{(M\text{-}\mathbf{Set})_{\mathrm{fs}}}$ is a natural transformation. □

Remark 3.11. Suppose X is a finitely supported Cb-set. If $M = \mathrm{Perm}(\mathbb{D})$ and $N = Cb$, then

(i) the relation $\sim$ over X, defined after Corollary 3.8, is given as follows:

$$t \sim t' \iff \exists \pi \in \mathrm{Perm}(\mathbb{D}), \quad \pi t = t',$$

for every $t, t' \in X$, and it is a congruence relation.

(ii) $K(X) = X/{\sim}$ is a nominal set.

(iii) furthermore, $K \dashv \Delta \dashv Z$, in which $K, Z : (Cb\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Nom}$ and $\Delta :\ \mathbf{Nom} \to (Cb\text{-}\mathbf{Set})_{\text{fs}}$.

Definition 3.12. [12, Remark 2.3.1] Let X be a set. We consider $\prod_{m\in M} X$ to be an M-set equipped with the action $m' * (x_m)_{m\in M} = (x_{m\cdot m'})_{m\in M}$, for every $m' \in M$ and $(x_m)_{m\in M} \in \prod_{m\in M} X$.

The finitely supported elements of the above defined M-set, $(\prod_{m\in M} X)_{\text{fs}}$, is a finitely supported M-set and denoted by $R(X)$. For every map $f : X \to Y$ we define $R(f) : R(X) \to R(Y)$ by $R(f)((x_m)_{m\in M}) = (f(x_m))_{m\in M}$, for every $(x_m)_{m\in M} \in R(X)$. One can easily see that $R : \mathbf{Set} \to (M\text{-}\mathbf{Set})_{\text{fs}}$ forms a functor. Now, for every $m' \in M$, we consider the natural projection map $\rho_{m'} : R(X) \to X$ mapping $(x_m)_{m\in M}$ to $x_{m'}$ and we have $\rho_m(m' * u) = \rho_{mm'}(u)$, for all $u \in R(X)$. It is worth noting that for every $m', m'' \in M$ and $u = (x_m)_{m\in M} \in \prod_{m\in M} X$ we have

$$\rho_{m'}(m'' * u) = \rho_{m'}(x_{m\cdot m''})_{m\in M} = x_{m'\cdot m''} = \rho_{m'\cdot m''}((x_m)_{m\in M}).$$

Theorem 3.13. *The forgetful functor $V : (M\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Set}$ is a left adjoint to the functor $R : \mathbf{Set} \to (M\text{-}\mathbf{Set})_{\text{fs}}$.*

Proof. The proof is similar to [12, Lemma 2.3.2]. Indeed, it is enough to show that $\eta_X : X \to RVX$, mapping each $x \in X$ to $(m \cdot x)_{m\in M}$, is an R-universal arrow, for each $X \in (M\text{-}\mathbf{Set})_{\text{fs}}$. We first show that η_X is equivariant. For every $m' \in M$ and $x \in X$ we have:

$$\begin{aligned}
\eta_X(m' \cdot x) &= (m(m'x))_{m\in M} \\
&= ((m \cdot m') \cdot x)_{m\in M} \\
&= ((m \cdot x)_{m\cdot m'})_{m\in M} \\
&= m' * (m \cdot x)_{m\in M} \\
&= m' * \eta_X(x).
\end{aligned}$$

Also η_X is R-universal because for each set Y and each equivariant map $h :\ X \to\ R(Y)$, there is $\bar{h} : V(X) \to Y$ defined by $x \mapsto \rho_{1_M}(h(x))$ in such a way that

$$R(\bar{h}) \circ \eta_X(x) = R(\bar{h})(m \cdot x)_{m\in M}$$

$$
\begin{aligned}
&= (\bar{h}(m \cdot x))_{m \in M} \\
&= \rho_{1_M}(h(m \cdot x))_{m \in M} \\
&= h(1_M \cdot x) \\
&= h(x).
\end{aligned}
$$

This means that $R(\bar{h})$ completes the following commutative triangle.

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & R(V(X)) \\
& \searrow^{h} & \downarrow{\scriptstyle R(\bar{h})} \\
& & R(Y)
\end{array}
\qquad\qquad
\begin{array}{c}
V(X) \\
\downarrow{\scriptstyle \exists \bar{h}} \\
Y
\end{array}
$$

To prove uniqueness of $\bar{h}$, suppose there exists $\bar{g} : V(X) \to Y$ with $R(\bar{g}) \circ \eta_X = h$. Then if we denote $h(x) = (y_m)_{m \in M}$, for each $x \in X$, we have $h(x) = (y_m)_{m \in M} = R(\bar{g}) \circ \eta_X(x) = (\bar{g}(m \cdot x))_{m \in M}$ and hence $\bar{h}(x) = y_{1_M} = \bar{g}(x)$, for each $x \in X$. $\square$

The diagram below shows the summary of the adjunctions given in this subsection.

$$
\begin{array}{ccc}
(N\text{-}\mathbf{Set})_{\mathrm{fs}} & \underset{U}{\overset{F}{\leftrightarrows}}\ (\perp) & (M\text{-}\mathbf{Set})_{\mathrm{fs}} \\
K \dashv \Delta \dashv Z & & \\
(M\text{-}\mathbf{Set})_{\mathrm{fs}} & \underset{R}{\overset{V}{\rightleftarrows}}\ (\perp) & \mathbf{Set}
\end{array}
\tag{1}
$$

3.2 Free functor from Nom to $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ To construct the free functor from the category of nominal sets to the category of finitely supported Cb-sets, we first show that the monoid Cb is isomorphic to a submonoid of $\mathrm{End}(\mathbb{D})$, and then use the results from the previous subsection.

Let A be a finite subset of $\mathbb{D}$. Then the set

$$M = \{m \in \mathrm{End}(\mathbb{D}) \mid m|_A = id|_A,\ m \text{ is an injective (one-one) map on } \mathbb{D} \backslash A\}$$

is a submonoid of $\mathrm{End}(\mathbb{D})$.

Example 3.14. Define an action on $\mathbb{D}$ as: $m * d = m(d)$, $m * a = a$, for all $a \in A$ and $d \in \mathbb{D} \setminus A$. The set $\mathbb{D}$ is an M-set with $Z(\mathbb{D}) = A$.

Lemma 3.15. *The monoid Cb is isomorphic to M where $|A| = 2$.*

Proof. Suppose $A = \{d_1, d_2\}$. First, notice that, since A is finite and $\mathbb{D}$ is countable, we get that $\mathbb{D} \setminus A$ is countable. Thus, there exists a bijective map $g : \mathbb{D} \setminus A \to \mathbb{D}$. So $f : (\mathbb{D} \setminus A) \cup A \to \mathbb{D} \cup 2$ defined as $f|_{\mathbb{D}\setminus A} = g$, $f(d_1) = 0$ and $f(d_2) = 1$ is a bijective map. Now, $\varphi : Cb \to M$ defined by $\varphi(\sigma) = f^{-1}\hat{\sigma} f$ is an isomorphism between two monoids, as required. □

Corollary 3.16. *The monoid Cb is isomorphic to a submonoid of* $\mathrm{End}(\mathbb{D})$.

It is also worth noting that the relation R_X, given in Definition 3.1, is a congruence over $N \times X$ if $M = \mathrm{Perm}(\mathbb{D})$ and $N = Cb$.

Lemma 3.17. *Let X be a nominal set. Then*

(i) *for every $(\sigma, x), (\sigma', x') \in Cb \times X$, $(\sigma, x)\ R_X\ (\sigma', x')$, if and only if there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi x = x'$ and $\sigma'\pi|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$.*

(ii) *the relation R_X is a congruence on $Cb \times X$.*

(iii) *the Cb-set $(Cb \times X)/R_X$ is finitely supported.*

Proof. (i) It immediately follows from Definition 3.1, when $N = Cb$, $M = \mathrm{Perm}(\mathbb{D})$ and $S = \mathrm{supp}\, x$.

(ii) It is clear that R_X is reflexive. To prove the symmetry property of R_X, suppose $((\sigma, x), (\sigma', x')) \in R_X$. Then there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi x = x'$ and $\sigma'\pi|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$. So $\pi^{-1}x' = x$. Let $d \in \mathrm{supp}\, x'$. Then $\pi^{-1}d \in \mathrm{supp}\, x$ and so $\sigma' d = \sigma'\pi(\pi^{-1}d) = \sigma(\pi^{-1}d)$. Thus, $\sigma'|_{\mathrm{supp}\, x'} = \sigma\pi^{-1}|_{\mathrm{supp}\, x'}$.

To show that R_X is transitive, let $((\sigma, x), (\sigma', x')) \in R_X$ and $((\sigma', x'), (\sigma'', x'')) \in R_X$. Then there exist $\pi, \pi' \in \mathrm{Perm}(\mathbb{D})$ with $\pi x = x'$ and $\pi' x' = x''$. Also, $\sigma'\pi|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$ and $\sigma'\pi'|_{\mathrm{supp}\, x'} = \sigma''|_{\mathrm{supp}\, x'}$. So $\pi'\pi x = \pi' x' = x''$. If $d \in \mathrm{supp}\, x$, then $\pi d \in \mathrm{supp}\, x'$ and so $\sigma''\pi'\pi(d) = \sigma'(\pi(d)) = \sigma(d)$. Thus, $\sigma''\pi'\pi|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$.

Now, given $\sigma_1 \in Cb$ and $((\sigma, x), (\sigma', x')) \in R_X$, we have $\pi x = x'$ and $\sigma'\pi|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$. So for all $d \in \mathrm{supp}\, x$, we have $\sigma_1\sigma'\pi(d) = \sigma_1\sigma(d)$ which implies that $\sigma_1\sigma'\pi|_{\mathrm{supp}\, x} = \sigma_1\sigma|_{\mathrm{supp}\, x}$. Consequently, we have $((\sigma_1\sigma, x), (\sigma_1\sigma', x')) \in R_X$, as desired.

(iii) The set $Cb \times X$ together with the action $(\sigma, (\sigma', x)) \mapsto (\sigma\sigma', x)$ is a Cb-set, for each nominal set X. Now since R_X is a congruence on $Cb \times X$, by (ii), $(Cb \times X)/R_X$ is a Cb-set. On the other hand, by Lemma 3.5, a finite support of $x_\sigma = [(\sigma, x)]_{R_X} \in (Cb \times X)/R_X$ is $\sigma(\operatorname{supp} x)$. Therefore, $(Cb \times X)/R_X$ is a finitely supported Cb-set. □

Using Corollary 3.16, one can consider the monoid Cb as a submonoid of End($\mathbb{D}$). Also, the monoid Perm($\mathbb{D}$) is a submonoid of Cb. So applying Theorem 3.7, we have free finitely supported Cb-sets over nominal sets.

Corollary 3.18. *A left adjoint to the forgetful functor* $U : (Cb\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Nom}$ *is given by the functor* $F : \mathbf{Nom} \to (Cb\text{-}\mathbf{Set})_{\text{fs}}$ *mapping each nominal set* X *to* $F(X) = (Cb \times X)/R_X$*, and each equivariant map* $f : X \to Y$ *to* $F(f) : F(X) \to F(Y)$ *with* $F(f)(x_\sigma) = (f(x))_\sigma$*. So we have the first row of Diagram* (1) *as follows.*

$$(Cb\text{-}\mathbf{Set})_{\text{fs}} \underset{\xrightarrow{\;U\;}}{\overset{\xleftarrow{\;F\;}}{\bot}} \mathbf{Nom}$$

4 A generator in the category $(Cb\text{-}\mathbf{Set})_{\text{fs}}$

A set $\mathcal{G} = \{G_i\}_{i\in I}$ of objects of a category is called a set of *generators* or a *generating set* provided that for each pair $f_1, f_2 : K \to K'$ of distinct morphisms there exist $i \in I$ and a morphism $g : G_i \to K$ with $f_1 g \neq f_2 g$. When a generating set is reduced to a singleton set $\{G\}$, we say that G is a *generator* in the category; this means that the associated homfunctor $(G, -) : \mathcal{C} \to \mathbf{Set}$ detects differences between objects of the category, see [2, Definition 0.6].

In this brief section, we present a generator in the category $(Cb\text{-}\mathbf{Set})_{\text{fs}}$. To do so, remember that the free Cb-set over X, for a given set X, is $Cb \times X$ together with the action $\sigma \cdot (\delta, x) = (\sigma \circ \delta, x)$, for every $\sigma, \delta \in Cb$ and $x \in X$.

Lemma 4.1. *The intersection of all nontrivial congruences of the free Cb-set over a singleton set* $\{x\}$ *is trivial.*

Proof. To prove the desired statement, first we note that the free Cb-set over $\{x\}$, $Cb \times \{x\}$, is not finitely supported. Then, by adding an element θ to $\mathbb{D}$ (as a zero element), we consider the set $(\mathbb{D} \cup \{\theta\}) \times \{x\}$ and define the following action.

$$\begin{aligned} Cb \times (\mathbb{D} \cup \{\theta\}) \times \{x\} &\to (\mathbb{D} \cup \{\theta\}) \times \{x\} \\ (\sigma, (d, x)) &\mapsto (\sigma(d), x) \\ (\sigma, (\theta, x)) &\mapsto (\theta, x) \end{aligned}$$

One can easily see that $(\mathbb{D} \cup \{\theta\}) \times \{x\}$ together with the above action is a Cb-set. Also, for every $d \in \mathbb{D}$, $\mathrm{supp}\,(d, x) = \{d\}$, and $\mathrm{supp}\,(\theta, x) = \emptyset$, since (θ, x) is a zero element. Hence $(\mathbb{D} \cup \{\theta\}) \times \{x\}$ is a finitely supported Cb-set.

Now for each $d \in \mathbb{D}$ we defined the map $f_d : \{x\} \to (\mathbb{D} \cup \{\theta\}) \times \{x\}$ by $x \mapsto (d, x)$. Therefore, by the universal property of the free *Cb-set* over $\{x\}$, $Cb \times \{x\}$, there exists a unique equivariant map $\bar{f}_d : Cb \times \{x\} \to (\mathbb{D} \cup \{\theta\}) \times \{x\}$ defined by $(\sigma, x) \mapsto (\sigma(d), x)$. It is worth noting that $\bar{f}_d$ is not injective, for every $d \in \mathbb{D}$, because otherwise, $Cb \times \{x\}$ will be isomorphic to a finitely supported Cb-set while it is not finitely supported, and this is a contradiction. Hence $\ker \bar{f}_d$ is a nontrivial congruence over $Cb \times \{x\}$, for every $d \in \mathbb{D}$. Now since the intersection of all nontrivial congruences over the free Cb-set $Cb \times \{x\}$ is a subset of $\bigcap_{d \in \mathbb{D}} \ker \bar{f}_d$, it is sufficient to show that $\bigcap_{d \in \mathbb{D}} \ker \bar{f}_d = \Delta$. Suppose $((\sigma_1, x), (\sigma_2, x)) \in \bigcap_{d \in \mathbb{D}} \ker \bar{f}_d$. Then $\bar{f}_d(\sigma_1) = \bar{f}_d(\sigma_2)$. Therefore $\sigma_1(d) = \sigma_2(d)$, for each $d \in \mathbb{D}$, hence $\sigma_1 = \sigma_2$ as required. □

Theorem 4.2. *The category* $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ *has a generating set.*

Proof. To construct a generating set, first we define a set $\mathcal{A}$ as follows.

$$\mathcal{A} \doteq \{\rho \in \mathsf{Con}(Cb \times \{x\}) \mid (Cb \times \{x\})/\rho \in (Cb\text{-}\mathbf{Set})_{\mathrm{fs}}\}.$$

Of course $\mathcal{A}$ is non-empty, because the kernel of the unique equivariant map to the trivial one-point Cb-set $\{\theta\}$ lies in $\mathcal{A}$, see Lemma 4.1. Now take

$$\mathcal{G} \doteq \{(Cb \times \{x\})/\rho \mid \rho \in \mathcal{A}\}$$

We show that $\mathcal{G}$ is a generating set in $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ To do so, let $g, f : A \to B$ be two different equivariant maps in $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$. Then there exists some $a \in A$ such that $f(a) \neq g(a)$. Hence there is the equivariant map $h : Cb \times \{x\} \to A$, which maps $(\iota, x) \in Cb \times \{x\}$ to a, with $fh \neq gh$, since $(Cb \times \{x\})$ is a generator in the category of Cb-sets. Therefore, by the First Isomorphism Theorem, there exists the equivariant injective map $\bar{h} : (Cb \times \{x\})/\ker(h) \to A$ such that $f\bar{h} \neq g\bar{h}$. Therefore, $\mathcal{G}$ is a generating set. □

Corollary 4.3. *The category* $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ *has a generator.*

Proof. To construct a generator in the category $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$, first we note that this category is cocomplete (for more information see [6, 14]). Now we define the finitely supported Cb-set $G \doteq \coprod_{\rho \in \mathcal{A}} (Cb \times \{x\})/\rho$ and show that G is a generator in the category $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$. To do so, consider two different equivariant maps $g, f : A \to B$, then as it is shown in the proof of Theorem 4.2, there exists the equivariant injective map $\bar{h} : (Cb \times \{x\})/\ker(h) \to A$ such that $f\bar{h} \neq g\bar{h}$. Since every finitely supported Cb-set has a zero, see Remark 2.17(iii), we define the set $\{k_\rho : (Cb \times \{x\})/\rho \to A\}_{\rho \in \mathcal{A}}$ to be:

$$\begin{cases} k_\rho = \theta & \text{for every } \rho \in \mathcal{A} \text{ with } \rho \neq \ker(h) \\ k_\rho = \bar{h} & \text{for } \rho = \ker(h), \end{cases}$$

in which θ is supposed to be a fixed zero element of A, and get the equivariant map $k : G \to A$, by the universal property of the coporoduct, with $f(k([(\iota, x)]_{\ker(h)})) \neq g(k([(\iota, x)]_{\ker(h)}))$. □

Definition 4.4. [2, Definition 0.6] We recall from category theory that: (a) an epimorphism $f : A \to B$ is called *extremal* (or *strong*) when it does not factor through any proper subobject of B; that is $f = i \circ e$ in which i is a monomorphism implies that i is an isomorphism.

(b) A generating set $\mathcal{G}$ in a cocomplete category is called *strong* if for every object K in the category there exists an extremal epimorphism from a coproduct of $\mathcal{G}$-objects to K.

Theorem 4.5. *The obtained generating set in Theorem* 4.2 *is strong.*

Proof. Suppose $K \in (Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ is generated by $\{a_1, a_2, \dots\}$. For $a_i \in \{a_1, a_2, \dots\}$, we define a map $l_i : \{x\} \to K$ mapping $x \mapsto a_i$. By the universal property of free object, there exists a unique equivariant map $l_{a_i} : Cb \times \{x\} \to K$ with $(\iota, x) \mapsto a_i$. Take $\ker l_{a_i} = \mathrm{k_i}$. Therefore, by the First Isomorphism Theorem, there exists the equivariant injective map $\bar{l}_{a_i} : (Cb \times \{x\})/\mathrm{k_i} \to K$ with $[(\sigma, x)]_{\mathrm{k_i}} \mapsto \sigma a_i$. We define $\varphi : \coprod_i (Cb \times$

$\{x\})/\mathrm{k_i} \to K$ with $[(\sigma, x)]_{\mathrm{k_i}} \mapsto \sigma a_i$. Clearly φ is an epimorphism from a coproduct of $\mathcal{G}$-objects to K. Now we show φ is extremal. So let $\varphi = me$ in which $m : A \to K$ is an injective equivariant map.

$$\begin{array}{ccc} \coprod_i (Cb \times \{x\})/\mathrm{k_i} & \xrightarrow{\varphi} & K \\ & \searrow^{e} & \uparrow m \\ & & A \end{array}$$

Since φ is epic (surjective), so is m, and we are done. □

5 Finitely presentable finitely supported *Cb*-sets

In a general algebraic category, an object A is said to be *finitely presentable* if its homfunctor $(A, -)$ preserves filtered colimits, and it is said to be *finitely presented* if it can be presented by a finite set of generators and a finite set of relations, see [2, Definition 1.1]. This means that there exists a finite set X (of generators) such that A can be obtained as a quotient of the free algebra $F(X)$ by a finitely generated congruence. We also recall that an algebra is said to be *finitely generated* if it is generated by a finite subset $X = \{x_1, \cdots, x_n\} \subseteq A$, see [4, Definition II.3.4]. In general, finitely generated and finitely presented are not equivalent concepts. Nonetheless, it is demonstrated [13, Theorem 5.16] that the classes of finitely generated objects and finitely presented objects coincide in the category **Nom**. Now the question is: is this statement true in the category $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$? So here, we describe finitely presentable objects in the category $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$.

But first we recall that a Cb-set X is called *decomposable* if there exist two Cb-subsets X_1, X_2 of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Otherwise X is called *indecomposable*, see [10, Definition I.5.7].

Theorem 5.1. (i) *If a finitely supported Cb-set X is finitely generated, then it is finitely generated as a nominal set.*

(ii) *Every finitely presented M-set is finitely generated.*

(iii) *Every cyclic M-set is indecomposable.*

(iv) *Let $X_i \subseteq X$ $(i \in I)$ be a family of indecomposable M-subsets of an M-set X such that $\bigcap_{i\in I} X_i \neq \emptyset$. Then $\bigcup X_i$ is an indecomposable M-subset of X.*

(v) *Every M-set X has a unique decomposition into indecomposable M-subsets.*

Proof. (i) Let X be generated by $\{x_1, \dots, x_n\}$. Then $X = \bigcup_{i=1}^{n} Cbx_i$. Since for every $i = 1, \dots, n$, by Lemma 2.23, $Cbx_i = \mathrm{Perm}(\mathbb{D})S'_{x_i}x_i \cup \mathrm{Perm}(\mathbb{D})x_i$ in which $S'_{x_i}x_i$ is finite, Cbx is a union of disjoint cyclic nominal sets and so X is a finite union of cyclic nominal sets. Hence X as a nominal set is finitely generated.

(ii) Since X is finitely presented, there exist a finitely generated free M-set $F(B)$ over a finite set B and a finitely generated congruence ρ such that $\psi : F(B)/\rho \to X$ is an isomorphism. Since $F(B)$ is finitely generated, $F(B)/\rho$ is finitely generated and we get the result.

(iii) By [10, Proposition I.5.8].

(iv) By [10, Lemma I.5.9].
(v) By [10, Theorem I.5.10]. □

Before presenting the following proposition, it will be useful to keep in mind Remark 2.22(ii), which states that a cyclic finitely supported Cb-set X is in the form of $X = Cbx$, for some $x \in X$, and Corollary 3.18, which states that $F(X) = (Cb \times X)/R_X$.

Proposition 5.2. *Let X be a finitely supported Cb-set. Then*
(i) *for all $\sigma \in Cb$, there exists an epimorphism (surjective equivariant map) $\varphi : F(X) \to X$ defined by $\varphi(x_\sigma) = \sigma x$.*
(ii) *if X is finitely generated, then $F(X)$ is finitely generated.*

Proof. (i) First we note that φ is well-defined. For, if $x_\sigma = x'_{\sigma'}$, then there exists $\pi_1 \in \mathrm{Perm}(\mathbb{D})$ with $\pi_1 x = x'$ and $\sigma'\pi_1|_{\mathrm{supp}\, x} = \sigma|_{\mathrm{supp}\, x}$. Thus, $\sigma'x' = \sigma'\pi_1 x = \sigma x$. Also φ is equivariant, because $\sigma_1\varphi(x_\sigma) = \sigma_1\sigma x = \varphi(x_{\sigma_1\sigma}) = \varphi(\sigma_1 x_\sigma)$, for all $\sigma_1 \in Cb$. Furthermore, $\varphi(x_\iota) = x$, for all $x \in X$, that is φ is surjective, and we are done.

(ii) Let X be finitely generated. Then by Theorem 5.1(i), X is finitely generated as a nominal set. Thus, $X = \coprod_{i=1}^{n} \mathrm{Perm}(\mathbb{D})x_i$ is a finite coproduct of cyclic nominal sets $\mathrm{Perm}(\mathbb{D})x_i$. Now we show that $F(X) = \dot{\bigcup}_{i=1}^{n} Cb(x_i)_\iota$ is a finite disjoint union of cyclic finitely supported Cb-sets. (Note that $Cb(x_i)_\iota$ is an instance of Cbx for $x = (x_i)_\iota$, where $(x_i)_\iota$ is an instance of the notation x_n for $x = x_i$ and $n = \iota$). To prove the nontrivial part, let $a \in F(X)$. Then there exist $\sigma \in Cb$ and $x \in X$ with $a = x_\sigma$. Since $x \in X$, there exist $\pi \in \mathrm{Perm}(\mathbb{D})$ and $1 \leq i \leq n$ with $x = \pi x_i$. Now, $a = (\pi x_i)_\sigma = (x_i)_{\sigma\pi} = \sigma\pi(x_i)_\iota \in Cb(x_i)_\iota$ as required. □

In the sequel, we show that the finitely presentable finitely supported Cb-sets are exactly finitely generated ones, see Theorem 5.18.

Definition 5.3. Let X be a finitely supported Cb-set. For all non-zero $x, x' \in X$, define

$$G_{x,x'} = \{\pi \in \mathrm{Perm}(\mathbb{D}) \mid \pi x = x',\ \mathrm{supp}\,\pi \subseteq \mathrm{supp}\,x \cup \mathrm{supp}\,x'\}.$$

Lemma 5.4. *The set $G_{x,x'}$ is empty or isomorphic with a subset of* $\mathrm{Sym}(C)$ *where $C = \mathrm{supp}\,x \cup \mathrm{supp}\,x'$.*

Proof. Let $G_{x,x'}$ be non-empty and $C = \mathrm{supp}\,x \cup \mathrm{supp}\,x'$. Then define the assignment $\varphi : G_{x,x'} \to \mathrm{Sym}(C)$ by $\varphi(\pi) = \pi|_C$. Notice that, since $\mathrm{supp}\,\pi \subseteq C$, we have $\pi|_C \in \mathrm{Sym}(C)$. So φ is well-defined. Now we show that φ is an injective map. Let $\varphi(\pi_1) = \varphi(\pi_2)$. Then $\pi_1|_C = \pi_2|_C$. If $d \notin C$, then since $\mathrm{supp}\,\pi_1, \mathrm{supp}\,\pi_2 \subseteq C$, we get that $\pi_1 d = d = \pi_2 d$. Therefore, $\pi_1 = \pi_2$. □

Remark 5.5. It is worth noting that since $\mathrm{Sym}(C)$ is finite, $G_{x,x'}$ is finite.

Corollary 5.6. *Suppose X is a finitely supported Cb-set. If $\pi\delta x = \delta' x'$, for some non-zero elements $x, x' \in X$, $\pi \in$* $\mathrm{Perm}(\mathbb{D})$*, and $\delta, \delta' \in S$, then there exists $\pi_1 \in G_{\delta x, \delta' x'}$ with $\pi_1|_{\mathrm{supp}\,\delta x} = \pi|_{\mathrm{supp}\,\delta x}$ and $\pi_1 \delta x = \pi\delta x = \delta' x'$.*

Proof. Since $\pi\delta x = \delta' x'$, we get $\pi \mathrm{supp}\,\delta x = \mathrm{supp}\,\delta' x'$. So $\pi|_{\mathrm{supp}\,\delta x} : \mathrm{supp}\,\delta x \to \mathrm{supp}\,\delta' x'$ is a bijective map. Now using the Homogeneity Lemma 2.12, there exists $\pi_1 \in \mathrm{Perm}(\mathbb{D})$ with $\pi_1|_{\mathrm{supp}\,\delta x} = \pi|_{\mathrm{supp}\,\delta x}$, and $\pi_1 d = d$ for all $d \notin \mathrm{supp}\,\delta x \cup \mathrm{supp}\,\delta' x'$. Thus, $\mathrm{supp}\,\pi_1 \subseteq \mathrm{supp}\,\delta x \cup \mathrm{supp}\,\delta' x'$ and $\pi_1 \delta x = \pi\delta x = \delta' x'$. □

Proposition 5.7. *Let X be a finitely supported Cb-set and $x \in X, \delta, \delta'_1 \in S$. Then, for the equivalence class $(\delta x)_{\pi\delta'_1}$ we have the following cases:*

(i) *If $\delta \in S'_x$ and $\delta'_1 \in S'_{\delta x}$, then $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_{\pi\delta''_1}$ with $\mathbb{D}_{\delta_1} \subseteq \mathrm{supp}\,x$ and $\mathbb{D}_{\delta''_1} \subseteq \mathrm{supp}\,\delta_1 x$.*

(ii) *If* $\delta \in S'_x$ *and* $\delta'_1 \in S_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_\pi$ *with* $\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} x$.

(iii) *If* $\delta \in S_x$ *and* $\delta'_1 \in S'_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (x)_{\pi\delta''_1}$ *with* $\mathbb{D}_{\delta''_1} \subseteq \operatorname{supp} x$.

(iv) *If* $\delta \in S_x$ *and* $\delta'_1 \in S_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (x)_\pi$.

Proof. (i) If $\delta \in S'_x$ and $\delta'_1 \in S'_{\delta x}$, then by Corollary 2.24 there exist $\delta_1 \in S'_x$ and $\delta''_1 \in S'_{\delta x}$ with

$$\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} x,\ \delta x = \delta_1 x,\ \mathbb{D}_{\delta''_1} \subseteq \operatorname{supp} \delta_1 x,\ \delta'_1|_{\operatorname{supp}\delta_1 x} = \delta''_1|_{\operatorname{supp}\delta_1 x}.$$

So applying Lemma 3.17(i) we have $(\delta x)_{\pi\delta'} = (\delta_1 x)_{\pi\delta''_1}$.

(ii) If $\delta'_1 \in S_{\delta x}$, then by Lemma 2.21(i), $\mathbb{D}_{\delta'_1} \cap \operatorname{supp}\delta x = \emptyset$. So $\delta'_1|_{\operatorname{supp}\delta x} = \iota|_{\operatorname{supp}\delta x}$. Also, by (i), $\delta x = \delta_1 x$. Now, applying (i) and Lemma 3.17(i), we have $\iota\delta x = \delta_1 x$ and $\pi|_{\operatorname{supp}\delta_1 x} = \pi\delta'_1|_{\operatorname{supp}\delta_1 x}$. So $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_\pi$.

Items (iii) and (iv) follow from items (i) and (ii). □

Corollary 5.8. *Let* X *be a finitely supported Cb-set and* $x \in X, \delta, \delta' \in S$. *Then,*

(i) *the set* $\overline{S'_x} = \{\delta \in S'_x \mid \mathbb{D}_\delta \subseteq \operatorname{supp} x\}$ *is a finite set.*

(ii) *If* $\delta \in S'_x$ *and* $\delta' \in S'_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_{\pi\delta''_1}$ *with* $\delta_1 \in \overline{S'_x}$ *and* $\delta''_1 \in \overline{S'_{\delta_1 x}}$.

(iii) *If* $\delta \in S'_x$ *and* $\delta'_1 \in S_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_\pi$ *with* $\delta_1 \in \overline{S'_x}$.

(iv) *If* $\delta \in S_x$ *and* $\delta'_1 \in S'_{\delta x}$*, then* $(\delta x)_{\pi\delta'_1} = (x)_{\pi\delta''_1}$ *with* $\delta''_1 \in \overline{S'_{\delta_1 x}}$.

Proof. (i) If $|\operatorname{supp} x| = n$, then we get that $|\overline{S'_x}| = \sum_{i=1}^n 2^i\binom{n}{i}$. So $\overline{S'_x}$ is finite. The other parts follows from (i) and Proposition 5.7. □

Lemma 5.9. *Let* X *be a finitely supported Cb-set,* $x, x' \in X$*, and*

$$B_{x,x'} \doteq \{((\delta x)_{\pi_1\delta'_1}, (\delta' x')_{\delta'_2}) \in F(X)\times F(X) \mid\ \pi_1 \in G_{\delta'_1\delta x, \delta'_2\delta' x'}, \delta, \delta_1, \delta'_1, \delta'_2 \in S\}.$$

Then $B_{x,x'}$ *is a finite subset of* $\ker\varphi$*, where* $\varphi : F(X) \to X$ *is given in Proposition* 5.2.

Proof. First we note that, by Remark 5.5, $G_{x,x'}$ is finite. Now, we show that $B_{x,x'}$ is finite, for possible cases which occur for $\delta, \delta'_1, \delta', \delta'_2 \in S$. According to Proposition 5.7 we have four cases for $(\delta x)_{\pi_1\delta'_1}$ (similarly for $(\delta' x')_{\delta'_2}$). In each case, one can prove the number of the equivalence classes $(\delta x)_{\pi_1\delta'_1}$ and $(\delta' x')_{\delta'_2}$ are finite. For instance, when $\delta \in S'_x$ and $\delta'_1 \in S'_{\delta x}$, by Corollary 5.8(ii), we have $(\delta x)_{\pi\delta'_1} = (\delta_1 x)_{\pi\delta''_1}$ with $\delta_1 \in \overline{S'_x}$ and $\delta''_1 \in \overline{S'_{\delta_1 x}}$. Now since by Corollary 5.8(i), $\overline{S'_x}$ is finite, in this case the number of the equivalence classes $(\delta x)_{\pi_1\delta'_1}$ is finite. The other cases are analogous.

We show that $B_{x,x'} \subseteq \ker\varphi$. Suppose $((\delta x)_{\pi_1\delta'_1}, (\delta' x')_{\delta'_2}) \in B_{x,x'}$. Since $\pi_1 \in G_{\delta'_1\delta x, \delta'_2\delta' x'}$, $\pi_1\delta'_1\delta x = \delta'_2\delta' x$, we get that $\varphi((\delta x)_{\pi_1\delta'_1})) = \varphi((\delta' x)_{\delta'_2})$. □

Lemma 5.10. *Let X be a finitely generated finitely supported Cb-set. Then* $\ker\varphi$ *is finitely generated, where* $\varphi : F(X) = (Cb \times X)/R_X \to X$ *is given in Proposition* 5.2.

Proof. By the hypothesis, one can suppose $X = \bigcup_{i=1}^{k} Cbx_i$. Now take $B \doteq \bigcup_{1\le i,j\le k} B_{x_i,x_j}$, where the B_{x_i,x_j} are defined in Lemma 5.9. Then by Lemma 5.9, B is a finite subset of $F(X) \times F(X)$. We show that $\ker\varphi = \rho(B)$ and so $\ker\varphi$ is finitely generated. Indeed, since by Lemma 5.9, $B_{x_i,x_j} \subseteq \ker\varphi$, for every $1 \le i, j \le k$, $B \subseteq \ker\varphi$ and hence $\rho(B) \subseteq \ker\varphi$.

To prove the reverse inclusion, suppose $((\sigma_1 x_1)_\sigma, (\sigma_2 x_2)_{\sigma'}) \in \ker\varphi$. Then by Theorem 2.15, $\sigma, \sigma', \sigma_1, \sigma_2 \in \mathrm{Perm}(\mathbb{D}) \cup \mathrm{Perm}(\mathbb{D})S = Cb$. Hence, several cases may occur for $\sigma, \sigma', \sigma_1, \sigma_2$. We take $\sigma, \sigma', \sigma_1, \sigma_2 \in \mathrm{Perm}(\mathbb{D})S$ and show that $((\sigma_1 x_1)_\sigma, (\sigma_2 x_2)_{\sigma'}) \in \rho(B)$, other cases will be proved analogously. Let $\sigma = \pi\delta$, $\sigma' = \pi'\delta'$, $\sigma_1 = \pi_1\delta_1$, and $\sigma_2 = \pi_2\delta_2$ in which $\delta_1 \in S'_{x_1}$, $\delta_2 \in S'_{x_2}$, $\pi, \pi', \pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$. Then since $((\sigma_1 x_1)_\sigma, (\sigma_2 x_2)_{\sigma'}) \in \ker\varphi$, we get that $\varphi((\pi_1\delta_1 x_1)_{\pi\delta}) = \varphi((\pi_2\delta_2 x_2)_{\pi'\delta'})$, and hence we have $\pi\delta\pi_1\delta_1 x_1 = \sigma\sigma_1 x_1 = \sigma'\sigma_2 x_2 = \pi'\delta'\pi_2\delta_2 x_2$. By Remark 2.14, $\delta\pi_1 = \pi_1\delta'_1$ and $\delta'\pi_2 = \pi_2\delta'_2$ where $\mathbb{D}_{\delta'_1} = \{\pi_1^{-1}d : d \in \mathbb{D}_\delta\}$ and $\mathbb{D}_{\delta'_2} = \{\pi_2^{-1}d : d \in \mathbb{D}_{\delta'}\}$. Hence $\pi\pi_1\delta'_1\delta_1 x_1 = \pi'\pi_2\delta'_2\delta_2 x_2$ and so $\pi_2^{-1}\pi'^{-1}\pi\pi_1\delta'_1\delta_1 x_1 = (\pi'\pi_2)^{-1}\pi\pi_1\delta'_1\delta_1 x_1 = \delta'_2\delta_2 x_2$. Now, applying Corollary 5.6 to $\pi_2^{-1}\pi'^{-1}\pi\pi_1 \in \mathrm{Perm}(\mathbb{D})$ and $\delta'_1\delta_1, \delta'_2\delta_2 \in S$, there exists $\pi_3 \in G_{\delta'_1\delta_1 x_1, \delta'_2\delta_2 x_2}$ with $\pi_3\delta'_1\delta_1 x_1 = \delta'_2\delta_2 x_2$. We have

$$\begin{aligned}(\sigma_1 x_1)_\sigma &= [(\sigma, \sigma_1 x_1)]_R = [(\pi\delta, \pi_1\delta_1 x_1)]_R = [(\pi\pi_1\delta, \delta_1 x_1)]_R \\ &= [(\pi\pi_1\pi_3^{-1}\pi_3\delta, \delta_1 x_1)]_R = \pi\pi_1\pi_3^{-1}[(\pi_3\delta, \delta_1 x_1)]_R = \pi\pi_1\pi_3^{-1}(\delta_1 x_1)_{\pi_3\delta}.\end{aligned}$$

Since $\pi_2^{-1}\pi'^{-1}\pi\pi_1\delta'_1\delta_1 x_1 = \delta'_2\delta_2 x_2$, we get

$$(\delta'_1\delta_1 x_1)_{\pi\pi_1} = [(\pi\pi_1, \delta'_1\delta_1 x_1)]_R = [(\pi'\pi_2, \delta'_2\delta_2 x_2)]_R = (\delta'_2\delta_2 x_2)_{\pi'\pi_2},$$

and so

$$\pi\pi_1\pi_3^{-1}(\delta_1'\delta_1x_1)_{\pi_3} = \pi\pi_1\pi_3^{-1}[(\pi_3, \delta_1'\delta_1x_1)]_R = \pi'\pi_2[(\iota, \delta_2'\delta_2x_2)]_R = \pi'\pi_2(\delta_2'\delta_2x_2)_\iota.$$

Also we have

$$\begin{aligned}(\sigma_2x_2)_{\sigma'} &= [(\sigma', \sigma_2x_2)]_R = [(\pi'\delta', \pi_2\delta_2x_2)]_R = [(\pi'\pi_2\delta', \delta_2x_2)]_R \\ &= \pi'\pi_2[(\delta', \delta_2x_2)]_R = \pi'\pi_2(\delta_2x_2)_{\delta'}.\end{aligned}$$

Hence we have the following equalities:

$$a = (\sigma_1x_1)_\sigma = \pi\pi_1\pi_3^{-1}(\delta_1x_1)_{\pi_3\delta}, \quad \pi\pi_1\pi_3^{-1}(\delta_1'\delta_1x_1)_{\pi_3} = \pi'\pi_2(\delta_2'\delta_2x_2)_\iota,$$

$$\pi'\pi_2(\delta_2x_2)_{\delta'} = (\sigma_2x_2)_{\sigma'}.$$

Notice that $((\delta_1x_1)_{\pi_3\delta}, (\delta_1'\delta_1x_1)_{\pi_3})$, $((\delta_2'\delta_2x_2)_\iota, (\delta_2x_2)_{\delta'}) \in B$. Thus

$$\begin{array}{ccc} a = (\sigma_1x_1)_\sigma = \pi\pi_1\pi_3^{-1}(\delta_1x_1)_{\pi_3\delta} & & \pi'\pi_2(\delta_2'\delta_2x_2)_\iota \\ \Big| B & & \Big| B \\ \pi\pi_1\pi_3^{-1}(\delta_1'\delta_1x_1)_{\pi_3} & = & \pi'\pi_2(\delta_2x_2)_{\delta'} = b \end{array}$$

and so by Lemma 2.1 we get the desired result. □

Theorem 5.11. *Let X be a finitely supported Cb-set. Then X is a finitely generated Cb-set if and only if X is finitely presented.*

Proof. (⇒) Since X is finitely generated, by Proposition 5.2(ii), $L(X)$ is finitely generated. Also Lemma 5.10 implies that $\ker\varphi$ is finitely generated. Thus, $X(\simeq L(X)/\ker\varphi)$ is finitely presented.

(⇐) This part holds by Theorem 5.1(ii). □

In the sequel, we give a characterization of finitely presentable objects in Cb-**Set**. But first, we mention a number of facts in the following theorem.

Theorem 5.12. (i) *If $X = \coprod_{i\in I} X_i$ is a coproduct of indecomposable finitely supported Cb-sets X_i, then the X_i's are retracts of X.*

(ii) *Finitely presentable objects are closed under finite colimits.*

(iii) *A nominal set is a finitely presentable object of* **Nom** *if and only if it is orbit-finite.*

Proof. (i) First notice that, by Remark 2.17, every finitely supported Cb-set X_i has a zero element θ. Now define $\varphi : X \to X_i$ by $\varphi(x) = \theta$ if $x \notin X_i$ and $\varphi(x) = x$ if $x \in X_i$. Clearly φ is equivariant. Also $\varphi|_{X_i} = id_{X_i}$. Therefore X_i is a retract of X, for every $i \in I$.

(ii) By [3, Lemma 5.11].

(iii) By [13, Theorem 5.16]. □

Lemma 5.13. *Every finitely presentable finitely supported Cb-set is finitely generated.*

Proof. Let X be an arbitrary finitely presentable finitely supported Cb-set and $D : \mathcal{I} \to (Cb\text{-}\mathbf{Set})_{\text{fs}}$ be a functor in which $\mathcal{I}$ is a small filtered category. Then since by Remark 3.11(iii) $K \dashv \Delta \dashv Z$, we have

$$\begin{aligned} Hom_{\mathbf{Nom}}(K(X), \text{colim}_j D(j)) &\cong Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, \Delta(\text{colim}_j D(j))) \\ &\cong Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, \text{colim}_j \Delta D(j)) \\ &\cong \text{colim}_j Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, \Delta D(j)). \end{aligned}$$

On the other hand, since $K \dashv \Delta$, we have

$$Hom_{\mathbf{Nom}}(K(X), D(j)) \cong Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, \Delta D(j)),$$

for every $j \in \mathcal{I}$. Thus,

$$Hom_{\mathbf{Nom}}(K(X), \text{colim}_j D(j)) \cong \text{colim}_j Hom_{\mathbf{Nom}}(K(X), D(j)),$$

meaning that $K(X) = X/\!\sim$ is a finitely presentable nominal set and hence, by Theorem 5.12(iii), $X/\!\sim\, = \bigcup_{i=1}^{n} \text{Perm}(\mathbb{D})([t_i]_\sim)$ in which $t_i \in X$. Now we show that $A = \{t_1, \ldots, t_n\}$ is a finite generator for the Cb-set X and so X is finitely generated. It is clear that $Cbt_i \subseteq X$, for every $i = 1, \ldots, n$. Let $y \in X = \bigcup_{x \in X} Cbx$. Then there exists $x \in X$ with $y = \pi x$ or $y = \pi\delta x$. So $y \sim x$ or $y \sim \delta x$. Thus $[y]_\sim \in X/\!\sim$ and so there exist $t_{i_0} \in X$ and $\pi_1 \in \text{Perm}(\mathbb{D})$ with $[y]_\sim = \pi_1[t_{i_0}]_\sim = [(\pi_1 t_{i_0})]_\sim$. Therefore, there exists $\pi_2 \in \text{Perm}(\mathbb{D})$ with $y = \pi_2\pi_1 t_{i_0}$ which implies that $y \in Cbt_{i_0}$. □

Remark 5.14. Let X be a finitely supported Cb-set generated by $\{x_1, \ldots, x_k\}$.

(i) For every $i, j = 1, \dots, k$, we define

$$\begin{aligned}B_{i,j} = & \{(\pi\delta_i, \delta_j) \mid \pi \in G_{\delta_i x_i, \delta_j x_j}, \mathbb{D}_{\delta_i} \subseteq \operatorname{supp} x_i, \mathbb{D}_{\delta_j} \subseteq \operatorname{supp} x_j\} \\ & \cup \{(\pi, \iota)) \mid \pi \in G_{x_i, x_j}\} \\ & \cup \{(\pi, \delta_j)) \mid \pi \in G_{x_i, \delta_j x_j}, \mathbb{D}_{\delta_j} \subseteq \operatorname{supp} x_j\} \\ & \cup \{(\pi\delta_i, \iota)) \mid \pi \in G_{\delta_i x_i, x_j}, \mathbb{D}_{\delta_i} \subseteq \operatorname{supp} x_i\}.\end{aligned}$$

By Remark 5.5, the $G_{x,x'}$'s are finite, for all $x, x' \in X$. Also by Corollary 5.8(i) we have:

$$\begin{aligned}B_{i,j} = & \{(\pi\delta_i, \delta_j) \mid \pi \in G_{\delta_i x_i, \delta_j x_j}, \delta_t \in \overline{S'_{x_t}}, t = i, j\} \\ & \cup \{(\pi, \iota)) \mid \pi \in G_{x_i, x_j}\} \\ & \cup \{(\pi, \delta_j)) \mid \pi \in G_{x_i, \delta_j x_j}, \delta_j \in \overline{S'_{x_j}}\} \\ & \cup \{(\pi\delta_i, \iota)) \mid \pi \in G_{\delta_i x_i, x_j}, \delta_i \in \overline{S'_{x_i}}\}.\end{aligned}$$

Since the $\overline{S'_x}$'s are finite, we get that $B_{i,j}$, for $i, j = 1, \dots, k$, is finite.

(ii) For given $x, x' \in X$, define $A_{x,x'} = \{(\sigma, \sigma') \in Cb \times Cb \mid \sigma x = \sigma' x'\}$.

Lemma 5.15. *Suppose $X \in (Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ is generated by $\{x_1, \dots, x_k\}$. For each finitely supported Cb-set Y and $y_1, \dots, y_k \in Y$, there exists at most one equivariant map $f : X \to Y$ with $f(x_i) = y_i$; and exactly one if and only if*

$$B_{i,j} \subseteq A_{y_i, y_j}, \quad \operatorname{supp} y_i \subseteq \operatorname{supp} x_i, \quad \operatorname{supp} \delta y_i \subseteq \operatorname{supp} \delta x_i, \quad (*)$$

where $\delta \in S'_{x_i}$.

Proof. Suppose there exist the equivariant maps $f, f' : X \to Y$ with $f(x_i) = f'(x_i) = y_i$, for all $i = 1, \dots, k$. Then since $\{x_1, \dots, x_k\}$ is a generator for X, for every $x \in X$ there exist $\sigma \in Cb$ and $i \in \{1, \dots, k\}$ with $x = \sigma x_i$. So

$$f(x) = f(\sigma x_i) = \sigma f(x_i) = \sigma f'(x_i) = f'(\sigma x_i) = f'(x).$$

This proves that $f = f'$.

Now suppose $f : X \to Y$ is an equivariant map with $f(x_i) = y_i$, for every $i = 1, \dots, k$. Assume $(\sigma, \sigma') \in B_{i,j}$ such that $\sigma = \pi\delta_i$ and $\sigma' = \delta_j$.

We prove that $(\sigma, \sigma') \in A_{y_i, y_j}$. Other cases will be proved analogously. The assumption $(\pi\delta_i, \delta_j) \in B_{i,j}$ implies that $\pi \in G_{\delta_i x_i, \delta_j x_j}$. So $\pi\delta_i x_i = \delta_j x_j$. Now, since f is equivariant, we get

$$\pi\delta_i y_i = \pi\delta_i f(x_i) = f(\pi\delta_i x_i) = f(\delta_j x_j) = \delta_j f(x_j) = \delta_j y_j.$$

Thus $(\sigma, \sigma') = (\pi\delta_i, \delta_j) \in A_{y_i, y_j}$. We also have $\operatorname{supp} y_i = \operatorname{supp} f(x_i) \subseteq \operatorname{supp} x_i$. Since f is equivariant and $f(\delta x_i) = \delta y_i$, we get $\operatorname{supp} \delta y_i \subseteq \operatorname{supp} \delta x_i$, for all $\delta \in S'_{x_i}$.

Conversely, suppose x_i's and y_i's satisfy $(*)$. We show that the equivariant subset $f = \{(\sigma x_i, \sigma y_i) \mid \sigma \in Cb\} \subseteq X \times Y$ is single-valued. Let $\sigma, \sigma' \in Cb$ with $\sigma x_i = \sigma' x_j$ where $i, j = 1, \ldots, k$. By Remark 2.22, we have the following cases;

Case (1): $\sigma = \pi\delta$ and $\sigma' = \pi'\delta'$ where $\delta \in S'_{x_i}$ and $\delta' \in S'_{x_j}$.
Case (2): $\sigma = \pi\delta$ and $\sigma' = \pi'\delta'$ where $\delta \in S'_{x_i}$ and $\delta' \in S_{x_j}$.
Case (3): $\sigma = \pi\delta$ and $\sigma' = \pi'\delta'$ where $\delta \in S_{x_i}$ and $\delta' \in S'_{x_j}$.
Case (4): $\sigma = \pi\delta$ and $\sigma' = \pi'\delta'$ where $\delta \in S_{x_i}$ and $\delta' \in S_{x_j}$.

Here we prove the first case. The other cases are proved analogously. If Case (1) holds, then by Corollary 2.24 there are $\delta_1 \in S'_{x_i}$ and $\delta'_1 \in S'_{x_j}$ with

$$\delta|_{\operatorname{supp} x_i} = \delta_1|_{\operatorname{supp} x_i}, \delta'|_{\operatorname{supp} x_j} = \delta'_1|_{\operatorname{supp} xji}, \ \delta x_i = \delta_1 x_i, \ \delta' x_j = \delta'_1 x_j, \text{and}$$

$$\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} x_i, \ \mathbb{D}_{\delta'_1} \subseteq \operatorname{supp} x_j.$$

Since $\pi\delta_1 x_i = \pi'\delta'_1 x_j$ we have $\pi'^{-1}\pi\delta_1 x_i = \delta'_1 x_j$. Applying Corollary 5.6, there exists $\pi_1 \in G_{\delta_1 x_i, \delta'_1 x_j}$ with $\pi_1\delta_1 x_i = \pi'^{-1}\pi\delta_1 x_i = \delta'_1 x_j$ and $\pi_1|_{\operatorname{supp} \delta_1 x_i} = \pi'^{-1}\pi|_{\operatorname{supp} \delta_1 x_i}$. Thus, $(\pi_1\delta_1, \delta'_1) \in B_{i,j}$. So $(\pi_1\delta_1, \delta'_1) \in A_{y_i, y_j}$. By $(*)$, since $\operatorname{supp} y_i \subseteq \operatorname{supp} x_i$ and since $\delta|_{\operatorname{supp} x_i} = \delta_1|_{\operatorname{supp} x_i}$, $\delta'|_{\operatorname{supp} x_j} = \delta'_1|_{\operatorname{supp} x_j}$ we get that $\delta|_{\operatorname{supp} y_i} = \delta_1|_{\operatorname{supp} y_i}$, $\delta'|_{\operatorname{supp} y_j} = \delta'_1|_{\operatorname{supp} y_j}$. Thus $\pi_1\delta y_i = \pi_1\delta_1 y_i = \delta'_1 y_i = \delta' y_i$.

Notice that, by $(*)$, we have $\operatorname{supp} \delta y_i \subseteq \operatorname{supp} \delta x_i$. Hence, $\pi_1|_{\operatorname{supp} \delta y_i} = \pi'^{-1}\pi|_{\operatorname{supp} \delta y_i}$. Therefore, $\delta' y_j = \pi_1\delta y_i = \pi'^{-1}\pi\delta y_i$. Now, for every $x \in X$ there exist $\sigma \in Cb$ and x_i with $x = \sigma x_i$. So $f(x) = f(\sigma x_i) = \sigma y_i$ and also for every $\sigma_1 \in Cb$ we have $\sigma_1 f(x) = \sigma_1 f(\sigma x_i) = \sigma_1\sigma y_i = f(\sigma_1\sigma x_i) = f(\sigma_1 x)$. □

Remark 5.16. By [14, Lemma 3.5], the functor M-**Set** $\to$ $(M\text{-}\mathbf{Set})_{\text{fs}}, X \mapsto X_{\text{fs}}$ is a right adjoint to the inclusion functor $(M\text{-}\mathbf{Set})_{\text{fs}} \hookrightarrow M\text{-}\mathbf{Set}$. Hence the inclusion functor preserves all colimits. So one can infer that the colimits in the category of finitely supported Cb-sets are computed at the level of the category of Cb-sets.

Proposition 5.17. *Given a small filtered category $\mathcal{I}$ and a functor $D : \mathcal{I} \to (Cb\text{-}\mathbf{Set})_{\text{fs}}$, consider the colimit of the filtered diagram $D(\mathcal{I})$, $\text{colim}_{i\in\mathcal{I}} D(i)$, with colimit injections denoted by the ι_i's and connecting morphism from $D(i)$ to $D(j)$ denoted by ι_{ij}. Then for all $y \in \text{colim}_{i\in\mathcal{I}} D(i)$ there exist $k \in \mathcal{I}$ and $x \in D(k)$ with $y = \iota_k(x)$ such that* $\operatorname{supp} y = \operatorname{supp} x$.

Proof. Existence of $k \in \mathcal{I}$ and $x \in D(k)$ immediately follow from the definition of filtered colimit in $(Cb\text{-}\mathbf{Set})_{\text{fs}}$. Notice that, since ι_k is equivariant, by Theorem 2.6(i), $\operatorname{supp} y = \operatorname{supp} \iota_k(x) \subseteq \operatorname{supp} x$. Now if $\operatorname{supp} y \subsetneq \operatorname{supp} x$, then take $\delta \in S$ with $\mathbb{D}_\delta = (\operatorname{supp} x) \setminus \operatorname{supp} y$. We show that $\operatorname{supp} y = \operatorname{supp} \delta x$, where $\delta x \in D(k)$. Indeed, since $\mathbb{D}_\delta \cap \operatorname{supp} y = \emptyset$, we have $\delta y = y$. So $\operatorname{supp} y = \operatorname{supp} \delta y = \operatorname{supp} \delta \iota_k(x) = \operatorname{supp} \iota_k(\delta x) \subseteq \operatorname{supp} \delta x$. If $d \in \operatorname{supp} \delta x$ and $d \notin \operatorname{supp} y$, then $d \in (\operatorname{supp} x) \setminus \operatorname{supp} y$ and $d \notin \mathbb{D}_\delta$ which is impossible because, $(\operatorname{supp} x) \setminus \operatorname{supp} y = \mathbb{D}_\delta$. Therefore, in this case, there exist $k \in \mathcal{I}$ and $x_0 = \delta x \in D(k)$ with $y = \iota_k(x_0)$ and $\operatorname{supp} y = \operatorname{supp} x_0$. □

Theorem 5.18. *Let X be a finitely supported Cb-set. Then X is finitely presentable if and only if X is finitely generated.*

Proof. The 'only if' direction follows from Lemma 5.13. For the 'if' direction, let X be generated by $\{x_1, \ldots, x_k\}$, and $D : \mathcal{I} \to (Cb\text{-}\mathbf{Set})_{\text{fs}}$ be a filtered diagram in $(Cb\text{-}\mathbf{Set})_{\text{fs}}$. Then the diagram

$$\mathcal{I} \xrightarrow{D} (Cb\text{-}\mathbf{Set})_{\text{fs}} \xrightarrow{Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X,-)} \mathbf{Set}$$

is filtered in the category of sets in which each connecting morphism $\iota_{ij} : D(i) \to D(j)$ is assigned to the connecting map $\mathsf{u}_{ij} : Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, D(i)) \to Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, D(j))$, defined by $\mathsf{u}_{ij}(h) = \iota_{ij} \circ h$, for every $h \in Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, D(i))$. Now, consider the colimit cocone

$$(\mathsf{u}_l : Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, D(l)) \to \text{colim}_{l\in\mathcal{I}} Hom_{(Cb\text{-}\mathbf{Set})_{\text{fs}}}(X, D(l)))_{l\in\mathcal{I}},$$

where every element of $\mathrm{colim}_{l\in\mathcal{I}} Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(l))$ is of the form of $\mathsf{u}_j(h)$, for some $j \in \mathcal{I}$ and $h : X \to D(j)$. So we can define the map

$$\varphi : \mathrm{colim}_{l\in\mathcal{I}} Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(l)) \to Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, \mathrm{colim}_{l\in\mathcal{I}} D(l))$$

by $\varphi(\mathsf{u}_j(h)) = \iota_j \circ h$ where $\iota_j : D(j) \to \mathrm{colim}_{l\in\mathcal{I}} D(l)$ denotes the jth colimit injection. First we show that φ is well-defined. Indeed, if $\mathsf{u}_j(h_1) = \mathsf{u}_t(h_2)$, for some $j, t \in \mathcal{I}$, $h_1 \in Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(j))$, and $h_2 \in Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(t))$, then using the filteredness of the diagram $Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(\mathcal{I}))$, there exist $k \in \mathcal{I}$ and connecting $\mathsf{u}_{jk} : j \to k$ and $\mathsf{u}_{tk} : j \to k$ such that

$$\mathsf{u}_{jk}(h_1) = \iota_{jk} \circ h_1 = \iota_{tk} \circ h_2 = \mathsf{u}_{tk}(h_2),$$

and hence

$$\iota_j \circ h_1 = \iota_k \circ \iota_{jk} \circ h_1 = \iota_k \circ \iota_{tk} \circ h_2 = \iota_t \circ h_2.$$

Therefore

$$\varphi(\mathsf{u}_j(h_1)) = \iota_j \circ h_1 = \iota_t \circ h_2 = \varphi(\mathsf{u}_t(h_2)).$$

Now, to prove that φ is a bijection, we use Lemma 5.15 and show that each member of $Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, \mathrm{colim}_{l\in\mathcal{I}} D(l))$ is the image of exactly one element of $\mathrm{colim}_{l\in\mathcal{I}} Hom_{(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}}(X, D(l))$ under φ. Indeed, since for each equivariant map $g : X \to \mathrm{colim}_{l\in\mathcal{I}} D(l)$ we have $g(x_i) \in \mathrm{colim}_{l\in\mathcal{I}} D(l)$, by Proposition 5.17, there exist $k_i \in \mathcal{I}$ and $y_i \in D(k_i)$ with $g(x_i) = \iota_{k_i}(y_i)$ and $\mathrm{supp}\, y_i = \mathrm{supp}\, g(x_i)$, for all $i = 1, \dots, k$. Now for every $(\pi\delta_i, \delta_j) \in B_{i,j}$ with $\pi \in G_{\delta_i x_i, \delta_j x_j}$, $\mathbb{D}_{\delta_i} \subseteq \mathrm{supp}\, x_i$, and $\mathbb{D}_{\delta_j} \subseteq \mathrm{supp}\, x_j$, we have:

$$\begin{aligned}\iota_{k_i}(\pi\delta_i y_i) &= \pi\delta_i \iota_{k_i}(y_i) = \pi\delta_i g(x_i) = g(\pi\delta_i x_i) = g(\delta_j x_j) = \delta_j g(x_j)\\ &= \delta_j \iota_{k_j}(y_j) = \iota_{k_j}(\delta_j y_j).\end{aligned}$$

Hence, using the filteredness of the given diagram $D(\mathcal{I})$, there exist an object $t^1_{ij} \in \mathcal{I}$ and connecting $i \to t^1_{ij}$ and $j \to t^1_{ij}$ in $\mathcal{I}$ such that $\iota_{i,t^1_{ij}}(\pi\delta_i y_i) = \iota_{j,t^1_{ij}}(\delta_j y_j)$. Also for every $(\pi, \iota) \in B_{i,j}$ with $\pi \in G_{x_i,x_j}$, we have:

$$\iota_{k_i}(\pi y_i) = \pi\iota_{k_i}(y_i) = \pi g(x_i) = g(\pi x_i) = g(x_j) = \iota_{k_j}(y_j),$$

and hence, using the filteredness of the diagram $D(\mathcal{I})$, there exist an object $t^2_{ij} \in \mathcal{I}$ and connecting $i \to t^2_{ij}$ and $j \to t^2_{ij}$ in $\mathcal{I}$ such that $\iota_{i,t^2_{ij}}(\pi y_i) = \iota_{j,t^2_{ij}}(y_j)$. By the same argument, one can also see that for every $(\pi, \delta_j) \in B_{i,j}$ with $\pi \in G_{x_i,\delta_j x_j}$ and $\mathbb{D}_{\delta_j} \subseteq \operatorname{supp} x_j$, and every $(\pi\delta_i, \iota) \in B_{i,j}$ with $\pi \in G_{\delta_i x_i, x_j}$ and $\mathbb{D}_{\delta_i} \subseteq \operatorname{supp} x_i$ we, respectively, have:

$$\iota_{k_i}(\pi y_i) = \iota_{k_j}(\delta_j y_j), \text{ and } \iota_{k_i}(\pi\delta_i y_i) = \iota_{k_j}(y_j),$$

and hence there exist $t^3_{ij}, t^4_{ij} \in \mathcal{I}$ such that $\iota_{i,t^3_{ij}}(\pi y_i) = \iota_{j,t^3_{ij}}(\delta_j y_j)$ and $\iota_{i,t^4_{ij}}(\pi\delta_i y_i) = \iota_{j,t^4_{ij}}(y_j)$, respectively. Since $\mathcal{I}$ is filtered and $B = \bigcup_{i,j} B_{i,j}$ is finite, one can find $t_1 \in \mathcal{I}$ with

$$\iota_{k_i t_1}(\sigma y_i) = \iota_{k_j t_1}(\sigma' y_j), \tag{5.1}$$

for all $(\sigma, \sigma') \in B_{i,j}$ and $i, j = 1, \ldots, k$. So

$$B_{i,j} \subseteq A_{\iota_{k_i t_1}(y_i), \iota_{k_j t_1}(y_j)}, \quad \text{for every } i, j = 1, \ldots, k. \tag{5.2}$$

We also have

$$\operatorname{supp} \iota_{k_i t_1}(y_i) \subseteq \operatorname{supp} y_i = \operatorname{supp} g(x_i) \subseteq \operatorname{supp} x_i. \tag{5.3}$$

Also for every $\delta \in S'_{x_i}$, by Corollary 2.24, without loss of generality one can assume that $\mathbb{D}_\delta \subseteq \operatorname{supp} x_i$, and we have $g(\delta x_i) = \delta g(x_i) = \delta \iota_{k_i}(y_i) = \iota_{k_i}(\delta y_i)$. Since $g(\delta x_i) \in \operatorname{colim}_{l \in \mathcal{I}} D(l)$, by Proposition 5.17, there exist $j_i \in \mathcal{I}$ and $y \in D(j_i)$ with $g(\delta x_i) = \iota_{j_i}(y)$ and $\operatorname{supp} g(\delta x_i) = \operatorname{supp} y$. Hence $\iota_{k_i}(\delta y_i) = \iota_{j_i}(y)$. So there exist $t' \in \mathcal{I}$ and connecting $k_i \to t'$ and $j_i \to t'$ in $\mathcal{I}$ with $\iota_{j_i t'}(y) = \iota_{k_i t'}(\delta y_i)$, for every $i = 1, \ldots, k$. Notice that, by Corollary 5.8(i), number of such $\delta \in S'_{x_i}$ with $\mathbb{D}_\delta \subseteq \operatorname{supp} x_i$ is finite. Hence, using the filteredness of $\mathcal{I}$, one can find $t_2 \in \mathcal{I}$ and connecting $k_i \to t_2$ and $j_i \to t_2$ with $\iota_{j_i t_2}(y) = \iota_{k_i t_2}(\delta y_i)$, for every $i = 1, \ldots, k$, and every $\delta \in S'_{x_i}$, and

$$\begin{aligned} \operatorname{supp} \delta \iota_{k_i t_2}(y_i) &= \operatorname{supp} \iota_{k_i t_2}(\delta y_i) \\ &= \operatorname{supp} \iota_{j_i t_2}(y) \\ &\subseteq \operatorname{supp} y \\ &= \operatorname{supp} g(\delta x_i) \\ &\subseteq \operatorname{supp} \delta x_i. \end{aligned} \tag{5.4}$$

Now, using the filteredness of $\mathcal{I}$ for objects t_1 and t_2, one can find an object $t \in \mathcal{I}$ and connecting $t_1 \to t$ and $t_2 \to t$ such that, for every $i, j = 1, \ldots, k$, the followings hold:

$$\begin{aligned} B_{i,j} &\subseteq A_{\iota_{k_i t}(y_i), \iota_{k_j t}(y_j)}, && \text{(by 5.2)} \\ \operatorname{supp} \iota_{k_i t}(y_i) &\subseteq \operatorname{supp} x_i, && \text{(by 5.3)} \\ \operatorname{supp} \delta\iota_{k_i t}(y_i) &\subseteq \operatorname{supp} \delta x_i, && \text{(by 5.4).} \end{aligned}$$

So, by Lemma 5.15, there exists a unique equivariant map $g' : X \to D(t)$ with $g'(x_i) = \iota_{k_i t}(y_i)$. Also, $g(x_i) = \iota_{k_i}(y_i) = \iota_t(\iota_{k_i t}(y_i)) = (\iota_t \circ g')(x_i)$ and hence $\iota_t \circ g' = g$. Thus $\varphi(\mathsf{u}_t(g')) = \iota_t \circ g' = g$, and we are done. □

As an application of Theorem 5.18 we give the following corollary. But first, we recall that a category is called *locally finitely presentable* provided that it is cocomplete and has a set $\mathcal{A}$ of finitely presentable objects such that every object is a directed colimit of objects from $\mathcal{A}$, see [2, Definition 1.9]. We also recall the fact that a category is locally finitely presentable if it is cocomplete and has a strong generating set formed by finitely presentable objects [2, Theorem 1.11].

Corollary 5.19. (i) *The finitely supported Cb-sets in the given generating set $\mathcal{G}$ in Theorem* 4.2 *are finitely presentable.*

(ii) *The category* $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ *is a locally finitely presentable category.*

(iii) *Every quotient of a finitely presentable finitely supported Cb-set is finitely presentable.*

It is worth noting that, because in the adjunction $F \dashv U : (Cb\text{-}\mathbf{Set})_{\mathrm{fs}} \to \mathbf{Nom}$, given in Corollary 3.18, U is finitary, one can apply Corollary 5.19(ii) and [1, Lemma 2.4] to the adjunction $F \dashv U$, and obtain the following corollary.

Corollary 5.20. *If X is a finitely presentable nominal set, then $F(X)$ is a finitely presentable finitely supported Cb-set.*

Theorem 5.21. *Let X be a finitely supported Cb-set. Then X is finitely presentable if and only if X is a finite disjoint union of indecomposable finitely presentable finitely supported Cb-sets.*

Proof. (⇐) Since, by Remark 5.16, coproducts in the category of finitely supported Cb-sets are given by disjoint union, the result follows from Theorem 5.12(ii).

(⇒) Suppose X is finitely presentable. By Theorem 5.18, X is finitely generated. So $X = \bigcup_{i=1}^{n} Cbx_i$, for some $x_1, \ldots, x_n \in X$. Hence, by Theorem 5.1(v), X is a finite disjoint union of indecomposable Cb-subsets. Now, by Theorem 5.12(i), each of these Cb-subsets is a retract of X and hence each of them is finitely presentable by Corollary 5.19(iii). □

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