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# Another closure operator on preneighbourhood spaces

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Dedicated to Themba Dube on the occasion of his $65^{th}$ birthday.

**Abstract.** The notions of dense, proper, separated or perfect morphisms and hence of compact, Hausdorff or compact Hausdorff are all consequent to *good* properties of a family of closed morphisms is well known in literature. Deeper consequences like the Tychonoff product theorem or the Stone Čech compactifications follow from *richer* properties of the set of closed morphisms. The purpose of this paper is to provide a closure operation on a preneighbourhood space so that the resulting set of closed morphisms possess all the properties mentioned above.

## 1 Introduction

The notion of a preneighbourhood space $(X, \mu)$ was initiated in [5]; in [6] a closure operation $\mathrm{cl}_\mu$ (see Definition 3.1, [6]) is investigated, the corresponding set $\mathbb{A}_{\mathsf{cl}}$ of closed morphisms (see Definition 4.1, [6]) exhibited to have *good properties* (see §4 and Table 1 on page 218, [6]). This facilitated in-

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vestigation into dense morphisms (see §5, [6]), proper morphisms and hence compact preneighbourhood spaces (see §6, [6]), separated morphisms and hence Hausdorff spaces (see §7, [6]), perfect morphisms and hence compact Hausdorff spaces (see §8, [6]). The investigation of proper, separated and perfect morphisms depended only on *good properties* of the set $\mathbb{A}_{\mathsf{cl}}$.

The dependence of notions of proper, separated, perfect morphisms on *good* properties of the set of closed morphism is inspired from [3]. In [3], the *good properties* were formulated as axioms on a set of morphisms, called *closed morphisms*, (see conditions ($\mathsf{F3}$), ($\mathsf{F4}$) and ($\mathsf{F5}$), [3]). More involved topological facts like the Tychonoff's product theorem or the Stone Čech compactification of a Tychonoff space required an additional list of four axioms (see conditions ($\mathsf{F6}$), ($\mathsf{F7}$), ($\mathsf{F8}$) and ($\mathsf{F9}$) in §11, [3]).

The purpose of this paper is to provide another closure operation $\mathsf{cl}^{\mathsf{F}}_{\mu}$ on each preneighbourhood space $(X, \mu)$ (see Definition 3.1). It is shown $\mathsf{cl}^{\mathsf{F}}_{\mu}$ has, in comparison to $\mathrm{cl}_{\mu}$ in [6], many more properties facilitating its set $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ (see Definition 4.1) of closed morphisms having analogues of all the seven axioms for closed sets referred above. Thus it is expected that most of the notions dealt in [3] would be achieved with this specific closure operation. The paper provides list of properties for the consequent notions which only depend on the analogues of conditions ($\mathsf{F3}$)-($\mathsf{F5}$) of [3]; consequences involving analogues of later axioms ($\mathsf{F6}$)-($\mathsf{F9}$) shall be dealt in later papers.

At this point it must be emphasised that *categorical neighbourhood operators* have already been considered earlier, especially in [7, 10, 11, 15–22] as well as in the references therein. The closure operation in this paper was initially described in [10] and investigated there. However, there are two major points of divergence between the approaches in these papers or their references and the work in this paper or [5, 6]:

(i) Firstly, it is standard in literature to consider *operators* and thereby the notion of continuity of morphisms is hard-wired in the definition. In this paper, as in [5, 6] the situation is transversal: a suitable category with an ambient structure is considered, wherein each object is assigned a *neighbourhood system*; a morphism between objects endowed with *neighbourhood systems* may or may not *preserve* the neighbourhood structure, and hence *continuity* becomes an extra property that a morphism may or may not possess. Often, the objects endowed with the *neighbourhood systems* and morphisms *preserving* them (i.e., *continuous* ones in that sense) make a (full) subcategory and there is a *forgetful* functor from this subcategory to the base category, which in many cases is topological (see [5] for details).

(ii) To include a large class of examples, minimum structure is assumed on the category. Usually, apart from finite completeness and existence of finite coproducts, the category is assumed to have a proper factorisation structure $(\mathsf{E}, \mathsf{M})$ (see Definition 2.2). In most of the papers, for instance in [10, 11], it is assumed that $\mathsf{E}$ is stable under pullbacks of $\mathsf{M}$-morphisms — a restriction that does not hold for locales (see for instance in [23]).

However, a comparison between these approaches is necessary to place them in their proper perspective and is done in the paper.

The paper is organised as follows: §2 recalls all the facts necessary for this paper, §3 investigates closure operation $\mathsf{cl}^{\mathsf{F}}_{\mu}$, §4 investigates the properties of the set $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ , §5 provides a list of properties for proper, separated and perfect morphisms and hence properties for compact, Hausdorff and compact Hausdorff preneighbourhood spaces.

The categorical notation used in this paper is akin to usage in [12] or [1]. The paper is based on $\mathsf{NBG}$ set theory (see [13] for details); in this paper a set $x$ is *small* if $(\exists y)(x \in y)$, else it is *large*[1] and $2^{X}_{<\aleph_0}$ is the set of all finite subsets of $X$.

## 2 Preliminaries

This section recalls facts that are necessary for the paper.

**2.1 Closure operations** Let $P$ be a poset with smallest element 0 and largest element 1; an order preserving endomap $P \xrightarrow{f} P$ is *extensional* if $x \leq f(x)$ $(x \in P)$ and *grounded* if $f(0) = 0$. In this paper, a *closure operation* is an extensional grounded order preserving endomap and $\mathsf{EGM}(P)$ is the set of all closure operations on $P$. Evidently, $\mathsf{EGM}(P)$ is ordered pointwise, i.e., $f \leq g$ if $f(x) \leq g(x)$ $(x \in P,\ f, g \in \mathsf{EGM}(P))$, the smallest closure

[1] [13] uses the term *proper class* to refer to *large sets* of this paper, *set* to refer to *small sets* of this paper, and the term *class* is used for a generic set.

operation is $\mathbf{1}_P$ and the largest is $P \xrightarrow{l} P$ where $l(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise} \end{cases}$.
A closure operation $f \in \mathsf{EGM}(P)$ is *idempotent* if $f \circ f = f$ and *additive* if $f(x \vee y) = f(x) \vee f(y)$, whenever the suprema exist. For any closure operation $f \in \mathsf{EGM}(P)$, $\texttt{Fix}[f]$ is the set of all its fixed points, i.e., $\texttt{Fix}[f] = \big\{x \in P : f(x) = x\big\}$. Furthermore, for any $S \subseteq \texttt{Fix}[f]$, $f(\bigwedge S) = \bigwedge S$, if the infima exist, i.e., $\texttt{Fix}[f]$ is closed under arbitrary meets.

**Proposition 2.1** (see §2.1, [6]). *For any complete lattice $L$ and a closure operation $L \xrightarrow{f} L$ the equation*:

$$\hat{f}(x) = \bigwedge \big\{k \in \texttt{Fix}[f] : x \leq k\big\} \tag{2.1}$$

*defines an idempotent closure operation on $L$, $\texttt{Fix}[\hat{f}] = \texttt{Fix}[f]$, $f \leq \hat{f}$ and for any idempotent closure operation $g$ on $L$, $f \leq g \Leftrightarrow \hat{f} \leq g$.*

*Proof.* See §2, [6]. □

The closure operation $\hat{f}$ of Proposition is called the *idempotent hull* of $f$, (see §4.6 in [4] for details).

**2.2 Contexts** The notion of a context is introduced in [5]; a closure operation and its consequences treated in [6].

**Definition 2.2** (see [5, 6]). (a) A *context* is $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ where $\mathbb{A}$ is a finitely complete category with finite coproducts, $(\mathsf{E}, \mathsf{M})$ is a proper factorisation system on $\mathbb{A}$[2] such that for each object $X \in \mathbb{A}_0$, the set $\texttt{Sub}_{\mathsf{M}}(X)$ is a complete lattice[3].

(b) An order preserving map $\texttt{Sub}_{\mathsf{M}}(X)^{\mathsf{op}} \xrightarrow{\mu} \texttt{Fil}\,[X]$[4] with the property $u \in \mu(p) \Rightarrow p \leq u$ is a *preneighbourhood system* on $X$; the pair $(X, \mu)$ is called a *preneighbourhood space*.

[2] A morphism in $\mathsf{M}$ is called an *admissible monomorphism*.

[3] An $m \in \texttt{Sub}_{\mathsf{M}}(X)$ is called an *admissible subobject* of $X$.

[4] Given any semilattice $L$, $\texttt{Fil}\,[L]$ is the complete lattice of all filters in $L$; in this paper, $\texttt{Fil}\,[X]$ is abbreviation for $\texttt{Fil}\,[\texttt{Sub}_{\mathsf{M}}(X)]$.

(c) A preneighbourhood system $\mu$ is a *weak neighbourhood system* on $X$ if for every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mu(p) \subseteq \bigcup_{x\in\mu(p)} \bigcap_{y\leq x} \mu(y)$ (or equivalently, $\mu(p) = \bigcup_{x\in\mu(p)} \mu(x)$, (see [5] for details); the pair $(X, \mu)$ is called a *weak neighbourhood space*.

(d) A weak neighbourhood system $\mu$ is a *neighbourhood system* on $X$ if it preserves all meets, i.e., $\mu(\bigvee S) = \bigcap_{s\in S} \mu(s)$ $(S \subseteq \mathtt{Sub}_{\mathsf{M}}(X))$; the pair $(X, \mu)$ is called a *neighbourhood space*.

(e) Given a preneighbourhood space $(X, \mu)$, the (possibly large) set of *$\mu$-open* subobjects of $X$ is $\mathfrak{O}_\mu = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : p \in \mu(p)\}$; for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, *$\mu$-interior of* $p$ is $\mathtt{int}_\mu p = \bigvee\{u \in \mathfrak{O}_\mu : u \leq p\}$.

(f) Given preneighbourhood spaces $(X, \mu)$ and $(Y, \phi)$, a morphism $X \xrightarrow{f} Y$ is a *preneighbourhood morphism*, written $(X, \mu) \xrightarrow{f} (Y, \phi)$, if $u \in \phi(y) \Rightarrow f^{-\mathbf{1}}u \in \mu(y^{-\mathbf{1}})$ for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$. The category of preneighbourhood spaces and preneighbourhood morphisms is $\mathtt{pNbd}[\mathbb{A}]$, $\mathtt{wNbd}[\mathbb{A}]$ is the full subcategory of weak neighbourhood spaces, and $\mathtt{Nbd}[\mathbb{A}]$ is the non-full subcategory of $\mathtt{wNbd}[\mathbb{A}]$ with neighbourhood spaces as objects and preneighbourhood morphisms $(X, \mu) \xrightarrow{f} (Y, \phi)$ for which $f^{-\mathbf{1}}$ preserves arbitrary joins.

Similar definition appears in literature:

A. In [10, Definition 3.1] the authors define a *neighbourhood operator* quite similarly; however, there are some differences. Firstly, the codomain of each neighbourhood operator is a *stack* of admissible subobjects of $X$ — a *stack* on a poset $P$ is alternatively known as a *upward closed subset* of $P$, i.e., $U \subseteq P$ such that $x \geq y \in U \Rightarrow x \in U$; secondly, their conditions ($\mathsf{nbh0}$) and ($\mathsf{nbh1}$) together define preneighbourhood system of this paper; condition ($\mathsf{nbh2}$) of their paper defines neighbourhood system of this paper without the weak neighbourhood condition. The condition ($\mathsf{nbh3}$) in their paper is used to effect *continuity* of each morphism with respect to the neighbourhood operations, contrary to the case here where being a preneighbourhood morphism is an additional property of a morphism which may or not be possessed.

A similar definition also appears in [11].

B. In [17, 19] neighbourhood operatons are defined as certain lax natural transformations, which are almost same as the preneighbourhood systems of this paper, with the only difference being in the present paper the codomains are filters while in the papers mentioned they are posets.

Contexts abound — every small complete and small cocomplete, if wellpowered category $\mathbb{A}$ provides the context $\mathcal{E} = (\mathbb{A}, \mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$, and if co-wellpowered provides the context $\mathcal{M} = (\mathbb{A}, \mathtt{ExtEpi}(\mathbb{A}), \mathtt{Mono}(\mathbb{A}))$; for other familiar examples see [5, 6].

Some notations need to be explained and hence standardised for use in this paper:

C. Every morphism $X \xrightarrow{f} Y$ factors uniquely as
$$\begin{tikzcd} X \arrow[r, two heads, "f^{\mathsf{E}}"] \arrow[rr, bend left, "f"] & \mathcal{I}_f \arrow[r, tail, "f^{\mathsf{M}}"] & Y \end{tikzcd}$$
with $f^{\mathsf{E}} \in \mathsf{E}$ and $f^{\mathsf{M}} \in \mathsf{M}$; morphisms from $\mathsf{E}$ shall be denoted as $\twoheadrightarrow$ while morphisms from $\mathsf{M}$ as $\rightarrowtail$ .

D. Given $P \overset{p}{\rightarrowtail} X \xrightarrow{f} Y \overset{q}{\leftarrowtail} Q$ , $\exists_f p = (f \circ p)^{\mathsf{M}}$ is the *image of* $p$ *under* $f$, $(f\big|_p) = (f \circ p)^{\mathsf{E}}$ is the *restriction of* $f$ *on* $p$ and in the pullback square
$$\begin{tikzcd} f^{-\mathbf{1}}Q \arrow[r, "f_q"] \arrow[d, tail, "f^{-\mathbf{1}}q"'] & Q \arrow[d, tail, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
, $f^{-\mathbf{1}}q$ is the *preimage of* $q$ *under* $f$, $f_q$ is the *corestriction of* $f$ *along* $q$.

E. The $(\mathsf{E}, \mathsf{M})$-factorisation induces the adjunction on the left

$$\mathtt{Sub}_{\mathsf{M}}(X) \underset{f^{-\mathbf{1}}}{\overset{\exists_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathtt{Sub}_{\mathsf{M}}(Y) \qquad \mathtt{Fil}\,[Y] \underset{\overrightarrow{f}}{\overset{\overleftarrow{f}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathtt{Fil}\,[X]$$

between complete lattices, whenever $X \xrightarrow{f} Y$ is a morphism of $\mathbb{A}$; furthermore, $f$ also induces the adjunction on the right between complete lattices of filters, where for any $A \in \mathtt{Fil}\,[X]$ and $B \in \mathtt{Fil}\,[Y]$:

$$\overrightarrow{f} A = \left\{ y \in \mathtt{Sub}_{\mathsf{M}}(Y) : (\exists x \in A)(\exists_f x \leq y) \right\}$$

$$= \{y \in \mathtt{Sub}_{\mathsf{M}}(Y) : f^{-\mathbf{1}}y \in A\}, \tag{2.2}$$

and

$$\overleftarrow{f} B = \{x \in \mathtt{Sub}_{\mathsf{M}}(X) : (\exists y \in B)(f^{-\mathbf{1}}y \leq x)\}. \tag{2.3}$$

F. $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism if and only if for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\overleftarrow{f}\,\phi(\exists_f x) \subseteq \mu(x)$ if and only if for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $\phi(y) \subseteq \overrightarrow{f}\,\mu(f^{-\mathbf{1}}y)$. Incidentally, the assignment:

$$x \xmapsto{\overleftarrow{f}\,\phi\exists_f} \overleftarrow{f}\,\phi(\exists_f x)$$

is a preneighbourhood system on $X$, the smallest one making $f$ a preneighbourhood morphism.

G. The categories $\mathtt{pNbd}[\mathbb{A}]$ as well as $\mathtt{wNbd}[\mathbb{A}]$ are both topological over $\mathbb{A}$, [5, Theorem 4.8, for details].

H. In view of G., every small limit (respectively, small colimit) object is usually considered as a preneighbourhood space with the smallest (respectively, largest) preneighbourhood system making each component of the limiting cone (respectively, colimiting cone) preneighbourhood morphism. In particular:

(i) For a preneighbourhood space $(X, \mu)$ and an admissible monomorphism $M \overset{m}{\rightarrowtail} X$ of $X$, $M$ is usually considered a preneighbourhood space with preneighbourhood system $(\mu\big|_m)$, where for each $a \in \mathtt{Sub}_{\mathsf{M}}(M)$:

$$\begin{aligned}(\mu\big|_m)(a) &= \overleftarrow{m}\mu(m{\circ}a)\\ &= \{u \in \mathtt{Sub}_{\mathsf{M}}(M) : (\exists v \in \mu(m{\circ}a))(m^{-\mathbf{1}}v \leq u)\}\\ &= \{u \in \mathtt{Sub}_{\mathsf{M}}(M) : (\exists v \in \mu(m{\circ}a))(v \wedge m \leq m{\circ}u)\}.\end{aligned} \tag{2.4}$$

(ii) For preneighbourhood spaces $(X, \mu)$ and $(Y, \phi)$, the product object $X \times Y$ with product projections $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$ is usually considered a preneighbourhood space with preneighbourhood system $\mu \times \phi$, where for each $u = (u_X, u_Y) \in \mathtt{Sub}_{\mathsf{M}}(X \times Y)$:

$$\begin{aligned}(\mu \times \phi)(u) &= \overleftarrow{p_X}\mu(\exists_{p_X} u) \vee \overleftarrow{p_Y}\phi(\exists_{p_Y} u) \\ &= \big\{(v_X, v_Y) \in \mathtt{Sub}_{\mathsf{M}}(X \times Y) : (\exists w_X \in \mu(v_X^{\mathsf{M}}))(\exists w_Y \in \phi(v_Y^{\mathsf{M}})) \\ &\qquad (w_X \times w_Y \le v)\big\}.\end{aligned} \tag{2.5}$$

I. Given the family $\big((X_i, \mu_i)_i\big)_{i \in I}$ of preneighbourhood spaces, let $X = \prod_{i \in I} X_i$ with product projections $X \xrightarrow{p_i} X_i$ $(i \in I)$, $X_J = \prod_{j \in J} X_j$ with product projections $X_J \xrightarrow{p_j^J} X_j$ for any $j \in J \in 2^I_{<\aleph_0}$. Evidently, there are the unique morphisms $X \xrightarrow{p_J} X_J$ with $p_j^J \circ p_J = p_j$ $(j \in J \in 2^I_{<\aleph_0})$ such that for $J, K \in 2^I_{<\aleph_0}$ with $J \subseteq K$ there exists the unique *bonding morphism* $X_K \xrightarrow{p_{J,K}} X_J$ making the diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "p_J"'] \arrow[dr, "p_K"] \arrow[dd, bend left, "p_j"] & \\
X_J \arrow[dr, "p_j^J"'] & & X_K \arrow[ll, dashed, "!p_{J,K}"'] \arrow[dl, "p_j^K"] \\
 & X_j &
\end{tikzcd} \tag{2.6}$$

to commute for every $j \in J \subseteq K$. Hence $X \simeq \prod_{J \in 2^I_{<\aleph_0}} X_J$ with product projections $X \xrightarrow{p_J} X_J$ $(J \in 2^I_{<\aleph_0})$.

The object $X$ is usually considered a preneighbourhood space with preneighbourhood system $\mu$, where for each $u = \big(u_i\big)_{i \in I} \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\begin{aligned}\mu(u) &= \bigvee_{i \in I} \overleftarrow{p_i}\mu_i(\exists_{p_i} u) \\ &= \big\{v \in \mathtt{Sub}_{\mathsf{M}}(X) : (\exists J \in 2^I_{<\aleph_0})(\forall j \in J)(\exists w_j \in \mu_j(u_j^{\mathsf{M}})) \\ &\qquad (\bigwedge_{j \in J} {p_j}^{-\mathbf{1}} w_j \le v)\big\}\end{aligned}$$

$$= \bigvee_{J \in 2^{I}_{<\aleph_0}} \overleftarrow{p_J} \mu_J(\exists_{p_J} u), \tag{2.7}$$

where $\mu_J$ is the product preneighbourhood system on $X_J$ ($J \in 2^{I}_{<\aleph_0}$).

The product is a limit of an inverse sytem (see [3, §11.2]), and (2.7) is also mentioned in [19, Definition 4] or [17, §5].

**2.3 Morphisms reflecting zero** Let $\mathscr{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

A morphism $X \xrightarrow{f} Y$ is said to *reflect zero* if $f^{-\mathbf{1}}\sigma_Y = \sigma_X$, where $\sigma_X \in \mathtt{Sub}_{\mathsf{M}}(X)$ is the smallest admissible subobject of $X$.

**Proposition 2.3** ([6, Theorem 2.11])**.** *Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following statements hold.*

(a) *The following are equivalent*:

  (i) *$f$ reflects zero.*

  (ii) *For any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f p = \sigma_Y \Rightarrow p = \sigma_X$.*

  (iii) *For any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$ and $v \in \mathtt{Sub}_{\mathsf{M}}(Y)$:*

$$v \wedge \exists_f p = \sigma_Y \Rightarrow p \wedge f^{-\mathbf{1}} v = \sigma_X. \tag{2.8}$$

(b) *If $f^{-\mathbf{1}} \circ \exists_f = \mathbf{1}_{\mathtt{Sub}_{\mathsf{M}}(X)}$ then $f$ reflects zero; in particular, every admissible monomorphism reflects zero.*

(c) *The set of morphisms reflecting zero is closed under composition.*

(d) *If $g \circ f$ reflects zero then $f$ reflects zero.*

(e) *For any $n \in \mathtt{Sub}_{\mathsf{Y}}(Y)$, the corestriction $f_n$ of $f$ along $n$ reflects zero whenever $f$ reflects zero.*

(f) *In presence of pullbacks in $\mathbb{A}$*:

  (i) *if every morphism reflect zero then the initial object $\emptyset$ is strict.*

  (ii) *if the initial object is strict and the unique morphism $\emptyset \xrightarrow{\mathtt{i}_1} \mathtt{1}$ is an admissible monomorphism then every morphism reflects zero.*

*Proof.* [6, page 170]. □

J. The term *reflects* 0 is used in [11, Definition 2.1] and called *reflects least subobject* in [10, §2]. Also, a characterisation of morphisms reflecting zero appears in [11, Proposition 2.2] under closure of $\mathsf{E}$-morphisms along pullbacks along $\mathsf{M}$-morphisms, while some connection with Frobenius morphisms[5] established in [10, Lemma 2.1]. It seems the reflecting zero property is more about a *Frobenius reciprocity at the smallest subobject* — compare with Proposition 2.4.

K. A finitely complete category with an initial object is *quasi-pointed* if the unique morphism $\emptyset \xrightarrow{\mathtt{i}_1} \mathtt{1}$ is a monomorphism, (see [2, 8]). In many familiar contexts the unique morphism $\mathtt{i}_1$ is a regular monomorphism and hence an admissible monomorphism, [6, Remark 2.12]. A context $\mathscr{A}$ is called *admissibly quasi-pointed* if the unique morphism $\mathtt{i}_1$ in $\mathbb{A}$ is an admissible monomorphism. Thus: in admissibly quasi-pointed contexts, every morphism reflect zero if and only if the initial object is strict.

L. Henceforth the context $\mathscr{A}$ shall be called a *reflecting zero context* if every morphism of $\mathbb{A}$ reflects zero.

### 2.3.1 Converse of reflecting zero condition

With regards to converse implication of (2.8):

**Proposition 2.4.** *Given a morphism* $X \xrightarrow{f} Y$, $p \in \mathtt{Sub}_{\mathsf{M}}(X)$ *and* $v \in \mathtt{Sub}_{\mathsf{M}}(Y)$ *consider the statements*:

$$p \wedge f^{-\mathbf{1}}v = \sigma_X \Rightarrow v \wedge \exists_f p = \sigma_Y. \tag{2.9}$$

$$f^{-\mathbf{1}}v = \sigma_X \Rightarrow v \wedge f^{\mathsf{M}} = \sigma_Y. \tag{2.10}$$

*For each* $f$, $p$ *and* $v$, (2.9) *implies* (2.10)*; if for every* $f$ *and* $v$, (2.10) *is true then for every* $f, p, v$, (2.9) *holds good.*

*Proof.* Assuming (2.9), and taking $p = \mathbf{1}_X$ implies (2.10). Conversely, assuming (2.10) holding for each $f, v$, consider the diagram:

[5] A morphism $f$ is a *Frobenius morphism* if it satisfies the *Frobenius reciprocity law*, i.e., for all admissible subobjects $x$ of the domain and admissible subobjects $y$ of the codomain, $\exists_f \big(x \wedge f^{-\mathbf{1}}y\big) = y \wedge \exists_f x$.

$$\begin{tikzcd}
 & P \wedge f^{-\mathbf{1}}V \arrow[r, dashed, two heads, "(f\big|_{\exists_f(p\wedge f^{-\mathbf{1}}v)})"] \arrow[dl, tail, "f_1'"'] \arrow[dd, tail, "v_1'"] & \exists_f(P\wedge f^{-\mathbf{1}}V) \arrow[r, dashed, tail, "v_2"] & V \wedge \exists_f P \arrow[dl, tail, "f_1"'] \arrow[dd, tail, "v_1"] \\
P \arrow[rr, two heads, "(f\big|_p)"] \arrow[dd, tail, "p"'] & & \exists_f P \arrow[dd, tail, "\exists_f p"] & \\
 & f^{-\mathbf{1}}V \arrow[rr, "f_v"] \arrow[dl, tail, "f^{-\mathbf{1}}v"'] & & V \arrow[dl, tail, "v"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

where:

(i) the left hand and right hand upright squares are pullback squares depicting intersections,

(ii) the bottom horizontal square is a pullback square depicting preimage $f^{-\mathbf{1}}v$,

(iii) since $\exists_f p \circ (f\big|_p) \circ f_1' = v \circ f_v \circ v_1'$, $(P \wedge f^{-\mathbf{1}}V) \xrightarrow{w} (V \wedge \exists_f P)$ is the unique morphism such that $f_1 \circ w = (f\big|_p) \circ f_1'$ and $v_1 \circ w = f_v \circ v_1'$.

Since $\exists_f p \circ f_1 \circ w = f \circ (p \wedge f^{-\mathbf{1}}v)$, using the $(\mathsf{E},\mathsf{M})$-factorisation of $w$, $w^{\mathsf{E}} = (f\big|_{\exists_f(p\wedge f^{-\mathbf{1}}v)})$, $w^{\mathsf{M}} = v_2$ and $\exists_f(p \wedge f^{-\mathbf{1}}v) = (\exists_f p) \circ f_1 \circ v_2$. Furthermore, since the upright left hand square, the bottom horizontal square and the upright right hand square are pullbacks, the top horizontal square is also a pullback, i.e., $f_1' = (f\big|_p)^{-\mathbf{1}} f_1$. Consequently for non-trivial $p, v$:

$$\begin{aligned}
p \wedge f^{-\mathbf{1}}v = \sigma_X &\Leftrightarrow f_1' = \sigma_P \\
&\Leftrightarrow (f\big|_p)^{-\mathbf{1}} f_1 = \sigma_P \\
&\Rightarrow f_1 \wedge (f\big|_p)^{\mathsf{M}} = \sigma_{\exists_f P} \qquad \text{(from assumption)} \\
&\Leftrightarrow f_1 = \sigma_{\exists_f P} \qquad \text{(since } (f\big|_p) \in \mathsf{E} \Rightarrow (f\big|_p)^{\mathsf{M}} = \mathbf{1}_{\exists_f P}\text{)} \\
&\Rightarrow v \wedge \exists_f p = \sigma_Y,
\end{aligned}$$

completing the proof. $\square$

The property in equation (2.10) is reminiscent of sets and functions; however it also holds in the context $(\mathsf{Loc}, \mathsf{Epi}, \mathsf{RegMono})$ — if $X \xrightarrow{f} Y$ is a localic map then $f^{-\mathbf{1}}V$ is non-null if and only if there exists a $x \in X \setminus \{1\}$ with $f(x) \in V$, and hence $f^{-\mathbf{1}}V = \{1\} \Leftrightarrow \big(f(x) \in V \Rightarrow x = 1\big) \Leftrightarrow \exists_f X \cap V = \{1\}$[6]. Interesting to note: equation (2.8) also holds for locales: since for each localic map $f$, $f(x) = 1 \Leftrightarrow x = 1$, $f[P] = \{1\} \Leftrightarrow \big(p \in P \Rightarrow f(p) = 1\big) \Leftrightarrow \big(p \in P \Rightarrow p = 1\big) \Leftrightarrow P = \{1\}$.

**Definition 2.5.** A morphism $X \xrightarrow{f} Y$ is a *formal function* if (2.10) holds for every $v$.

Recall from [6, Definition 2.7(f)]: a morphism $X \xrightarrow{f} Y$ is *formally surjective* (or *semistable* as in [24]) if $\exists_f f^{-\mathbf{1}} = \mathbf{1}_{\mathsf{Sub}_{\mathsf{M}}(Y)}$.

**Corollary 2.6.** *Every admissible monomorphism or a formally surjective morphism is a formal function.*

M. Henceforth the context $\mathcal{A}$ shall be called a *formal function context* if each of its morphisms is a formal function.

### 2.4 Properties of products

Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context. This section collects results concerning product objects. In this connection, recall in a category morphisms $T \xrightarrow{f} X \times Y$ to a product correspond naturally to pairs of morphisms $\Big(T \xrightarrow{f_X} X, T \xrightarrow{f_Y} Y\Big)$; using this natural bijection morphisms like $f$ is alternatively be presented as $(f_X, f_Y)$; furthermore, if $T \xrightarrow{g} X$ and $S \xrightarrow{h} Y$ are morphisms then $T \times S \xrightarrow{g \times h} X \times Y$ is the unique morphism $g \times h = (g \circ p_T, h \circ p_S)$. Note: the subobjects in $\mathsf{Sub}_{\mathsf{M}}(X \times Y)$ are precisely given by pairs $U \xrightarrow{u=(u_X, u_Y)} X \times Y$ which are *jointly monic*, i.e., for all morphisms $S \underset{x'}{\overset{x}{\rightrightarrows}} U$, $u \circ x = u \circ x' \Leftrightarrow \left.\begin{array}{rcl} u_X \circ x & = & u_X \circ x' \\ u_Y \circ x & = & u_Y \circ x' \end{array}\right\} \Rightarrow x = x'$.

[6] In localic parlance, $f^{-\mathbf{1}}V$ is usually denoted by $f_{-1}V$, [14, §III.4.2.1], and $\exists_f P$ denoted by $f[P]$.

**Proposition 2.7.** *Given the product diagram* $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$ *the following statements are true.*

(a) *For any subobject* $u = (u_X, u_Y) \in \mathtt{Sub}_{\mathsf{M}}(X \times Y)$:

$$p_X{}^{-\mathbf{1}} u_X^{\mathsf{M}} \wedge p_Y{}^{-\mathbf{1}} u_Y^{\mathsf{M}} = u_X^{\mathsf{M}} \times u_Y^{\mathsf{M}} \text{ and } u \leq u_X^{\mathsf{M}} \times u_Y^{\mathsf{M}}. \tag{2.11}$$

(b) *If the initial object is strict then* $\emptyset \times Y \simeq \emptyset$.

(c) *If* $t \in \mathtt{Sub}_{\mathsf{M}}(X)$ *then* $p_X{}^{-\mathbf{1}} t = t \times \mathbf{1}_Y$*; in particular, if the initial object is strict then finite product projections reflect zero.*

(d) *The non-trivial finite product projections are formally surjective if and only if they are* $\mathsf{E}$*-morphisms.*

(e) *Given the morphisms* $A \xrightarrow{f} C$, $B \xrightarrow{g} D$, $U \xrightarrow{c} B$ *and* $U \xrightarrow{d} D$ *the pullback of* $(c, d)$ *along* $f \times g$ *is given by*:

$$\begin{tikzcd} U' \arrow[r, "w"] \arrow[d, "(c_f \circ u,\, d_g \circ v)"'] & C \arrow[d, "(c,d)"] \\ A \times B \arrow[r, "f \times g"'] & C \times D \end{tikzcd} \tag{2.12}$$

*where*:

$$\begin{tikzcd} A \times_C U \arrow[r, "f_c"] \arrow[d, "c_f"'] & U \arrow[d, "c"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \quad \begin{tikzcd} B \times_D U \arrow[r, "g_d"] \arrow[d, "d_g"'] & U \arrow[d, "d"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U' \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, "w"] & B \times_D U \arrow[d, "g_d"] \\ A \times_C U \arrow[r, "f_c"'] & U \end{tikzcd}$$

*with* $w = c_f \circ u = d_g \circ v$*, are pullback squares.*

*Hence if* $(c, d)$*,* $f$ *and* $g$ *are monomorphisms then*:

$$(f \times g) \wedge (c, d) = (c, d) \circ \big( (c^{-\mathbf{1}} f) \wedge (d^{-\mathbf{1}} g) \big). \tag{2.13}$$

*In particular,*

$$u \wedge p_1{}^{-\mathbf{1}} t = u \circ \big( u_1{}^{-\mathbf{1}} t \big), \text{ for } u \in \mathtt{Sub}_{\mathsf{M}}(X \times Y), t \in \mathtt{Sub}_{\mathsf{M}}(X). \tag{2.14}$$

(f) *For any $J, K \in 2^I_{<\aleph_0}$, $J \subseteq K$, each bonding morphism $p_{J,K}$ and finite product projection $p_J$ (see I., page 8) are formally surjective if the non-trivial finite product projections are formally surjective.*

*Furthermore for any $u \in \mathtt{Sub}_{\mathsf{M}}(X)$:*

$$\exists_{p_J} u = \left(\prod_{j\in J} u_j^{\mathsf{M}}\right) \circ \left(\left(u_j\right)_{j\in J}{}^{\mathsf{M}}\right) \leq \prod_{j\in J} u_j{}^{\mathsf{M}}.$$

*Proof.* Trivial. □

### 2.4.1 Product preneighbourhood systems

For preneighbourhood systems, let $\mu_J = \bigvee_{j\in J} \overleftarrow{p_j^J} \mu_j \exists_{p_j^J}$ $(J \in 2^I_{<\aleph_0})$ be the smallest preneighbourhood system on $X_J$ making each of the product projections $p_j^J$ preneighbourhood morphisms $(j \in J \in 2^I_{<\aleph_0})$ and $\mu = \bigvee_{i\in I} \overleftarrow{p_i} \mu_i \exists_{p_i}$ is the smallest preneighbourhood system on $X$ making each of the product projections $p_i$ $(i \in I)$ preneighbourhood morphisms. Consequently each $(X_K, \mu_K) \xrightarrow{p_{J,K}} (X_J, \mu_J)$ for $J \subseteq K, J, K \in 2^I_{<\aleph_0}$. Furthermore:

$$\begin{aligned}
\mu &= \bigvee_{i\in I} \overleftarrow{p_i} \mu_i \exists_{p_i} \\
&= \bigvee_{J\in 2^I_{<\aleph_0}} \bigvee_{j\in J} \overleftarrow{p_j} \mu_j \exists_{p_j} \\
&= \bigvee_{J\in 2^I_{<\aleph_0}} \bigvee_{j\in J} \overleftarrow{p_J} \overleftarrow{p_j^J} \mu_j \exists_{p_j^J} \exists_{p_J} \\
&= \bigvee_{J\in 2^I_{<\aleph_0}} \overleftarrow{p_J} \left(\bigvee_{j\in J} \overleftarrow{p_j^J} \mu_j \exists_{p_j^J}\right) \exists_{p_J} \qquad \text{(since } \overleftarrow{p_J} \dashv \overrightarrow{p_J}\text{)} \\
&= \bigvee_{J\in 2^I_{<\aleph_0}} \overleftarrow{p_J} \mu_J \exists_{p_J} = \bigcup_{J\in 2^I_{<\aleph_0}} \overleftarrow{p_J} \mu_J \exists_{p_J},
\end{aligned}$$

the last equality holds since the set $\{\overleftarrow{p_J}\mu_J\exists_{p_J} : J \in 2^I_{<\aleph_0}\}$ is an up-directed set of filters. This yields:

$$\mu = \bigvee_{i\in I} \overleftarrow{p_i}\mu_i\exists_{p_i} = \bigvee_{J\in 2^I_{<\aleph_0}} \overleftarrow{p_J}\mu_J\exists_{p_J} = \bigcup_{J\in 2^I_{<\aleph_0}} \overleftarrow{p_J}\mu_J\exists_{p_J}. \tag{2.15}$$

N. The computations above are based on the fact that the structure on products is an inverse limit, the preneighbourhood system is the smallest making the product morphisms preneighbourhood morphisms, and hence is a limit of the inverse system of the coordinate preneighbourhood systems, compare with [3, §11].

## 3 Closure operation from preneighbourhood

Let $\mathscr{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

**Definition 3.1.** Given a preneighbourhood system $(X, \mu)$ define:

$$\mathsf{cl}^{\mathsf{F}}_{\mu} p = \bigvee \{x \in \mathtt{Sub}_{\mathsf{M}}(X) : \sigma_X \neq t \leq x \ \ \mathsf{AND}\ \ u \in \mu(t) \Rightarrow u \wedge p \neq \sigma_X\}, \tag{3.1}$$

a *$\mu$-closure of* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} = \mathtt{Fix}[\mathsf{cl}^{\mathsf{F}}_{\mu}] = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : p = \mathsf{cl}^{\mathsf{F}}_{\mu} p\}$ is the (possibly large) set of all *$\mu$-closed admissible subobjects* of $X$.

Further, given the preneighbourhood spaces $(X, \mu)$, $(Y, \phi)$ and a morphism $X \xrightarrow{f} Y$, $f$ is *$\mu$-$\phi$ continuous* or simply *continuous* if for each $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\exists_f \ \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p \leq \ \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_f p; \tag{3.2}$$

$f$ is said to be *$\mu$-$\phi$ continuous with respect to closures* or simply *continuous with respect to closures* if for each $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\exists_f \mathsf{cl}^{\mathsf{F}}_{\mu} p \leq \mathsf{cl}^{\mathsf{F}}_{\phi} \exists_f p. \tag{3.3}$$

**Theorem 3.2.** *The following statements are true*:

(a) *For any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $p \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$, $\mathsf{cl}^{\mathsf{F}}_{\mu} \sigma_X = \sigma_X$ and the assignment $p \mapsto \mathsf{cl}^{\mathsf{F}}_{\mu} p$ is monotonic.*

(b) *If* $\mu$ *is a weak neighbourhood system then for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mathsf{cl}^{\mathsf{F}}_{\mu}\mathsf{cl}^{\mathsf{F}}_{\mu}p = \mathsf{cl}^{\mathsf{F}}_{\mu}p$.

(c) *If* $\mu$ *is a weak neighbourhood system then for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mathsf{cl}^{\mathsf{F}}_{\mu}p$ *is the smallest* $\mu$*-closed subobject containing* $p$, *i.e.,* $\mathsf{cl}^{\mathsf{F}}_{\mu}p = \bigwedge\big\{k \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} : p \leq k\big\}$.

(d) *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is distributive then* $\mathsf{cl}^{\mathsf{F}}_{\mu}$ *is* additive:

$$\mathsf{cl}^{\mathsf{F}}_{\mu}(p \vee p') = \mathsf{cl}^{\mathsf{F}}_{\mu}p \vee \mathsf{cl}^{\mathsf{F}}_{\mu}p'. \tag{3.4}$$

(e) *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented then*:

$$p \in \mathfrak{O}_{\mu} \Leftrightarrow p^* \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}, \tag{3.5}$$

$$p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} \Rightarrow p^* \in \mathfrak{O}_{\mu} \tag{3.6}$$

$$p^* \in \mathfrak{O}_{\mu} \Rightarrow p^{**} = \mathsf{cl}^{\mathsf{F}}_{\mu}p \tag{3.7}$$

*In particular, every closed subobject is* regular closed, *i.e., for any* $p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$, $p = p^{**}$.

(f) *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented then there exists an adjunction* $\mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} \underset{*}{\overset{*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} {\mathfrak{O}_{\mu}}^{\mathsf{op}}$ *restricting to a dual equivalence between* $\mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ *and* ${\mathfrak{O}_{\mu}}^{*} = \big\{p \in \mathfrak{O}_{\mu} : p = p^{**}\big\}$ *of* regular open *subobjects of* $X$.

(g) *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented,* $\mu$ *is weak neighbourhood with open interiors then*:

$$\big(\mathsf{cl}^{\mathsf{F}}_{\mu}p\big)^* = \mathtt{int}_{\mu}p^*. \tag{3.8}$$

(h) *Given* $A \overset{a}{\rightarrowtail} M \overset{m}{\rightarrowtail} X$ , $\mathsf{cl}^{\mathsf{F}}_{\mu}$ *is* hereditary:

$$\mathsf{cl}^{\mathsf{F}}_{(\mu|_{m})}a = m^{-\mathbf{1}}\mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a). \tag{3.9}$$

*Furthermore, if* $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ *then*:

$$\mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a) = m \circ \mathsf{cl}^{\mathsf{F}}_{(\mu|_{mu})}a. \tag{3.10}$$

(i) *Given the preneighbourhood spaces* $(X, \mu)$, $(Y, \phi)$ *and any morphism* $X \xrightarrow{f} Y$ *consider the statements*:

i. *$f$ is continuous with respect to closures*

ii. *$f$ is continuous*

iii. $t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}} \Rightarrow f^{-1}t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$

*the implications* i. $\Longrightarrow$ ii. $\Longleftrightarrow$ iii. *hold good. Evidently, if* $\mathsf{cl}^{\mathsf{F}}_{\mu}$ *is idempotent then all three are equivalent.*

(j) *Every reflecting zero preneighbourhood morphism which satisfies* (2.9) *is continuous with respect to closures.*

*Thus, in reflecting zero formal function context* $\mathcal{A}$ *all preneighbourhood morphisms are continuous with respect to closures.*

(k) *Assume*: $\mathcal{A}$ *is a reflecting zero formal function context. Given any family* $\big((X_i, \mu_i)_i\big)_{i \in I}$ *of preneighbourhood spaces with the earlier conventions for* $u \in \mathsf{Sub}_{\mathsf{M}}(X)$:

$$\mathsf{cl}^{\mathsf{F}}_{\mu} u = \bigwedge_{J \in 2^{I}_{<\aleph_0}} {p_J}^{-1} \mathsf{cl}^{\mathsf{F}}_{\mu_J} \exists_{p_J} u. \tag{3.11}$$

*Proof.* The statements in (a) are evident and the statement in (c) follows from properties of idempotent closure operators [6, Proposition 2.1] while (f) immediately follows from (e).

Clearly $\mathsf{cl}^{\mathsf{F}}_{\mu} p \leq \mathsf{cl}^{\mathsf{F}}_{\mu}\mathsf{cl}^{\mathsf{F}}_{\mu} p$, since $p \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$ from (a). If $\sigma_X \neq x \leq \mathsf{cl}^{\mathsf{F}}_{\mu}\mathsf{cl}^{\mathsf{F}}_{\mu} p$ then $u \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p \neq \sigma_X$, for each $\sigma_X \neq t \leq x$ and each $u \in \mu(t)$. Choose and fix a $\sigma_X \neq t \leq x \leq \mathsf{cl}^{\mathsf{F}}_{\mu}\mathsf{cl}^{\mathsf{F}}_{\mu} p$ and a $u \in \mu(t)$. Since $\mu$ is a weak neighbourhood system on $X$, there exists a $q \in \mu(t)$ such that $x' \leq q \Rightarrow u \in \mu(x')$. Since $q \in \mu(t)$, $q \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p \neq \sigma_X$, $u \in \mu(q \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p)$, and since $\sigma_X \neq q \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$, $u \wedge p \neq \sigma_X$, in particular. Hence: $\sigma_X \neq t \leq x, u \in \mu(t) \Rightarrow u \wedge p \neq \sigma_X$ implying $x \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$, proving (b).

From monotonicity in (a), $\mathsf{cl}^{\mathsf{F}}_{\mu} p \vee \mathsf{cl}^{\mathsf{F}}_{\mu} p' \leq \mathsf{cl}^{\mathsf{F}}_{\mu}(p \vee p')$; if $x \leq \mathsf{cl}^{\mathsf{F}}_{\mu}(p \vee p')$ then $u \wedge (p \vee p') \neq \sigma_X$ for each $u \in \mu(t)$, $\sigma_X \neq t \leq x$. If $\mathsf{Sub}_{\mathsf{M}}(X)$ is distributive then $u \wedge (p \vee p') = (u \wedge p) \vee (u \wedge p') \neq \sigma_X$ implies either $u \wedge p \neq \sigma_X$ or $u \wedge p' \neq \sigma_X$, proving (d).

Towards a proof of (e) assume $\mathtt{Sub}_{\mathsf{M}}(X)$ is pseudocomplemented. Since the statements are trivially true for $p = \sigma_X, \mathbf{1}_X$, consider $p$ otherwise in proof of this part. If $p \in \mathfrak{O}_\mu$ and $\sigma_X \neq t \leq x$ $\textsf{AND}$ $u \in \mu(t) \Rightarrow u \wedge p^* \neq \sigma_X$ then $x \wedge p = \sigma_X \Leftrightarrow x \leq p^*$ — for if $x \wedge p \neq \sigma_X$, then since $p \in \mathfrak{O}_\mu$, $p \in \mu(x \wedge p)$ and $p \wedge p^* = \sigma_X$, contradicting the property for $x$. Hence $x \leq \mathsf{cl}^{\mathsf{F}}_\mu p^* \Rightarrow x \leq p^*$, proving $p^* \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_\mu}$. Conversely, if $p^* \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_\mu}$ then:

$$\Big[\big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \in \mu(t) \Rightarrow u \wedge p^* \neq \sigma_X\big) \Rightarrow x \leq p^*\Big]$$
$$\Leftrightarrow \Big[\big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \wedge p^* = \sigma_X \Rightarrow u \notin \mu(t)\big) \Rightarrow x \wedge p = \sigma_X\Big],$$

and each of the statements are true. Since $\sigma_X \neq p \leq p$ $\textsf{AND}$ $p \wedge p^* = \sigma_X$ is a true statement, the truth of the last implication forces the truth of $p \in \mu(p)$ from $p \neq \sigma_X$; hence $p \in \mathfrak{O}_\mu$, proving (3.5). If $p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_\mu}$ then:

$$\Big[\big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \in \mu(t) \Rightarrow u \wedge p \neq \sigma_X\big) \Rightarrow x \leq p\Big]$$
$$\Leftrightarrow \Big[x \nleq p \Rightarrow \big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \in \mu(t) \ \textsf{AND}\ u \leq p^*\big)\Big]$$

and each of the statements are true. Since $p^* \leq p \Leftrightarrow p^* = \sigma_X$, if $p^* \neq \sigma_X$, then the truth of:

$$p^* \nleq p \Rightarrow \big(\sigma_X \neq p^* \leq p^* \ \textsf{AND}\ p^* \in \mu(p^*) \ \textsf{AND}\ p^* \leq p^*\big)$$

forces the truth of $p^* \in \mu(p^*)$, proving $p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_\mu} \Rightarrow p^* = \sigma_X$ $\textsf{OR}$ $p^* \in \mathfrak{O}_\mu$, and hence (3.6) stands proved. Finally, since $k \wedge p = \sigma_X \Leftrightarrow k \leq p^* \Leftrightarrow k \wedge p^{**} = \sigma_X$ and $p^* \in \mathfrak{O}_\mu \Leftrightarrow p^{**} \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_\mu}$ (by (3.5)) the following list of equivalent statements emerge:

$$\begin{aligned} x \leq p^{**} &\Leftrightarrow \big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \in \mu(t) \Rightarrow u \wedge p^{**} \neq \sigma_X\big) \\ &\Leftrightarrow \big(\sigma_X \neq t \leq x \ \textsf{AND}\ u \in \mu(t) \Rightarrow u \wedge p \neq \sigma_X\big) \\ &\Leftrightarrow x \leq \mathsf{cl}^{\mathsf{F}}_\mu p, \end{aligned}$$

yielding a proof of (3.7). Hence for any $p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$, $p^* \in \mathfrak{O}_\mu \Rightarrow p^{**} = \mathsf{cl}^{\mathsf{F}}_{\mu} p = p$, completing the proof of (e).

Since $\mu$ is a weak neighbourhood, $\mathsf{cl}^{\mathsf{F}}_{\mu}$ is idempotent, using (b); since $\mathsf{cl}^{\mathsf{F}}_{\mu} p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$, $\left(\mathsf{cl}^{\mathsf{F}}_{\mu} p\right)^* \in \mathfrak{O}_\mu$ using (3.6), and from $p \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$, $\left(\mathsf{cl}^{\mathsf{F}}_{\mu} p\right)^* \leq \mathtt{int}_\mu p^*$. If $\mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p \neq \sigma_X$ then for any $u \in \mu(\mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p)$, $u \wedge p \neq \sigma_X$. Since $\mathtt{int}_\mu p^*$ is $\mu$-open, $\mathtt{int}_\mu p^* \in \mu(\mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p) \Rightarrow p^* \in \mu(\mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p)$ and $p \wedge p^* = \sigma_X$, contradicting $\sigma_X \neq \mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$. Hence $\mathtt{int}_\mu p^* \wedge \mathsf{cl}^{\mathsf{F}}_{\mu} p = \sigma_X \Leftrightarrow \mathtt{int}_\mu p^* \leq \left(\mathsf{cl}^{\mathsf{F}}_{\mu} p\right)^*$, proving (3.8).

Since every admissible monomorphism satisfies the conditions of the first part of (j), $\mathsf{cl}^{\mathsf{F}}_{(\mu|_m)} a \leq m^{-1}\left(\mathsf{cl}^{\mathsf{F}}_{\mu} \exists_m a\right) = m^{-1} \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a)$. If $\sigma_M \neq t \leq x \leq m^{-1} \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a) \Leftrightarrow \sigma_X \neq m \circ t \leq m \circ x \leq \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a)$, and hence $v \in \mu(m \circ t) \Rightarrow v \wedge m \circ a \neq \sigma_X$; if $u \in (\mu|_m)(t)$, then there exists a $v \in \mu(m \circ t)$ such that $m^{-1} v \leq u$, and hence $u \wedge a \geq a \wedge m^{-1} v = m^{-1}(v \wedge m \circ a) \neq \sigma_M$, implying $x \leq \mathsf{cl}^{\mathsf{F}}_{(\mu|_m)} a$, proving (3.9). Furthermore, $\mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a) \leq \mathsf{cl}^{\mathsf{F}}_{\mu} m$, so that $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} \Rightarrow \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a) \leq m$; hence from (3.9), $m \circ \mathsf{cl}^{\mathsf{F}}_{(\mu|_a)} m = m \wedge \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a) = \mathsf{cl}^{\mathsf{F}}_{\mu}(m \circ a)$ if $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$, proving (h).

If $f$ is continuous, then for any $t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}$, $\widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} f^{-1} t \leq f^{-1}\, \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_f f^{-1} t \leq f^{-1}\, \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} t = f^{-1} t$, proving $f^{-1} t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$. Conversely, if $f^{-1}$ preserves closed subobjects, then:

$$
\begin{aligned}
f^{-1}\, \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_f p &= f^{-1} \bigwedge \left\{ k \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}} : \exists_f p \leq k \right\} \\
&= \bigwedge \left\{ f^{-1} k : k \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}} \text{ AND } p \leq f^{-1} k \right\} \\
&\geq \bigwedge \left\{ t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} : p \leq t \right\} \qquad (\text{since } k \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}} \Rightarrow f^{-1} k \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}) \\
&= \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p,
\end{aligned}
$$

proving continuity for $f$. Finally if $f$ is continuous with respect to closures, then for any $t \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}$, $\mathrm{cl}_\mu f^{-1} t \leq f^{-1} \mathsf{cl}^{\mathsf{F}}_{\phi} \exists_f f^{-1} t \leq f^{-1} \mathsf{cl}^{\mathsf{F}}_{\phi} t \leq f^{-1}\, \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} t = f^{-1} t$, proving $f^{-1}$ preserves closed subobjects. This proves (i).

Towards a proof of (j), given $(X,\mu) \xrightarrow{f} (Y,\phi)$ a preneighbourhood morphism with $f$ reflecting zero and satisfying (2.9), choose and fix a $x \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p$; if $\sigma_Y \neq s \leq \exists_f x$ then from (2.9) $x \wedge f^{-\mathbf{1}} s \neq \sigma_X$; further since $f$ is a preneighbourhood morphism, $v \in \phi(s) \Rightarrow f^{-\mathbf{1}} v \in \mu(f^{-\mathbf{1}} s)$. Since $\sigma_X \neq x \wedge f^{-\mathbf{1}} s \leq x$, $u \wedge p \neq \sigma_X$ for each $u \in \mu(x \wedge f^{-\mathbf{1}} s) \supseteq \mu(f^{-\mathbf{1}} s)$; in particular $p \wedge f^{-\mathbf{1}} v \neq \sigma_X \Rightarrow v \wedge \exists_f p \neq \sigma_Y$, since $f$ reflects zero [6, page 169, Theorem 2.11]. Hence $\exists_f x \leq \mathsf{cl}^{\mathsf{F}}_{\phi} \exists_f p$, proving (j).

Since for preneighbourhood systems $\mu \leq \phi$, $\mathsf{cl}^{\mathsf{F}}_{\phi} \leq \mathsf{cl}^{\mathsf{F}}_{\mu}$, using (j) on (2.15) one obtains for any $u \in \mathsf{Sub}_{\mathsf{M}}(X)$:

$$\mathsf{cl}^{\mathsf{F}}_{\mu} u \leq \bigwedge_{J \in 2^{I}_{<\aleph_0}} \mathsf{cl}^{\mathsf{F}}_{\overleftarrow{p_J} \mu_J \exists_{p_J}} u \leq \bigwedge_{J \in 2^{I}_{<\aleph_0}} {p_J}^{-\mathbf{1}} \mathsf{cl}^{\mathsf{F}}_{\mu_J} \exists_{p_J} u,$$

the last one following from (j) and the preneighbourhood morphisms ($J \in 2^{I}_{<\aleph_0}$) $(X,\mu) \xrightarrow{p_J} (X_j, \mu_J)$. Choose and fix $x \leq \bigwedge_{J \in 2^{I}_{<\aleph_0}} {p_J}^{-\mathbf{1}} \mathsf{cl}^{\mathsf{F}}_{\mu_J} \exists_{p_J} u$ and a $v \in \mu(t)$. Then each of the preceding statement in the list below is equivalent to the next:

(i) for each $J \in 2^{I}_{<\aleph_0}$, $\exists_{p_J} x \leq \mathsf{cl}^{\mathsf{F}}_{\mu_J} \exists_{p_J} u$

(ii) for each $J \in 2^{I}_{<\aleph_0}$, $\sigma_{X_J} \neq t \leq \exists_{p_J} x$, $v \in \mu_J(t)$, the condition $v \wedge \exists_{p_J} u \neq \sigma_{X_J}$ holds good

(iii) for each $J \in 2^{I}_{<\aleph_0}$, $\sigma_{X_J} \neq t \leq \exists_{p_J} x$, $v_j \in \mu_j(t^{\mathsf{M}}_j)$ $(j \in J)$, the condition $\times_{j \in J} v_j \wedge \exists_{p_J} u \neq \sigma_X$ holds good

(iv) for each $J \in 2^{I}_{<\aleph_0}$, $\sigma_{X_J} \neq t \leq \exists_{p_J} x$, $v_j \in \mu_j(t^{\mathsf{M}}_j)$ $(j \in J)$, the condition ${p_J}^{-\mathbf{1}}(\times_{j \in J} v_j) \wedge u \neq \sigma_X$ holds good,

where the last one follows from each morphism being formal functions. The proof of (k) now follows on an application of (2.15) to the last statement.

□

O. The preservation of closed subobjects by preimage in part (i).iii. of Theorem is characteristic of continuity; in the additional special case when the closure operation is idempotent, this is equivalent to continuity with respect to closures, [6, compare with Proposition 3.12 & Remark 3.14].

P. The assumption of *weak neighbourhood system* inducing idempotence in (b) can also be seen for $\mathrm{cl}_\mu$ [6, Definition 3.1]: if $x \leq \mathrm{cl}_\mu \mathrm{cl}_\mu p$ then $x \not> \mathrm{cl}_\mu p$ and $u \in \mu(x) \Rightarrow u \wedge \mathrm{cl}_\mu p \neq \sigma_X$. Since $\mu$ is a weak neighbourhood, one can choose a $u \in \mu(x)$ such that $y \leq u \Rightarrow u \in \mu(y)$. Choose and fix a $\sigma_X \neq y \not> p$ and $y \leq u \wedge \mathrm{cl}_\mu p$; hence $v \in \mu(y) \Rightarrow v \wedge p \neq \sigma_X$. Since $u \wedge v \in \mu(y)$, it follows that $u \wedge v \wedge p \neq \sigma_X$ and hence $u \wedge p \geq u \wedge v \wedge p \neq \sigma_X$. Consequently, $x \leq \mathrm{cl}_\mu p$, proving idempotence.

Q. In case when $\mathtt{Sub}_{\mathsf{M}}(X)$ is pseudocomplemented, $x \leq \mathsf{cl}^{\mathsf{F}}_\mu p \Leftrightarrow \big(\sigma_X \neq t \leq x \Rightarrow p^* \notin \mu(t)\big)$, [6, compare (3.5) & (3.6) with Theorem 3.5(c)].

R. The term *additive* for a closure operator is adopted from [4, condition $(\mathsf{AD})$ in §2.6]; (d) shows it holds whenever subobject lattice is distributive, [6, compare with Theorem 3.5(d), Proposition 3.12 & Remark 3.13].

S. The term *hereditary* for a closure operator is adopted from [4, condition $(\mathsf{HE})$ in §2.5]; the hereditary property for $\mathsf{cl}^{\mathsf{F}}_\mu$ is easily established unlike the case for $\mathrm{cl}_\mu$ [6, see Theorem 3.18 and ensuing discussion in §3.3]. Incidentally, this property appears as an axiom on the set of closed morphisms, [3, see condition $(\mathsf{F8})$, §11.1].

T. The statement in (k) essentially states that the closure operation in product preneighbourhood spaces is completely determined by the closures in finite products. Incidentally this property appears as an axiom for the set of closed morphisms, [3, see condition $(\mathsf{F9})$, §11.2].

U. Since Definition 3.1 appeared earlier and its properties investigated [10, see Proposition 2] some comparison between the two discussions is imperative:

(i). A blanket assumption of $\mathsf{E}$-morphisms satisfying Frobenius reciprocity law is tacitly present [10, see page 2357, after the proof of Lemma 1]; incidentally, this assumption is also present in [11]. In the present paper $(\mathsf{E}, \mathsf{M})$ is merely a proper factorisation structure and no further assumptions are made. Hence $(\mathtt{Loc}, \mathtt{Epi}(\mathtt{Loc}), \mathtt{RegMon}(\mathtt{Loc}))$ is an example of the theory developed herein, since in the category of locales the Frobenius reciprocity law holds for complemented sublocales only [23, see Proposition 1.4]. In this regard, the context of locales is known to possess a functorial neighbourhood system for a specific choice of neighbourhood system, namely the $T$-neighbourhood system $\mathtt{Sub}_{\mathtt{RegMon}(\mathtt{Loc})}(X) \xrightarrow{\tau_X} \mathtt{Fil}\,[X]$ on each locale $X$

— see §3.1 below and [5, see Example 3.37, Theorem 3.38 and Definition 4.3].

(ii). The paper [10] investigates closure operators on a category consequent to a neighbourhood operator. On the other hand, the approach in this paper is transversal: each object is assigned with a preneighbourhood system and thereby the preneighbourhood system provides a method to *measure closeness* of admissible subobjects. Thus *preservation* of this *measure of closeness* by morphisms is an extra property that a morphism may or may not have — statements (i) and (j) of Theorem 3.2 address extra conditions, either on the category or on morphisms to ensure possession of this property.

(iii). The paper [10] does not mention about the additivity, heredity and idempotence of the closure operations; here (b), (d), (h) of Theorem 3.2 discusses them. However, discussion on these properties do appear in [11], but in the restricted set up as already observed.

(iv). The paper discusses compatibility of closure and interior operators [10, see Proposition 3, page 2360]; statements (e)-(g) extend the observations of [10] to the pseudocomplemented case.

**3.1 Examples** If $\mathtt{Sub}_{\mathsf{M}}(X)$ is join generated by $\mathcal{G} \subseteq \mathtt{Sub}_{\mathsf{M}}(X)$, i.e., $\mathcal{G}$ is a set of non-zero admissible subobjects such that $p = \bigvee\{a \in \mathcal{G} : a \leq p\}$, then:

$$\mathsf{cl}^{\mathsf{F}}_{\mu} p = \bigvee\{a \in \mathcal{G} : a \leq \mathsf{cl}^{\mathsf{F}}_{\mu} p\}.$$

Thus, for instance in $(\mathsf{Loc}, \mathsf{Epi}, \mathsf{RegMono})$ with each object given the $T$-neighbourhood system $\mathtt{Sub}_{\mathsf{RegMono}}(X)^{\mathsf{op}} \xrightarrow{\tau_X} \mathtt{Fil}\,[X]$ [6, Example 2.20]:

$$\tau_X(S) = \{T \in \mathtt{Sub}_{\mathsf{RegMono}}(X) : (\exists a \in X)(S \subseteq \mathfrak{o}(a) \subseteq T)\},$$

where $\mathfrak{o}(a) = \{\ (a \Longrightarrow x) : x \in X\}$ is the *open sublocale* of $X$ for $a$ [14, §III.6.1], the computation for closure is:

$$\begin{aligned} x \in \mathsf{cl}^{\mathsf{F}}_{\tau_X} S &\Leftrightarrow [x] \subseteq \mathsf{cl}^{\mathsf{F}}_{\tau_X} S \\ &\Leftrightarrow \Big(\big([x'] \subseteq [x] \text{ AND } T \in \tau_X([x'])\big) \Rightarrow T \cap S \neq \{1\}\Big) \\ &\Leftrightarrow \Big(\big([x'] \subseteq [x] \text{ AND } [x'] \subseteq \mathfrak{o}(a)\big) \Rightarrow \mathfrak{o}(a) \cap S \neq \{1\}\Big) \end{aligned}$$

$$
\begin{aligned}
&\qquad\text{(since open sublocales generate } T\text{-neighbourhoods)}\\
&\Leftrightarrow \big([x] \subseteq \mathfrak{o}(a) \Rightarrow \mathfrak{o}(a) \cap S \neq \{1\}\big)\\
&\Leftrightarrow \big(\mathfrak{o}(a) \cap S = \{1\} \Rightarrow [x] \nsubseteq \mathfrak{o}(a)\big)\\
&\Leftrightarrow \big(S \subseteq {\uparrow}\, a \Rightarrow [x] \nsubseteq \mathfrak{o}(a)\big)\\
&\qquad\text{((ii) below)}\\
&\Leftrightarrow \big(a \leq \bigwedge S \Rightarrow [x] \nsubseteq \mathfrak{o}(a)\big)\\
&\Leftrightarrow x \geq \bigwedge S\\
&\qquad\text{((iii)\&(iv) below),}
\end{aligned}
$$

where:

(i) $[x] = \big\{\ (t \Longrightarrow x) : t \in X\big\}$ is the smallest sublocale of $X$ containing $x$.

(ii) For each $a \in X$, ${\uparrow}\, a$ is the closed sublocale for $a$; ${\uparrow}\, a$ and $\mathfrak{o}(a)$ are complementary sublocales, [14, Proposition III.6.1.3].

(iii) Assume $a \leq \bigwedge S \leq x \neq 1$; if for $s \in S$, $(s \Longrightarrow x) \in \mathfrak{o}(a)$ then $(s \Longrightarrow x) = (a \Longrightarrow (s \Longrightarrow x)) = (a \wedge s \Longrightarrow x) = (a \Longrightarrow x) = 1$ implying $s \leq x$. Hence $x = (1 \Longrightarrow x) \notin \mathfrak{o}(a)$ proving $[x] \nsubseteq \mathfrak{o}(a)$

(iv) Since the assignment $a \mapsto \mathfrak{o}(a)$ is order preserving [14, Corollary III.6.1.4], $(a \leq \bigwedge S \Rightarrow [x] \nsubseteq \mathfrak{o}(a)) \Leftrightarrow [x] \nsubseteq \mathfrak{o}(\bigwedge S)$; hence the implication is equivalent to proving the statement $[x] \nsubseteq \mathfrak{o}(\bigwedge S) \Rightarrow x \geq \bigwedge S$, which in turn is equivalent to proving the truth of the statement $[x] \subseteq \mathfrak{o}(\bigwedge S)$ $\mathsf{OR}$ $x \geq \bigwedge S$, and is immediate.

Hence: $\mathsf{cl}^{\mathsf{F}}_{\tau_X} S = {\uparrow} \bigwedge S = \mathrm{cl}_\mu S$, [6, page 194] and agrees with the closure for locales, [14, §III.8.1].

Furthermore, if $\mathcal{G}$ is a set of atoms then the formula becomes simpler:

$$
\mathsf{cl}^{\mathsf{F}}_{\mu} p = \bigvee \big\{a \in \mathsf{atom}(X) : u \in \mu(a) \Rightarrow u \wedge p \neq \sigma_X\big\}; \tag{3.12}
$$

in particular, in all such cases $\mathsf{cl}^{\mathsf{F}}_{\mu} p = \mathrm{cl}_\mu p$ [6, Remark 3.4].

## 4 Closed morphisms

This section deals with morphisms which preserve closure operation of Definition 3.1.

**Definition 4.1.** Given preneighbourhood spaces $(X,\mu)$ and $(Y,\phi)$, a morphism $X \xrightarrow{f} Y$ is a $\mu$-$\phi$ *closed morphism*, or simply *closed morphism* when the preneighbourhood systems are evident[7], if $\exists_f p \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}$ whenever $p \in \mathsf{cl}^{\mathsf{F}}_{\mu}$. The (possibly large) set of closed morphisms is $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ .

**Theorem 4.2.** *The following statements are true for preneighbourhood spaces $(X,\mu)$, $(Y,\phi)$, $(Z,\psi)$ and morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.*

(a) *The morphism $f$ is closed if and only if for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:*

$$\widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi}\exists_f p \leq \exists_f \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p. \tag{4.1}$$

(b) *If $f$ is continuous then $f$ is closed if and only if $\widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi}\exists_f p = \exists_f \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p$ for each $p \in \mathtt{Sub}_{\mathsf{M}}(X)$.*

*In particular, $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ if and only if $m$ is a closed morphism.*

(c) $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ *contains all isomorphisms and is closed under compositions.*

(d) *If $f \in \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ is continuous then for each $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}$ the corestriction $f_m \in \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ and continuous.*

(e) *If $g \circ f \in \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ and $f$ is formally surjective and continuous then $g \in \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$.*

*Proof.* The statement of the first part in (b) follows immediately from (a) and definition of continuity, while the second part follows from (3.10); evidently, every isomorphism is a closed morphism and closed morphisms are closed under composition follows from (a), proving (c).

If $f$ is a closed morphism then since $\widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p$ is a closed subobject containing $p$, $\exists_f \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} p$ is a closed subobject containing $\exists_f p$; hence (4.1) follows. Conversely, if $p$ is a closed subobject then (4.1) forces $\widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi}\exists_f p \leq \exists_f p$ and hence $\exists_f p$ is closed from extensionality of $\widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu}$ . This proves (a).

[7]For instance, for an $m \in \mathtt{Sub}_{\mathsf{M}}(X)$, to say $m$ is a closed morphism would mean $m$ is $(\mu|_m)$-$\mu$ closed morphism, where $\mu$ is a preneighbourhood system on $X$.

If $\begin{tikzcd} P \arrow[r, tail, "p"] & f^{-\mathbf{1}}M \arrow[r, "f_m"] \arrow[d, tail, "f^{-\mathbf{1}}m"'] & M \arrow[d, tail, "m"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$ is a pullback and $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ then $f^{-\mathbf{1}}m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ (continuity of $f$, Theorem 3.2(i)) and:

$$\begin{aligned} m \circ \exists_{f_m} \widehat{\mathsf{cl}^{\mathsf{F}}}_{(\mu|_{f^{-\mathbf{1}}m})} p &= \exists_f \left( (f^{-\mathbf{1}}m) \circ \widehat{\mathsf{cl}^{\mathsf{F}}}_{(\mu|_{f^{-\mathbf{1}}m})} p \right) \\ &= \exists_f \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu}((f^{-\mathbf{1}}m) \circ p) \quad (\because, f^{-\mathbf{1}}m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} \text{ and } (3.10)) \\ &= \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_f ((f^{-\mathbf{1}}m) \circ p) \quad (\because,\ f \text{ is closed and continuous}) \\ &= \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi}(m \circ \exists_{f_m} p) \\ &= m \circ \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_{f_m} p \qquad (\because, m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}), \end{aligned}$$

implying $\exists_{f_m} \widehat{\mathsf{cl}^{\mathsf{F}}}_{(\mu|_{f^{-\mathbf{1}}m})} p = \widehat{\mathsf{cl}^{\mathsf{F}}}_{(\phi|_m)} \exists_{f_m} p$; hence the corestriction $f_m$ is closed and continuous, proving (d).

If $g \circ f$ is closed and $f$ is formally surjective and continuous then for any $y \in \mathsf{Sub}_{\mathsf{M}}(Y)$:

$$\begin{aligned} \widehat{\mathsf{cl}^{\mathsf{F}}}_{\psi} \exists_g y &= \widehat{\mathsf{cl}^{\mathsf{F}}}_{\psi} \exists_g \exists_f f^{-\mathbf{1}} y \quad (\because,\ f \text{ is formally surjective}) \\ &\le \exists_g \exists_f \widehat{\mathsf{cl}^{\mathsf{F}}}_{\mu} f^{-\mathbf{1}} y \quad (\because,\ g \circ f \text{ is closed}) \\ &\le \exists_g \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} \exists_f f^{-\mathbf{1}} y \quad (\because,\ f \text{ is continuous}) \\ &= \exists_g \widehat{\mathsf{cl}^{\mathsf{F}}}_{\phi} y, \end{aligned}$$

proving (e). □

V. Definition 4.1 appears as an axiom on the set of closed morphisms in [3, condition (F7), §11.1].

W. The (possibly large) set

$$\mathbb{A}^{\mathsf{F}}_{\text{clemb}} = \bigcup \{ \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}} : \mu \text{ is a preneighbourhood system} \} = \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}} \cap \mathsf{M}$$

is the set of all *closed embeddings*. Evidently $\mathbb{A}^{\mathsf{F}}_{\mathrm{clemb}}$ is closed under intersections, even large ones. The fact that closed embeddings are closed under intersections (even large ones) appears as an axiom on the set of closed morphisms in [3, condition (F6), §11.1].

## 5 Dense, proper, separated and perfect morphisms

Let $\mathscr{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

Recall: given the preneighbourhood morphism $(X,\mu) \xrightarrow{f} (X,\mu)$, consider the pullback square

$$\begin{array}{ccc} (\mathtt{k}_f, \mu \times_\phi \mu) & \xrightarrow{f_2} & (X,\mu) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ (X,\mu) & \xrightarrow[f]{} & (Y,\phi) \end{array},$$

the kernel pair

$$(\mathtt{k}_f, \mu \times_\phi \phi) \underset{f_2}{\overset{f_1}{\rightrightarrows}} (X,\mu)$$

and their equaliser

$$(X,\mu) \overset{d_f}{\rightarrowtail} (\mathtt{k}_f, \mu \times_\phi \mu) \underset{f_2}{\overset{f_1}{\rightrightarrows}} (X,\mu)$$

note: $((\mu \times_\phi \mu)\big|_{d_f}) = \mu$, [6, §7].

**Definition 5.1.** Given the preneighbourhood spaces $(X,\mu)$, $(Y,\phi)$ and a morphism $X \xrightarrow{f} Y$:

(a) $f$ is $\mu$-$\phi$ *dense*, or simply *dense*, if $f = m \circ h$ with $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\phi}}$ implies $m$ is an isomorphism; the (possibly large) set of dense morphisms is $\mathbb{A}^{\mathsf{F}}_{\mathsf{d}}$.

(b) If $(X,\mu) \xrightarrow{f} (Y,\phi)$ then $f$ is a *proper morphism* if for every $(T,\tau) \xrightarrow{h} (Y,\phi)$ and the pullback

$$\begin{array}{ccc} (X \times_Y T, \mu \times_\phi \tau) & \xrightarrow{f_h} & (T,\tau) \\ {\scriptstyle h_f}\downarrow & & \downarrow{\scriptstyle h} \\ (X,\mu) & \xrightarrow[f]{} & (Y,\phi) \end{array}$$

of $h$ along $f$, the

preneighbourhood morphism $f_h$ is a $(\mu \times_\phi \tau)$-$\tau$ closed morphism; the (possibly large) set of proper morphisms $\mathbb{A}^{\mathsf{F}}_{\mathsf{pr}}$.

(c) If $(X,\mu) \xrightarrow{f} (Y,\phi)$ then $f$ is a *separated morphism* if $d_f$ is a proper morphism; the (possibly large) set of all separated morphisms is $\mathbb{A}^{\mathsf{F}}_{\mathsf{sep}}$.

(d) If $(X,\mu) \xrightarrow{f} (Y,\phi)$ then $f$ is a *perfect morphism* if it is proper and separated; the (possibly large) set of all perfect morphisms is $\mathbb{A}^{\mathsf{F}}_{\mathsf{per}}$.

A preneighbourhood space $(X,\mu)$ is:

(e) *compact* if $(X,\mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1},\nabla_{\mathtt{1}})$ is proper; the full subcategory of compact preneighbourhood spaces is $\mathtt{K}(\ \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}})$.

(f) *Hausdorff* if $(X,\mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1},\nabla_{\mathtt{1}})$ is separated; the full subcategory of Hausdorff preneighbourhood spaces is $\mathtt{Haus}(\ \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}})$.

(g) *compact Hausdorff* if $(X,\mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1},\nabla_{\mathtt{1}})$ is perfect; the full subcategory of compact Hausdorff preneighbourhood spaces is $\mathtt{KHaus}(\ \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}})$.

**5.1 Properties of dense morphisms** The properties of closed morphisms and their allies put forward in Definition 5.1 are exhibited in Table 1. The proofs precisely depend on the statements in Theorem 4.2((c)-(e)) and the hereditary property established in (3.9), and hence is exactly obtained from proofs of similar statements in [6] or as detailed in [3].

Some terms need to be explained:

Given (possibly large) sets $\mathfrak{a}$, $\mathfrak{b}$ of morphisms of $\mathbb{A}$, the phrases $\mathfrak{b}$ *is composition closed* or $\mathfrak{b}$ *is* (*pullback*) *stable* are well known; the set $\mathfrak{b}$ shall be said to be *left* $\mathfrak{a}$ *cancellative* (respectively, *right* $\mathfrak{a}$ *cancellative*) if $g \circ f \in \mathfrak{b}$ and $g \in \mathfrak{a}$ (respectively, $f \in \mathfrak{a}$) implies $f \in \mathfrak{b}$ (respectively, $g \in \mathfrak{b}$). The set $\mathfrak{b}$ is *stably in* $\mathsf{E}$ if in the pullback

$$\begin{array}{ccc} \cdot & \xrightarrow{f_g} & \cdot \\ \downarrow & & \downarrow g \\ \cdot & \xrightarrow[f]{} & \cdot \end{array}$$

where $f \in \mathfrak{b}$ and $g$ is any morphism, $f_g \in \mathsf{E}$. If $\mathfrak{b}$ is a set of preneighbourhood morphisms then it is said to be *stably continuous* if for any $\mu$-$\phi$ continuous preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ in $\mathfrak{b}$, and for any preneighbourhood morphism

$(Z,\psi) \xrightarrow{g} (Y,\phi)$, the pullback $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z,\psi)$ of $f$ along $g$ is $(\mu \times_\phi \psi)$-$\psi$ continuous and is also in $\mathfrak{b}$.

The following definition appears in §2 [9]:

**Definition 5.2.** A pullback stable (possibly large) set $\mathfrak{a}$ of morphisms of $\mathbb{A}$ is called a *topology* if it contains isomorphisms and is closed under compositions.

If $\mathfrak{a}$ is a topology and right $\mathfrak{a}$ cancellative, a topology $\mathfrak{b}$ is called an $\mathfrak{a}$-*topology* if it is right $\mathfrak{a}$ cancellative.

Drawing inspiration from [3], it is observed in [9, §2] that in case when a finitely complete category $\mathbb{A}$ with a proper $(\mathsf{E},\mathsf{M})$-factorisation system has a set $\mathbb{A}_{\mathsf{cl}}$ of closed morphisms described by axioms (see Axioms (F3)-(F5) [3]), then the set of proper morphisms (i.e., morphisms stably in $\mathbb{A}_{\mathsf{cl}}$) is an $\mathfrak{s}$-topology, where $\mathfrak{s}$ is the set of morphisms stably in $\mathsf{E}$.

In terms of Definition, Table 1 shows the set $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ is a right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative topology and each of the sets $\mathbb{A}_{\mathsf{pr}}$, $\mathbb{A}_{\mathsf{sep}}$, $\mathbb{A}_{\mathsf{per}}$ are $\mathbb{A}_{\mathtt{st(E,c,cl)}}$-topologies. The difference between the two approaches arises from the fact that in the present case $\mathbb{A}_{\mathsf{cl}}$ is right $\mathbb{A}_{\mathtt{fsc}}$ $(\subset \mathsf{E})$ cancellative, while the axioms of [3] assert $\mathbb{A}_{\mathsf{cl}}$ is right $\mathsf{E}$ cancellative. In case when $\mathscr{A}$ is $\mathtt{RZC}$ and $\mathsf{E}$ is pullback stable the present case reduces to the situation considered in [3].

Regarding Table 1: the cells highlighted in this colour are the properties where the *continuity* condition is required; the others do not require *continuity* of the involved preneighbourhood morphism, and hence are purely consequences of the preneighbourhood morphism property.

### 5.2 Properties of compact preneighbourhood spaces

This section lists the properties of compact preneighbourhood spaces; proofs are similar as in [3] or [6].

**Theorem 5.3** (Compact preneighbourhood spaces)**.** *A preneighbourhood space $(X,\mu)$ is compact if and only if for every preneighbourhood space $(Y,\phi)$ the second product projection $(X \times Y, \mu \times \phi) \xrightarrow{p_Y} (Y,\phi)$ is a closed morphism. Furthermore*:

(a) *If $(Y,\phi)$ is a compact preneighbourhood space and $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a proper morphism then $(X,\mu)$ is a compact preneighbourhood space.*

(b) *If $(X,\mu)$ is a compact preneighbourhood space and $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a preneighbourhood morphism with $f$ continuously stably in $\mathsf{E}$ then $(Y,\phi)$ is a compact preneighbourhood space.*

(c) *If every preneighbourhood morphism is continuous, $(X,\mu)$ is a compact preneighbourhood space and $\left(M,(\mu\big|_m)\right) \overset{m}{\rightarrowtail} (X,\mu)$ with $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$ then $\left(M,(\mu\big|_m)\right)$ is a compact preneighbourhood space.*

(d) *The category $\mathtt{K}(\ \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}})$ is finitely productive.*

**5.3 Properties of Hausdorff preneighbourhood spaces** This section lists equivalents of Hausdorff preneighbourhood spaces and their properties; proofs are similar as in [3] or [6].

Note the presence of *proper*-ness condition in Theorem, in contrast to just *closed*-ness in [3].

**Theorem 5.4** (Hausdorff preneighbourhood spaces)**.** *The following are equivalent for any preneighbourhood space $(X,\mu)$:*

(a) *$(X,\mu)$ is a Hausdorff preneighbourhood space.*

(b) *The diagonal morphism $(X,\mu) \overset{d_X}{\rightarrowtail} (X \times X, \mu \times \mu)$ is a proper morphism.*

(c) *Every preneighbourhood morphism with $(X,\mu)$ as domain is separated.*

(d) *There exists a separated preneighbourhood morphism from $(X,\mu)$ to a Hausdorff preneighbourhood space.*

(e) *If $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a proper preneighbourhood morphism with $f$ continuously stably in $\mathsf{E}$ then $(Y,\phi)$ is a Hausdorff preneighbourhood space.*

(f) *The product projection $(X \times Y, \mu \times \phi) \xrightarrow{p_Y} (Y,\phi)$ is a separated preneighbourhood morphism.*

(g) *For every preneighbourhood space $(Y,\phi)$, $(X \times Y, \mu \times \phi)$ is a Hausdorff preneighbourhood space.*

(h) *If $\left(E,(\psi\big|_e)\right) \overset{e}{\rightarrowtail} (Z,\psi) \underset{g}{\overset{f}{\rightrightarrows}} (X,\mu)$ is the equaliser diagram then $e$ is a proper morphism.*

**Corollary 5.5.** *The category* $\mathtt{Haus}(\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}})$ *of Hausdorff preneighbourhood spaces is finitely complete. Furthermore, given the preneighbourhood morphisms* $(M, \overleftarrow{m}\mu\exists_m) \xrightarrow{m} (X, \mu) \xrightarrow{f} (Y, \phi)$ *the following statements are true*:

(a) *if* $m$ *is a monomorphism,* $f$ *is continuously stably in* $\mathsf{E}$ *and* $(X, \mu)$ *a Hausdorff preneighbourhood space, then both* $(M, \overleftarrow{m}\mu\exists_m)$ *and* $(Y, \phi)$ *are Hausdorff spaces.*

(b) *if* $(X, \mu)$ *is a compact preneighbourhood space and* $(Y, \phi)$ *is a Hausdorff preneighbourhood space then* $f$ *is proper.*

(c) *if* $(Y, \phi)$ *is a compact and Hausdorff preneighbourhood space then* $f$ *is proper if and only if* $(X, \mu)$ *is compact.*

(d) *if* $(X, \mu)$ *is a Hausdorff preneighbourhood space,* $m \in \mathsf{M}$ *and* $\Big(M, (\mu\big|_m)\Big)$ *is a compact preneighbourhood space then* $m \in \mathfrak{C}_{\mathsf{cl}^{\mathsf{F}}_{\mu}}$.

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| | contains | stability | closed under composition | cancellation properties |
|---|---|---|---|---|
| $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ | $\mathtt{Iso}(\mathbb{A})$ | $m \in \mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}, f \in \mathbb{A}_{\mathtt{clc}} \Rightarrow f_m \in \mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$<br>$m \in \mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}, f \in \mathbb{A}_{\mathsf{c}} \Rightarrow f^{-1}m \in \mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}$ | composition closed | right $\mathbb{A}_{\mathtt{fsc}}$ cancellative |
| $\mathbb{A}^{\mathsf{F}}_{\mathsf{d}}$ | $\mathsf{E}$ | | $g \in \mathbb{A}_{\mathtt{dc}}, f \in \mathbb{A}^{\mathsf{F}}_{\mathsf{d}} \Rightarrow g \circ f \in \mathbb{A}^{\mathsf{F}}_{\mathsf{d}}$ | right $\mathbb{A}_1$ cancellative |
| $\mathbb{A}^{\mathsf{F}}_{\mathsf{pr}}$ | $\mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}$, in RZC | pullback stable | composition closed | right $\mathbb{A}_{\mathtt{st(E,c)}}$ cancellative |
| $\mathbb{A}^{\mathsf{F}}_{\mathsf{sep}}$ | $\mathtt{Mono}(\mathbb{A})$ | pullback stable | composition closed | left $\mathtt{Mono}(\mathbb{A})$ cancellative<br>right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative<br>left $\mathbb{A}_1$ cancellative |
| $\mathbb{A}^{\mathsf{F}}_{\mathsf{per}}$ | $\mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}$, in RZC | pullback stable | composition closed | right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative<br>left $\mathbb{A}^{\mathsf{F}}_{\mathsf{per}}$ cancellative |

[1] $\mathbb{A}_1$ is the (possibly large) set of all morphisms
[2] $\mathbb{A}^{\mathsf{F}}_{\mathsf{cl}}$ is the (possibly large) set of all closed morphisms
[3] $\mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}}$ is the (possibly large) set of all closed embeddings
[4] $\mathbb{A}^{\mathsf{F}}_{\mathsf{d}}$ is the (possibly large) set of all dense preneighbourhood morphisms
[5] $\mathbb{A}^{\mathsf{F}}_{\mathsf{pr}}$ is the (possibly large) set of all proper preneighbourhood morphisms
[6] $\mathbb{A}^{\mathsf{F}}_{\mathsf{sep}}$ is the (possibly large) set of all separated preneighbourhood morphisms
[7] $\mathbb{A}^{\mathsf{F}}_{\mathsf{per}}$ is the (possibly large) set of all perfect preneighbourhood morphisms
[8] $\mathbb{A}_{\mathsf{c}}$ is the (possibly large) set of all continuous preneighbourhood morphisms
[9] $\mathbb{A}_{\mathtt{dc}}$ is the (possibly large) set of all dense and continuous preneighbourhood morphisms
[10] $\mathbb{A}_{\mathtt{fsc}}$ is the (possibly large) set of all formally surjective and continuous preneighbourhood morphisms
[11] $\mathbb{A}_{\mathtt{clc}}$ is the (possibly large) set of all closed and continuous preneighbourhood morphisms
[11] $\mathbb{A}_{\mathtt{st(E,c)}}$ is the (possibly large) set of all preneighbourhood morphisms which are stably continuous and stably in $\mathsf{E}$
[12] $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ is the (possibly large) set of all preneighbourhood morphisms which are stably continuous, stably in $\mathsf{E}$ and stably closed
[13] RZC abbreviates *reflecting zero context*
[14] the cells in *this colour* indicate the presence of *continuity* in the assertion
[15] additionally, every RZC with continuous preneighbourhood morphisms has $(\mathbb{A}^{\mathsf{F}}_{\mathsf{d}}, \mathbb{A}^{\mathsf{F}}_{\mathsf{clemb}})$ factorisation structure

Table 1: Comparative list of properties

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