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# Generalised geometric logic

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**Abstract.** This paper introduces a notion of generalised geometric logic. Connections of generalised geometric logic with the L-topological system and L-topological space are established.

## 1 Introduction

This work is motivated by S. Vickers's work on topology via logic [21]. To illustrate the relationship between topology and geometric logic, the concept of a topological system played a crucial role. A topological system is a triple $(X, \models, A)$, consisting of a non-empty set $X$, a frame $A$ and a binary relation $\models$ (known as satisfaction relation) between $X$ and $A$ satisfying certain conditions. The notion of a topological system was introduced by S. Vickers in 1989. A topological system is an interesting mathematical structure that unifies the concepts of topology, algebra, and logic in a single framework. In our earlier work [1], we introduced a notion of fuzzy geometric logic to answer the question viz. "From which logic can fuzzy topology be studied?". For this purpose first of all we introduced the notion of fuzzy topological

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system [9] which is a triple $(X, \models, A)$ consisting of a non-empty set $X$, a frame $A$ and a fuzzy relation $\models$ (i.e. $[0,1]$ valued relation) from $X$ to $A$. J. Denniston et al. introduced the notion of lattice valued topological system ($L$-topological system) by considering frame valued relation between $X$ and $A$. In [4], the categorical relationship of Lattice valued topological space ($L$-topological space) with the frame was established using the categorical relationships of them with the $L$-topological system. Moreover categorical equivalence between spatial $L$-topological system with $L$-topological space was shown. In this paper, the main focus is to answer the question viz. "From which logic can $L$-topology be studied?". From [1], it is clear that the satisfaction relation $\models$ of the fuzzy topological system reflects the notion of satisfiability (*sat*) of a geometric formula by a sequence over the domain of interpretation of the corresponding logic. Hence we considered the grade of satisfiability from $[0,1]$. As for the $L$-topological system the satisfaction relation is an $L$ (frame)-valued relation, the natural tendency is to consider the grade of satisfiability from $L$. Keeping this in mind, generalised geometric logic (c.f. Section 3) is proposed to provide the answer to the raised question successfully.

In [6], Michael Healy proposed a model theoretic study of a slightly primitive machine learning algorithm known as the LAPART architecture which consists of conjoining two ART systems. Healy helped establish the association of a topological system and the connections formed between the two ART systems in the LAPART architecture. In [3], we explained the connection of LAPART with fuzzy geometric logic (c.f. [1]) and fuzzy topological system. Consequently, it is quite expected to use the current study in the area of artificial intelligence.

The paper is organised as follows. Section 2, includes some of the preliminary definitions and results which are used in the sequel. Generalised geometric logic is proposed and studied in detail in Section 3. Section 4, explains the connection of the proposed logic with the $L$-topological system whereas Section 5, contains the study of the connection of the proposed logic with $L$-topological space. Section 6, concludes the work presented in this article and provides some of the future directions.

## 2 Preliminaries

In this section, we include a brief outline of relevant notions to develop our proposed mathematical structures and results. In [1, 2, 4, 7, 10, 12, 17, 21, 22] one may find the details of the notions stated here.

**Definition 2.1** (Frame)**.** A **frame** is a complete lattice such that,

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

i.e., the binary meet distributes over the arbitrary join.

**Definition 2.2** ($L$-topological space)**.** Let $X$ be a set, and $\tau$ be a collection of $L$-fuzzy subsets of $X$ i.e., $\tilde{A} : X \to L$, where $L$ is a frame, s.t.

1. $\tilde{\emptyset}$ , $\tilde{X} \in \tau$, where $\tilde{\emptyset}(x) = 0_L$, for all $x \in X$ and $\tilde{X}(x) = 1_L$, for all $x \in X$;
2. $\tilde{A}_i \in \tau$ for $i \in I$ implies $\bigcup_{i \in I} \tilde{A}_i \in \tau$, where $\bigcup_{i \in I} \tilde{A}_i(x) = \underset{i \in I}{sup}(\tilde{A}_i(x))$;
3. $\tilde{A}_1$ , $\tilde{A}_2 \in \tau$ implies $\tilde{A}_1 \cap \tilde{A}_2 \in \tau$, where $(\tilde{A}_1 \cap \tilde{A}_2)(x) = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$.

Then $(X, \tau)$ is called an $L$**-topological space**. $\tau$ is called an $L$**-topology** over $X$.

Elements of $\tau$ are called $L$**-open sets** of $L$-topological space $(X, \tau)$.

**Definition 2.3.** [21] A **topological system** is a triple, $(X, \models, A)$, consisting of a non-empty set $X$, a frame $A$ and a binary relation $\models \subseteq X \times A$ from $X$ to $A$ such that:

1. for **any finite subset** $S$ of $A$, $x \models \bigwedge S$ if and only if $x \models a$ for all $a \in S$;
2. for **any subset** $S$ of $A$, $x \models \bigvee S$ if and only if $x \models a$ for some $a \in S$.

**Definition 2.4** ($L$-topological system)**.** An $L$**-topological system** is a triple $(X, \models, A)$, where $X$ is a non-empty set, $A$ is a frame and $\models$ is an $L$-valued relation from $X$ to $A$ ($\models: X \times A \to L$) such that

1. if $S$ is a finite subset of $A$, then $\models (x, \bigwedge S) = inf\{\models (x, s) \mid s \in S\}$;
2. if $S$ is any subset of $A$, then $\models (x, \bigvee S) = sup\{\models (x, s) \mid s \in S\}$.

Note that if $L = [0, 1]$ then the triple is known as a fuzzy topological system. Similarly considering $L = \{0, 1\}$, we arrive at the notion of a topological system.

**Definition 2.5** (Spatial)**.** An $L$-topological system $(X, \models, A)$ is said to be **spatial** if and only if (for any $x \in X$, $\models (x, a) = \models (x, b)$) implies $(a = b)$, for any $a, b \in A$.

**Theorem 2.6.** [4] *The category of spatial $L$-topological systems, for a fixed $L$, is equivalent to the category of $L$-topological spaces.*

## 3 Generalised geometric logic

In this section, we will introduce the notion of generalised geometric logic which may be considered as a generalisation of fuzzy geometric logic and consequently of so-called geometric logic. Detailed studies on fuzzy logic, geometric logic and fuzzy geometric logic may be found in [1, 5, 13–16, 18–21].

The **alphabet** of the language $\mathscr{L}$ of generalised geometric logic comprises of the connectives $\wedge$, $\bigvee$, the existential quantifier $\exists$, parentheses ) and ( as well as:

- countably many individual constants $c_1, c_2, \ldots$;
- denumerably many individual variables $x_1, x_2, \ldots$;
- propositional constants $\top$, $\bot$;
- for each $i > 0$, countably many $i$-place predicate symbols $p_j^i$'s, including at least the 2-place symbol "=" for identity;
- for each $i > 0$, countably many $i$-place function symbols $f_j^i$'s.

**Definition 3.1** (Term)**. Terms** are recursively defined in the usual way.

- every constant symbol $c_i$ is a term;
- every variable $x_i$ is a term;
- if $f_j$ is an $i$-place function symbol, and $t_1, t_2, \ldots, t_i$ are terms then $f_j^i t_1 t_2 \ldots t_i$ is a term;
- nothing else is a term.

**Definition 3.2** (Geometric formula)**. Geometric formulae** are recursively defined as follows:

- $\top$, $\perp$ are geometric formulae;
- if $p_j$ is an $i$-place predicate symbol, and $t_1, t_2, \ldots, t_i$ are terms then $p^i_j t_1 t_2 \ldots t_i$ is a geometric formula;
- if $t_i$, $t_j$ are terms then $(t_i = t_j)$ is a geometric formula;
- if $\phi$ and $\psi$ are geometric formulae then $(\phi \wedge \psi)$ is a geometric formula;
- if $\phi_i$'s $(i \in I)$ are geometric formulae then $\bigvee\{\phi_i\}_{i\in I}$ is a geometric formula, when $I = \{1, 2\}$ then the above formula is written as $\phi_1 \vee \phi_2$;
- if $\phi$ is a geometric formula and $x_i$ is a variable then $\exists x_i \phi$ is a geometric formula;
- nothing else is a geometric formula.

**Definition 3.3** (Interpretation)**.** An **interpretation** $I$ consists of

- a set $D$, called the domain of interpretation;
- an element $I(c_i) \in D$ for each constant $c_i$;
- a function $I(f^i_j) : D^i \longrightarrow D$ for each function symbol $f^i_j$;
- an $L$-fuzzy relation $I(p^i_j) : D^i \longrightarrow L$, where $L$ is a frame, for each predicate symbol $p^i_j$ i.e. it is an $L$-fuzzy subset of $D^i$.

**Definition 3.4** (Graded Satisfiability)**.** Let $s = (s_1, s_2, \ldots)$ be a sequence over $D$ where $s_1, s_2, \ldots$ are all elements of $D$. Let $d$ be an element of $D$. Then $s(d/x_i)$ is the result of replacing $i$'th coordinate of $s$ by $d$ i.e., $s(d/x_i) = (s_1, s_2, \ldots, s_{i-1}, d, s_{i+1}, \ldots)$. Let $t$ be a term. Then $s$ assigns an element $s(t)$ of $D$ as follows:

- if $t$ is the constant symbol $c_i$ then $s(c_i) = I(c_i)$;
- if $t$ is the variable $x_i$ then $s(x_i) = s_i$;
- if $t$ is the function symbol $f^i_j t_1 t_2 \ldots t_i$ then $s(f^i_j t_1 t_2 \ldots t_i) = I(f^i_j)(s(t_1), s(t_2), \ldots, s(t_i))$.

Now we define grade of satisfiability of $\phi$ by $s$ written as $gr(s\ sat\ \phi)$, where $\phi$ is a geometric formula, as follows:

- $gr(s\ sat\ p^i_j t_1 t_2 \ldots t_i) = I(p^i_j)(s(t_1), s(t_2), \ldots, s(t_i))$;
- $gr(s\ sat\ \top) = 1_L$;
- $gr(s\ sat\ \bot) = 0_L$;
- $gr(s\ sat\ t_i = t_j) = \begin{cases} 1_L & if\ s(t_i) = s(t_j) \\ 0_L & otherwise; \end{cases}$
- $gr(s\ sat\ \phi_1 \wedge \phi_2) = gr(s\ sat\ \phi_1) \wedge gr(s\ sat\ \phi_2)$;
- $gr(s\ sat\ \phi_1 \vee \phi_2) = gr(s\ sat\ \phi_1) \vee gr(s\ sat\ \phi_2)$;
- $gr(s\ sat\ \bigvee\{\phi_i\}_{i\in I}) = sup\{gr(s\ sat\ \phi_i) \mid i \in I\}$;
- $gr(s\ sat\ \exists x_i \phi) = sup\{gr(s(d/x_i)\ sat\ \phi) \mid d \in D\}$.

Throughout this article, $\wedge$ and $\vee$ in $L$ will stand for the meet and join of the frame $L$ respectively. The expression $\phi \vdash \psi$, where $\phi$ and $\psi$ are geometric formulae, is called a sequent. We now define the satisfiability of a sequent.

**Definition 3.5.** 1. $s\ sat\ \phi \vdash \psi$ iff $gr(s\ sat\ \phi) \leq gr(s\ sat\ \psi)$.
2. $\phi \vdash \psi$ is valid in $I$ iff $s\ sat\ \phi \vdash \psi$ for all $s$ in the domain of $I$.
3. $\phi \vdash \psi$ is universally valid iff it is valid in all interpretations.

**Theorem 3.6** (Substitution Theorem)**.** *Let $D$ be the domain of interpretation $I$:*

1. *Let $t$ and $t'$ be terms. For every sequence $s$ over $D$,* $s(t[t'/x_k]) = s(s(t')/x_k)(t)$.
2. *Let $\phi$ be a geometric formula and $t$ be a term. For every sequence $s$ over $D$, $gr(s \text{ sat } \phi[t/x_k]) = gr(s(s(t)/x_k) \text{ sat } \phi)$.*

*Proof.* By induction on $t$ and $\phi$ respectively. □

### 3.1 Rules of inference

The rules of inference for generalised geometric logic are as follows.

1. $\phi \vdash \phi$,
2. $\dfrac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi}$,

3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\dfrac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$,
4. (i) $\phi \vdash \bigvee S$ $(\phi \in S)$, (ii) $\dfrac{\phi \vdash \psi \qquad \text{all } \phi \in S}{\bigvee S \vdash \psi}$,
5. $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$,
6. $\top \vdash (x = x)$,
7. $((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n) \mid (x_1, \ldots, x_n)]$,
8. (i) $\dfrac{\phi \vdash \psi[x \mid y]}{\phi \vdash \exists y \psi}$, (ii) $\dfrac{\exists y \phi \vdash \psi}{\phi[x \mid y] \vdash \psi}$, $y$ is not free in $\psi$,
9. $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$, $y$ is not free in $\phi$.

Note that the rule of inference (9) is known as Frobenius axiom [11] and the converse of the Frobenius axiom is derivable in geometric logic [11].

**Theorem 3.7.** *The rules of inference for generalised geometric logic are universally valid.*

*Proof.* To show the universal validity of the rules of inference is kind of a routine check but for the sake of clarity we would like to provide the proof in detail up to a certain extent.

1. $gr(s \text{ sat } \phi) = gr(s \text{ sat } \phi)$, for any $s$. Hence $\phi \vdash \phi$ is valid.
2. Given $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \chi)$ for any $s$. Therefore $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \chi)$ for any $s$. Hence $\phi \vdash \chi$ is valid when $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid.
3. (i) $gr(s \text{ sat } \phi) \leq 1_L = gr(s \text{ sat } \top)$ for any $s$. Hence $\phi \vdash \top$ is valid.
   (ii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \phi)$ for any $s$. Hence $\phi \wedge \psi \vdash \phi$ is valid.
   (iii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \psi)$ for any $s$. Hence $\phi \wedge \psi \vdash \psi$ is valid.
   (iv) Given $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \chi)$ for any $s$. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi) \wedge gr(s \text{ sat } \chi) = gr(s \text{ sat } \psi \wedge \chi)$ for any $s$. Hence $\phi \vdash \psi \wedge \chi$ is valid when $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid.
4. (i) $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \bigvee S(\phi \in S))$ for any $s$. Hence $\phi \vdash \bigvee S(\phi \in S)$ is valid. (ii) Given $\phi \vdash \psi$ is valid for all $\phi \in S$. So $gr(s \text{ sat } \phi) \leq$

$gr(s \text{ sat } \psi)$ for all $\phi \in S$ and any $s$. So, $sup_{\phi\in S}\{gr(s \text{ sat } \phi)\} \leq gr(s \text{ sat } \psi)$ for any $s$. Hence $gr(s \text{ sat } \bigvee S) \leq gr(s \text{ sat } \psi)$ for any $s$. So, $\bigvee S \vdash \psi$ is valid when $\phi \vdash \psi$ is valid for all $\phi \in S$.

5. We have, $gr(s \text{ sat } \phi \wedge \bigvee S) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S) = gr(s \text{ sat } \phi) \wedge sup_{\psi\in S}\{gr(s \text{ sat } \psi)\} = sup_{\psi\in S}\{gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi)\} = sup\{gr(s \text{ sat } \phi \wedge \psi) \mid \psi \in S\}$, for any $s$. Hence $\phi \wedge \bigvee S \vdash sup\{\phi \wedge \psi \mid \psi \in S\}$ is valid.

6. $gr(s \text{ sat } \top) = 1_L = gr(s \text{ sat } x = x)$, for any $s$. Hence $\top \vdash x = x$ is valid.

7. $gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi)$
$= gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n))) \wedge gr(s \text{ sat } \phi)$.
Now $gr(s \text{ sat } \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)])$
$= gr(s(s((y_1, \ldots, y_n))/(x_1, \ldots, x_n)) \text{ sat } \phi)$.
When $s((y_1, \ldots, y_n)) = s((x_1, \ldots, x_n))$
then $gr(s(s((y_1, \ldots, y_n))/(x_1, \ldots, x_n)) \text{ sat } \phi) = gr(s \text{ sat } \phi)$.
Hence, $gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi)$
$\leq gr(s \text{ sat } \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)])$, for any $s$. So,
$((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)]$ is valid.

8. (i) $\phi \vdash \psi[x \mid y]$ is valid so, $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi[x \mid y])$, for any $s$. Using Theorem 3.6(2) $gr(s \text{ sat } \phi) \leq gr(s(s(x)/y) \text{ sat } \psi)$, for any $s$, which implies that $gr(s \text{ sat } \phi) \leq sup\{gr(s(d/y) \text{ sat } \psi) \mid d \in D\}$, for any $s$. So, $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \exists y\psi)$ and hence $\phi \vdash \exists y\psi$ is valid.
(ii) $\exists y\phi \vdash \psi$ is valid if and only if $gr(s \text{ sat } \exists y\phi) \leq gr(s \text{ sat } \psi)$, for any $s$. Hence $sup\{gr(s(d/y) \text{ sat } \phi) \mid d \in D\} \leq gr(s \text{ sat } \psi)$, for any $s$. So, $gr(s(s(x)/y) \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$, for any $s$, using Theorem 3.6(2). Therefore $gr(s \text{ sat } \phi[x/y]) \leq gr(s \text{ sat } \psi)$, for any $s$ and hence $\phi[x/y] \vdash \psi$ is valid provided $\exists y\phi \vdash \psi$ is valid.

9. Finally $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$ is valid because of the following:

$$\begin{aligned}
gr(s \text{ sat } \phi \wedge (\exists y)\psi) &= gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \exists y\psi) \\
&= gr(s \text{ sat } \phi) \wedge \underset{d\in D}{sup}\{gr(s(d/y) \text{ sat } \psi)\} \\
&= \underset{d\in D}{sup}\{gr(s \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\} \\
&\leq \underset{d\in D}{sup}\{gr(s(d/y) \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\}
\end{aligned}$$

$$= \sup_{d \in D}\{gr(s \text{ sat } \phi \wedge \psi)\}$$
$$= gr(s \text{ sat } (\exists y)\phi \wedge \psi), \text{ for any } s.$$

□

**3.2 Generalised logic of finite observations** In this subsection, we will consider the propositional fragment of the proposed generalised geometric logic and call it the generalised logic of finite observations. Throughout this part, the justification for choosing the name will be provided. In [21], one may notice that the logic of finite observations is nothing but the logic of affirmative assertions. Recall that an assertion is affirmative if and only if the assertion is true precisely in the circumstances where it can be affirmed. Note that we need to do the job in finite time, a finite amount of work and based on what we can observe. In [21], it is nicely explained why the logic of affirmative assertions allows the connectives $\wedge$, $\bigvee$, $\top$, $\bot$ but not $\neg$ and $\rightarrow$. If we wish to make the idea of the logic of affirmative assertions closer to real-life situations then discussing the validity (truth value) of affirmative assertions up to some extent instead of, whether affirmative assertions are valid (true) or not valid (false), is a better idea. To address this issue we need to concentrate on the notion of valuation function. In this stage, it is better to quickly recapture the propositional part of our proposed generalised logic for better understanding. Let $\Phi$ be a set of propositional variables. The language $GGL(\Phi)$ of generalised logic of finite observations of the propositional generalised geometric formula is given by

$$\phi ::= \top \mid \bot \mid p \mid \phi_1 \wedge \phi_2 \mid \bigvee\{\phi_i\}_{i \in I}$$

where $p \in \Phi$ and $I$ is some index set. The rules are given by

1. $\phi \vdash \phi$,
2. $\dfrac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi}$ ,
3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\dfrac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$ ,
4. (i) $\phi \vdash \bigvee S$ $(\phi \in S)$, (ii) $\dfrac{\phi \vdash \psi \qquad \text{all } \phi \in S}{\bigvee S \vdash \psi}$ ,

5. $\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}$.

**Proposition 3.8.** *$\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ is derivable.*

*Proof.*

$$\dfrac{\dfrac{\dfrac{\phi \wedge \psi \vdash \phi}{\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi} \qquad \dfrac{\dfrac{\phi \wedge \psi \vdash \psi \qquad \psi \vdash \bigvee S \ (\psi \in S)}{\phi \wedge \psi \vdash \bigvee S}}{\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \bigvee S}}{\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S \text{ (all } \psi \in S)}}{\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S}$$

□

The valuation function $v : \Phi \to L$ can be extended to $\hat{v} : GGL(\Phi) \to L$ defined by

1. $\hat{v}(\top) = 1_L$;
2. $\hat{v}(\bot) = 0_l$;
3. $\hat{v}(\phi \wedge \psi) = \hat{v}(\phi) \wedge \hat{v}(\psi)$;
4. $\hat{v}(\bigvee\{\phi_i\}_{i \in I}) = sup\{\hat{v}(\phi_i) \mid i \in I\}$.

Now notice that if we consider the range of the valuation function a frame ($L$) instead of $\{0, 1\}$ then mathematically we can reach our goal. Consideration of the range as any frame allows us to think about the incomparable truth values of affirmative assertions, which is a natural phenomenon that arises in our daily life situations. If we think in this line then it is not very hard to understand how the definition of the extended valuation function considered here is the expected one. In this sense, we will be able to generalise the notion of the so-called logic of affirmative assertions or logic of finite observations to address real-life situations in a better way. Moreover, this generalised version of the logic connects the desired mathematical structures ($L$-topological space, $L$-topological system and frame) as well.

**Definition 3.9.** $\phi \vdash \psi$ is valid if and only if $\hat{v}(\phi) \leq \hat{v}(\psi)$ for all $\hat{v} : GGL(\Phi) \to L$.

**Proposition 3.10.** *The rules of inference are valid.*

The proposition stated above implies that $\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ is valid. We will use this piece of information in the next section.

## 4 $L$-Topological system via generalised geometric logic

In this section, the way to get an $L$-topological system from generalised geometric logic is provided. In this respect, it is to be noted that the propositional fragment of the proposed generalised geometric logic is enough to serve our purpose. Let us consider the triplet $(X, \models, A)$ where $X$ is the non-empty set of extended valuation functions, $A$ is the set of geometric formulae and $\models$ defined as $\models (\hat{v}, \phi) = \hat{v}(\phi)$.

**Proposition 4.1.** (i) $\models (\hat{v}, \phi \wedge \psi) = \models (\hat{v}, \phi) \wedge \models (\hat{v}, \psi)$.
(ii) $\models (\hat{v}, \bigvee\{\phi_i\}_{i\in I}) = \underset{i\in I}{sup}\{\models (\hat{v}, \phi_i)\}$.

*Proof.* (i) $\models (\hat{v}, \phi \wedge \psi) = \hat{v}(\phi \wedge \psi) = \hat{v}(\phi) \wedge \hat{v}(\psi) = \models (\hat{v}, \phi) \wedge \models (\hat{v}, \psi)$.
(ii) $\models (\hat{v}, \bigvee\{\phi_i\}_{i\in I}) = \hat{v}(\bigvee\{\phi_i\}_{i\in I}) = \underset{i\in I}{sup}\{\hat{v}(\phi_i)\} = \underset{i\in I}{sup}\{\models (\hat{v}, \phi_i)\}$. □

**Definition 4.2.** $\phi \approx \psi$ iff $\models (\hat{v}, \phi) = \models (\hat{v}, \psi)$ for any $\hat{v} \in X$ and $\phi, \psi \in A$.

The above defined "$\approx$" is an equivalence relation. Thus we get $A/_{\approx}$.

**Proposition 4.3.** $(A/_{\approx}, \leq, \wedge, \bigvee)$ *is a frame, where* $[\phi] \leq [\psi]$ *holds when* $\hat{v}(\phi) \leq \hat{v}(\psi)$ *for all* $\hat{v} : GGL(\Phi) \to L$, $[\phi] \wedge [\psi] = [\phi \wedge \psi]$ *and* $\bigvee\{[\phi_i]\}_{i\in I} = [\bigvee\{\phi_i\}_{i\in I}]$.

*Proof.* First of all $\hat{v}(\phi) = \hat{v}(\phi)$, for all $\hat{v}$. So $[\phi] \leq [\phi]$ holds for any $[\phi] \in A/_{\approx}$. Let $[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$. Then $\hat{v}(\phi) = \hat{v}(\psi)$, for all $\hat{v}$. Hence $\phi \approx \psi$, which indicates that $[\phi] = [\psi]$ whenever $[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$ holds. Similarly, if $[\phi] \leq [\psi]$ and $[\psi] \leq [\chi]$ holds then $[\phi] \leq [\chi]$ holds. Hence $(A/_{\approx}, \leq)$ is a poset. It is easy to observe that $A/_{\approx}$ is closed under $\wedge$ and $\bigvee$ (follows from Proposition 4.1). Moreover from the previous section we have $\bigvee\{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ and $\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}$ are valid. Hence $\hat{v}(\phi \wedge \bigvee S) = \hat{v}(\bigvee\{\phi \wedge \psi \mid \psi \in S\})$ for all $\hat{v} : GGL(\Phi) \to L$. Therefore $\phi \wedge \bigvee S \approx \bigvee\{\phi \wedge \psi \mid \psi \in S\}$. Consequently we have $[\phi \wedge \bigvee S] = [\bigvee\{\phi \wedge \psi \mid \psi \in S\}]$, i.e., $[\phi] \wedge \bigvee\{[\psi]\}_{\psi\in S} = \bigvee\{[\phi] \wedge [\psi]\}_{\psi\in S}$. Consequently, we arrive at the conclusion that finite meet distributes over arbitrary join, i.e., the frame distributive property holds good. □

Propositions 4.1 and 4.3 provide the following theorem.

**Theorem 4.4.** *$(X, \models', A/_{\approx})$ is an $L$-topological system, where $\models'$ is defined by $\models' (\hat{v}, [\phi]) = \models (\hat{v}, \phi)$.*

**Proposition 4.5.** *The $L$-topological system $(X, \models', A/_{\approx})$ defined as above, is spatial.*

*Proof.* Let for any $\hat{v} \in X$, $\models (\hat{v}, a) = \models (\hat{v}, b)$. Then $\hat{v}(a) = \hat{v}(b)$ for all $\hat{v} \in X$, which implies that $a = b$. Therefore for any $\hat{v} \in X$, $\models (\hat{v}, a) = \models (\hat{v}, b)$ implies $(a = b)$, for any $a, b \in A$. □

It is to be noted that in [11] it is remarked that "if we take $\mathbb{T}$ to be a geometric propositional theory, then $\mathbb{T}$ has a model in its Lindenbaum algebra $L$, or equivalently in the topos $\mathbf{Sh}(X)$, where $X$ is the locale corresponding to $L$, and so it is consistent whenever $L$ is non-degenerate. But its models in **Set** are the same things as points of $X$, so if $X$ is a nontrivial locale without points, then $\mathbb{T}$ is a consistent geometric theory having no models in **Set**.

Nevertheless, there is a 'classical completeness theorem' for geometric logic, which asserts that any geometric theory has enough models to determine provability (not in **Set** but) in Boolean toposes..."

Therefore, considering the completeness issue for the proposed logic here will be of great interest but it is in our future goal.

## 5 $L$-Topology via generalised geometric logic

We first construct the $L$-topological system $(X, \models', A/_{\approx})$ from generalised geometric logic. Then $(X, ext(A/_{\approx}))$ is constructed as follows:
$ext(A/_{\approx}) = \{ext([\phi])\}_{[\phi] \in A/_{\approx}}$ where $ext([\phi]) : X \longrightarrow L$ is such that, for each $[\phi] \in A/_{\approx}$, $ext([\phi])(\hat{v}) = \models' (\hat{v}, [\phi]) = \models (\hat{v}, \phi)$.
It can be shown that $ext(A/_{\approx})$ forms an $L$-topology on $X$ as follows. Let $ext([\phi]), ext([\psi]) \in ext(A/_{\approx})$. Then

$$\begin{aligned}
(ext([\phi]) \cap ext([\psi]))(\hat{v}) &= (ext([\phi]))(\hat{v}) \wedge (ext([\psi]))(\hat{v}) \\
&= \models' (\hat{v}, [\phi]) \wedge \models' (\hat{v}, [\psi]) \\
&= \models (\hat{v}, \phi) \wedge \models (\hat{v}, \psi) \\
&= \models (\hat{v}, \phi \wedge \psi) \\
&= \models' (\hat{v}, [\phi \wedge \psi])
\end{aligned}$$

$$= (ext([\phi \wedge \psi]))(\hat{v}).$$

Hence $ext([\phi]) \cap ext([\psi]) = ext([\phi \wedge \psi]) \in ext(A/_{\approx})$. Similarly, it can be shown that $ext(A/_{\approx})$ is closed under arbitrary union. Hence $(X, ext(A/_{\approx}))$ is an $L$-topological space obtained via generalised geometric logic.

Proposition 4.5 indicates that $(X, \models', A/_{\approx})$ is a spatial $L$-topological system and hence from Theorem 2.6 we arrive at the conclusion that $(X, \models', A/_{\approx})$, $(A, ext(A/_{\approx}))$ are equivalent to each other. That is, $(X, \models', A/_{\approx})$ and $(X, \in, ext(A/_{\approx}))$ represent the same $L$-topological system. The following diagram summarises all the facts that we have proved till this stage:

Generalised logic of finite observations

$\downarrow$

Spatial $L$ – Topological system $\longleftrightarrow$ $L$ – Topological space

Let $X$ be an $L$-topological space, $\tau$ is its $L$-topology. Then the corresponding generalised geometric theory can be defined as follows:

- for each $L$-open set $\tilde{T}$, a proposition $P_{\tilde{T}}$
- if $\tilde{T_1} \subseteq \tilde{T_2}$, then an axiom

$$P_{\tilde{T_1}} \vdash P_{\tilde{T_2}}$$

- if $S$ is a family of $L$-open sets, then an axiom

$$P_{\bigcup S} \vdash \bigvee_{\tilde{T} \in S} P_{\tilde{T}}$$

- if $S$ is finite collection of $L$-open sets, then an axiom

$$\bigwedge_{\tilde{T} \in S} P_{\tilde{T}} \vdash P_{\bigcap S}$$

All other axioms for the (propositional) generalised geometric logic will follow from the above clauses.

If $x \in X$, then $x$ gives a model of the theory in which the truth value of the interpretation of $P_{\tilde{T}}$ will be $\tilde{T}(x)$.

Hence one may study $L$-topology via generalised geometric logic.

## 6 Concluding remarks

In this paper, the notion of generalised geometric logic is introduced and studied in detail. The connection between $L$-topological system and $L$-topological space establishes the strong connection between the proposed logic and $L$-topological space. The interpretation of the predicate symbols for the generalised geometric logic is $L$ (frame)-valued relations, so the proposed logic is more expressible. The proposed logic can interpret situations where the truth values are incomparable.

In [8], we have already indicated how the extension of 3-valued geometric logic (a fragment of generalised geometric logic) can be seen as the stepping stone to propose the logic of ethics. Hence, it will be helpful to use the proposed generalised geometric logic to enhance the result of the work in [8].

As stated in the introduction, our future agenda is to enhance the results of M. J. Healy in [3] for the LAPART architecture. We would like to expand the work to see whether geometric logic can be useful in extracting knowledge from standard neural network structures or not, for example feed-forward neural nets. Also, it is possible to use several metrics in knowledge extraction to evaluate our method considered in [3], for example, fidelity and expressiveness. Generalising the proposed logic considering graded consequence relation is also on the future agenda.

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