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Characterization of monoids by $(U\text{-})GPW$-flatness of right acts

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Abstract. The authors in 2020 introduced GPW-flatness and gave a characterization of monoids by this property of their right acts. In this article we continue this investigation and will give a characterization of monoids by this condition of their right Rees factor acts. Also we give a characterization of monoids by comparing this property of their right acts with other properties. We also introduce U-GPW-flatness of acts, which is an extension of GPW-flatness and give some general properties and a characterization of monoids when this property of acts implies some others and vice versa.

1 Introduction and Preliminaries

In 1970, Kilp [10] initiated the study of flatness of acts. A right S-act A_S is called *flat* if the functor $A_S \otimes {}_S-$ preserves all monomorphisms. In 1983, Kilp [11] further investigated the (principal) weak version of flatness under the name of (principal) weak flatness. A right S-act A_S is called (*principally*)

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weakly flat if the functor $A_S \otimes {}_S-$ preserves all embeddings of (principal) left ideals into S.

It was shown in [13] and [2] that, if we require either bijectivity or surjectivity of φ for pullback diagrams of certain types, we obtain new properties such as pullback flatness (PF), weak pullback flatness (WPF), Condition (WP), Condition (PWP), weakly kernel flatness (WKF), principally weakly kernel flatness $(PWKF)$ and translation kernel flatness (TKF).

In [15], we introduced GPW-flatness property as a generalization of principal weak flatness, and characterized monoids by this property of their right acts in some cases.

In this article we give a characterization of monoids S for which all GPW-flat right Rees factor S-acts satisfy other flatness properties. Also we give a characterization of monoids by comparing this property of their right acts with other properties.

We also introduce U-GPW-flatness of acts, which is an extension of GPW-flatness and give some general properties. Then we give a characterization of monoids when this property of acts implies some others and vice versa.

Throughout this paper S always will stand for a monoid and $\mathbb{N}$ the set of natural numbers. Recall that a monoid S is called *right* (*left*) *reversible* if for every $s, t \in S$, there exist $u, v \in S$ such that $us = vt$ $(su = tv)$. A monoid S is called *left* (*right*) *collapsible* if for every $s, t \in S$ there exists $z \in S$ such that $zs = zt$ $(sz = tz)$. Also a monoid S is called *regular* if for every $s \in S$, there exists $x \in S$ such that $s = sxs$. A right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$. A nonempty set A is called a *right S-act*, usually denoted A_S, if S acts on A unitary from the right, that is, there exists a mapping $A \times S \to A; (a, s) \mapsto as$, satisfying conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. An act A_S is called *weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of left ideals into S, or equivalently if for every $s, t \in S, a, a' \in A_S$, $a \otimes s = a' \otimes t$ in $A_S \otimes S$ implies $a \otimes s = a' \otimes t$ in $A_S \otimes {}_S(Ss \bigcup St)$ [12, III, Lemma 11.1]. An act A_S is called *principally weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of principal left ideals into S, or equivalently, a right S-act A_S is principally weakly flat if and only if $a \otimes s = a' \otimes s$ in $A_S \otimes {}_S S$ implies $a \otimes s = a' \otimes s$ in $A_S \otimes {}_S(Ss)$ for all $s \in S, a, a' \in A_S$ [12, III, Lemma 10.1]. A right S-act A_S is *torsion free* if for $a, b \in A_S$ and a right cancellable element c of S the equality $ac = bc$ implies that $a = b$. A right S-act A_S satisfies *Condition* (E) if for every $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (P) if for every $a, a' \in A_S$, $s, s' \in S$, $as = a't$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (PWP) if for every $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs$.

Definition 1.1. [15] A right S-act A_S is called *GPW-flat* if for every $s \in S$, there exists a natural number $n = n(s, A_S) \in \mathbb{N}$ such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_S S$.

Clearly, every principally weakly flat right S-act is GPW-flat, but not the converse (see [15, Example 2.2]).

Also every GPW-flat right S-act is torsion free, but not the converse (see [15, Proposition 2.5 and Example 2.6]).

We recall from [14] that a right S-act A_S is called *GP-flat* if the equality $a \otimes s = a' \otimes s$ in $A_S \otimes {}_S S$, for every $s \in S$ and $a, a' \in A_S$ implies that there exists a natural number $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$.

Clearly GPW-flatness implies GP-flatness.

Thus we have

free $\Rightarrow$ projective $\Rightarrow$ projective generator $\Rightarrow$ strongly flat $\Rightarrow$ WPF $\Rightarrow$ condition (P) $\Rightarrow$ flat$\Rightarrow$ weakly flat $\Rightarrow$ principally weakly flat $\Rightarrow$ GPW-flat $\Rightarrow$ GP-flat $\Rightarrow$ torsion free.

Definition 1.2. [15] An element $s \in S$ is called *eventually regular* if s^n is regular for some $n \in \mathbb{N}$. That is, $s^n = s^n x s^n$ for some $n \in \mathbb{N}$ and $x \in S$. A monoid S is called *eventually regular* if every $s \in S$ is eventually regular.

Obviously every regular monoid is eventually regular.

Definition 1.3. [15] An element $s \in S$ is called *eventually left almost regular* if

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_mc_m &= s_{m-1}r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. If every element of a monoid S is eventually left almost regular, then S is called *eventually left almost regular*.

It is clear that every left almost regular monoid is eventually left almost regular, and also every eventually regular monoid is eventually left almost regular.

2 Characterization of monoids by GPW-flatness of acts

In [15], we characterized monoids over which all right S-acts are GPW-flat and also monoids over which some other properties imply GPW-flatness and vice versa. We showed that all right S-acts are GPW-flat if and only if S is an eventually regular monoid. Also we showed that GPW-flatness implies torsion freeness, but not the converse. Then we proved that all torsion free right S-acts are GPW-flat if and only if S is an eventually left almost regular monoid.

Now we give a characterization of monoids by comparing this property of their right acts with other properties.

Definition 2.1. Let S be a monoid and K be a proper right ideal of S. The right ideal K of a monoid S is called *GPW-left stabilizing* if for every $s \in S$ there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$, implies that $ls^n = ks^n$ for some $k \in K$.

It is clear that every left stabilizing right ideal of S is GPW-left stabilizing.

Remark 2.2. If for $s \in S$ there exists $n \in \mathbb{N}$ such that the right ideal $s^n S$ is GPW-left stabilizing, then s is eventually regular.

Proof. Let for $s \in S$ there exists $n \in \mathbb{N}$ such that the right ideal $s^n S$ be GPW-left stabilizing. Since $s^n \in s^n S$, so there exists $k \in s^n S$ such that $s^n = ks^n$. Since $k \in s^n S$, there exists $x \in S$ such that $k = s^n x$, and so s is eventually regular. □

Theorem 2.3. *For any monoid S, the following statements are equivalent*:

(1) *All GPW-flat right S-acts are free.*

(2) *All finitely generated GPW-flat right S-acts are free.*

(3) *All cyclic GPW-flat right S-acts are free.*

(4) *All monocyclic GPW-flat right S-acts are free.*

(5) *All GPW-flat right S-acts are projective generator.*

(6) *All finitely generated GPW-flat right S-acts are projective generator.*

(7) *All cyclic GPW-flat right S-acts are projective generator.*

(8) *All monocyclic GPW-flat right S-acts are projective generator.*

(9) *All GPW-flat right S-acts are projective.*

(10) *All finitely generated GPW-flat right S-acts are projective.*

(11) *All GPW-flat right S-acts are strongly flat.*

(12) *All finitely generated GPW-flat right S-acts are strongly flat.*

(13) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (8), (1) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (8), (1) ⇒ (9) ⇒ (10) ⇒ (12) and (1) ⇒ (11) ⇒ (12) are obvious.

(8) ⇒ (13) If all monocyclic GPW-flat right S-acts are projective generator, then all monocyclic right S-acts satisfying Condition (P) are projective generator and so by [12, IV, Theorem 12.8], $S = \{1\}$.

(12) ⇒ (13) Assume all finitely generated GPW-flat right S-acts are strongly flat. Then all finitely generated right S-acts satisfying Condition (P) are strongly flat and so S is aperiodic by [12, IV, Theorem 10.2]. Let $1 \neq s \in S$, then there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$ and so $e = s^n$

is an idempotent different from 1. It is easy to see that eS is a GPW-left stabilizing right ideal and so the right S-act $S_S \coprod^{eS} S_S$ is GPW-flat by [15, Theorem 2.10]. Thus by the assumption it is strongly flat (satisfies Condition (P)), which is a contradiction [12, III, Proposition 13.14]. So $S = \{1\}$.

(13) $\Rightarrow$ (1) This is obvious. □

Lemma 2.4. *If all monocyclic GPW-flat right S-acts are strongly flat, then all monocyclic right S-acts are strongly flat.*

Proof. Suppose that all GPW-flat monocyclic right S-acts are strongly flat, then all monocyclic right S-acts satisfying Condition (P) are strongly flat and so S is aperiodic by [12, IV, Theorem 10.2]. Thus for every $s \in S$ there exists $n \in N$ such that s^n is an idempotent, which gives that S is eventually regular. Now by [15, Theorem 4.5], all right S-acts are GPW-flat. □

Theorem 2.5. *For any monoid S, the following statements are equivalent*:

(1) *All cyclic GPW-flat right S-acts are projective.*

(2) *All monocyclic GPW-flat right S-acts are projective.*

(3) *All cyclic GPW-flat right S-acts are strongly flat.*

(4) *All monocyclic GPW-flat right S-acts are strongly flat.*

(5) $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By the assumption all monocyclic GPW-flat right S-acts are strongly flat, and so by Lemma 2.4, all monocyclic right S-acts are strongly flat. Thus by [12, IV, Proposition 10.10], $S = \{1\}$ or $S = \{0, 1\}$.

(2) $\Leftrightarrow$ (4) It follows from [12, III, Lemma 17.13].

(5) $\Rightarrow$ (1) It follows from [12, IV, Theorem 11.14]. □

Theorem 2.6. *For any monoid S, the following statements are equivalent*:

(1) *All GPW-flat right S-acts are generator.*

(2) *All finitely generated GPW-flat right S-acts are generator.*

(3) *All cyclic GPW-flat right S-acts are generator.*

(4) *All GPW-flat right Rees factor S-acts are generator.*

(5) $S = \{1\}$

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) Since $\Theta_S \cong S/S_S$ is GPW-flat by [15, Proposition 2.8], thus by the assumption, $\Theta_S \cong S/S_S$ is a generator. Therefore there exists an epimorphism $\pi : \Theta_S \longrightarrow S_S$, and so $S = \{1\}$.

(5) $\Rightarrow$ (1) Since $S = \{1\}$, all right S-acts are generators, as desired. □

Theorem 2.7. *For any monoid S, the following statements are equivalent:*

(1) *All GPW-flat right S-acts are regular.*

(2) *All finitely generated GPW-flat right S-acts are regular.*

(3) *All cyclic GPW-flat right S-acts are regular.*

(4) $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. This is obvious by [5, Theorem 1.12]. □

Theorem 2.8. *For any monoid S, the following statements are equivalent:*

(1) *All GPW-flat right S-acts satisfy Condition (E).*

(2) *All finitely generated GPW-flat right S-acts satisfy Condition (E).*

(3) *All cyclic GPW-flat right S-acts satisfy Condition (E).*

(4) *All monocyclic GPW-flat right S-acts satisfy Condition (E).*

(5) $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) It follows from Theorem 2.5 and [12, IV, Proposition 10.10].

(5) $\Rightarrow$ (1) It is straightforward. □

We recall from [12] that a right S-act A_S is (strongly) faithful, if for $s, t \in S$ the equality $as = at$, for all (some) $a \in A_S$, implies that $s = t$. It is obvious that every strongly faithful act is faithful.

Theorem 2.9. *For any monoid S, the following statements are equivalent:*

(1) *All GPW-flat right S-acts are (strongly) faithful.*

(2) *All finitely generated GPW-flat right S-acts are (strongly) faithful.*

(3) *All cyclic GPW-flat right S-acts are (strongly) faithful.*

(4) *All GPW-flat right Rees factor S-acts are (strongly) faithful.*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5) By [15, Proposition 2.8], the one-element right S-act $\Theta_S \cong S/S_S$ is GPW-flat, and so by the assumption it is (strongly) faithful. Thus $S = \{1\}$, as required.

(5) ⇒ (1) This is obvious. □

We recall from [12] that a right S-act A_S is called simple if it contains no subacts other than A_S itself, and A_S is called completely reducible if it is a disjoint union of simple subacts.

Theorem 2.10. *For any monoid S, the following statements are equivalent*:

(1) *All GPW-flat right S-acts are completely reducible*

(2) *All finitely generated GPW-flat right S-acts are completely reducible.*

(3) *All cyclic GPW-flat right S-acts are completely reducible.*

(4) *All monocyclic GPW-flat right S-acts are completely reducible.*

(5) *S is a group.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5) By [15, Proposition 2.8], $S_S \cong S/\rho(1,1)$ is GPW-flat as a monocyclic right S-act, and so by the assumption S_S is completely reducible. Thus S is a group by [12, I, Lemma 5.33].

(5) ⇒ (1) It follows from [12, I, Proposition 5.34]. □

We recall from [17] that a right S-act A_S is $\mathcal{R}$-torsion free if $ac = a'c$ and $a\mathcal{R}a'$, for $a, a' \in A_S$, $c \in S$, c right cancellable, imply that $a = a'$.

Theorem 2.11. *For any monoid S, the following statements are equivalent*:

(1) *All $\mathcal{R}$-torsion free right S-acts are GPW-flat.*

(2) *S is eventually regular.*

Proof. It follows from [15, Theorem 4.5] and [17, Lemma 4.1]. □

Example 2.12. Let $S = (\mathbb{N}, .)$ be the monoid of natural numbers with multiplication and let $A_S = S_S \coprod^{S \setminus \{1\}} S_S$. Since there exist no $x \in S \setminus \{1\}, n \in \mathbb{N}$ such that $2^n = x2^n$, A_S is not GPW-flat by [15, Theorem 2.10]. But A_S satisfies Condition (E) by [12, III, Exercise 14.3(3)]. Thus it is natural to ask for monoids over which Condition (E) implies GPW-flatness.

Recall from [6, 7, 13] that a right S-act A_S satisfies Condition (E') if for all $a \in A_S$, $s, s', z \in S$, $as = as'$ and $sz = s'z$ imply that there exist $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S satisfies Condition (EP) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies Condition $(E'P)$ if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

Theorem 2.13. *For any monoid S, the following statements are equivalent*:

(1) *All right S-acts satisfying Condition $(E'P)$ are GPW-flat.*

(2) *All right S-acts satisfying Condition (EP) are GPW-flat.*

(3) *All right S-acts satisfying Condition (E') are GPW-flat.*

(4) *All right S-acts satisfying Condition (E) are GPW-flat.*

(5) *S is eventually regular.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$ are obvious because $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

$(4) \Rightarrow (5)$ Let $s \in S$. Since S_S is GPW-flat by [15, Proposition 2.8], there exists a natural number $n \in \mathbb{N}$ such that the functor $S_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. If $s^nS = S$, then there exists $x \in S$ such that $s^nx = 1$, and so $s^nxs^n = s^n$. Thus s is an eventually regular element. Now assume that $s^nS \neq S$. Consider $A_S = S \coprod^{s^nS} S$. Then by [12, III, Exercise 14.3(3)], A_S satisfies Condition (E) and by the assumption it is GPW-flat. Now by [15, Theorem 2.10], the right ideal s^nS is GPW-left stabilizing and so s is eventually regular by Remark 2.2.

$(5) \Rightarrow (1)$ This is obvious, by [15, Theorem 4.5]. □

Note that above theorem is also true for finitely generated (at most (exactly) by two elements) right S-acts.

An element $s \in S$ is *right semi-cancellable* if the equality $xs = ys$ for any $x, y \in S$ implies that there exists $r \in S$ such that $rs = s$ and $xr = yr$. A monoid S is called *left PSF* if every element $s \in S$ is right semi-cancellable.

Theorem 2.14. *Let S be a left PSF monoid. Then the following statements are equivalent:*

(1) *All divisible right S-acts are GPW-flat.*

(2) *All principally weakly injective right S-acts are GPW-flat.*

(3) *All fg-weakly injective right S-acts are GPW-flat.*

(4) *All weakly injective right S-acts are GPW-flat.*

(5) *All injective right S-acts are GPW-flat.*

(6) *All cofree right S-acts are GPW-flat.*

(7) *S is eventually regular.*

Proof. Since Cofree $\Rightarrow$ Injective $\Rightarrow$ weakly injective $\Rightarrow$ finitely generated weakly injective $\Rightarrow$ principally weakly injective $\Rightarrow$ divisible, implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obtained immediately.

(6) $\Rightarrow$ (7) We know that every right S-act can be embedded into a corfree right S-act. So by the assumption, every right S-act is a subact of a GPW-flat right S-act. Since S is left PSF, every subact of a GPW-flat right S-act is GPW-flat (see [15, Proposition 2.12]). Therfore all right S-acts are GPW-flat and so by [15, Theorem 4.5], S is eventually regular.

(7) $\Rightarrow$ (1) It follows from [15, Theorem 4.5]. □

Theorem 2.15. *For any monoid S, the following statements are equivalent:*

(1) *All faithful right S-acts are GPW-flat.*

(2) *All finitely generated faithful right S-acts are GPW-flat.*

(3) *All faithful right S-acts generated by two elements are GPW-flat.*

(4) *S is eventually regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Let $s \in S$. Since S_S is GPW-flat by [15, Proposition 2.8], there exists a natural number $n \in \mathbb{N}$ such that the functor $S_S \otimes_S -$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. If $s^nS = S$, then

there exists $x \in S$ such that $s^n x = 1$, and so $s^n x s^n = s^n$. Thus s is an eventually regular element. Now assume that $I = s^n S \neq S$. Consider $A_S = S \coprod^{s^n S} S$. As we know A_S is a faithful right S-act generated by two elements, and so by the assumption it is GPW-flat. Thus $s^n S$ is GPW-left stabilizing by [15, Theorem 2.10], and so s is eventually regular by Remark 2.2.

(4) $\Rightarrow$ (1) This is obvious, by [15, Theorem 4.5]. □

Theorem 2.16. *For any monoid S, the following statements are equivalent*:

(1) *All strongly faithful right S-acts are GPW-flat.*

(2) *All finitely generated strongly faithful right S-acts are GPW-flat.*

(3) *All strongly faithful right S-acts generated by two elements are GPW-flat.*

(4) *S is not left cancellative or S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Let S be a left cancellative monoid and $s \in S$. Since S_S is GPW-flat, there exists a natural number $n \in \mathbb{N}$ such that the functor $S_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. If $s^n S = S$, then there exists $x \in S$ such that $s^n x = 1$, and so $s^n x s^n = s^n$. Thus s is an eventually regular element.

Now assume that $I = s^n S \neq S$. Set

$$A_S = S \amalg^{s^n S} S = \{(l,x) |\ l \in S \setminus s^n S\} \ \dot\cup\ s^n S \ \dot\cup\ \{(t,y) |\ t \in S \setminus s^n S\}.$$

Clearly

$$B_S = \{(l,x) |\ l \in S \setminus s^n S\} \ \dot\cup\ s^n S \ \cong S_S \cong \ \{(t,y) |\ t \in S \setminus s^n S\} \ \dot\cup\ s^n S = C_S.$$

Since $A_S = B \bigcup C$, so A_S is generated by different two elements $(1,x)$ and $(1,y)$ and also $B_S \cong S_S \cong C_S$ and $A_S = B_S \bigcup C_S$, where B_S and C_S are subacts of A_S. Since S is left cancellative, so by [1, Lemma 2.10], S_S is strongly faithful. Hence by above isomorphism subacts B_S and C_S of A_S are strongly faithful. Therefore the equality $A_S = B_S \bigcup C_S$ implies that A_S is strongly faithful and so A_S is GPW-flat by the assumption. Thus $s^n S$ is GPW-left stabilizing by [15, Theorem 2.10], and so s is eventually regular by Remark 2.2.

Hence in two cases, s is eventually regular. Now since by the assumption S is left cancellative, so s is left invertible and we can say that every $s \in S$ is left invertible. Thus S is a group.

(4) $\Rightarrow$ (1) If S is not left cancellative, then there exists no strongly faithful right S-act, by [1, Lemma 2.10]. Thus (1) is satisfied. If S is left cancellative, then there exists at least a strongly faithful right S-act, by [1, Lemma 2.10]. Since S is a group, it is eventually regular and so (1) is satisfied, by [15, Theorem 4.5]. □

We recall from [12] that an S-act A_S is called decomposable if there exist two subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$.

Theorem 2.17. *For any monoid S, the following statements are equivalent*:

(1) *All indecomposable right S-acts are GPW-flat.*

(2) *All finitely generated indecomposable right S-acts are GPW-flat.*

(3) *All indecomposable right S-acts generated by two elements are GPW-flat.*

(4) *S is eventually regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) By a similar argument which used in the proof (3) $\Rightarrow$ (4) of Proposition 2.16, we can conclude that S is eventually regular

(4) $\Rightarrow$ (1) It follows from [15, Theorem 4.5]. □

3 Characterization of monoids by GPW-flatness property of right Rees factor S-acts

In this section we give a characterization of monoids by GPW-flatness property of their right Rees factor acts.

Lemma 3.1. *Let S be a monoid and K be a proper right ideal of S. Then S/K is GPW-flat if and only if K is GPW-left stabilizing.*

Proof. This is obvious by [15, Theorem 3.3]. □

Definition 3.2. Let S be a monoid . The right ideal K of a monoid S is called *GP-left stabilizing* if $ls \in K$ for $l \in S \setminus K$ and $s \in S$, implies that there exists $n \in \mathbb{N}$ such that $ls^n = ks^n$ for some $k \in K$.

Lemma 3.3. *Let S be a monoid and K be a proper right ideal of S. Then S/K is GP-flat if and only if K is a GP-left stabilizing right ideal.*

Proof. This is obvious by [14, Proposition 2.7]. □

Theorem 3.4. *Let S be a monoid. Then all GP-flat right Rees factors of S are GPW-flat if and only if every GP-left stabilizing right ideal of S is GPW-left stabilizing.*

Proof. Suppose that all GP-flat right Rees factor S-acts are GPW-flat and let K be a GP-left stabilizing right ideal of S. Then by Lemma 3.3, S/K is GP-flat, and so by the assumption S/K is GPW-flat. Hence by Lemma 3.1, K is GPW-left stabilizing.
Conversely, suppose that for the right ideal K of S, S/K is GP-flat. Then there are two cases as follows:
Case 1. $K = S$. Then $S/K \cong \Theta_S$ is GPW-flat by [15, Proposition 2.8].
Case 2. $K \neq S$. Then by Lemma 3.3, K is GP-left stabilizing, and so by the assumption K is GPW-left stabilizing. Thus S/K is GPW-flat by Lemma 3.1. □

The proofs of the following theorems are similar to Theorem 3.4.

Theorem 3.5. *Let S be a monoid. Then all GPW-flat right Rees factors of S are principally weakly flat if and only if every GPW-left stabilizing right ideal of S is left stabilizing.*

Theorem 3.6. *Let S be a monoid. Then all GPW-flat right Rees factors of S are (weakly) flat if and only if S is right reversible and the existence of a GPW-left stabilizing proper right ideal K of S implies that K is a left stabilizing ideal.*

Recall from [13] that a right ideal K of a monoid S is called *left annihilating* if

$$(\forall t \in S)(\forall x, y \in S \setminus K)(xt, yt \in K \Rightarrow xt = yt).$$

Theorem 3.7. *Let S be a monoid. Then all GPW-flat right Rees factors of S satisfy Condition (PWP) if and only if every GPW-left stabilizing right ideal of S is left annihilating and left stabilizing.*

Proof. This is obvious by Lemma 3.1 and [13, Lemma 2.8]. □

Recall from [13] that a right S-act A_S satisfies *Condition* (WP) if $af(s) = a'f(t)$, for $a, a' \in A_S$, $s, t \in S$, and homomorphism $f : {}_S(Ss \cup St) \to {}_SS$, implies that there exist $a'' \in A_S$, $u, v \in S$ and $s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$. Also, we recall from [13] that a right ideal K of a monoid S is called *strongly left annihilating* if $f(s), f(t) \in K$, $s, t \in S \setminus K$ and homomorphism $f : {}_S(Ss \cup St) \to {}_SS$ imply that $f(s) = f(t)$.

From Lemma 3.1 and [13, Lemma 2.13], we have the following theorem.

Theorem 3.8. *Let S be a monoid. Then all GPW-flat right Rees factors of S satisfy Condition (WP) if and only if S is right reversible and every GPW-left stabilizing right ideal of S is strongly left annihilating and left stabilizing.*

Theorem 3.9. *Let S be a monoid. Then all GPW-flat right Rees factors of S satisfy Condition (P) if and only if S is right reversible and there is no GPW-left stabilizing proper right ideal K of S with $|K| \geq 2$.*

Proof. Necessity. Suppose that all GPW-flat right Rees factor S-acts satisfy Condition (P) and let K be a GPW-left stabilizing proper right ideal of S. Then by Lemma 3.1, S/K is GPW-flat, and so by the assumption S/K satisfies Condition (P). Hence by [12, III, Proposition 13.9], $|K| = 1$. Since $\Theta_S \cong \dfrac{S}{S_S}$ is GPW-flat, it satisfies Condition (P) by the assumption, and so S is right reversible by [12, III, Corollary 13.7].
Sufficiency. Suppose that S/K is GPW-flat, for the right ideal K of S. Then there are two cases:
Case 1. $K = S$. Since S is right reversible and $S/K \cong \Theta_S$, S/K satisfies Condition (P) by [12, III, Corollary 13.7].
Case 2. $K \neq S$. Then by Lemma 3.1, K is GPW-left stabilizing. Thus by the assumption $|K| = 1$. Thus S/K satisfies Condition (P) by [12, III, Proposition 13.9]. □

Recall from [13] that a right S-act A_S is *weakly pullback flat* if and only if it satisfies Conditions (P) and (E'). Also we recall that a monoid S is *weakly left collapsible* if for every $s, t, z \in S$, the equality $sz = tz$, implies the existence of $u \in S$, such that $us = ut$.

The proof of following theorems are similar in nature as to that of Theorem 3.9.

Theorem 3.10. *Let S be a monoid. Then all GPW-flat right Rees factors of S are weakly pullback flat if and only if S is weakly left collapsible and right reversible, and there exist no GPW-left stabilizing proper right ideal K of S with $|K| \geq 2$.*

Theorem 3.11. *Let S be a monoid. Then all GPW-flat right Rees factors of S are strongly flat if and only if S is left collapsible and there exist no GPW-left stabilizing proper right ideal K of S with $|K| \geq 2$.*

Theorem 3.12. *Let S be a monoid. Then all GPW-flat right Rees factors of S are projective if and only if S contains a left zero, and there exist no GPW-left stabilizing proper right ideal K of S with $|K| \geq 2$.*

Theorem 3.13. *All GPW-flat right Rees factors of S are free if and only if $S = \{1\}$.*

Proof. This is obvious by [12, IV, Theorem 13.9]. □

Recall from [4] that a right S-act A_S satisfies *Condition* (P_E) if whenever $a, a' \in A, s, s' \in S$, and $as = a's'$, there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue, a'f = a''vf, es = s, fs' = s'$ and $us = vs'$. A right ideal K of a monoid S is called (P_E)- *left annihilating* if for all $x, y, t, t' \in S$,

$$(xt \neq yt') \Rightarrow [(x \in K) \vee (y \in K) \vee (xt \notin K) \vee (yt' \notin K) \vee$$
$$(\exists u, v \in S, e, f \in E(S), et = t, ft' = t', ut = vt'$$
$$xe \neq ue \Rightarrow ue, xe \in K, yf \neq vf \Rightarrow yf, vf \in K)]$$

It is clear that every right S-act satisfying Condition (P_E) is GPW-flat, but not the converse.

Theorem 3.14. *Let S be a monoid. Then all GPW-flat right Rees factors of S satisfy Condition (P_E) if and only if S is right reversible and every GPW-left stabilizing right ideal of S is (P_E)-left annihilating.*

Proof. This is obvious by Lemma 3.1 and [4, Theorem 3.5]. □

Recall from [3] that a right S-act A_S satisfies *Condition* (PWP_E) if whenever $a, a' \in A, s \in S$, and $as = a's$, there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue, a'f = a''vf, es = s = fs$ and $us = vs$. A right ideal K of a monoid S is called (E)-*left annihilating*, if for all $x, y, t, \in S$,

$$\begin{aligned}(xt \neq yt) \Rightarrow [(x \in K) \vee (y \in K) \vee (xt \notin K) \vee (yt \notin K) \vee \\ (\exists u, v \in S, e, f \in E(S), et = t = ft, ut = vt) \\ (xe \neq ue) \Rightarrow ue, xe \in K, yf \neq vf \Rightarrow yf, vf \in K)]\end{aligned}$$

It is clear that every right S-act satisfying Condition (PWP_E) is GPW-flat, but not the converse.

Theorem 3.15. *Let S be a monoid. Then all GPW-flat right Rees factors of S satisfy Condition (PWP_E) if and only if every GPW-left stabilizing right ideal of S is left stabilizing and (E)-left annihilating.*

Proof. This is obvious by Lemma 3.1 and [3, Theorem 4.2]. □

Recall from [9] that a right S-act A_S is called strongly (P)-cyclic if for every $a \in A_S$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P). Because freeness does not imply strong (P)-cyclic property, so GPW-flatness does not imply strong (P)-cyclic.

Theorem 3.16. *Let S be a monoid. Then all GPW-flat right Rees factors of S are strongly (P)-cyclic if and only if S contains a left zero, there is no GPW-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$ and every principal right ideal of S satisfies Condition (P).*

Proof. This is obvious by Lemma 3.1 and [9, Theorem3.1]. □

Recall from [8] that a right S-act A_S is called P-regular if all cyclic subacts of A_S satisfy Condition (P). We know that a right S-act A_S is regular if every cyclic subact of A_S is projective. It is obvious that every regular right act is P-regular

Lemma 3.17. [8] Θ_S *is* P*-regular if and only if* S *is right reversible.*

Theorem 3.18. *For any monoid* S *the following statements are equivalent*:

1) *All GPW-flat right Rees factors of* S *are* P*-regular.*
2) S *is right reversible, no proper right ideal* K_S *of* S *with* $|K_S| \geq 2$ *is GPW-left stabilizing and all principal right ideals of* S *satisfy Condition* (P)*.*

Proof. This is obvious by Lemma 3.17 and [8, Theorem 3.1]. □

4 Characterization of monoids by U-GPW-flatness of right acts

In this section, we introduce property U-GPW-flatness of acts and give some general properties. Then we give a characterization of monoids when this property of acts implies some others.

Definition 4.1. Let S be a monoid. A right S-act A_S is *U-GPW-flat* if there exists a family $\{B_i \mid i \in I\}$ of subacts of A_S such that $A = \cup_{i\in I} B_i$ and B_i, $i \in I$ is GPW-flat.

Theorem 4.2. *Let* S *be a monoid. Then*

(1) *Every GPW-flat right* S*-act is U-GPW-flat.*

(2) *If* $\{B_i \mid i \in I\}$ *is a family of subacts of a right* S*-act* A_S *such that for every* $i \in I$, B_i *is U-GPW-flat, then* $\cup_{i\in I} B_i$ *is U-GPW-flat.*

(3) *A right* S*-act* A_S *is U-GPW-flat if and only if for every* $a \in A_S$ *there exists a subact* B *of* A_S *such that* $a \in B$ *and* B *is GPW-flat.*

(4) *Every cyclic right* S*-act* A_S *is GPW-flat if and only if* A_S *is U-GPW-flat.*

(5) *For every proper right ideal* I *of* S, $A_S = S \coprod^I S$ *is U-GPW-flat, where it is indecomposable and is generated by exactly two elements, but it is not locally cyclic.*

Proof. The proofs of (1), (2), (3) and (4) are straightforward.
(5) Let I be a proper right ideal of S and let

$$A_S = S\coprod^{I} S = \{(l,x)|\ l \in S\setminus I\} \dot\cup I \dot\cup \{(t,y)|\ t\in S\setminus I\},$$

$$B = \{(l,x)|\ l\in S\setminus I\} \dot\cup I, \quad C = \{(t,y)|\ t\in S\setminus I\}\dot\cup I.$$

It is easy to show that B and C are cyclic subacts of A_S such that

$$B = (1,x)S \cong S_S \cong (1,y)S = C,$$

$$A_S = \langle (1,x),(1,y)\rangle = (1,x)S\cup(1,y)S = B\cup C.$$

Now, since S_S is GPW-flat, subacts B and C are GPW-flat too, and so $A_S = B\cup C$ is U-GPW-flat.
Also since

$$A_S = (1,x)S\cup(1,y)S,\ (1,x)S\cap(1,y)S = I$$

it is easy to show that A_S is indecomposable, but it is not locally cyclic. □

We know that GPW-flatness implies torsion freeness, but the following example shows that U-GPW-flatness of acts does not imply torsion freeness in general.

Example 4.3. Let $(\mathbb{N},.)$ be the monoid of natural numbers under multiplication, and consider $A_S = \mathbb{N}\coprod^{2\mathbb{N}}\mathbb{N}$. Then A_S is U-GPW-flat by Theorem 4.2. But $(1,x)\neq(1,y)$ and $(1,x)2 = 2 = (1,y)2$ and so A_S is not torsion free.

Using the above example we can also show that for a commutative monoid S, there exists an indecomposable right S-act A_S generated by exactly two elements, such that A_S is U-GPW-flat, but it is neither locally cyclic nor torsion free.

Now it is natural to ask for monoids over which U-GPW-flatness of acts implies torsion freeness and other properties. In the following we answer these questions.

Theorem 4.4. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts are torsion free.*

(2) *All U-GPW-flat right S-acts are torsion free.*

(3) *All finitely generated U-GPW-flat right S-acts are torsion free.*

(4) *All indecomposable right S-acts which are U-GPW-flat are torsion free.*

(5) *All finitely generated indecomposable right S-acts which are U-GPW-flat are torsion free.*

(6) *All right cancellable elements of S are right invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) and (3) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6) Let $c \in S$ be a right cancellable element such that $cS \neq S$ and consider $A_S = S \coprod^{cS} S$. Obviously, A_S is indecomposable which is generated by two elements $(1, x)$ and $(1, y)$. So A_S is U-GPW-flat by Theorem 4.2 and so by the assumption it is torsion free. Hence the equality $(1, x)c = c = (1, y)c$ implies $(1, x) = (1, y)$, which is a contradiction. Thus $cS = S$ and so c is right invertible as required.

(6) $\Rightarrow$ (1) It is obvious by [12, IV, Theorem 6.1]. □

Theorem 4.5. *For any monoid S the following statements are equivalent*:

(1) *All U-GPW-flat right S-acts are WPF.*

(2) *All U-GPW-flat right S-acts are WKF.*

(3) *All U-GPW-flat right S-acts are $PWKF$.*

(4) *All U-GPW-flat right S-acts are TKF.*

(5) *All U-GPW-flat right S-acts satisfy Condition (P).*

(6) *All U-GPW-flat right S-acts satisfy Condition (WP).*

(7) *All U-GPW-flat right S-acts satisfy Condition (PWP).*

(8) *S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7) and (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (8) Suppose for $s \in S$, $sS \neq S$. Consider $A_S = S \coprod^{sS} S$. By Theorem 4.2, A_S is U-GPW-flat and so by the assumption A_S satisfies Condition (PWP). Thus the equality $(1, x)s = (1, y)s$ implies that there

exist $a \in A_S$ and $u, v \in S$ such that $(1, x) = au$, $(1, y) = av$ and $us = vs$. Then the equalities $(1, x) = au$ and $(1, y) = av$ imply respectively that there exist $l, l' \in S \setminus I$ such that $a = (l, x)$ and $a = (l', y)$, a contradiction. Thus $sS = S$ and so S is a group as required.

(8) $\Rightarrow$ (1) This is obvious by [2, Proposition 9]. □

Theorem 4.6. *For any monoid S the following statements are equivalent*:

(1) *All U-GPW-flat right S-acts are free.*

(2) *All U-GPW-flat right S-acts are projective generator.*

(3) *All U-GPW-flat right S-acts are projective.*

(4) *All U-GPW-flat right S-acts are strongly flat.*

(5) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (5) By the assumption, all U-GPW-flat right S-acts are WPF and so S is a group by Theorem 4.5. Thus all right S-acts satisfy Condition (PWP) by [2, Proposition 9] and so all right S-acts are GPW-flat, thus they are U-GPW-flat. Hence by the assumption all right S-acts are strongly flat and so $S = \{1\}$ by [12, IV, Theorem 10.5]. □

Theorem 4.7. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts are principally weakly flat.*

(2) *All U-GPW-flat right S-acts are principally weakly flat.*

(3) *All finitely generated U-GPW-flat right S-acts are principally weakly flat.*

(4) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Let $s \in S$. If $sS = S$, then it is obvious that s is regular. Thus we suppose that $sS \neq S$ and let $A_S = S \coprod^{sS} S$. By Theorem 4.2, A_S is U-GPW-flat, and so by the assumption A_S is principally weakly flat. Thus by [12, III, Proposition 12.19], sS is left stabilizing, and so there exists $l \in sS$ such that $s = ls$. Hence there exists $x \in S$ such that $l = sx$, and so $s = ls = sxs$, that is, S is regular.

(4) $\Rightarrow$ (1) This is obvious by [12, IV, Theorem 6.6] □

Recall from [14] that a monoid S is said to be *generally regular* if for every $s \in S$, there exist $x \in S, n \in \mathbb{N}$ such that $s^n = sxs^n$.

Theorem 4.8. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are GP-flat.*

(2) *All U-GPW-flat right S-acts are GP-flat.*

(3) *All finitely generated U-GPW-flat right S-acts are GP-flat.*

(4) *S is generally regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Let $s \in S$. If $sS = S$, then it is obvious that s is generally regular. Thus we suppose that $sS \neq S$ and let $A_S = S \coprod^{sS} S$. By Theorem 4.2, A_S is U-GPW-flat, and so by the assumption A_S is GP-flat. Thus by [14, Lemma 2.4], for $s \in sS$ there exist $n \in \mathbb{N}$ and $j \in sS$ such that $s^n = js^n$. Hence there exists $x \in S$ such that $j = sx$, that is, $s^n = sxs^n$.

(4) $\Rightarrow$ (1) Since S is generally regular, by [14, Theorem 3.4], all right S-acts are GP-flat. □

We recall from [12] that a right S-act A_S is *divisible* if for every element $a \in A_S$ and any left cancellable element $c \in S$ there exists $b \in A_S$ such that $a = bc$.

Theorem 4.9. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are divisible.*

(2) *All U-GPW-flat right S-acts are divisible.*

(3) *All left cancellable elements of S are left invertible.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Since S_S is U-GPW-flat, so by the assumption it is divisible. Hence all left cancellable elements of S are left invertible by [12, III, Proposition 2.2].

(3) $\Rightarrow$ (1) It is true by [12, III, Proposition 2.2]. □

Clearly for a non-trivial monoid S, Θ_S is U-GPW-flat but it is not faithful, because $|S| > 1$. Thus U-GPW-flatness of acts does not imply faithfulness in general.

Theorem 4.10. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts are (strongly)faithful.*

(2) *All U-GPW-flat right S-acts are (strongly)faithful.*

(3) *All U-GPW-flat finitely generated right S-acts are (strongly)faithful.*

(4) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4) For any monoid S, $A_S = \Theta_1 \dot{\cup} \Theta_2$ is a U-GPW-flat finitely generated right S-act and so by the assumption A_S is (strongly)faithful. If $S \neq \{1\}$ then there exist $s, t \in S$ such that $s \neq t$. But it is obvious that for any $a \in A_S$, $as = at$ which is a contradiction. Thus $S = \{1\}$ as required.

(4) ⇒ (1) It is obvious. □

Recall from [16] that a right S-act A_S is called *strongly torsion free* if the equality $as = a's$, for $a, a' \in A_S$ and $s \in S$ implies $a = a'$. It is clear that every strongly torsion free right S-act is GPW-flat, but not the converse.

Theorem 4.11. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts are strongly torsion free.*

(2) *All U-GPW-flat right S-acts are strongly torsion free.*

(3) *All U-GPW-flat finitely generated right S-acts are strongly torsion free.*

(4) *S is a group.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4) Let $s \in S$ be such that $sS \neq S$ and suppose $A_S = S \coprod^{sS} S$. By Theorem 4.2, A_S is U-GPW-flat and so by the assumption A_S is strongly torsion free.
Now let

$$B = \{(l, x) \mid l \in S \setminus sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) \mid t \in S \setminus sS\} \dot{\cup} sS = C.$$

Clearly $A_S = \langle (1, x), (1, y) \rangle = (1, x)S \cup (1, y)S = B \cup C$. Then by [16, Proposition 2.1], B as a subact of A_S is strongly torsion free, and so S_S is strongly torsion free. Hence S is right cancellative by [16, Proposition

2.1]. But in the case of right cancellability of S, strong torsion freeness and torsion freeness are the same. So by Theorem 4.4, every right cancellable element of S is right invertible, hence $sS = S$, which is a contradiction. Thus for every $s \in S$, $sS = S$ and so S is a group as required.

(4) ⇒ (1) It is true by [16, Theorem 6.1]. □

Theorem 4.12. *Let S be a right cancellative monoid. Then following statements are equivalent*:

(1) *All right S-acts are flat.*

(2) *All U-GPW-flat right S-acts are flat.*

(3) *All finitely generated U-GPW-flat right S-acts are flat.*

(4) *All right S-acts are weakly flat.*

(5) *All U-GPW-flat right S-acts are weakly flat.*

(6) *All finitely generated U-GPW-flat right S-acts are weakly flat.*

(7) *S is a group.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (6) and (1) ⇒ (4) ⇒ (5) ⇒ (6) are obvious.

(6) ⇒ (7) Since, for right cancellative monoids, torsion freeness and strong torsion freeness of right acts coincide, and also weak flatness implies torsion freeness, thus all finitely generated U-GPW-flat right acts are strongly torsion free, and so S is a group by Theorem 4.11.

(7) ⇒ (1) This is obvious by [2, Proposition 9]. □

Now we consider monoids over which other properties of acts are U-GPW-flat.

Theorem 4.13. *Let S be a monoid. Then*:

(1) *All strongly faithful right S-acts are U-GPW-flat.*

(2) *All P-regular right S-acts are U-GPW-flat.*

(3) *All strongly P-cyclic right S-acts are U-GPW-flat.*

(4) *All regular right S-acts are U-GPW-flat.*

Proof. (1). Let A_S be a strongly faithful right S-act. For every $\alpha \in A_S$ define the mapping $\psi_\alpha : \alpha S \to S_S$ as $\psi_\alpha(\alpha s) = s$. It is obvious that ψ_α is an isomorphism and so for every $\alpha \in A_S$, $\alpha S \cong S_S$. Since S_S is GPW-flat by [15, Theorem 2.8], thus all cyclic subacts of A_S are GPW-flat. But $A_S = \cup_{\alpha \in A_S} \alpha S$, and so A_S is U-GPW-flat as required.

(2). Let A_S be a P-regular right S-act. By definition every cyclic subact of A_S satisfy Condition (P). Thus for every $\alpha \in A_S$, αS is GPW-flat and so $A_S = \cup_{\alpha \in A_S} \alpha S$ is U-GPW-flat as required.

Implications (3) and (4) are obvious from (2), because every strongly P-cyclic or regular right S-act is P-regular. □

Theorem 4.14. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts are U-GPW-flat.*

(2) *All finitely generated right S-acts are U-GPW-flat.*

(3) *All cyclic right S-acts are U-GPW-flat.*

(4) *S is an eventually regular monoid.*

Proof. (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇔ (4) It follows by (4) of Theorem 4.2 and [15, Theorem 4.5].

(3) ⇒ (1) It is clear. □

Theorem 4.15. *For any monoid S the following statements are equivalent*:

(1) *All torsion free right S-acts are U-GPW-flat.*

(2) *All torsion free finitely generated right S-acts are U-GPW-flat.*

(3) *All torsion free cyclic right S-acts are GPW-flat.*

(4) *S is an eventually left almost regular monoid.*

Proof. Implication (1) ⇒ (2) is obvious.

(2) ⇒ (3) It is true by (4) of Theorem 4.2.

(3) ⇒ (1) Let the right S-act A_S be torsion free. It is obvious that every subact of A_S is also torsion free. Thus by the assumption, αS is GPW-flat for every $\alpha \in A_S$. Hence $A_S = \cup_{\alpha \in A_S} \alpha S$ is U-GPW-flat.

(3) ⇔ (4) It is obvious by [15, Theorem 4.4]. □

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