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Direct products of cyclic semigroups and left zero semigroups in $\beta\mathbb{N}$

Yuliya Zelenyuk

Dedicated to Themba Dube on the occasion of his 65^{th} birthday.

Abstract. We show that for every $n \in \mathbb{N}$, the direct product of the cyclic semigroup of order n and period 1 and the left zero semigroup $2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$.

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation $\lambda_a : \beta\mathbb{N} \ni x \mapsto a + x \in \beta\mathbb{N}$ is continuous, and for each $q \in \beta\mathbb{N}$, the right translation $\rho_q : \beta\mathbb{N} \ni x \mapsto x + q \in \beta\mathbb{N}$ is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ and $A^* = \overline{A} \setminus A$. The subsets $\overline{A}$, where $A \subseteq \mathbb{N}$, form a base for the

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topology of $\beta\mathbb{N}$, and $\overline{A}$ is the closure of A. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p+q$ has a base consisting of subsets of the form $\bigcup_{x\in A}(x+B_x)$, where $A \in p$ and for each $x \in A$, $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal, the intersection of a minimal right ideal and a minimal left ideal is a group, and the idempotents in a minimal right (left) ideal form a right (left) zero semigroup, that is, $x + y = y$ $(x + y = x)$ for all x, y.

An elementary introduction to $\beta\mathbb{N}$ can be found in [4].

In 1979, E. van Douwen asked (in [3], published much later) whether there are topological and algebraic copies of $\beta\mathbb{N}$ contained in $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. This question was answered in the negative by D. Strauss in [6], where it was in fact established that continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$ have finite images. It follows that if $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ is a continuous homomorphism, then $p = \varphi(1)$ is an element of a finite order n. That is, all $ip = \underbrace{p + \ldots + p}_{i}$, where $i \in \{1, \ldots, n\}$, are distinct and $(n + 1)p = mp$ for some $m \in \{1, \ldots, n\}$. Conversely, every element $p \in \mathbb{N}^*$ of finite order determines a continuous homomorphism $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ by $\varphi(1) = p$. In 1996, Y. Zelenyuk proved that $\beta\mathbb{N}$ contains no nontrivial finite groups (see [4, Theorem 7.17]). Consequently, if $p \in \beta\mathbb{N}$ is an element of order n, then $(n + 1)p = np$.

As distinguished from finite groups, $\beta\mathbb{N}$ does contain bands (semigroups of idempotents): for example, left zero semigroups, right zero semigroups, chains of idempotents (with respect to the order $x \le y$ if and only if $x+y = y + x = x$), and rectangular bands (direct products of a left zero semigroup and a right zero semigroup). To ask whether $\beta\mathbb{N}$ contains a finite semigroup distinct from bands is the same as asking whether $\beta\mathbb{N}$ contains an element of order 2 which is the same as asking whether there exists a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ [4, Question 10.19].

The question whether $\beta\mathbb{N}$ contains an element of order 2 was solved in the affirmative in [7, Theorem 1]. This result has an interesting Ramsey theoretic consequence, the implication itself was established in [2, Corollary 3.5], see also [1, 8]. In [8], some further finite semigroups in $\beta\mathbb{N}$ consisting of idempotents and elements of order 2 were constructed, in particular null semigroups ($x+y=0$ for all x,y). In [10], it was shown that for every $m\geq 1$, the direct product of the m-element null semigroup and the rectangular band $2^{\mathfrak{c}}\times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ (that the rectangular band $2^{\mathfrak{c}}\times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ was established in [5]).

The question whether $\beta\mathbb{N}$ contains an element of finite order $n>2$ was solved in the affirmative in [9, Theorem 3]. In fact it was shown that for every $m\geq 1$ and every $n\geq 2$, there are distinct elements $p=p_1,p_2,\ldots,p_m$ in $\beta\mathbb{N}$ of order n such that $p_s+p_t=2p$ for all $s,t\in\{1,\ldots,m\}$. The subsemigroup generated by $p_1,\ldots,p_m$ consists of the elements $p_1,\ldots,p_m,2p,\ldots,np$ and has defining relations $(n+1)p=np$ and $p_s+p_t=2p$. We denote this semigroup by $C_{m,n}$. If $m=1$, this is the cyclic semigroup of order n and period 1, and if $n=2$, this is the m-element null semigroup.

In this paper we combine and modify constructions in [10] and [9] and prove that for every $m\geq 1$ and every $n\geq 2$, the direct product of the semigroup $C_{m,n}$ and the left zero semigroup $2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$. In particular, the direct product of the cyclic semigroup of order n and period 1 and the left zero semigroup $2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$.

Theorem 1.1. *Let $m\geq 1$ and $n\geq 2$. There is an isomorphic embedding $\varepsilon: C_{m,n}\times 2^{\mathfrak{c}}\to\beta\mathbb{N}$. Furthermore, ε can be chosen so that $\varepsilon(C_{m,n}\times 2^{\mathfrak{c}})\subseteq \overline{K(\beta\mathbb{N})}$ and $\varepsilon(np,\alpha)\in K(\beta\mathbb{N})$ for all $\alpha<2^{\mathfrak{c}}$.*

In the rest of the paper we prove Theorem 1.1.

Let $l=m+n-1$. For every $x\in\mathbb{N}$, supp x is a unique finite nonempty subset of $\omega=\mathbb{N}\cup\{0\}$ such that

$$x=\sum_{k\in\operatorname{supp} x}2^k.$$

Pick an increasing sequence $I_0\subseteq I_1\subseteq\ldots\subseteq I_l=\omega$ of subsets of ω such that $I_i\setminus I_{i-1}$ is infinite for each $i\in\{0,1,\ldots,l\}$ (with $I_{-1}=\emptyset$). Define a function h from $\mathbb{N}$ onto the decreasing chain $0>1>\ldots>l$ of idempotents (with the operation $i*j=\max\{i,j\}$) by

$$h(x)=\min\{i\leq l:\operatorname{supp} x\subseteq I_i\}=\max\{i\leq l:(\operatorname{supp} x)\cap(I_i\setminus I_{i-1})\neq\emptyset\}$$

and let the same letter h denote its continuous extension $\beta\mathbb{N} \to \{0, 1, \ldots, l\}$. If $x, y \in \mathbb{N}$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $h(x + y) = h(x) * h(y)$. It then follows (see [4, Theorem 4.21]) that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, where

$$\mathbb{H} = \bigcap_{n=0}^{\infty} \overline{2^n\mathbb{N}},$$

one has $h(u + v) = h(u) * h(v)$, in particular, the restriction of h to $\mathbb{H}$ is a homomorphism. For each $i \in \{0, 1, \ldots, l\}$, let

$$T_i = h^{-1}(\{0, 1, \ldots, i\}) \cap \mathbb{H}.$$

Then $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_l = \mathbb{H}$ is an increasing sequence of closed subsemigroups of $\mathbb{H}$ such that $h(K(T_i)) = \{i\}$ for each $i \leq l$, and so $T_i \cap \overline{K(T_{i+1})} = \emptyset$ for each $i < l$ and $K(T_l) = K(\beta\mathbb{N}) \cap T_l$ [8, Lemma 3.1], in particular, all $K(T_0), K(T_1), \ldots, K(T_l)$ are pairwise disjoint. Moreover, $h(K(\beta\mathbb{N})) = \{l\}$, and so $T_{l-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$.

To see this, let $u \in K(\beta\mathbb{N})$. Then $u + \beta\mathbb{N}$ is the minimal right ideal of $\beta\mathbb{N}$ containing u and $\beta\mathbb{N} + u$ the minimal left ideal containing u. Let v be the identity of the group $(u+\beta\mathbb{N})\cap(\beta\mathbb{N}+u)$. Then $u = u+v$ and $v \in K(\mathbb{H})$, so $h(u) = h(u + v) = h(u) * h(v) = h(u) * l = l$.

For each $i \in \{0, 1, \ldots, l\}$, let

$$X_i = \{x \in \mathbb{N} : (\operatorname{supp} x) \cap (I_i \setminus I_{i-1}) \neq \emptyset\}.$$

Notice that for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $u + v \in \overline{X_i}$, and for any $v \in \overline{X_i}$ and $w \in \mathbb{H}$, $v + w \in \overline{X_i}$.

Define $\phi_i : X_i \to \omega$ by

$$\phi_i(x) = \max((\operatorname{supp} x) \cap (I_i \setminus I_{i-1}))$$

and let the same letter ϕ_i denote its continuous extension $\overline{X_i} \to \beta\omega$. Notice that $\overline{\{2^k : k \in I_i \setminus I_{i-1}\}} \subseteq X_i$ and, since $\phi_i(2^k) = k$, ϕ_i homeomorphically maps $\overline{\{2^k : k \in I_i \setminus I_{i-1}\}}$ onto $\overline{I_i \setminus I_{i-1}}$. If $x \in \mathbb{N}$, $y \in X_i$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $x+y \in X_i$ and $\phi_i(x+y) = \phi_i(y)$. And if $y \in X_i$, $z \in \mathbb{N}\setminus X_i$ and $\max\operatorname{supp} y < \min\operatorname{supp} z$, then $\phi_i(y + z) = \phi_i(y)$. It then follows that for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_i(u + v) = \phi_i(v)$, and for any $v \in \overline{X_i}$ and $w \in \mathbb{H} \setminus \overline{X_i}$, $\phi_i(v + w) = \phi_i(v)$.

To see for example the first statement, we first note that for any $x \in \mathbb{N}$ and $v \in \overline{X_i} \cap \mathbb{H}$, $\phi_i(x+v) = \phi_i(v)$ because the continuous functions $\phi_i \circ \lambda_x$ and ϕ_i agree on $X_i \cap 2^n\mathbb{N}$, where $n = (\max\operatorname{supp} x) + 1$. Then for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_i(u+v) = \phi_i(v)$ because the continuous function $\phi_i \circ \rho_v$ is constantly equal to $\phi_i(v)$ on $\mathbb{N}$.

Notice that $K(T_i) \subseteq \overline{X_i} \cap \mathbb{H}$ and $T_{i-1} \subseteq \mathbb{H} \setminus \overline{X_i}$ (with $T_{-1} = \emptyset$).

We shall construct

(i) a chain $e_0 > e_1 > \ldots > e_l$ of idempotents with $e_i \in K(T_i)$,

(ii) for each $i \in \{0, 1, \ldots, l\}$, a left zero semigroup $\{e_{i,\alpha} : \alpha < 2^{\mathfrak{c}}\} \subseteq K(T_i)$ such that $e_{i,0} = e_i$ and $e_{i,\alpha} = e_{0,\alpha} + e_i$ for all $\alpha < 2^{\mathfrak{c}}$, and

(iii) for each $i \in \{1, m+1, \ldots, l-1\}$, a right zero semigroup $\{e_i(j) : j \in \omega\} \subseteq K(T_i)$ such that $e_i(0) = e_i$, $e_i(j) < e_{i-1}$ for all $j \in \omega$, and $\phi_i(e_i(j)) \neq \phi_i(e_i(k))$ if $j \neq k$.

Notice that (i) and (ii) imply that

$$e_{i,\alpha} + e_{j,\beta} = e_{i*j,\alpha}$$

for all $i, j \in \{0, 1, \ldots, l\}$ and $\alpha, \beta < 2^{\mathfrak{c}}$.

Indeed,

$$\begin{aligned} e_{i,\alpha} + e_{j,\beta} &= e_{0,\alpha} + e_i + e_{0,\beta} + e_j = e_{0,\alpha} + (e_i + e_0) + e_{0,\beta} + e_j \\ &= e_{0,\alpha} + e_i + (e_0 + e_{0,\beta}) + e_j = e_{0,\alpha} + e_i + e_0 + e_j \\ &= e_{0,\alpha} + e_{i*j} = e_{i*j,\alpha}. \end{aligned}$$

The construction goes by induction on $i \in \{0, 1, \ldots, l\}$.

For $i = 0$, pick an injective $2^{\mathfrak{c}}$-sequence $\{r_{0,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ in $\{2^k : k \in I_0\}^*$.

Lemma 1.2. *$(r_{0,\alpha} + T_l) \cap (r_{0,\beta} + T_l) = \emptyset$ if $\alpha \neq \beta$.*

Proof. Consider the function $\mathbb{N} \ni x \mapsto \min\operatorname{supp} x \in \omega$ and let θ denote its continuous extension $\beta\mathbb{N} \to \beta\omega$. If $x, y \in \mathbb{N}$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $\theta(x+y) = \theta(x)$. It then follows that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, $\theta(u+v) = \theta(u)$. Consequently, $\theta(r_{0,\alpha} + T_l) = \{\theta(r_{0,\alpha})\}$ and $\theta(r_{0,\beta} + T_l) = \{\theta(r_{0,\beta})\}$. Since $\theta(2^k) = k$, $\theta(r_{0,\alpha}) \neq \theta(r_{0,\beta})$, so $(r_{0,\alpha} + T_l) \cap (r_{0,\beta} + T_l) = \emptyset$. □

For every $\alpha < 2^{\mathfrak{c}}$, choose a minimal right ideal $R_{0,\alpha}$ of T_0 contained in $r_{0,\alpha} + T_0$. Pick a minimal left ideal L_0 of T_0, and for every $\alpha < 2^{\mathfrak{c}}$, let $e_{0,\alpha}$

be the identity of the group $R_{0,\alpha} \cap L_0$. By Lemma 1.2, $e_{0,\alpha} \neq e_{0,\beta}$ if $\alpha \neq \beta$. Put $e_0 = e_{0,0}$.

For $i = 1$, choose a minimal right ideal $R_{1,\alpha}$ of T_1 contained in $e_0 + T_1$. Pick an injective sequence $(r_{1,j})_{j=0}^{\infty}$ in $\{2^k : k \in I_1 \setminus I_0\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{1,j}$ of T_1 contained in $T_1 + r_{1,j} + e_0$. For every $j \in \omega$, let $e_1(j)$ be the identity of the group $R_{1,0} \cap L_{1,j}$. Then $\phi_1(e_{1,j}) = \phi_1(r_{1,j} + e_0) = \phi_1(r_{1,j})$. Since $e_1(j) \in e_0 + T_1$, one has $e_0 + e_1(j) = e_1(j)$, and since $e_1(j) \in T_1 + r_{1,j} + e_0$, one has $e_1(j) + e_0 = e_1(j)$, so $e_1(j) < e_0$. Put $e_1 = e_1(0)$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{1,\alpha} = e_{0,\alpha} + e_1$. Then $e_{1,\alpha} + e_{1,\beta} = e_{0,\alpha} + e_1 + e_{0,\beta} + e_1 = e_{0,\alpha} + (e_1 + e_0) + e_{0,\beta} + e_1 = e_{0,\alpha} + e_1 + (e_0 + e_{0,\beta}) + e_1 = e_{0,\alpha} + e_1 + e_0 + e_1 = e_{0,\alpha} + e_1 = e_{1,\alpha}$, so $\{e_{1,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ is a left zero semigroup (in $K(T_1)$). Since $e_{1,\alpha} = e_{0,\alpha} + e_1 \in r_{0,\alpha} + T_0 + e_1 \subseteq r_{0,\alpha} + T_1$, by Lemma 1.2, $e_{1,\alpha} \neq e_{1,\beta}$ if $\alpha \neq \beta$.

For $i \in \{2, \ldots, m\}$, pick a minimal right ideal R_i of T_i contained in $e_{i-1} + T_i$ and a minimal left ideal L_i of T_i contained in $T_i + e_{i-1}$ and let e_i be the identity of the group $R_i \cap L_i$. For every $\alpha < 2^{\mathfrak{c}}$, let $e_{i,\alpha} = e_{0,\alpha} + e_i$. Then $\{e_{l,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ is a left zero semigroup and $e_{i,\alpha} \neq e_{i,\beta}$ if $\alpha \neq \beta$.

For $i \in \{m + 1, \ldots, l - 1\}$ (for $n \geq 3$), choose a minimal right ideal R_i of T_i contained in $e_{i-1} + T_i$. Pick an injective sequence $(r_{i,j})_{j=0}^{\infty}$ in $\{2^k : k \in I_i \setminus I_{i-1}\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{i,j}$ of T_i contained in $T_i + r_{i,j} + e_{i-1}$, and let $e_i(j)$ be the identity of the group $R_i \cap L_{i,j}$. Then $\phi_i(e_i(j)) = \phi_i(r_{i,j} + e_0) = \phi_i(r_{i,j})$ and $e_i(j) < e_{i-1}$ for all j. Put $e_i = e_i(0)$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{i,\alpha} = e_{0,\alpha} + e_i$. Then $\{e_{i,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ a left zero semigroup and $e_{i,\alpha} \neq e_{i,\beta}$ if $\alpha \neq \beta$.

For $i = l$, pick a minimal right ideal R_l of T_l contained in $e_{l-1} + T_l$ and a minimal left ideal L_l of T_l contained in $T_l + e_{l-1}$ and let e_l be the identity of the group $R_l \cap L_l$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{l,\alpha} = e_{0,\alpha} + e_l$.

Now let

$$D_{l-1} = \begin{cases} \{e_l + e_1(j) : j < \omega\} & \text{if } n = 2 \\ \{e_l + e_{l-1}(j) : j < \omega\} & \text{if } n \geq 3 \end{cases}$$

and pick $q_{l-1} \in \overline{D_{l-1}} \setminus D_{l-1}$. Then inductively, for each $i \in \{l-2, \ldots, m+1\}$ (for $n \geq 4$), let

$$D_i = \{e_{i+1} + q_{i+1} + e_i(j) : j < \omega\}$$

and pick $q_i \in \overline{D_i} \setminus D_i$. For $i = m$ (for $n \geq 3$), let

$$D_m = \{e_{m+1} + q_{m+1} + e_1(j) : j < \omega\}$$

and pick $q_m \in \overline{D_m} \setminus D_m$.

Since $e_l \in K(\beta\mathbb{N})$ and $\overline{K(\beta\mathbb{N})}$ is an ideal of $\beta\mathbb{N}$ [4, Theorem 4.44], we have inductively that for each $i \in \{l-1, \dots, m\}$, $D_i \subseteq \overline{K(\beta\mathbb{N})}$ and $q_i \in \overline{K(\beta\mathbb{N})}$.

For each $s \in \{0, 1, \dots, l\}$, $e_l = e_s + e_l$ and $e_s \in \overline{X_s}$, so $e_l \in \overline{X_s}$. It then follows inductively that for each $i \in \{l-1, \dots, m\}$, $D_i \subseteq \overline{X_s} \cap \mathbb{H}$ and $q_i \in \overline{X_s} \cap \mathbb{H}$. Notice that for each $i \in \{l-1, \dots, m+1\}$ (for $n \geq 3$), ϕ_i is injective on D_i (because $\phi_{l-1}(e_l + e_{l-1}(j)) = \phi_{l-1}(e_{l-1}(j))$ and $\phi_i(e_{i+1} + q_{i+1} + e_i(j)) = \phi_i(e_i(j))$), and ϕ_1 is injective on D_m ($\phi_1(e_{m+1} + e_1(j)) = \phi_1(e_1(j))$ for $n = 2$ and $\phi_1(e_{m+1} + q_{m+1} + e_1(j)) = \phi_1(e_1(j))$ for $n \geq 3$).

An ultrafilter $q \in \mathbb{N}^*$ is *right cancelable* (*in* $\beta\mathbb{N}$) if the right translation of $\beta\mathbb{N}$ by q is injective. An ultrafilter $q \in \mathbb{N}^*$ is right cancelable if and only if $q \notin \mathbb{N}^* + q$ [4, Theorem 8.18]. From the next lemma we obtain that all $q_m, \dots, q_{l-1}$ are right cancelable.

Lemma 1.3. *Let $i \in \{0, 1, \dots, l\}$. Also, let D be a countable subset of $\overline{X_i} \cap \mathbb{H}$, and suppose that ϕ_i is injective on D. Then every $q \in \overline{D} \setminus D$ is right cancelable.*

Proof. This is [9, Lemma 5]. □

The next lemma gives us relations between $q_m, \dots, q_{l-1}$ and $e_{i,\alpha}$.

Lemma 1.4. *For every $\alpha < 2^{\mathfrak{c}}$,*

(1) $q_{l-1} + e_{l-1,\alpha} = e_l$,

(2) *if $n = 2$, then for each $s \in \{1, \dots, l\}$, $q_{l-1} + e_{s,\alpha} = e_l$,*

(3) *if $n \geq 3$, then for each $i \in \{m+1, \dots, l-1\}$, $q_i + e_{i-1,\alpha} = q_i$,*

(4) *if $n \geq 3$, then for each $i \in \{m, \dots, l-2\}$, $q_i + e_{i,\alpha} = e_{i+1} + q_{i+1}$, and*

(5) *if $n \geq 3$, then for each $s \in \{1, \dots, m\}$, $q_m + e_{s,\alpha} = e_{m+1} + q_{m+1}$.*

Proof. (1) For $n \geq 3$, $(e_l + e_{l-1}(j)) + e_{l-1,\alpha} = e_l + (e_{l-1}(j) + e_{l-2}) + e_{l-1,\alpha} = e_l + e_{l-1}(j) + ((e_{l-2} + e_{l-1,\alpha})) = e_l + e_{l-1}(j) + e_{l-1} = e_l + e_{l-1} = e_l$, and since $\rho_{e_{l-1,\alpha}}$ is constantly equal to e_l on D_{l-1}, $\rho_{e_{l-1,\alpha}}(q_{l-1}) = e_l$, so $q_{l-1} + e_{l-1,\alpha} = e_l$. The case $n = 2$ is included in (2).

(2) $(e_l + e_1(j)) + e_{s,\alpha} = e_l + (e_1(j) + e_0) + e_{s,\alpha} = e_l + e_1(j) + (e_0 + e_{s,\alpha}) = e_l + e_1(j) + e_s = e_l + e_1(j) + (e_1 + e_s) = e_l + (e_1(j) + e_1) + e_s = e_l + e_1 + e_s = e_l$.

(3) For $i = l-1$, $(e_l + e_{l-1}(j)) + e_{l-2,\alpha} = e_l + (e_{l-1}(j) + e_{l-2}) + e_{l-2,\alpha} = e_l + e_{l-1}(j) + (e_{l-2} + e_{l-2,\alpha}) = e_l + e_{l-1}(j) + e_{l-2} = e_l + e_{l-1}(j)$, and for $i \leq l-2$,

$(e_{i+1}+q_{i+1}+e_i(j))+e_{i-1,\alpha}=e_{i+1}+q_{i+1}+(e_i(j)+e_{i-1})+e_{i-1,\alpha}=e_{i+1}+q_{i+1}+e_i(j)+(e_{i-1}+e_{i-1,\alpha})=e_{i+1}+q_{i+1}+e_i(j)+e_{i-1}=e_{i+1}+q_{i+1}+e_i(j)$.

(4) For $i\geq m+1$, $(e_{i+1}+q_{i+1}+e_i(j))+e_{i,\alpha}=e_{i+1}+q_{i+1}+(e_i(j)+e_{i-1})+e_{i,\alpha}=e_{i+1}+q_{i+1}+e_i(j)+(e_{i-1}+e_{i,\alpha})=e_{i+1}+q_{i+1}+e_i(j)+e_i=e_{i+1}+q_{i+1}+e_i=e_{i+1}+q_{i+1}$. The case $i=m$ is included in (5).

(5) $e_{m+1}+q_{m+1}+e_1(j)+e_{s,\alpha}=e_{m+1}+q_{m+1}+(e_1(j)+e_0)+e_{s,\alpha}=e_{m+1}+q_{m+1}+e_1(j)+(e_0+e_{s,\alpha})=e_{m+1}+q_{m+1}+e_1(j)+e_s=e_{m+1}+q_{m+1}+e_1(j)+(e_1+e_s)=e_{m+1}+q_{m+1}+(e_1(j)+e_1)+e_s=e_{m+1}+q_{m+1}+e_1+e_s=e_{m+1}+q_{m+1}+e_s=e_{m+1}+q_{m+1}$. □

Now for each $s\in\{1,\dots,m\}$ and each $\alpha<2^{\mathfrak{c}}$, let

$$p_s(\alpha)=e_{s,\alpha}+q_m.$$

Lemma 1.5. *For all $i\geq 2$, $s_1,\dots,s_i\in\{1,\dots,m\}$, and $\alpha_1,\dots,\alpha_i<2^{\mathfrak{c}}$,*

$$p_{s_1}(\alpha_1)+\ldots+p_{s_i}(\alpha_i)=\begin{cases}e_{m+i-1,\alpha_1}+q_{m+i-1}+\ldots+q_m & \text{if } i\leq n-1\\ e_{l,\alpha_1}+q_{l-1}+\ldots+q_m & \text{otherwise.}\end{cases}$$

Proof. We use Lemma 1.4. If $n=2$, then

$$\begin{aligned}p_{s_1}(\alpha_1)+p_{s_2}(\alpha_2)&=e_{s_1,\alpha_1}+q_m+e_{s_2,\alpha_2}+q_mx\\&=e_{s_1,\alpha_1}+(q_m+e_{s_2,\alpha_2})+q_m\\&=e_{s_1,\alpha_1}+e_l+q_m\\&=e_{l,\alpha_1}+q_m,\ \text{and}\\p_{s_1}(\alpha_1)+p_{s_2}(\alpha_2)+p_{s_3}(\alpha_3)&=(p_{s_1}(\alpha_1)+p_{s_2}(\alpha_2))+p_{s_3}(\alpha_3)\\&=e_{l,\alpha_1}+q_m+e_{s_3,\alpha_3}+q_m\\&=e_{l,\alpha_1}+(q_m+e_{s_3,\alpha_3})+q_m\\&=e_{l,\alpha_1}+e_l+q_m\\&=e_{l,\alpha_1}+q_m.\end{aligned}$$

Let $n\geq 3$. We first notice that for each $j\in\{m,\dots,l-2\}$,

$$\begin{aligned}q_j+\ldots+q_m+e_{s,\alpha}&=e_{j+1}+q_{j+1}+\ldots+q_{m+1}\ \text{and}\\q_{l-1}+\ldots+q_m+e_{s,\alpha}&=e_l+q_{l-1}+\ldots+q_{m+1}.\end{aligned}$$

Indeed, inductively, $q_m + e_{s,\alpha} = e_{m+1} + q_{m+1}$, and for $j \geq m+1$,

$$\begin{aligned} q_j + \ldots + q_m + e_{s,\alpha} &= q_j + (q_{j-1} + \ldots + q_m + e_{s,\alpha}) \\ &= q_j + e_j + q_j + \ldots + q_{m+1} \\ &= e_{j+1} + q_{j+1} + q_j + \ldots + q_{m+1}, \end{aligned}$$

and then

$$\begin{aligned} q_{l-1} + \ldots + q_m + e_{s,\alpha} &= q_{l-1} + (q_{l-2} + \ldots + q_m + e_{s,\alpha}) \\ &= q_{l-1} + e_{l-1} + q_{l-1} + \ldots + q_{m+1} \\ &= e_l + q_{l-1} + \ldots + q_{m+1}. \end{aligned}$$

Now by induction on $i \in \{2, \ldots, n-1\}$,

$$\begin{aligned} p_{s_1}(\alpha_1) + p_{s_2}(\alpha_2) &= e_{s_1,\alpha_1} + q_m + e_{s_2,\alpha_2} + q_m \\ &= e_{s_1,\alpha_1} + (q_m + e_{s_2,\alpha_2}) + q_m \\ &= e_{s_1,\alpha_1} + e_{m+1} + q_{m+1} + q_m \\ &= e_{m+1,\alpha_1} + q_{m+1} + q_m, \end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned} p_{s_1}(\alpha_1) + \ldots + p_{s_i}(\alpha_i) &= (p_{s_1}(\alpha_1) + \ldots + p_{s_{i-1}}(\alpha_{i-1})) + p_{s_i}(\alpha_i) \\ &= e_{m+i-2,\alpha_1} + q_{m+i-2} + \ldots + q_m + e_{s_i,\alpha_i} + q_m \\ &= e_{m+i-2,\alpha_1} + e_{m+i-1} + q_{m+i-1} + \ldots + q_{m+1} + q_m \\ &= e_{m+i-1,\alpha_1} + q_{m+i-1} + \ldots + q_m, \end{aligned}$$

and then

$$\begin{aligned} p_{s_1}(\alpha_1) + \ldots + p_{s_n}(\alpha_n) &= (p_{s_1}(\alpha_1) + \ldots + p_{s_{n-1}}(\alpha_{n-1})) + p_{s_n}(\alpha_n) \\ &= e_{l-1,\alpha_1} + q_{l-1} + \ldots + q_m + e_{s_n,\alpha_i} + q_m \\ &= e_{l-1,\alpha_1} + e_l + q_{l-1} + \ldots + q_{m+1} + q_m \\ &= e_{l,\alpha_1} + q_{l-1} + \ldots + q_m \end{aligned}$$

and

$$p_{s_1}(\alpha_1) + \ldots + p_{s_{n+1}}(\alpha_{n+1}) = (p_{s_1}(\alpha_1) + \ldots + p_{s_n}(\alpha_n)) + p_{s_{n+1}}(\alpha_{n+1})$$

$$\begin{aligned}&= e_{l,\alpha_1} + q_{l-1} + \ldots + q_m + e_{s_{n+1},\alpha_{n+1}} + q_m\\&= e_{l,\alpha_1} + e_l + q_{l-1} + \ldots + q_{m+1} + q_m\\&= e_{l,\alpha_1} + q_{l-1} + \ldots + q_m.\end{aligned}$$

□

It follows from Lemma 1.5 that for each $i \geq 2$, $p_{s_1}(\alpha_1) + \ldots + p_{s_i}(\alpha_i) = ip(\alpha_1)$, where $p(\alpha) = p_1(\alpha)$, and for $i \geq n$, $ip(\alpha) = np(\alpha)$.

Lemma 1.6. *All elements $p_s(\alpha)$ and $ip(\alpha)$, where $\alpha < 2^{\mathfrak{c}}$, $s \in \{1, \ldots, m\}$, and $i \in \{2, \ldots, n\}$, are pairwise distinct.*

Proof. Since all $e_{s,\alpha}$ are distinct and q_m is right cancelable (Lemma 1.3), it follows that all $p_s(\alpha) = e_{s,\alpha} + q_m$ are distinct. Suppose that $ip_s(\alpha) = jp_t(\beta)$ for some $\alpha, \beta < 2^{\mathfrak{c}}$, $s, t \in \{1, \ldots, m\}$, and $i, j \in \{1, \ldots, n\}$ with $i + j \geq 3$. We show that $i = j$ and $\alpha = \beta$.

Without loss of generality one may suppose that $i \geq j$ and $i = n$ (by adding $(n-i)p_s(\alpha)$ to both sides of the equality from the right), and consequently, we have

$$e_{l,\alpha} + q_{l-1} + \ldots + q_m = \begin{cases} e_{s,\beta} + q_m & \text{if } j = 1\\ e_{m+j-1,\beta} + q_{m+j-1} + \ldots + q_m & \text{if } 2 \leq j < n\\ e_{l,\beta} + q_{l-1} + \ldots + q_m & \text{if } j = n.\end{cases}$$

If $j = 1$, then canceling the equality by q_m we obtain $e_{l,\alpha} + q_{l-1} + \ldots + q_{m+1} = e_{s,\beta}$ in the case $n \geq 3$ or $e_{l,\alpha} = e_{s,\beta}$ in the case $n = 2$. The second possibility is impossible, and the first also gives a contradiction because q_{m+1} is in $\overline{K(\beta\mathbb{N})}$ and so is $e_{l,\alpha} + q_{l-1} + \ldots + q_{m+1}$, and $e_{s,\beta} \in T_s$ (and $T_s \cap \overline{K(\beta\mathbb{N})} = \emptyset$). Thus $j \geq 2$.

If $j = n-1$, then canceling by $q_m, \ldots, q_{l-1}$ we obtain $e_{l,\alpha} = e_{l-1,\beta}$ which is impossible, and if $j \leq n-2$, then canceling we obtain

$$e_{l,\alpha} + q_{l-1} + \ldots + q_k = e_{m+j-1,\beta},$$

where $k = l - (i - j - 1)$, which also gives a contradiction because q_k is in $\overline{K(\beta\mathbb{N})}$ and so is $e_{l,\alpha} + q_{l-1} + \ldots + q_k$, and $e_{m+j-1,\beta} \in T_{l-1}$ (and $T_{l-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$). Hence $j = n = i$. Then canceling we obtain $e_{l,\alpha} = e_{l,\beta}$, whence $\alpha = \beta$. □

Define $\varepsilon : C_{m,n} \times 2^{\mathfrak{c}} \to \beta\mathbb{N}$ by

$$\varepsilon(ip_s, \alpha) = ip_s(\alpha).$$

By Lemma 1.6, ε is injective, and

$$\varepsilon((ip_s, \alpha) + (jp_t, \beta)) = \varepsilon(ip_s + jp_t, \alpha + \beta) = \varepsilon((i+j)p_s, \alpha) = (i+j)p_s(\alpha)$$

and

$$\varepsilon(ip_s, \alpha) + \varepsilon(jp_t, \beta) = ip_s(\alpha) + jp_t(\beta) = (i+j)p_s(\alpha),$$

so ε is an isomorphic embedding.

Since q_m is in $\overline{K(\beta\mathbb{N})}$, so are $\varepsilon(p_s, \alpha) = p_s(\alpha) = e_{s,\alpha} + q_m$ and $\varepsilon(ip, \alpha) = i\varepsilon(p, \alpha)$, and since $e_{l,\alpha}$ are in $K(\beta\mathbb{N})$, so are $\varepsilon(np, \alpha) = np(\alpha) = e_{l,\alpha} + q_{l-1} + \ldots + q_m$.

This finishes the proof of Theorem 1.1.

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Yuliya Zelenyuk *School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa.*
Email: yuliya.zelenyuk@wits.ac.za