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The coherator Θ^{∞}_{W} of cubical weak ∞-categories with connections

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Abstract. This work exhibits two applications of the combinatorial approach in [12] of the small category Θ_0 which objects are cubical pasting diagrams. First we provide an accurate description of the monad $\mathbb{S} = (S, \lambda, \mu)$ acting on the category $\mathbb{C}\mathbb{S}$ets of cubical sets (without degeneracies and connections), which algebras are cubical strict ∞-categories with connections, and show that this monad is cartesian, which solve a conjecture in [16]. Secondly we give a precise construction of the cubical coherator Θ^{∞}_{W} which set-models are cubical weak ∞-categories with connections, and we also give a precise construction of the cubical coherator $\Theta^{\infty}_{W^0}$ which set-models are cubical weak ∞-groupoids with connections.

1 Introduction

In globular higher category theory two important algebraic steps were necessary for a deeper understanding of some of its facets: the construction of its category Θ_0 *of arities* (which leads a good ambients for operations

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for the globular theories) [3, 18], and the definition of *globular contractions* (which permits to weakened globular theories) [3, 20]. For cubical higher category theory expressed with algebraic structures we also need the cubical analogue of these two ingredients: but the *cubical* Θ_0 *of arities* for the cubical theories is described in [12], and the *cubical contractions* for weakened cubical structures were described accurately in [14, 16]. With these two ingredients in hands this work exhibits two applications which aim is to understand deeper importants facets of cubical higher category theory:

- First we use the cubical Θ_0 [12] which objects are cubical pasting diagrams to provide an accurate description of the monad $\mathbb{S} = (S, \lambda, \mu)$ acting on the category $\mathbb{C}\mathbb{S}$ets of cubical sets (without degeneracies and connections), which algebras are cubical strict ∞-categories with connections, and show that this monad is cartesian, which solve a conjecture in [16]. In [16] we used this cartesianity to build the theory of *cubical operads* which leads to a fundamental cubical weak ∞-groupoid functor:

  $$\mathbb{T}\text{op} \xrightarrow{\Pi_\infty} \infty\text{-}\mathbb{C}\mathbb{G}\text{rp},$$

  where ∞-$\mathbb{C}\mathbb{G}$rp is here the category of cubical weak ∞-groupoids with connections (this category is described in [15]), and this functor Π_∞ is the cubical analogue of the fundamental globular weak ∞-groupoid functor:

  $$\mathbb{T}\text{op} \xrightarrow{\Pi_\infty} \infty\text{-}\mathbb{G}\mathbb{G}\text{rp},$$

  where ∞-$\mathbb{G}\mathbb{G}$rp denotes the category of globular weak ∞-groupoids (this category is described in [10]), built by Michael Batanin in [3].

  If X is a rectangular n-divisor, it means that it is an object of Θ_0 and also it is an n-cell of the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv built in [12]. In particular X is described as an n-configuration of coordinates:

  $$C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!],$$

  weighted by n-cells $A_l \in R(n)$ (where R denotes the underlying endofunctor of the monad $\mathbb{R} = (R, i, m)$ acting on the category of cubical sets, which algebras are cubical sets equipped with degeneracies), i.e for each coordinates $dx^i_{k^l_i} \in C_n$ $(l \in [\![1; m_1 \cdots m_n]\!])$ is attached an n-cells $A_l \in R(n)$, and this weighted is subject to compatibility of faces (see below). Such attachments of $A_l \in R(n)$ with $dx^i_{k^l_i} \in C_n$ is denoted by $A_l dx^i_{k^l_i}$[1] and is called a basic n-divisor. Thus X is described as a formal sum $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$ equipped with a configuration C_n as above, and if $A_l dx^i_{k^l_i}$ is a basic n-divisor of X then the cell $A_l \in R(n)$ is just a degeneracy $A_l = z(1(q))$ of the unique q-cell $1(q)$ of the cubical site $\mathbb{C}$, and z here denotes a zigzag of degeneracies. In [12] the rectangular n-divisor X are first of all seen as terms of a language which set of variables are objects $1(q)$ of the cubical site $\mathbb{C}$, thus we can replace these variables by many kinds of cubical entities, and such replacement are called *decorations* of X. More precisely we are going to see that any rectangular n-divisor X can be *decorated* by cubical object, like: a fixed cubical set $C \in \mathbb{C}\mathbb{S}\text{ets}$, then such decorations are denoted by $\langle X, C\rangle$, and we use its associated inductive sketch $\mathcal{E}_{\langle X,C\rangle}$ to build the free cubical strict ∞-category with connections $S(C)$; or by a fixed functor:

$$\mathbb{C} \xrightarrow{\quad F \quad} \mathcal{C},$$

where such decoration is denoted by $\langle X, F\rangle$, and we use its associated inductive sketch $\mathcal{E}_{\langle X,F\rangle}$ to define *cubical extension* and *cubical theory* (see 3.2.3), and if $F = Y$ is the Yoneda:

$$\mathbb{C} \xrightarrow{\quad Y \quad} \mathbb{C}\mathbb{S}\text{ets},$$

then the sketches $\mathcal{E}_{\langle X,Y\rangle}$ are going to be the most important ingredients to exhibit the underlying combinatorics of the monad $\mathbb{S} = (S, \lambda, \mu)$: the colimit colim $\mathcal{E}_{\langle X,Y\rangle}$ is a kind of gluing of representables, more precisely this is a gluing of degenerate representables, and by the Yoneda lemma an n-cell $x \in S(C)$ is given by a natural transformation:

$$\text{colim } \mathcal{E}_{\langle X,Y\rangle} \xrightarrow{\quad x \quad} C,$$

[1]These labellings of $A_l \in R(n)$ with the coordinates $dx^i_{k^l_i} \in C_n$ need in fact to be more sophisticated: in [12] we enriched these coordinates with a notion of *basic boxes*, which have the advantage to capture the entire information of the cubical shape of each cells in $R(1)$.

where $X \in \Theta_0$ is here an n-cell in $\bullet$-$\mathbb{R}$ectDiv; thus the action of S on $C \in \mathbb{CS}$ets is described by the formula:

$$S(C) := \coprod_{X \in \Theta_0} \hom_{\mathbb{CS}\text{ets}}(\operatorname{colim} \mathcal{E}_{\langle X,\mathrm{Y}\rangle}, C).$$

To achieve the description of $\mathbb{S} = (S, \lambda, \mu)$ we need to describe its multiplication μ; this is done using another decoration of the $X \in \Theta_0$, by means of a fixed cubical strict ∞-category with connections $C \in \infty$-$\mathbb{CCAT}$, and such decoration is still denoted by $\langle X, C\rangle$. Consider now the full subcategory $\Theta_0[C] \subset \mathbb{CS}$ets, where objects of $\Theta_0[C]$ are these C-decorated rectangular divisors equipped with their cubical set structure; with it we get a functor:

$$\Theta_0[C] \xrightarrow{\mathrm{Subst}_C} (1 \downarrow C),$$

defined on objects as follow: $\mathrm{Subst}_C(\langle X, C\rangle) := \operatorname{colim} \mathcal{E}_{\langle X,C\rangle}$. When $C = \bullet$-$\mathbb{R}$ectDiv we obtain different decorations $\langle X, \bullet\text{-}\mathbb{R}\text{ectDiv}\rangle$ and a functor:

$$\Theta_0[\bullet\text{-}\mathbb{R}\text{ectDiv}] \xrightarrow{\mathrm{Subst}} (1 \downarrow \bullet\text{-}\mathbb{R}\text{ectDiv}) = \Theta_0,$$

which is a key ingredient to build the multiplication μ of the monad $\mathbb{S}$.

Another important aspect of this work is to have a well defined realization of the sketch $\mathcal{E}_X$ in each decorated sketches. We usually prove these realization by induction with the first floor, and more, we shall see that it is enough to deal just with fragments of cocones of the first floor of $\mathcal{E}_{\langle X,Y\rangle}$ or $\mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\text{ectDiv}\rangle}$.

- In a second part 3.2 of our work we provide another application of the cubical Θ_0 followed with main ideas of *weakened cubical algebraic structures* which first ideas were developed in [14, 16]. Let us mention some historical facts related to Section 3, 3.1, and 3.2: in [8] Alexander Grothendieck has proposed an algebraic definition of globular weak ∞-groupoids, as $\mathbb{S}$ets-models of a sketch $\Theta^{\infty}_{W^0}$ named by him *coherator*. In [19] the author showed that a small variation of this coherator $\Theta^{\infty}_{W^0}$ leads to a coherator Θ^{∞}_{W} which $\mathbb{S}$ets-models are globular weak ∞-categories. In [5] it is proved that Batanin and Grothendieck approaches of globular weak ∞-categories are both equivalent. More precisely if we denote by $\mathbb{B}^0$ the globular operad of Batanin such that $\mathbb{B}^0$-algebras are Batanin's globular weak ∞-categories [3], then these $\mathbb{B}^0$-algebras are equivalent to $\mathbb{S}$ets-models of the globular coherator described in [19]. Thus our second main application of the cubical Θ_0 description in [12], is going to mimic the construction in [19] and in [8], but for the cubical paradigm and inspired by our method of weakened cubical algebraic structures in [14, 16]: more precisely we give a precise construction of the cubical coherator Θ^{∞}_{W} which set-models are cubical weak ∞-categories with connections, and we also give a precise construction of the cubical coherator $\Theta^{\infty}_{W^0}$ which set-models are cubical weak ∞-groupoids with connections. The main tools to gain these definitions are inspired by the following works:

1. The globular coherators as in [8, 13, 19];
2. The category Θ_0 of cubical pasting diagrams defined in [12];
3. The cubical contractions defined in [14, 16];
4. The cubical weak ∞-groupoids with connections defined in [15].

In order to grasp some good comprehensions about *coherators* we start Section 3 by defining in 3.1 the globular coherators $\Theta^{\infty}_{\mathbb{M}^m}$ for globular weak (∞, m)-categories ($m \in \mathbb{N}$) and the category $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^m})$ of $\mathbb{S}$ets-models for $\Theta^{\infty}_{\mathbb{M}^m}$. Surprisingly these models of globular weak (∞, m)-categories have never been described before despite their simplicity and despite that some literatures surrounding coherators were already available 20 years ago (see the publications [8, 19]). These models of globular weak (∞, m)-categories permit to recast the *Grothe-ndieck Conjecture for Homotopy Theory* in the wider context of the (∞, m)-categories:

Conjecture 1.1 (Grothendieck's Conjecture for Homotopy Theory)**.** The category $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^m})$ is Quillen equivalent to categories of simplicial models of weak (∞, m)-categories (for all $m \in \mathbb{N}$).

See for example [4] for such existing simplicial models. It is also important to notice that the author had developed other globular algebraic models of weak (∞, m)-categories (for all $m \in \mathbb{N}$) in [10].

2 The monad of cubical strict ∞-categories with connections

Let us recall some terminology from [12]. With a family of languages $\mathbb{L}_n = (\mathcal{L}_n, (\circ^n_j)_{j\in[\![1;n]\!]})$ $(n \in \mathbb{N})$ we exhibit cubical pasting diagrams as elements X of a set n-$\mathbb{R}$ectDiv of $\mathbb{L}_n$-terms (for each $n \in \mathbb{N}$), which lead to a free[2] cubical strict ∞-category (with connections) $\bullet$-$\mathbb{R}$ectDiv on a terminal object $1 \in \mathbb{C}\mathbb{S}$ets. The full subcategory of $\mathbb{C}\mathbb{S}$ets which objects are elements $X \in n$-$\mathbb{R}$ectDiv (for all $n \in \mathbb{N}$) is denoted by Θ_0 and it is the small category of cubical pasting diagrams, which are synonymous in [12] with rectangular divisors. For each cubical pasting diagram $X \in n$-$\mathbb{R}$ectDiv we associate an inductive sketch $\mathcal{E}_X$ and thus we can alternatively see Θ_0 as a small category which objects are such inductive sketches, but also $\bullet$-$\mathbb{R}$ectDiv can be replaced by the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv$'$ which cells are then these inductive sketches. It is important to recall the description of the inductive sketch $\mathcal{E}_X$: if $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$ then its underlying n-configuration is usually written $C_n = dx^i_{k^1_i} + \cdots + dx^i_{k^r_i}$, but its precise definition is just the following finite set of coordinates:

$$C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!].$$

The sketch $\mathcal{E}_X$ associated to X is provided by several sketches $\mathcal{E}_{C_n}$ associated to its n-configuration C_n (see [12]) called the $(\circ_{j_n}, \circ_{j_{n-1}}, \cdots, \circ_{j_1})$-decompositions of C_n where $j_i \in [\![1; n]\!]$ and $j_i \neq j_k$ if $i \neq k$. Thanks to the congruences of associativities and of interchange laws [12], all these sketches provide equivalent formulation of $\mathcal{E}_X$, and in order to simplify the theory we use the $(\circ_n, \circ_{n-1}, \cdots, \circ_1)$-decomposition of C_n which has been accurately described in the beginning of Section 5 in [12]. The sketch $\mathcal{E}_X$ is thus given by the $(\circ_n, \circ_{n-1}, \cdots, \circ_1)$-decomposition of C_n weighted by the basic n-divisors of X. In fine $\mathcal{E}_X$ is described by a *stepped structure of cocones*, that we prefer to call a *floor structure on cocones*, and it is interesting

[2] Thus $\bullet$-$\mathbb{R}$ectDiv $= S(1)$ where S is the underlying endofunctor of the monad $\mathbb{S} = (S, \lambda, \mu)$ acting on the category $\mathbb{C}\mathbb{S}$ets of cubical sets (without degeneracies and connections), which algebras are cubical strict ∞-categories with connections.

to know that such *structured sketches* have been studied under the name *Trames* in [17]; $\mathcal{E}_X$ precisely consists of:

- The $m_2 \cdots m_n$ $\circ_1$-cocones $\mathcal{E}_X^{l_n,\cdots,l_2}$ (first floor cocones), where $(l_2, \cdots, l_n) \in [\![1; m_2]\!] \times \cdots \times [\![1; m_n]\!]$:

$$
\begin{tikzcd}
 & & & X^{l_n,\cdots,l_2} & & & \\
X^{l_n,\cdots,l_2,1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_2,2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & t^n_{n-1,1} = s^n_{n-1,1} \arrow[ul, "t^n_{n-1,1}"] \arrow[ur, "s^n_{n-1,1}"'] & & & & t^n_{n-1,1} = s^n_{n-1,1} \arrow[ul, "t^n_{n-1,1}"] \arrow[ur, "s^n_{n-1,1}"'] &
\end{tikzcd}
$$

  here $t^n_{n-1,1} = s^n_{n-1,1}$ at the bottom left means:

$$t^n_{n-1,1}(X^{l_n,\cdots,l_2,1}) = s^n_{n-1,1}(X^{l_n,\cdots,l_2,2}),$$

  and $t^n_{n-1,1} = s^n_{n-1,1}$ at the bottom right means:

$$t^n_{n-1,1}(X^{l_n,\cdots,l_2,m_1-1}) = s^n_{n-1,1}(X^{l_n,\cdots,l_2,m_1}).$$

- The $m_{j+1} \cdots m_n$ $\circ_j$-cocones $\mathcal{E}_X^{l_n,\cdots,l_{j+1}}$ (j-floor cocones), where $(l_{j+1}, \cdots, l_n) \in [\![1; m_{j+1}]\!] \times \cdots \times [\![1; m_n]\!]$:

$$
\begin{tikzcd}
 & & & X^{l_n,\cdots,l_{j+1}} & & & \\
X^{l_n,\cdots,l_{j+1},1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_{j+1},2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_{j+1},m_j-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_{j+1},m_j} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,j} = \sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] & & & & \tau^n_{n-1,j} = \sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] &
\end{tikzcd}
$$

  here $\tau^n_{n-1,j} = \sigma^n_{n-1,j}$ at the bottom left means:

$$\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},1}) = \sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},2}),$$

and $\tau^n_{n-1,j} = \sigma^n_{n-1,j}$ at the bottom right means:

$$\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j-1}) = \sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j}).$$

- The unique $\circ_n$-cocone of X (n-floor cocone):

$$\begin{tikzcd}
& & & X & & & \\
X^1 \arrow[urrr, dotted] & & X^2 \arrow[ur, dotted] & \cdots & X^{m_n-1} \arrow[ul, dotted] & & X^{m_n} \arrow[ulll, dotted] \\
& \tau^n_{n-1,n} = \sigma^n_{n-1,n} \arrow[ul, "\tau^n_{n-1,n}"] \arrow[ur, "\sigma^n_{n-1,n}"'] & & & & \tau^n_{n-1,n} = \sigma^n_{n-1,n} \arrow[ul, "\tau^n_{n-1,n}"] \arrow[ur, "\sigma^n_{n-1,n}"'] &
\end{tikzcd}$$

here $\tau^n_{n-1,n} = \sigma^n_{n-1,n}$ at the bottom left means:

$$\tau^n_{n-1,n}(X^1) = \sigma^n_{n-1,n}(X^2),$$

and $\tau^n_{n-1,n} = \sigma^n_{n-1,n}$ at the bottom right means:

$$\tau^n_{n-1,n}(X^{m_n-1}) = \sigma^n_{n-1,n}(X^{m_n}).$$

In the sequel, many of our reasonings will appeal to induction starting with the first floor of $\mathcal{E}_X$, and these reasonings shall need only *fragment of cocones*[3] of this first floor, i.e we shall use only sub diagrams of the following shape:

$$\begin{tikzcd}
x & & x' \\
& t^p_{p-1,j}(x) = s^p_{p-1,j}(x') \arrow[ul, "t^p_{p-1,j}"'] \arrow[ur, "s^p_{p-1,j}"] &
\end{tikzcd}$$

[3]These cocones are called *j-gluing datas* in [12].

where x and x' are basic n-divisors of X (because this fragment is taken from the first floor of $\mathcal{E}_X$).

The monad acting on $\mathbb{C}\mathbb{S}$ets which algebras are cubical strict ∞-categories with connections is denoted by $\mathbb{S}=(S,\lambda,\mu)$ and the existence of this monad is proved in [14]. In this section we will accurately describe $\mathbb{S}$ with the cubical Θ_0. For that we need to *decorate* cubical pasting diagrams. Indeed consider a rectangular n-divisor $X=A_1dx^i_{k^1_i}+\cdots+A_rdx^i_{k^r_i}$ and a cubical set $C\in\mathbb{C}\mathbb{S}$ets. In fact each cell $1(q)$ in each basic divisors of X, which are formally degenerate or not, play the role of variable, and thus we can substitute them by cells of C, which lead to the following definition:

Definition 2.1. A *decoration* of X by cells of C is given by a *C-decorated rectangular n-divisor*:

$$\langle X,C\rangle=c_1dx^i_{k^1_i}+\cdots+c_rdx^i_{k^r_i}$$

i.e a filling of X with cells x of C; more precisely we substitute the $1(q)$'s in each basic divisors of X, which are formally degenerate or not, with q-cells x of C, such that for all directions $j\in[\![1,n]\!]$ if $(A_ldx^i_{k^l_i},A_{l'}dx^i_{k^{l'}_i})$ are j-gluing datas for X, i.e are such that $t^n_{n-1,j}(A_ldx^i_{k^l_i})=s^n_{n-1,j}(A_{l'}dx^i_{k^{l'}_i})$ then

$$t^n_{n-1,j}(c_l)=s^n_{n-1,j}(c_{l'}).$$

A warning is important here: for $l\in[\![1;r]\!]$ if $c_ldx^i_{k^l_i}$ is a basic n-divisor (decorated) of $\langle X,C\rangle$, then c_l is not necessarily a cell of the cubical set C, but in general it is a formal degeneracy $c_l=z(x_l)$ (where z is a zigzag of degeneracies, see [12]) of a cell x_l of C, because basic divisors in X are first of all boxes which contain degeneracies of the $1(q)$'s.

Because X is a rectangular n-divisor, the decoration $\langle X,C\rangle$ above is an n-cell of the free cubical strict ∞-category with connections $S(C)$. Here we see that rectangular divisors may be used to give an accurate description of the action of the monad $\mathbb{S}=(S,\lambda,\mu)$ on cubical sets. But there is a more elegant way to use these rectangular divisors for the description of $S(C)$, by means of using gluing of representables[4]; and for this purpose we need to define another type of decoration.

[4]More precisely we will use gluing of degenerate representables.

Definition 2.2. Consider a rectangular n-divisor $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$, a category $\mathcal{C}$, and a functor

$$\mathbb{C} \xrightarrow{\quad F \quad} \mathcal{C}$$

where we denote $F(1(n)) = I^n$, $F(s^n_{n-1,j}) = \mathbf{s^n_{n-1,j}}$, and $F(t^n_{n-1,j}) = \mathbf{t^n_{n-1,j}}$. The *$F$-decorated rectangular n-divisor*

$$\langle X, F\rangle = c_1 dx^i_{k^1_i} + \cdots + c_l dx^i_{k^l_i} + \cdots + c_r dx^i_{k^r_i}$$

is a filling of X by the objects I^q of $\mathcal{C}$, in the sense that in each occurrence of the $1(q)$'s in the boxes of X (degenerate or not), we substitute $1(q)$ by I^q, such that for all directions $j \in [\![1, n]\!]$ if $(A_l dx^i_{k^l_i}, A_{l'} dx^i_{k^{l'}_i})$ are j-gluing datas for X, i.e are such that $t^n_{n-1,j}(A_l dx^i_{k^l_i}) = s^n_{n-1,j}(A_{l'} dx^i_{k^{l'}_i})$ then

$$t^n_{n-1,j}(c_l) = s^n_{n-1,j}(c_{l'}).$$

Remark 2.3. It is easy to see that for each rectangular n-divisor X, corresponds one and only one decoration $\langle X, F\rangle$ for each such functor F.

Remark 2.4. This is important to notice that the expressions $c_l dx^i_{k^l_i}$ are formal degenerate terms build with the objects I^n of $\mathcal{C}$. This kind of terms can be disturbing for logicians because terms for a language must be built with sets and not by objects of a category, but here we will fall back on our feet, because these terms have a sense in any elementary topos, and our main example, the category $\mathbb{C}\mathbb{S}$ets, feats well for this paradigm.

We can associate to the decoration $\langle X, F\rangle$ above, a sketch $\mathcal{E}_{\langle X,F\rangle}$, exactly as in [12]. Now we are going to define *cubical sums* for functors $\mathbb{C} \xrightarrow{F} \mathcal{C}$, as above, and for that we need that the terms $c_l dx^i_{k^l_i}$ and the sketches $\mathcal{E}_{\langle X,F\rangle}$ are well realized in the category $\mathcal{C}$, i.e that these terms and these sketches, have both realizations which are well defined in the category $\mathcal{C}$.

Definition 2.5. Consider a rectangular n-divisor $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$, a category $\mathcal{C}$, a functor $\mathbb{C} \xrightarrow{F} \mathcal{C}$, and the *$F$-decorated rectangular n-divisor*

$$\langle X, F\rangle = c_1 dx^i_{k^1_i} + \cdots + c_l dx^i_{k^l_i} + \cdots + c_r dx^i_{k^r_i}$$

as in 2.2, such that the terms $c_l dx^i_{k^l_i}$ are well realized in $\mathcal{C}$, and the associated sketch $\mathcal{E}_{\langle X,F\rangle}$ of $\langle X,F\rangle$, is well realized in $\mathcal{C}$[5]. If the colimit: colim $\mathcal{E}_{\langle X,F\rangle}$ exists in $\mathcal{C}$, then we say that F has the *cubical sum* associated to the F-decoration $\langle X,F\rangle$, or it has X-*cubical sum* for short. If F has X-cubical sum for all rectangular divisors $X\in\Theta_0$, then we say that F has all cubical sums, or just F is a *cubical extension.*

The functor i :

$$\mathbb{C} \xrightarrow{\quad i \quad} \Theta_0$$

which sends each objects $1(n)$ of $\mathbb{C}$ to the basic box $1(n)dx^i_{k_i}$ is a trivial example of cubical extension. Indeed for the rectangular n-divisor $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$, the i-decorated rectangular divisor $\langle X,i\rangle$ is X itself, and by definition the terms $c_l dx^i_{k^l_i} = A_l dx^i_{k^l_i}$ are objects of Θ_0; also we have $\mathcal{E}_{\langle X,i\rangle} = \mathcal{E}_X$, and indeed the sketch $\mathcal{E}_X$ is by definition a diagram inside Θ_0 (or a subcategory of Θ_0).

The Yoneda embedding Y:

$$\mathbb{C} \xrightarrow{\qquad \mathrm{Y} \qquad} \mathbb{C}\mathbb{S}\mathrm{ets}$$

$$1(n) \longmapsto \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(-,1(n))$$

is a crucial example of cubical extension. Indeed for the rectangular n-divisor $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$, we have the Y-decorated rectangular divisor $\langle X,Y\rangle$:

$$\langle X,\mathrm{Y}\rangle = c_1 dx^i_{k^1_i} + \cdots + c_l dx^i_{k^l_i} + \cdots + c_r dx^i_{k^r_i}$$

where here $c_l dx^i_{k^l_i}$ are decorated basic divisor, and the terms c_l are obtained with $A_l = z(1(q))$ (z is a zigzag of degeneracies and necessarily $q\leq n$) by substituting $1(q)$ with the representable $\mathrm{Y}(1(q)) = \hom_{\mathbb{C}}(-,1(q))$; now we need to show that its associated sketch $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ is well realized in the category $\mathbb{C}\mathbb{S}$ets; we will treat only cocones of first floor of $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ (see Section 6 in [12]) because for the other floors of $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ the arguments are similar; arrows of the

[5] Just below we provide example of well realizations.

cocones in the first floor[6] of $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ have the following shapes (coordinates are unnecessary at this stage):

$$\begin{tikzcd} c_l & & c_l \\ s^p_{p-1,i}(c_l) \arrow[u, "s^p_{p-1,i}"] & & t^p_{p-1,i}(c_l) \arrow[u, "t^p_{p-1,i}"] \end{tikzcd}$$

where $1 \leq p \leq n$ and $1 \leq i \leq p$; these arrows are themselves built with the following arrows of the cocones of the sketch $\mathcal{E}_X$:

$$\begin{tikzcd} A_l & & A_l \\ s^p_{p-1,i}(A_l) \arrow[u, "s^p_{p-1,i}"] & & t^p_{p-1,i}(A_l) \arrow[u, "t^p_{p-1,i}"] \end{tikzcd}$$

where each term c_l is obtained with $A_l = z(1(q))$ (z is a zigzag of degeneracies where necessarily $q \leq p$) by substituting $1(q)$ with the representable $\mathrm{Y}(1(q)) = \hom_{\mathbb{C}}(-,1(q))$; but these terms $c_l = z(\hom_{\mathbb{C}}(-,1(q)))$ are presheaves[7]:

$$\mathbb{C}^{\mathrm{op}} \xrightarrow{z(\hom_{\mathbb{C}}(-,1(q)))} \mathbb{S}\mathrm{ets}$$

defined pointwise as follow

$$\begin{aligned} z(\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(-,1(q)))(1(n)) &= z(\hom_{\mathbb{C}}(1(n),1(q))) \\ &:= \{\text{terms } z(x)/x \in \hom_{\mathbb{C}}(1(n),1(q))\} \end{aligned}$$

i.e $z(\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(-,1(q)))(1(n))$ is a set of terms. Also if

$$1(n) \xrightarrow{f} 1(p)$$

[6]In Section 6 in [12] we built the first floor with $\circ_1$-cocones, i.e they were built along the direction 1; here we treat instead the more general situation where the first floor consists of $\circ_i$-cocones, and this doesn't affect at all our arguments because the choice to use $\circ_1$-cocones in [12] was for simplicity. See the Remark 9 in the end of Section 5 in [12].

[7]These kinds of *formal presheaves*, i.e which are built pointwise with terms for a language, out of other presheaves, are standard in topos theory.

is a morphism in $\mathbb{C}^{\mathrm{op}}$, then $z(\hom_{\mathbb{C}}(-,1(q)))(f)$ is the map in $\mathbb{S}$ets:

$$z(\hom_{\mathbb{C}}(1(n),1(q))) \xrightarrow{z(\hom_{\mathbb{C}}(-,1(q)))(f)} z(\hom_{\mathbb{C}}(1(p),1(q)))$$

and if $z(x) \in z(\hom_{\mathbb{C}}(1(n),1(q)))$ (i.e $x \in \hom_{\mathbb{C}}(1(n),1(q))$), then

$$z(\hom_{\mathbb{C}}(-,1(q)))(f)(z(x)) := z(x \circ f).$$

This shows that the terms c_l $(1 \leq p \leq n)$ are well realized in $\mathbb{CS}$ets.
Now we need to show that the arrows:

$$\begin{tikzcd} c_l & c_l \\ s^p_{p-1,i}(c_l) \arrow[u, "s^p_{p-1,i}"] & t^p_{p-1,i}(c_l) \arrow[u, "t^p_{p-1,i}"] \end{tikzcd}$$

are morphisms in $\mathbb{CS}$ets, i.e are natural transformations; the only interesting cases are when c_l are degenerates (otherwise this is trivial), i.e $c_l = z(\hom_{\mathbb{C}}(-,1(q)))$; but here z can be a zigzag z' of degeneracies of $\hom_{\mathbb{C}}(-,1(q))$ followed by $1^{p-1}_{p,j}$ $(1 \leq j \leq p)$, or z can be a zigzag z' of degeneracies of $\hom_{\mathbb{C}}(-,1(q))$ followed by $1^{p-1,\gamma}_{p,j}$ $(1 \leq j \leq p-1$ and $\gamma \in \{-,+\})$; if we want to have an accurate description of the natural transformations $s^p_{p-1,i}$ and $t^p_{p-1,i}$ above, we need these different possibilities of c_l:

- $c_l = 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))$, where $1 \leq j \leq p$;
- $c_l = 1^{p-1,-}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))$, where $1 \leq j \leq p-1$;
- $c_l = 1^{p-1,+}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))$, where $1 \leq j \leq p-1$.

For example, if $c_l = 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))$, where $1 \leq j \leq p$, then we get:

$$\begin{tikzcd} 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q)))) & 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q)))) \\ s^p_{p-1,i}(1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))) \arrow[u, "s^p_{p-1,i}"] & t^p_{p-1,i}(1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))) \arrow[u, "t^p_{p-1,i}"] \end{tikzcd}$$

but we have the following relations for $1^{p-1}_{p,j}$ (see [14]):

- $s^p_{p-1,i}1^{p-1}_{p,j} = 1^{p-2}_{p-1,j-1}s^{p-1}_{p-2,i}$ and $t^p_{p-1,i}1^{p-1}_{p,j} = 1^{p-2}_{p-1,j-1}t^{p-1}_{p-2,i}$, if $1 \leq i < j \leq p$;
- $s^p_{p-1,i}1^{p-1}_{p,j} = 1^{p-2}_{p-1,j}s^{p-1}_{p-2,i-1}$ and $t^p_{p-1,i}1^{p-1}_{p,j} = 1^{p-2}_{p-1,j}t^{p-1}_{p-2,i-1}$, if $1 \leq j < i \leq p$;
- $s^p_{p-1,j}1^{p-1}_{p,j} = \mathrm{id}(p-1)$ and $t^p_{p-1,j}1^{p-1}_{p,j} = \mathrm{id}(p-1)$.

We treat only the case $1 \leq i < j \leq p$, because the other cases are completely similar; if $1 \leq i < j \leq p$, then thanks to the relations just above, we have got:

$$\begin{tikzcd} 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q)))) \\ 1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(-,1(q))))) \arrow[u, "s^p_{p-1,i}"] \end{tikzcd} \qquad \begin{tikzcd} 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q)))) \\ 1^{p-2}_{p-1,j-1}(t^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(-,1(q))))) \arrow[u, "t^p_{p-1,i}"] \end{tikzcd}$$

For the object $1(n) \in \mathbb{C}^{\mathrm{op}}$ we have the following sets:

$$1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(1(n),1(q)))) = \{\text{terms } 1^{p-1}_{p,j}(z'(x))/x \in \hom_{\mathbb{C}}(1(n),1(q))\},$$

$$\begin{aligned} &1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n),1(q))))) \\ &= \{\text{terms } 1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(x)))/x \in \hom_{\mathbb{C}}(1(n),1(q))\}, \end{aligned}$$

$$\begin{aligned} &1^{p-2}_{p-1,j-1}(t^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n),1(q))))) \\ &= \{\text{terms } 1^{p-2}_{p-1,j-1}(t^{p-1}_{p-2,i}(z'(x)))/x \in \hom_{\mathbb{C}}(1(n),1(q))\}. \end{aligned}$$

Now the natural transformations $s^p_{p-1,i}$ and $t^p_{p-1,i}$ become more transparent: for the object $1(n) \in \mathbb{C}^{\mathrm{op}}$ we have two maps in $\mathbb{S}\mathrm{ets}$:

$$\begin{tikzcd}
1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(1(n),1(q)))) & 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(1(n),1(q)))) \\
1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n),1(q))))) \arrow[u, "s^{p}_{p-1,i}(1(n))"] & 1^{p-2}_{p-1,j-1}(t^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n),1(q))))) \arrow[u, "t^{p}_{p-1,i}(1(n))"]
\end{tikzcd}$$

where

$$s^{p}_{p-1,i}(1(n))(1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(x))))$$

$:= 1^{p-1}_{p,j}(z'(x))$ and $t^{p}_{p-1,i}(1(n))(1^{p-2}_{p-1,j-1}(t^{p-1}_{p-2,i}(z'(x)))) := 1^{p-1}_{p,j}(z'(x))$.

If $1(n) \xrightarrow{f} 1(n')$ is a morphism in $\mathbb{C}^{\mathrm{op}}$ then it is easy to see that we have the expected naturality for $s^{p}_{p-1,i}$:

$$\begin{tikzcd}
1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(1(n),1(q)))) \arrow[r, "1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(f,1(q))))"] & 1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(1(n'),1(q)))) \\
1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n),1(q))))) \arrow[u, "s^{p}_{p-1,i}(1(n))"] \arrow[r, "1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(f,1(q)))))"'] & 1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(1(n'),1(q))))) \arrow[u, "s^{p}_{p-1,i}(1(n'))"']
\end{tikzcd}$$

because

$$\begin{aligned}
1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(f,1(q))))(s^{p}_{p-1,i}(1(n))(1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(x))))) = \\
1^{p-1}_{p,j}(z'(\hom_{\mathbb{C}}(f,1(q))))(1^{p-1}_{p,j}(z'(x))) = 1^{p-1}_{p,j}(z'(x\circ f)),
\end{aligned}$$

and also

$$\begin{aligned}
s^{p}_{p-1,i}(1(n'))(1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(\hom_{\mathbb{C}}(f,1(q)))))(1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(x)))))) \\
= s^{p}_{p-1,i}(1(n'))(1^{p-2}_{p-1,j-1}(s^{p-1}_{p-2,i}(z'(x\circ f))) = 1^{p-1}_{p,j}(z'(x\circ f)).
\end{aligned}$$

Of course, the naturality of $t^{p}_{p-1,i}$ is proven similarly, and for the cases $c_l = 1^{p-1,\gamma}_{p,j}(z'(\hom_{\mathbb{C}}(-,1(q))))$ where $1 \leq j \leq p-1$ and $\gamma \in \{-,+\}$, we use similar arguments (the relations expressing faces of $1_{p,j}^{p-1,\gamma}$ are in [14]), thus we have proved that the sketch $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ is well realized in $\mathbb{C}\mathbb{S}\mathrm{ets}$ for any rectangular divisor $X \in \Theta_0$.

Now the Yoneda embedding:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad \mathrm{Y} \quad} & \mathbb{C}\mathbb{S}\mathrm{ets} \\ 1(n) & \longmapsto & \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(-,1(n)) \end{array}$$

is a cubical extension because colim $\mathcal{E}_{\langle X,Y\rangle}$ exists in $\mathbb{C}\mathbb{S}\mathrm{ets}$ for any rectangular divisors $X \in \Theta_0$, just because $\mathbb{C}\mathbb{S}\mathrm{ets}$ is cocomplete. An interesting fact here is that the faces of the cubical set colim $\mathcal{E}_{\langle X,Y\rangle}$, thanks to the formalism of the objects of Θ_0, have a direct description: for all directions $j \in [\![1,n]\!]$, $s_{n-1,j}^{n}(\text{colim } \mathcal{E}_{\langle X,\mathrm{Y}\rangle}) = \text{colim } \mathcal{E}_{\langle \sigma_{n-1,j}^{n}(X),\mathrm{Y}\rangle}$, $t_{n-1,j}^{n}(\text{colim } \mathcal{E}_{\langle X,\mathrm{Y}\rangle}) = \text{colim } \mathcal{E}_{\langle \tau_{n-1,j}^{n}(X),\mathrm{Y}\rangle}$, and any zigzag of sources-targets of colim $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$ is equal to the colimit colim $\mathcal{E}_{\langle x,\mathrm{Y}\rangle}$ where x is the face of X obtained by this zigzag.

Now let us define the category $\mathbb{C}$-$\mathbb{E}\mathrm{xt}$ of cubical extensions:

A morphism in $\mathbb{C}$-$\mathbb{E}\mathrm{xt}$ is given by a commutative triangle

$$\begin{tikzcd} & \mathcal{C} \arrow[dd, "f"] \\ \mathbb{C} \arrow[ur, "F"] \arrow[dr, "F'"'] & \\ & \mathcal{C}' \end{tikzcd}$$

such that the functor f preserves cubical sums, i.e we have $f(\text{colim } \mathcal{E}_{\langle X,F\rangle}) = \text{colim } \mathcal{E}_{\langle X,F'\rangle}$ for each F-decoration $\langle X,F\rangle$ of each rectangular divisor X. It is easy to see that the functor i :

$$\mathbb{C} \xrightarrow{\quad i \quad} \Theta_0$$

which sends each objects $1(n)$ of $\mathbb{C}$ to the basic box $1(n)dx_{k_i}^{i}$ is an initial object of $\mathbb{C}$-$\mathbb{E}\mathrm{xt}$, such that the unique map is given by the functor:

$$\begin{tikzcd} & \Theta_0 \arrow[dd, "\mathrm{colim}\ \mathcal{E}_{\langle -,F\rangle}"] \\ \mathbb{C} \arrow[ur, "i"] \arrow[dr, "F"'] & \\ & \mathcal{C} \end{tikzcd}$$

and thus the small category Θ_0 inherits a universal property. In particular we have the unique map colim $\mathcal{E}_{\langle -,Y\rangle}$:

$$\begin{tikzcd} & \Theta_0 \arrow[dd, "\mathrm{colim}\ \mathcal{E}_{\langle -,Y\rangle}"] \\ \mathbb{C} \arrow[ur, "i"] \arrow[dr, "Y"'] & \\ & \mathbb{CS}\mathrm{ets} \end{tikzcd}$$

where colim $\mathcal{E}_{\langle -,Y\rangle}$ is also the left Kan extension of Y along the functor i. Thus our formalism shows how to compute this left Kan extension $\mathrm{Lan}_i(Y)$ by using sketches for objects in Θ_0.

Now we are ready to describe the monad $\mathbb{S} = (S, \lambda, \mu)$ of cubical strict ∞-categories with connections: as we wrote in [14] the forgetful functor:

$$\infty\text{-}\mathbb{CCAT} \xrightarrow{\quad U \quad} [\mathbb{C}^{op}, \mathbb{S}\mathrm{ets}]\,,$$

which sends cubical strict ∞-categories with connections to cubical sets is right adjoint and its induced monad is written $\mathbb{S} = (S, \lambda, \mu)$ where $1_{\mathbb{CS}\mathrm{ets}} \xrightarrow{\ \lambda\ } S$ is its unit and $S^2 \xrightarrow{\ \mu\ } S$ is its multiplication.

If $C \in \mathbb{CS}\mathrm{ets}$ is a cubical set, then we put:

$$S(C) := \coprod_{X\in\Theta_0} \mathrm{colim}\ \mathcal{E}_{\langle X,C\rangle} = \coprod_{X\in\Theta_0} \hom_{\mathbb{CS}\mathrm{ets}}(\mathrm{colim}\ \mathcal{E}_{\langle X,\mathrm{Y}\rangle}, C).$$

Remark 2.6. In [18] Tom Leinster has defined before, a similar description of the monad on globular sets which algebras are globular strict ∞-categories.

This description shows immediately that S preserved fiber products and $S(C)$ is a cubical strict ∞-categories with connections.

The unit λ of $\mathbb{S}$ is the map:

$$C \xrightarrow{\quad\lambda\quad} S(C),$$

such that if $c \in C(1(n))$, then $\lambda(1(n))(c) := cdx^i_{k_i}$, where here $dx^i_{k_i}$ is just the non-degenerate basic n-divisor (see [12]).

The multiplication μ of $\mathbb{S}$:

$$S^2(C) \xrightarrow{\quad\mu\quad} S(C),$$

is more subtle and in order to describe it we need first to define a kind of decoration for cubical strict ∞-categories (with connections).

Definition 2.7. Consider an object $C \in \infty\text{-}\mathbb{CCAT}$ and a rectangular n-divisor $X = A_1 dx^i_{k^1_i} + \cdots + A_r dx^i_{k^r_i}$. A *decoration* of X by cells of C is given by a *C-decorated rectangular n-divisor*:

$$\langle X, C\rangle = c_1 dx^i_{k^1_i} + \cdots + c_r dx^i_{k^r_i},$$

i.e a replacement of each $A_l dx^i_{k^l_i}$ in X by $c_l dx^i_{k^l_i}$ where c_l is an n-cell in $C(n)$ ($l \in [\![1;r]\!]$), such that for all directions $j \in [\![1,n]\!]$ if $(A_l dx^i_{k^l_i}, A_{l'} dx^i_{k^{l'}_i})$ are j-gluing datas for X, then

$$t^n_{n-1,j}(c_l) = s^n_{n-1,j}(c_{l'}).$$

Here we must pay attention about this decoration: this decoration must be seen as a decoration of X by the underlying cubical set of C, i.e in each basic n-divisor $A_l dx^i_{k^l_i}$ in X such that $A_l = z(1(q))$ is a zigzag of degeneracies of the variable $1(q)$, we replace $1(q)$ with a q-cell $x \in C(q)$; the resulting decorated basic n-divisor $c_l dx^i_{k^l_i} = z(x) dx^i_{k^l_i}$ is such that $c_l = z(x) \in C(n)$ because we use the degeneracies z of C[8] to produce the n-cell $c_l \in C(n)$.

Of course they are a multitude of C-decorated rectangular n-divisors $\langle X, C\rangle$ for each X; each $\langle X, C\rangle$ can be thought in the following way: each

[8] The z of the expression $A_l = z(1(q))$ are formal degeneracies and the realization of z in C is still denoted by z.

coordinate of the underlying rectangular configuration of X is weighted by (or inhabited by) an n-cell of C, such that all pair (x, y) of cells in it which are j-adjacents, are such that $t^n_{n-1,j}(x) = s^n_{n-1,j}(y)$ in C; also each $\langle X, C\rangle$ is (as X) an n-cube (see [12]); thus each $\langle X, C\rangle$ is an n-cubical set such that its faces are themselves C-decorated rectangular p-divisors, where $0 \leq p \leq n$; consider now the full subcategory $\Theta_0[C] \subset \mathbb{C}\mathbb{S}\mathrm{ets}$, where objects of $\Theta_0[C]$ are these C-decorated rectangular divisors equipped with their cubical set structure. We shall use this construction just below when $C = \bullet\text{-}\mathbb{R}\mathrm{ectDiv}$, the cubical strict ∞-category of cubical pasting diagrams (described in [12]). Also for any C-decorated rectangular n-divisors $\langle X, C\rangle$ we can associate its sketch $\mathcal{E}_{\langle X,C\rangle}$ (as in [12]), where fragments of cocones of its first floor are as follow

$$\begin{tikzcd} c_l & & c_{l'} \\ & t^p_{p-1,j}(c_l) = s^p_{p-1,j}(c_{l'}) \arrow[ul, "t^p_{p-1,j}"] \arrow[ur, "s^p_{p-1,j}"'] & \end{tikzcd}$$

It is enough to work with these fragments to show that the first floor of the sketch $\mathcal{E}_{\langle X,C\rangle}$ is well realized, and then by a straightforward induction it shows that the entire sketch $\mathcal{E}_{\langle X,C\rangle}$ is indeed well realized. Here c_l and $c_{l'}$ are p-cells in C such that $t^p_{p-1,j}(c_l) = s^p_{p-1,j}(c_{l'})$, and the realization of this cocone is then just the composition:

$$c_l \circ^p_{p-1,j} c_{l'},$$

with respect to the structure of cubical strict ∞-category with connections on C. As it is written in Section 5 in [12], the decoration $\langle X, C\rangle$ is called in [7] page 350 a *multi-dimensional arrays*, and in [7] they did not provide a procedure to compute the n-cell colim $\mathcal{E}_{\langle X,C\rangle} \in C(n)$. The sketch $\mathcal{E}_{\langle X,C\rangle}$ comes from the sketch $\mathcal{E}_X$ which itself is described in [12] with the help of the sketch $\mathcal{E}_{C_n}$ of the underlying n-configuration C_n of X, and this accurate description of $\mathcal{E}_{\langle X,C\rangle}$ permits a procedure to compute this n-cell colim $\mathcal{E}_{\langle X,C\rangle}$. Thus we have the functor:

$$\Theta_0[C] \xrightarrow{\mathrm{Subst}_C} (1 \downarrow C),$$

defined on objects as follow: $\mathrm{Subst}_C(\langle X,C\rangle) := \mathrm{colim}\ \mathcal{E}_{\langle X,C\rangle}$. Now we consider the important case $C = \bullet\text{-}\mathbb{R}\mathrm{ectDiv}$, thus a decoration of X by cells of $\bullet\text{-}\mathbb{R}\mathrm{ectDiv}$, i.e we start with a $\bullet\text{-}\mathbb{R}\mathrm{ectDiv}$-decorated rectangular n-divisor

$$\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle = X_1 dx^i_{k^1_i} + \cdots + X_l dx^i_{k^l_i} + \cdots + X_r dx^i_{k^r_i},$$

i.e a filling of X with cells X_l of n-$\mathbb{R}\mathrm{ectDiv}$ such that for all directions $j \in [\![1,n]\!]$ if $(A_l dx^i_{k^l_i}, A_{l'} dx^i_{k^{l'}_i})$ are j-gluing datas for X, then

$$\tau^n_{n-1,j}(X_l dx^i_{k^l_i}) = \sigma^n_{n-1,j}(X_{l'} dx^i_{k^{l'}_i}).$$

Here sources $\sigma^n_{n-1,j}$ and targets $\tau^n_{n-1,j}$ are the sources and the targets for rectangular divisors.

From $\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle \in \Theta_0[\bullet\text{-}\mathbb{R}\mathrm{ectDiv}]$ (see just above) we obtain then a new rectangular n-divisor $X(X_1,\cdots,X_r) \in \Theta_0$ that we obtain by reindexing all coordinates of basic divisors inside each rectangular n-divisors X_l of it. This construction

$$\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle \longmapsto X(X_1,\cdots,X_r),$$

produces a functor:

$$\Theta_0[\bullet\text{-}\mathbb{R}\mathrm{ectDiv}] \xrightarrow{\mathrm{Subst}} \Theta_0,$$

called functor of *substitution*. Here $\Theta_0 = (1 \downarrow \bullet\text{-}\mathbb{R}\mathrm{ectDiv})$, and as we wrote above this functor is precisely defined on objects by

$$\mathrm{Subst}(\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle) = \mathrm{colim}\ \mathcal{E}_{\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}.$$

Indeed, consider a fragment of a cocone of the first floor in $\mathcal{E}_X$:

$$\begin{tikzcd}
A_l & & A_{l'} \\
& t^p_{p-1,j}(A_l) = s^p_{p-1,j}(A_{l'}) \arrow[ul, "t^p_{p-1,j}"] \arrow[ur, "s^p_{p-1,j}"'] &
\end{tikzcd}$$

it is realized to the following fragment of a cocone of the first floor of the sketch $\mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}$:

$$\begin{tikzcd}
X_l & & X_{l'} \\
& \tau^p_{p-1,j}(X_l)=\sigma^p_{p-1,j}(X_{l'}) \arrow[ul, "\tau^p_{p-1,j}"] \arrow[ur, "\sigma^p_{p-1,j}"'] &
\end{tikzcd}$$

Here X_l and $X_{l'}$ are p-cells in $\bullet$-$\mathbb{R}$ectDiv such that $\tau^p_{p-1,j}(X_l)=\sigma^p_{p-1,j}(X_{l'})$, and the realization of this cocone is then just the composition: $X_l\circ^p_{p-1,j}X_{l'}$ with respect to the structure of cubical strict ∞-category with connections on $\bullet$-$\mathbb{R}$ectDiv. Thus the functor Subst is first of all a colimit functor, in the sense that Subst sends each $\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle\in\Theta_0[\bullet\text{-}\mathbb{R}\mathrm{ectDiv}]$ to the colimit of the realized sketch $\mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}$:

$$\mathrm{Subst}(\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle)=X(X_1,\cdots,X_r)=\mathrm{colim}\ \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},$$

but Subst can also be seen as a re-indexation of coordinates for basic divisors inside each X_l in $\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle$.

Remark 2.8. It is interesting to notice that this realization transforms the maps s and t of the first floor of $\mathcal{E}_X$ to the maps σ and τ of the first floor of $\mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}$. Here we see the interplay between internal sources and internal targets of individual rectangular n-divisor X with sources and targets of the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv which contains X as object.

The multiplication μ of $\mathbb{S}$:

$$S^2(C)\xrightarrow{\quad\mu\quad}S(C),$$

is then given by

$$S^2(C)=\coprod_{X\in\Theta_0}\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(\mathrm{colim}\ \mathcal{E}_{\langle X,\mathrm{Y}\rangle},S(C))\xrightarrow{\mu(C)}\coprod_{X\in\Theta_0}\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(\mathrm{colim}\ \mathcal{E}_{\langle X,\mathrm{Y}\rangle},C),$$

where $S^2(C)$ is unpacked as follow

$$S^2(C)=\coprod_{X\in\Theta_0}\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}\Big(\mathrm{colim}\ \mathcal{E}_{\langle X,\mathrm{Y}\rangle},\coprod_{X\in\Theta_0}\hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(\mathrm{colim}\ \mathcal{E}_{\langle X,\mathrm{Y}\rangle},C)\Big).$$

Thus an element $t \in S^2(C)(1(n))$ is described by a natural transformation:

$$\text{colim } \mathcal{E}_{\langle X,\mathrm{Y}\rangle} \xrightarrow{\quad t \quad} \coprod_{X\in\Theta_0} \hom_{\mathbb{CS}\mathrm{ets}}(\text{colim } \mathcal{E}_{\langle X,\mathrm{Y}\rangle}, C),$$

where X is a rectangular n-divisor, and thus t can be presented with a decoration X by C-decorated[9] rectangular n-divisors $\langle X_l, C\rangle$, and such decoration is written

$$t = \langle X, S(C)\rangle = \langle X_1, C\rangle dx^i_{k^1_i} + \cdots + \langle X_l, C\rangle dx^i_{k^l_i} + \cdots + \langle X_r, C\rangle dx^i_{k^r_i},$$

which itself underlies the $\bullet$-$\mathbb{R}$ectDiv-decorated rectangular n-divisor

$$\langle X, \bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle = X_1 dx^i_{k^1_i} + \cdots + X_l dx^i_{k^l_i} + \cdots + X_r dx^i_{k^r_i}.$$

Also the elements $\langle X_l, C\rangle \in S(C)(1(n))$ are described by natural transformations:

$$\text{colim } \mathcal{E}_{\langle X_l,\mathrm{Y}\rangle} \xrightarrow{\quad t_l \quad} C,$$

and thus the action of $\mu(C)$ on the n-cell $t \in S^2(C)(1(n))$ are written

$$\mu(C)(1(n))(t) = t(t_1, \cdots, t_r) \in S(C)(1(n)).$$

As we saw above, the rectangular n-divisor $X(X_1, \cdots, X_r)$ is equal to

$$\text{colim } \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},$$

thus its associated sketch

$$\mathcal{E}_{\text{colim } \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}},$$

is the colimits (in the category $\mathbb{S}$ketch of sketches) of the sketches $\mathcal{E}_{X_l}$, or in other words the sketch $\mathcal{E}_{\text{colim } \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle}}$ is the gluing of the sketches $\mathcal{E}_{X_l}$ along the sketch $\mathcal{E}_X$. Thus the induced sketch

$$\mathcal{E}_{\langle \text{colim } \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},\mathrm{Y}\rangle},$$

[9] This is a decoration by cubical sets as in 2.1

is also the colimits of the sketches $\mathcal{E}_{\langle X_l,Y\rangle}$, or we could say that it is the gluing of the sketches $\mathcal{E}_{\langle X_l,Y\rangle}$ along the sketch $\mathcal{E}_{\langle X,Y\rangle}$. Also this colimit cocone in the category $\mathbb{S}$ketch of sketches which base is built with the sketches $\mathcal{E}_{\langle X_l,Y\rangle}$ and which colimit is the sketch $\mathcal{E}_{\langle \mathrm{colim}\ \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},\mathrm{Y}\rangle}$, is a diagram of representables[10] (they are objects in $\mathbb{CS}$ets), and thus we can take the colimit in $\mathbb{CS}$ets, on each vertices of this colimit cocone: the resulting diagram is again a colimit diagram, and takes place in the category $\mathbb{CS}$ets, where each vertices of this diagram are now gluing of representables: colim $\mathcal{E}_{\langle X_l,\mathrm{Y}\rangle}$ form the bases of this diagram and colim $\mathcal{E}_{\langle \mathrm{colim}\ \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},\mathrm{Y}\rangle}$ is the colimit of this diagram. The arrows of this cocone colimit in $\mathbb{CS}$ets are denoted by

$$\mathrm{colim}\ \mathcal{E}_{\langle X_l,\mathrm{Y}\rangle} \xrightarrow{\quad q_l \quad} \mathrm{colim}\ \mathcal{E}_{\langle \mathrm{colim}\ \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},\mathrm{Y}\rangle}.$$

Now we are ready to give a description of $\mu(C)$:

$$S^2(C) \xrightarrow{\quad \mu(C) \quad} S(C).$$

We saw that $t \in S^2(C)(1(n))$ just above is given by the natural transformations:

$$\mathrm{colim}\ \mathcal{E}_{\langle X_l,\mathrm{Y}\rangle} \xrightarrow{\quad t_l \quad} C,$$

which fill X, i.e $t \in S^2(C)(1(n))$ has the following presentation:

$$t = \langle X, S(C)\rangle = t_1 dx^i_{k^1_i} + \cdots + t_l dx^i_{k^l_i} + \cdots + t_r dx^i_{k^r_i}.$$

But by the universality of colim $\mathcal{E}_{\langle \mathrm{colim}\ \mathcal{E}_{\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle},\mathrm{Y}\rangle}$ we get a unique natural transformation:

$$\mathrm{colim}\ \mathcal{E}_{\langle \mathrm{Subst}(\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle),\mathrm{Y}\rangle} \xrightarrow{\quad t(t_1,\cdots,t_r) \quad} C,$$

such that for all $l \in [\![1,r]\!]$ we have the commutativity

$$\begin{tikzcd}
\mathrm{colim}\ \mathcal{E}_{\langle X_l,\mathrm{Y}\rangle} \arrow[r, "t_l"] \arrow[d, "q_l"'] & C \\
\mathrm{colim}\ \mathcal{E}_{\langle \mathrm{Subst}(\langle X,\bullet\text{-}\mathbb{R}\mathrm{ectDiv}\rangle),\mathrm{Y}\rangle} \arrow[ur, "t(t_1,\cdots,t_r)"'] &
\end{tikzcd}$$

[10]We call it representables but in fact they are degeneracies of it; see above when we tried to describe arrows of the cocones in $\mathcal{E}_{\langle X,\mathrm{Y}\rangle}$.

and by definition we have $\mu(C)(1(n))(t) := t(t_1, \cdots, t_r)$.

Theorem 2.9. *The monad $\mathbb{S} = (S, \lambda, \mu)$ acting on $\mathbb{CS}$ets which algebras are cubical strict ∞-categories with connections (described in [14, 16]) is cartesian.*

Proof. The description of the monad $\mathbb{S} = (S, \lambda, \mu)$ above shows that its underlying endofunctor S preserves fibred products. We are going to prove that the multiplication μ is cartesian, i.e we are going to prove that if $C \in \mathbb{CS}$ets is a cubical set then the commutative diagram

$$\begin{tikzcd} S^2(C) \arrow[r, "S^2(!)"] \arrow[d, "\mu(C)"'] & S^2(1) \arrow[d, "\mu(1)"] \\ S(C) \arrow[r, "S(!)"'] & S(1) \end{tikzcd}$$

is a cartesian square. Consider the commutative diagram in $\mathbb{CS}$ets:

$$\begin{tikzcd} C' \arrow[r, "g"] \arrow[d, "f"'] & S^2(1) \arrow[d, "\mu(1)"] \\ S(C) \arrow[r, "S(!)"'] & S(1) \end{tikzcd}$$

If x is an n-cell of C' then $f(x) \in S(C)(1(n))$ is given by a natural transformation:

$$\operatorname{colim} \mathcal{E}_{\langle X', Y\rangle} \xrightarrow{f(x)} C,$$

where X' is a rectangular n-divisor; also $S(!)(1(n))(f(x)) \in S(1)(1(n))$ is given by the composition:

$$\operatorname{colim} \mathcal{E}_{\langle X', Y\rangle} \xrightarrow{f(x)} C \xrightarrow{!} 1,$$

i.e $S(!)(1(n))(f(x))$ is the unique map

$$\operatorname{colim} \mathcal{E}_{\langle X', Y\rangle} \xrightarrow{!} 1.$$

Also $g(x) \in S^2(1)(1(n))$ is given by the family of natural transformations:

$$\text{colim } \mathcal{E}_{\langle X_i, Y\rangle} \xrightarrow{\quad ! \quad} 1,$$

for $i \in [\![1;r]\!]$, and where X_i (for all $i \in [\![1;r]\!]$) are rectangular n-divisors which decorate another rectangular n-divisor X, where this decoration is given by a natural transformation:

$$\text{colim } \mathcal{E}_{\langle X, Y\rangle} \xrightarrow{\quad t \quad} S(C),$$

and thus $\mu(1)(1(n))(g(x))$ is given by the unique map $t(!, \cdots, !)$:

$$\begin{tikzcd}
\text{colim } \mathcal{E}_{\langle X_i, \mathrm{Y}\rangle} \arrow[r, "!"] \arrow[d, "q_i"'] & 1 \\
\text{colim } \mathcal{E}_{X(X_1, \cdots, X_r), \mathrm{Y}\rangle} \arrow[ur, "{t(!, \cdots, !)}"'] &
\end{tikzcd}$$

such that for all $i \in [\![1;r]\!]$, $t(!, \cdots, !) \circ q_l =!$, which means that $\mu(1)(1(n))(g(x)) = t(!, \cdots, !)$ is just the unique map:

$$\text{colim } \mathcal{E}_{\langle X(X_1, \cdots, X_r), Y\rangle} \xrightarrow{\quad ! \quad} 1.$$

But by the hypothesis $S(!)(1(n))(f(x)) = \mu(1)(1(n))(g(x))$, thus $X' = X(X_1, \cdots, X_r)$, and thus we can rewrite $f(x)$ as follow

$$\text{colim } \mathcal{E}_{\langle X(X_1, \cdots, X_r), Y\rangle} \xrightarrow{\quad f(x) \quad} C.$$

Now consider the morphism l in the diagram

$$\begin{tikzcd}
C' \arrow[dr, dotted, "l"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& S^2(C) \arrow[r, "S^2(!)"] \arrow[d, "\mu(C)"'] & S^2(1) \arrow[d, "\mu(1)"] \\
& S(C) \arrow[r, "S(!)"'] & S(1)
\end{tikzcd}$$

defined as follow: for $x \in C'(1(n))$, $l(x) \in S^2(C)(1(n))$ is given by the following family of natural transformations:

$$\text{colim } \mathcal{E}_{\langle X_i,Y\rangle} \xrightarrow{f(x)\circ q_i} C,$$

for $i \in [\![1;r]\!]$, where this family decorate a rectangular n-divisor X by the hypothesis, and where this decoration were given by the natural transformation:

$$\text{colim } \mathcal{E}_{\langle X,Y\rangle} \xrightarrow{t} S(C),$$

then $\mu(C)(1(n))(l(x)) = t(f(x)\circ q_1, \cdots, f(x)\circ q_r)$ is the unique natural transformation such that for all $i \in [\![1;r]\!]$ the following diagrams are commutative

$$\begin{tikzcd}
\text{colim } \mathcal{E}_{\langle X_i,Y\rangle} \arrow[r, "f(x)\circ q_i"] \arrow[d, "q_i"'] & 1 \\
\text{colim } \mathcal{E}_{X(X_1,\cdots,X_r),Y\rangle} \arrow[ur, "t(f(x)\circ q_1,\cdots,f(x)\circ q_r)"'] &
\end{tikzcd}$$

which shows by unicity that $\mu(C)(1(n))(l(x)) = f(x)$; also $S^2(!)(1(n))(l(x)) \in S^2(1)(1(n))$ is given by the following compositions of natural transformations:

$$\text{colim } \mathcal{E}_{\langle X_i,Y\rangle} \xrightarrow{f(x)\circ q_i} C \xrightarrow{!} 1,$$

and for all $i \in [\![1;r]\!]$, $!\circ f(x)\circ q_i =!$, thus $S^2(!)(1(n))(l(x)) = g(x)$; the unicity of l is evident.

The cartesianity of the unit:

$$C \xrightarrow{\lambda} S(C),$$

is easier; we need to prove that the following diagram is cartesian.

$$\begin{tikzcd}
C \arrow[r, "!"] \arrow[d, "\lambda(C)"'] & 1 \arrow[d, "\lambda(1)"] \\
S(C) \arrow[r, "S(!)"'] & S(1)
\end{tikzcd}$$

We start with a commutative diagram in $\mathbb{C}\mathbb{S}$ets:

$$\begin{tikzcd}
C' \arrow[r, "!"] \arrow[d, "f"'] & 1 \arrow[d, "\lambda(1)"] \\
S(C) \arrow[r, "S(!)"'] & S(1)
\end{tikzcd}$$

Let x be an n-cell of C'. Then $f(x) \in S(C)(1(n))$ is given by a natural transformation:

$$\operatorname{colim} \mathcal{E}_{\langle X',Y\rangle} \xrightarrow{f(x)} C,$$

for a rectangular n-divisor X', and thus $S(!)(1(n))(f(x)) \in S(1)(1(n))$ is the unique natural transformation:

$$\operatorname{colim} \mathcal{E}_{\langle X',Y\rangle} \xrightarrow{f(x)} C \xrightarrow{!} 1,$$

given by $! \circ f(x) = !$; but by the hypothesis we have

$$\begin{aligned}
S(!)(1(n))(f(x)) &= \lambda(1)(1(n))(!(1(n)(x))) \\
&= \lambda(1)(1(n))(1(n)) \\
&= 1(n)dx^{i}_{k_i},
\end{aligned}$$

by the definition of λ, this mean that $X' = 1(n)dx^{i}_{k_i}$ is just the basic non-degenerate n-divisor (see [12]); thus $f(x)$ may be written $f(x) = f_x dx^{i}_{k_i}$, where f_x is an n-cell of C; and thus we put $l(x) := f_x$, and the unicity of l becomes trivial.

$$\begin{tikzcd}
C' \arrow[dr, dotted, "l"] \arrow[drr, "!"] \arrow[ddr, "f"'] & & \\
& C \arrow[r, "!"] \arrow[d, "\lambda(C)"'] & 1 \arrow[d, "\lambda(1)"] \\
& S(C) \arrow[r, "S(!)"'] & S(1)
\end{tikzcd}$$

□

With this theorem we solved the conjecture in [16] for the monad of cubical strict ∞-categories with connections which provides a complete description of the cubical operad $\mathbb{B}^0_C$ of cubical weak ∞-categories with connections.

Proposition 2.10. *The monad* $\mathbb{S}=(S,\lambda,\mu)$ *acting on* $\mathbb{CS}ets$ *which algebras are cubical strict* ∞*-categories* (*without connections*) *is cartesian.*

Proof. This is easy. Here, we just use the previous proof using only rectangular divisors build with classical degeneracies $1^n_{n+1,j}$ ($n\in\mathbb{N}$ and $j\in [\![1,n+1]\!]$) and their associated rectangular sketches. □

With this proposition we can easily use the materials in [16] to build the cubical operad $\mathbb{B}^0_C$ of cubical weak ∞-categories without connections. In particular it is interesting to know that $\mathbb{B}^0_C$-algebras of dimensions 2 are exactly *double categories of Verity* [21]. Indeed, in [3] Michael Batanin proved that with globular operads, $\mathbb{B}^0_C$-algebras of dimensions 2 are exactly bicategories, and we can adapt this proof for this cubical paradigm.

3 The cubical coherators

Coherators have been initiated by Alexander Grothendieck [8] in order to define properly globular weak ∞-groupoids. A coherator $\Theta^{\infty}_{\mathbb{M}^0}$ for globular weak ∞-groupoids is a theory in the sense of [6] such that $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^0})$ is the category of globular weak ∞-groupoids. These theories are kinds of generalization of Lawvere theories and are powerful enough and simple to capture many higher structures. For example a slight modification of the definition of the globular coherator $\Theta^{\infty}_{\mathbb{M}^0}$ (see [19]) leads to the definition of an other globular coherator $\Theta^{\infty}_{\mathbb{M}}$ which set-models are globular weak ∞-categories, and such models are thus called *Grothendieck's globular weak* ∞*-categories.* In [1] it is conjectured that these models are equivalent to Batanin's globular weak ∞-categories [3], and this conjecture has been proved in [5].

3.1 The globular coherators $\Theta^{\infty}_{\mathbb{M}^m}$ for globular weak (∞,m)-categories ($m\in\mathbb{N}$) In order to have a first good smell of the simplicity of this Grothendieck's approach for *weakened algebraic structures*, we first describe globular models of weak (∞,m)-categories ($m\in\mathbb{N}$). Thus for each $m\in\mathbb{N}$ we build a globular coherator $\Theta^{\infty}_{\mathbb{M}^m}$ which set-models are globular weak (∞,m)-categories. The author believes these models of (∞,m)-categories (for all $m\in\mathbb{N}$) are probably the simplest one encounter in the literature (see for example [4, 10]).

3.1.1 Globular magmatic structures

Consider the small category $\mathbb{G}$ with objects $1(n)$ for all $n\in\mathbb{N}$, with morphisms those generated for all $n\in\mathbb{N}$ by the *cosources* $1(n-1)\xrightarrow{s^n_{n-1}}1(n)$ and the *cotargets* $1(n-1)\xrightarrow{t^n_{n-1}}1(n)$, which satisfy the following coglobular relations:

(i) $s^n_{n-1}\circ s^{n+1}_n=t^n_{n-1}\circ s^{n+1}_n$,

(ii) $s^n_{n-1}\circ t^{n+1}_n=t^n_{n-1}\circ t^{n+1}_n$.

The small category $\mathbb{G}$ is called the *globe category* and we may represent it schematically with the following diagram

$$1(0)\overset{s^1_0}{\underset{t^1_0}{\rightrightarrows}}1(1)\overset{s^2_1}{\underset{t^2_1}{\rightrightarrows}}1(2)\overset{s^3_2}{\underset{t^3_2}{\rightrightarrows}}1(3)\cdots1(n-1)\overset{s^n_{n-1}}{\underset{t^n_{n-1}}{\rightrightarrows}}1(n)\cdots$$

Definition 3.1. Globular sets are presheaves on $\mathbb{G}^{\mathrm{op}}$. The category of globular sets is denoted by $\mathbb{G}\mathrm{lob}$.

A globular ∞-magma M is given by a globular set $\mathbb{G}^{\mathrm{op}}\xrightarrow{M}\mathbb{S}\mathrm{ets}$ equipped with operations $M_n\times_{M_p}M_n\xrightarrow{\circ^n_p}M_n$ for all $n\geq1$ and all $0\leq p\leq n-1$ such that

- for $0\leq p<q<m$, $s^m_q(y\circ^m_p x)=s^m_q(y)\circ^q_p s^m_q(x)$ and $t^m_q(y\circ^m_p x)=t^m_q(y)\circ^q_p t^m_q(x)$;
- for $0\leq q<p<m$, $s^m_q(y\circ^m_p x)=s^m_q(y)=s^m_q(x)$ and $t^m_q(y\circ^m_p x)=t^m_q(y)=t^m_q(x)$;
- for $0\leq p=q<m$, $s^m_q(y\circ^m_p x)=s^m_q(x)$ and $t^m_q(y\circ^m_p x)=t^m_q(x)$.

A globular reflexive ∞-magma is an ∞-magma equipped with map for reflexivity: $M_n \xrightarrow{1^n_{n+1}} M_{n+1}$, $n \geq 0$, such that

- $s^n_k(1^k_n(x)) = x = t^n_k(1^k_n(x))$;
- $1^q_n(1^p_q(x)) = 1^p_n(x)$.

Morphisms between reflexive ∞-magmas are morphisms of reflexive globular sets between their underlying reflexive globular set structure, i.e for: $M \xrightarrow{f} M'$, we have commutative diagrams

$$\begin{tikzcd} M_{n+1} \arrow[r, "f_{n+1}"] & M'_{n+1} \\ M_n \arrow[u, "1^n_{n+1}"] \arrow[r, "f_n"'] & M'_n \arrow[u, "1^n_{n+1}"'] \end{tikzcd}$$

which also preserve operations $\circ^n_p$; the category of reflexive ∞-magmas is denoted by ∞-$\mathbb{M}\mathrm{ag}_\mathrm{r}$.

An (∞, m)-globular set is a globular set X equipped with j^n_{n-1}-reversors, i.e with maps: $X_n \xrightarrow{j^n_{n-1}} X_n$, which satisfy the following equalities.

$$\begin{tikzcd} X_n \arrow[rr, "j^n_{n-1}"] \arrow[dr, "s^n_{n-1}"'] & & X_n \arrow[dl, "t^n_{n-1}"] \\ & X_{n-1} & \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[rr, "j^n_{n-1}"] \arrow[dr, "t^n_{n-1}"'] & & X_n \arrow[dl, "s^n_{n-1}"] \\ & X_{n-1} & \end{tikzcd}$$

A morphism of (∞, m)-globular sets is a morphism: $X \xrightarrow{f} X'$, of globular sets which satisfy for all $n \geq m$ the following equalities.

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "j^n_{n-1}"'] & X'_n \arrow[d, "j^n_{n-1}"] \\ X_n \arrow[r, "f_n"'] & X'_n \end{tikzcd}$$

The category of (∞, m)-globular sets is denoted by (∞, m)-$\mathbb{G}\mathrm{lob}$.

A globular reflexive (∞,m)-magma is a globular reflexive ∞-magma M equipped with a structure of globular (∞,m)-set; a morphism: $M \xrightarrow{f} M'$, of globular reflexive (∞,m)-magmas is a morphism of globular reflexive ∞-magmas which is also a morphism of (∞,m)-sets; the category of globular reflexive (∞,m)-magmas is denoted by (∞,m)-$\mathbb{M}\mathrm{ag}_{\mathrm{r}}$.

Remark 3.2. A globular strict ∞-category C is given by a globular reflexive ∞-magma C such that we have the following equalities:

- $x \circ^n_k 1^k_n(s^n_k(x)) = x$ and $1^k_n(t^n_k(x)) \circ^n_k x = x$;
- $1^q_n(y \circ^q_p x) = 1^q_p(y) \circ^n_p 1^q_p(x)$;
- $x \circ^n_k (y \circ^n_k z) = (x \circ^n_k y) \circ^n_k z$;
- $(y' \circ^n_q x') \circ^n_p (y \circ^n_q x) = (y' \circ^n_p y) \circ^n_q (x' \circ^n_p x)$.

The category of globular strict ∞-categories is denoted by ∞-$\mathbb{C}$AT. A globular strict (∞,m)-category is given by an (∞,m)-globular set C which is also a globular strict ∞-category such that if $\alpha \in C_n$ $(n \geq m)$ then $\alpha \circ^n_{n-1} j^n_{n-1}(\alpha) = 1^{n-1}_n(t^n_{n-1}(\alpha))$ and $j^n_{n-1}(\alpha) \circ^n_{n-1} \alpha = 1^{n-1}_n(s^n_{n-1}(\alpha))$. This n-cell $j^n_{n-1}(\alpha)$ of C_n is called a $\circ^n_{n-1}$-inverse of α and it is straightforward to see that such $\circ^n_{n-1}$-inverse is uniquely defined. The category of globular strict (∞,m)-categories is defined as the full subcategory of ∞-$\mathbb{C}$AT which objects are globular strict (∞,m)-categories and is denoted by (∞,m)-$\mathbb{C}$AT.

3.1.2 Globular Theories

We start to define *Globular Extensions*:

A globular tree t is given by a table of non-negative integers:

$$\begin{pmatrix} i_1 & & i_2 & & i_3 & \cdots & i_{k-1} & & i_k \\ & i'_1 & & i'_2 & & \cdots\cdots & & i'_{k-1} & \end{pmatrix}$$

where $k \geq 1$, $i_l > i'_l < i_{l+1}$, and $1 \leq l \leq k-1$.

Let $\mathcal{C}$ be a category and $\mathbb{G} \xrightarrow{F} \mathcal{C}$ be a functor. We denote $F(1(n)) = D^n$ and we shall keep the same notations for the image of cosources: $F(s^{i_l}_{i_{l'}}) =$

$s^{i_l}_{i_{l'}}$, and for the image of cotargets: $F(t^{i_l}_{i_{l'}}) = t^{i_l}_{i_{l'}}$, because no risk of confusion will occur. In this case $\mathbb{G} \xrightarrow{F} \mathcal{C}$ is called a globular extension if for all trees t as just above, the colimit of the following diagram exists in $\mathcal{C}$.

$$\begin{tikzcd}
D^{i_1} & & D^{i_2} & & \cdots & & D^{i_{k-1}} & & D^{i_k} \\
& D^{i'_1} \arrow[ul, "t^{i_1}_{i'_1}"] \arrow[ur, "s^{i_2}_{i'_1}"'] & & \arrow[ul, dotted] & \cdots & \arrow[ur, dotted] & & D^{i'_{k-1}} \arrow[ul, "t^{i_{k-1}}_{i'_{k-1}}"] \arrow[ur, "s^{i_k}_{i'_{k-1}}"']
\end{tikzcd}$$

Remark 3.3. In [8] Alexander Grothendieck called: *globular sums*, these colimits.

A morphism H of globular extensions, also called *globular functor*, is given by a commutative triangle in $\mathbb{C}$AT:

$$\begin{tikzcd}
& \mathcal{C} \arrow[dd, "H"] \\
\mathbb{G} \arrow[ur, "F"] \arrow[dr, "F'"'] & \\
& \mathcal{C}'
\end{tikzcd}$$

such that the functor H preserves globular sums. The category of globular extensions is denoted by $\mathbb{G}$-$\mathbb{E}$xt. In fact this category has an initial object denoted by $\mathbb{G} \xrightarrow{i} \Theta_0$. In fact the small category Θ_0 can be described as the full subcategory of $\mathbb{G}$lob which objects are globular trees, and its role of category of *globular arities* is central for describing different sketches which $\mathbb{S}$ets-models are globular higher structures. In particular this small category Θ_0 contains the basic inductive sketches we need to describe coherators $\Theta^{\infty}_{\mathbb{M}_m}$ which $\mathbb{S}$ets-models are globular weak (∞, m)-categories (for all $m \in \mathbb{N}$).

Now let us define *Globular theories*:

A globular theory is given by a small category Θ and a globular extension: $\mathbb{G} \xrightarrow{F} \Theta$, such that the unique induced functor $\overline{F}$ which makes commutative the diagram

$$\begin{tikzcd} & \Theta_0 \arrow[dd, "\overline{F}"] \\ \mathbb{G} \arrow[ur, "i"] \arrow[dr, "F"'] & \\ & \Theta \end{tikzcd}$$

induces a bijection between objects of Θ_0 and objects of Θ. The full subcategory of $\mathbb{G}$-$\mathbb{E}$xt which objects are globular theories is denoted by $\mathbb{G}$-$\mathbb{T}$h. Consider an object $\mathbb{G} \xrightarrow{F} \Theta$ of $\mathbb{G}$-$\mathbb{T}$h, in particular it induces the globular functor $\Theta_0 \xrightarrow{\overline{F}} \Theta$ as just above, which is a bijection on objects. A set model of (F, Θ) or for Θ for short, is given by a functor: $\Theta \xrightarrow{X} \mathbb{S}$ets, such that the functor $X \circ \overline{F}$:

$$\Theta_0 \xrightarrow{\overline{F}} \Theta \xrightarrow{X} \mathbb{S}\mathrm{ets},$$

sends globular sums to globular products[11], thus for all objects t of Θ_0:

$$\begin{pmatrix} i_1 & & i_2 & & i_3 & \cdots & i_{k-1} & & i_k \\ & i'_1 & & i'_2 & & \cdots\cdots & & i'_{k-1} & \end{pmatrix}$$

we have

$$\begin{aligned} X(\overline{F}(t)) &= X\left(\operatorname{colim}\left(\begin{tikzcd}[column sep=small] D^{i_1} & & D^{i_2} & \cdots & D^{i_{k-1}} & & D^{i_k} \\ & D^{i'_1} \arrow[ul, "t^{i_1}_{i'_1}"] \arrow[ur, "s^{i_2}_{i'_1}"'] & & \cdots \arrow[ul, dotted] \arrow[ur, dotted] & & D^{i'_{k-1}} \arrow[ul, "t^{i_{k-1}}_{i'_{k-1}}"] \arrow[ur, "s^{i_k}_{i'_{k-1}}"'] & \end{tikzcd}\right)\right) \\ &= X\left((D^{i_1}, t^{i_1}_{i'_1}) \coprod_{D^{i'_1}} (s^{i_2}_{i'_1}, D^{i_2}, t^{i_3}_{i'_2}) \coprod_{D^{i'_2}} \cdots \coprod_{D^{i'_{k-1}}} (s^{i_k}_{i'_{k-1}}, D^{i_k})\right) \\ &\simeq X(D^{i_1}) \underset{X(D^{i'_1})}{\times} \cdots \underset{X(D^{i'_{k-1}})}{\times} X(D^{i_k}). \end{aligned}$$

[11] Globular products are just dual to globular sums.

The category of $\mathbb{S}$ets-models of Θ is the full subcategory of the category of presheaves $[\Theta, \mathbb{S}\text{ets}]$ which objects are $\mathbb{S}$ets-models of Θ, and this category is denoted by $\mathbb{M}\text{od}(\Theta)$.

Here are some examples of globular theories:

Example 3.1. *The theory* $\Theta_{\mathbb{M}}$.

The forgetful functor U:

$$\begin{tikzcd} \infty\text{-}\mathbb{M}ag_r \arrow[d, shift left, "U"] \\ \mathbb{G}lob \arrow[u, dotted, shift left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

from the category ∞-$\mathbb{M}ag_r$ *of globular reflexive* ∞*-magmas to the category* $\mathbb{G}lob$ *of globular sets is right adjoint, which left adjoint is denoted by* F*, and it induces a monad* $\mathbb{M} = (M, \eta, \mu)$ *on* $\mathbb{G}lob$ *such that we have the equivalence of categories* ∞-$\mathbb{M}ag_r \simeq \mathbb{M}$-$\mathbb{A}lg$ *because* U *is monadic. The full subcategory* $\Theta_{\mathbb{M}} \subset \mathbb{K}l(\mathbb{M})$ *of the Kleisli category of* $\mathbb{M}$ *which objects are trees is called the theory of reflexive globular* ∞*-magmas. In fact we have the following equivalences of categories*:

$$\infty\text{-}\mathbb{M}ag_r \simeq \mathbb{M}\text{-}\mathbb{A}lg \simeq \mathbb{M}od(\Theta_{\mathbb{M}}).$$

Example 3.2. *The theories* $\Theta_{\mathbb{M}^m}$ $(m \in \mathbb{N})$.

The forgetful functor U^m $(m \in \mathbb{N})$:

$$\begin{tikzcd} (\infty, m)\text{-}\mathbb{M}ag_r \arrow[d, shift left, "U^m"] \\ \mathbb{G}lob \arrow[u, dotted, shift left, "F^m"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

from the category (∞, m)-$\mathbb{M}ag_r$ *of globular reflexive* (∞, m)*-magmas to the category* $\mathbb{G}lob$ *of globular sets is right adjoint, which left adjoint is denoted by* F^m*, and this induces a monad* $\mathbb{M}^m = (M^m, \eta^m, \mu^m)$ *on* $\mathbb{G}lob$ *such that we have the equivalence of categories* (∞, m)-$\mathbb{M}ag_r \simeq \mathbb{M}^m$-$\mathbb{A}lg$*, because each* U^m *(for all* $m \in \mathbb{N}$*) is monadic. The full subcategory* $\Theta_{\mathbb{M}^m} \subset \mathbb{K}l(\mathbb{M}^m)$ *of* *the Kleisli category of* $\mathbb{M}^m$ *which objects are trees is called the theory of reflexive globular* (∞,m)*-magmas. In fact we have the following equivalences of categories*:

$$(\infty,m)\text{-}\mathbb{M}ag_r \simeq \mathbb{M}^m\text{-}\mathbb{A}lg \simeq \mathbb{M}od(\Theta_{\mathbb{M}^m}).$$

3.1.3 Globular coherators

We need to define the notion of *admissiblity*:

Let $\mathbb{G} \xrightarrow{F} \Theta$ be a globular theory, i.e an object of $\mathbb{G}$-$\mathbb{T}$h; two arrows: $D^n \underset{g}{\overset{f}{\rightrightarrows}} t$ in Θ are parallels if $fs^n_{n-1} = gs^n_{n-1}$ and $ft^n_{n-1} = gt^n_{n-1}$:

$$\begin{tikzcd} D^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & t \\ D^{n-1} \arrow[u, shift left, "s^n_{n-1}"] \arrow[u, shift right, "t^n_{n-1}"'] & \end{tikzcd}$$

Consider a couple (f,g) of parallels arrows in Θ as just above. We say that it is admissible or algebraic if they don't belong to the image of the globular functor $\overline{F}$:

$$\begin{tikzcd} & \Theta_0 \arrow[d, "\overline{F}"] \\ \mathbb{G} \arrow[ur, "i"] \arrow[r, "F"'] & \Theta \end{tikzcd}$$

Consider a couple (f,g) of arrows of Θ which is admissible as just above; a lifting of (f,g) is given by an arrow h:

$$\begin{tikzcd} D^{n+1} \arrow[dr, "h"] & \\ D^n \arrow[u, shift left, "s^{n+1}_n"] \arrow[u, shift right, "t^{n+1}_n"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & t \end{tikzcd}$$

such that $hs_n^{n+1}=f$ and $ht_n^{n+1}=g$.

We now define the *Batanin-Grothendieck sequence* associated to a globular theory $\mathbb{G}\xrightarrow{F}\Theta$. We build it with the following induction:

- If $n=0$ we start with the couple (Θ,E) where E denotes the set of admissible pairs of arrows of $\mathcal{C}$; we shall write $(\Theta^0,E_0)=(\Theta,E)$ this first step;
- if $n=1$ we consider then the couple (Θ^1,E_1) where Θ^1 is obtained by formally adding in $\Theta^0=\Theta$ the liftings of all elements $(f,g)\in E_0=E$, and E_1 is the set of admissible couples of arrows in Θ^1 which are not elements of the set E_0;
- if for $n\geq 2$ the couple (Θ^n,E_n) is well defined then Θ^{n+1} is obtained by formally adding in Θ^n the liftings of all elements of E_n, and E_{n+1} is the set of admissible couples of arrows of Θ^{n+1} which are not elements of E_n.

We can also give a slightly different but equivalent induction to build the Batanin-Grothendieck sequence for such globular theory $\mathbb{G}\xrightarrow{F}\Theta$:

- If $n=0$ we start with the couple (Θ,E) where E is the set of couples of arrows which are admissible of Θ; we denote $E=E_0=E'_0=E'_0\setminus\emptyset$ (we shall see soon the meaning of these notations), and $\Theta^0=\Theta$;
- if $n=1$ we consider the couple (Θ^1,E_1) where Θ^1 is obtained by formally adding in Θ^0 all liftings of the elements $(f,g)\in E_0$, E'_1 is the set of all pairs of arrows which are admissible in Θ^1, and $E_1=E'_1\setminus E_0$; remark that $E_0=E'_0\subset E'_1$;
- if $n=2$ we consider the couple (Θ^2,E_2) where Θ^2 is obtained by formally adding in Θ^1 all liftings of the elements $(f,g)\in E_1$, E'_2 is the set of all pairs of arrows which are admissible in Θ^2, and $E_2=E'_2\setminus E'_1$;
- for $n\geq 3$ we suppose that the couple (Θ^n,E_n) is well defined with $E_n=E'_n\setminus E'_{n-1}$, then Θ^{n+1} is obtained by formally adding in Θ^n all liftings of the elements $(f,g)\in E_n$, E'_{n+1} is the set of all pairs of arrows which are admissible in Θ^{n+1}, and $E_{n+1}=E'_{n+1}\setminus E'_n$;

An important fact is the globular theory Θ^n obtained by formally adding in Θ^{n-1} all liftings of elements of E_{n-1} is universal for this adding. To give a precise meaning of "formally adding" is just an application of the following theorem of Christian Lair:

Theorem 3.4 (Lair). *The category $\mathbb{S}ketch$ of Sketches is projectively sketchable, that is there is a projective sketch $\mathcal{E}_{\mathbb{S}ketch}$ such that the category $\mathbb{M}od(\mathcal{E}_{\mathbb{S}ketch})$ of set-models of $\mathcal{E}_{\mathbb{S}ketch}$ is equivalent to the category $\mathbb{S}ketch$.*

Also the category $\mathbb{C}\mathrm{at}$ of small categories is also projectively sketchable by a projective sketch $\mathcal{E}_{\mathbb{C}\mathrm{at}}$ and we have an easy morphism of projective sketches: $\mathcal{E}_{\mathbb{C}\mathrm{at}} \xrightarrow{i} \mathcal{E}_{\mathbb{S}\mathrm{ketch}}$ which induces a left adjunction F with the functor $\mathbb{M}\mathrm{od}(i)$: $\mathbb{S}\mathrm{ketch} \xrightarrow{F} \mathbb{C}\mathrm{at}$ This construction is called the *free prototype functor* in [2]. With these results in hands it is useful to see the globular theory Θ^n obtained by formally adding in Θ^{n-1} all liftings of elements of E_{n-1} as the free category (with the free prototype functor) generated by this adding. Thus we start with the object $\Theta^{n-1} + E_{n-1}$ of $\mathbb{S}$ketch, where $\Theta^{n-1} + E_{n-1}$ means the sketch obtained by formally[12] adding all liftings of elements of E_{n-1} in the sketch Θ^{n-1}; then Θ^n is just the free category $F(\Theta^{n-1} + E_{n-1})$ generated by the free prototype functor. This result also stands for cubical coherators developed in Section 3.2.3.

The Batanin-Grothendieck sequence of the globular theory $\mathbb{G} \xrightarrow{F} \Theta$ produces, inside the category $\mathbb{G}$-$\mathbb{T}$h, the following filtered diagram

$$\Theta^0 \xrightarrow{\quad i_1 \quad} \Theta^1 \xrightarrow{\quad i_2 \quad} \cdots \xrightarrow{\quad i_n \quad} \Theta^n \xrightarrow{\qquad} \cdots$$

that we denote by

$$(\mathbb{N}, \leq) \xrightarrow{\quad \Theta^{\bullet} \quad} \mathbb{G}\text{-}\mathbb{T}\mathrm{h}.$$

Now we have the materials to define *coherators for globular theories*:

We start with the datas of the previous subsection, i.e we start with the Batanin-Grothendieck sequence:

$$(\mathbb{N}, \leq) \xrightarrow{\quad \Theta\bullet \quad} \mathbb{G}\text{-}\mathbb{T}\mathrm{h},$$

for a globular theory $\mathbb{G} \xrightarrow{F} \Theta$.

[12] Here "formally" has an accurate logical sense.

Definition 3.5. The globular theory: $\mathbb{G} \xrightarrow{F\infty} \Theta^\infty$, induced by the colimit of the previous filtered diagram $\Theta^\bullet$:

$$\begin{tikzcd}
\Theta^0 \arrow[r, "i_1"] \arrow[drrrrr] & \Theta^1 \arrow[r, "i_2"] \arrow[drrrr] & \cdots \arrow[r, "i_n"] & \Theta^n \arrow[r] \arrow[drr] & \cdots & \\
 & & & & & \Theta^\infty
\end{tikzcd}$$

is called the globular coherator of the type Batanin-Grothendieck associated to the globular theory $\mathbb{G} \xrightarrow{F} \Theta$.

For shorter terminology we shall say that $\mathbb{G} \xrightarrow{F^\infty} \Theta^\infty$ is the coherator associated to the globular theory $\mathbb{G} \xrightarrow{F} \Theta$. It is straightforward to see that the Batanin-Grothendieck construction of coherators associated to globular theory is functorial, and the following endofunctor Φ is called the Batanin-Grothendieck functor:

$$\mathbb{G}\text{-}\mathbb{T}\mathrm{h} \xrightarrow{\quad\Phi\quad} \mathbb{G}\text{-}\mathbb{T}\mathrm{h}$$

$$(\mathbb{G} \xrightarrow{F} \Theta) \longmapsto (\mathbb{G} \xrightarrow{F^\infty} \Theta^\infty)$$

An important coherator is $\Theta^\infty_{\mathbb{M}}$ which is the coherator associated to the globular theory:

$$\mathbb{G} \xrightarrow{\quad j\quad} \Theta_{\mathbb{M}},$$

that we obtain with the composition

$$\mathbb{G} \xrightarrow{i} \Theta_0 \hookrightarrow \Theta_{\mathbb{M}},$$

This coherator is denoted by $\Theta^\infty_{\mathbb{M}}$ and $\mathbb{M}\mathrm{od}(\Theta^\infty_{\mathbb{M}})$ is the category of globular weak ∞-categories of Grothendieck.

Remark 3.6. In 2019 John Bourke has proved [5] the *Ara conjecture* [1] which says that the category of globular weak ∞-categories of Batanin is equivalent to the category of globular weak ∞-categories of Grothendieck:

$$\mathbb{M}\mathrm{od}(\Theta_{\mathbb{B}^0_C}) \simeq \mathbb{M}\mathrm{od}(\Theta^\infty_{\mathbb{M}}),$$

where here $\mathbb{B}^0_C$ denotes the globular operad of Batanin [3] which algebras are his models of globular weak ∞-categories and $\Theta_{\mathbb{B}^0_C}$ is its associated theory that we obtain with the nerve theorem of Weber [22].

Now let us define the coherators $\Theta^{\infty}_{\mathbb{M}^m}$ ($m \in \mathbb{N}$):

The coherator associated to the globular theory $\mathbb{G} \xrightarrow{j_m} \Theta_{\mathbb{M}^m}$ is denoted by $\Theta^{\infty}_{\mathbb{M}^m}$ and $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^m})$ is the category of globular weak (∞, m)-categories of Grothendieck ($m \geq 0$). If $m = 0$, the coherator $\Theta^{\infty}_{\mathbb{M}^0}$ is the one of globular weak ∞-groupoids of Grothendieck. It is easy to see that we have the following filtration in the category $\mathbb{G}$-$\mathbb{T}\mathrm{h}$:

$$\cdots \longrightarrow \Theta^{\infty}_{\mathbb{M}^{m+1}} \longrightarrow \Theta^{\infty}_{\mathbb{M}^m} \longrightarrow \cdots \longrightarrow \Theta^{\infty}_{\mathbb{M}^0},$$

and also the following diagram

$$\begin{tikzcd}
 & & \Theta^{\infty}_{\mathbb{M}} \arrow[dll] \arrow[dl] \arrow[d] \arrow[drr] & & \\
\cdots \arrow[r] & \Theta^{\infty}_{\mathbb{M}^{m+1}} \arrow[r] & \Theta^{\infty}_{\mathbb{M}^m} \arrow[r] & \cdots \arrow[r] & \Theta^{\infty}_{\mathbb{M}^0}
\end{tikzcd}$$

which shows that we have the following inclusion of functors when passing to $\mathbb{S}\mathrm{ets}$-models.

$$\begin{tikzcd}
\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^0}) \arrow[r, "i_1"] \arrow[drrrr] & \mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^1}) \arrow[r, "i_2"] \arrow[drrr] & \cdots \arrow[r, "i_n"] & \mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^m}) \arrow[r] \arrow[dr] & \cdots \\
 & & & & \mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}})
\end{tikzcd}$$

3.2 The Cubical Coherators Θ^{∞}_{W} and $\Theta^{\infty}_{W^0}$ In this section we are going to define *cubical theory*, *cubical coherators* and we shall see that this cubical paradigm can be translated from the globular one, but this translation is often non-trivial. In Section 3.2.3 we treat the important cases of the coherator Θ^{∞}_{W} which $\mathbb{S}\mathrm{ets}$-models are cubical weak ∞-categories with connections, and the coherator $\Theta^{\infty}_{W^0}$ which $\mathbb{S}\mathrm{ets}$-models are cubical weak ∞-groupoids with connections.

3.2.1 Cubical Theories and their Models

A *cubical theory* (F, Θ) (or a *cubical theory* Θ if there is not risk of confusion) is given by a cubical extension (defined in 2):

$$\mathbb{C} \xrightarrow{\quad F \quad} \Theta,$$

such that the induced unique functor $\bar{F}$:

$$\Theta_0 \xrightarrow{\quad \bar{F} \quad} \Theta,$$

is a bijection on objects; a chosen initial object in $\mathbb{C}$-$\mathbb{E}$xt:

$$\mathbb{C} \xrightarrow{\quad i \quad} \Theta_0,$$

is a specific cubical theory called the initial cubical theory; what we saw in 2 is that $\bar{F}$ above is defined (for any rectangular divisor $X \in \Theta_0$) by the formula:

$$\bar{F}(X) = \mathrm{Lan}_i(F) = \mathrm{colim}\ \mathcal{E}_{\langle X;F \rangle},$$

and thus a cubical theory (F, Θ) is in particular given by a small category Θ which objects are identify with rectangular divisors.

The full subcategory of $\mathbb{C}$-$\mathbb{E}$xt which objects are cubical theories is denoted by $\mathbb{C}$-$\mathbb{T}$h and the cubical theory (i, Θ_0) is initial in it. We note that morphisms G in $\mathbb{C}$-$\mathbb{T}$h:

$$\begin{tikzcd} & \Theta \arrow[dd, "G"] \\ \mathbb{C} \arrow[ur, "F"] \arrow[dr, "F'"'] & \\ & \Theta' \end{tikzcd}$$

induce, thanks to the universality of Θ_0, the following commutative triangles in the category $\mathbb{C}$at of small categories:

$$\begin{tikzcd} & \Theta \arrow[dd, "G"] \\ \Theta_0 \arrow[ur, "\bar{F}"] \arrow[dr, "\bar{F'}"'] & \\ & \Theta' \end{tikzcd}$$

and G is given (for any rectangular divisor $X \in \Theta_0$) by the formula:

$$G(\text{colim } \mathcal{E}_{\langle X;F\rangle}) = \text{colim } \mathcal{E}_{\langle X;F'\rangle}.$$

Now we are going to define the category $\mathbb{M}\text{od}(\Theta)$ of $\mathbb{S}$ets-models of a cubical theory (F, Θ). A presheaf:

$$\Theta^{\text{op}} \xrightarrow{\quad G \quad} \mathbb{S}\text{ets},$$

induces the following commutative diagram

$$\begin{tikzcd}
 & \Theta_0^{\text{op}} \arrow[d, "\bar{F}^{\text{op}}"] \\
\mathbb{G}^{\text{op}} \arrow[ur, "i^{\text{op}}"] \arrow[r, "F^{\text{op}}"] \arrow[dr, "G|_{\mathbb{G}^{\text{op}}}"'] & \Theta^{\mathbb{G}^{\text{op}}} \arrow[d, "G"] \\
 & \mathbb{S}\text{ets}
\end{tikzcd}$$

where we have the following formula for all rectangular divisors $X \in \Theta_0^{\text{op}}$:

$$\bar{F}^{\text{op}}(X) = \text{colim } \mathcal{E}_{\langle X;F\rangle}.$$

We need to describe the value $G(\text{colim } \mathcal{E}_{\langle X;F\rangle})$, and in fact what we want is G to send this colimit to a limit. But the diagram just above shows us that we need to decorate any rectangular divisor X with the functor GF^{op}, i.e in each occurence of $1(q)$ in each basic divisor of X, we substitute it with $G(F(1(q)))$; the resulting decoration is written $\langle X; GF^{\text{op}}\rangle$ and we know that with it we can associate its inductive sketch $\mathcal{E}_{\langle X;GF^{\text{op}}\rangle}$ and also its projective sketch $\mathcal{E}^{\text{op}}_{\langle X;GF^{\text{op}}\rangle}$. The limits: lim $\mathcal{E}^{\text{op}}_{\langle X;GF^{\text{op}}\rangle}$, are dual to cubical sums and thus they may be called *cubical products*. Also it is interesting to notice that, as $\mathcal{E}_X$, the projective sketch $\mathcal{E}^{\text{op}}_{\langle X;GF^{\text{op}}\rangle}$ is also a sketch with a notion of floors. Now we have the materials to define $\mathbb{S}$ets-models for (F, Θ):

Definition 3.7. The presheaf:

$$\Theta^{\text{op}} \xrightarrow{\quad G \quad} \mathbb{S}\text{ets},$$

is a $\mathbb{S}$ets-model of the cubical theory (F,Θ) is for all rectangular divisor $X\in\Theta_0$ we have the equality:

$$G(\text{colim }\mathcal{E}_{\langle X;F\rangle})=\lim\,\mathcal{E}^{\text{op}}_{\langle X;GF^{\text{op}}\rangle}.$$

Morphisms of $\mathbb{S}$ets-models are just natural transformations, and the category of $\mathbb{S}$ets-models of the cubical theory (F,Θ) is denoted by $\mathbb{M}\text{od}(\Theta)$. It is interesting to notice that we can define models of (F,Θ) for other category than $\mathbb{S}$ets. For example for any elementary topos $\mathcal{T}$ it is possible to define a similar category $\mathbb{M}\text{od}(\Theta,\mathcal{T})$ of $\mathcal{T}$-models of the cubical theory (F,Θ).

3.2.2 Examples of Cubical Theories

Now we are going to give two crucial examples of cubical theory:

- The cubical theory $\Theta_{\mathbb{M}}$ of cubical reflexive ∞-magmas;
- the cubical theory $\Theta_{\mathbb{M}^0}$ of cubical reflexive $(\infty,0)$-magmas;

The theory $\Theta_{\mathbb{M}}$ leads to the coherator Θ^{∞}_{W} which $\mathbb{S}$ets-models are cubical weak ∞-categories with connections, and the theory $\Theta_{\mathbb{M}^0}$ leads to the coherator $\Theta^{\infty}_{W^0}$ which $\mathbb{S}$ets-models are cubical weak ∞-groupoids with connections, see 3. In this section we recall the definition of $\Theta_{\mathbb{M}}$ and $\Theta_{\mathbb{M}^0}$, but the reader is also encouraged to see other details in [14]. Consider a cubical reflexive set:

$$(C,(1^n_{n+1,j})_{n\in\mathbb{N},j\in[\![1,n+1]\!]},(1^{n,\gamma}_{n+1,j})_{n\geq 1,j\in[\![1,n]\!]}),$$

equipped with partial operations $(\circ^n_j)_{n\geq 1,j\in[\![1,n]\!]}$ where: if $a,b\in C(n)$, then $a\circ^n_j b$ is defined for $j\in\{1,...,n\}$ if $s^n_j(b)=t^n_j(a)$. We also require these operations to follow the following *axioms of position*:

(i) For $1\leq j\leq n$ we have $s^n_{n-1,j}(a\circ^n_j b)=s^n_{n-1,j}(a)$ and $t^n_{n-1,j}(a\circ^n_j b)=t^n_{n-1,j}(a)$,

(ii) $s^n_{n-1,i}(a\circ^n_j b)=\begin{cases} s^n_{n-1,i}(a)\circ^{n-1}_{j-1} s^n_{n-1,i}(b) \text{ if } 1\leq i<j\leq n\\ s^n_{n-1,i}(a)\circ^{n-1}_{j} s^n_{n-1,i}(b) \text{ if } 1\leq j<i\leq n\end{cases}$

(iii) $t^n_{n-1,i}(a\circ^n_j b)=\begin{cases} t^n_{n-1,i}(a)\circ^{n-1}_{j-1} t^n_{n-1,i}(b) \text{ if } 1\leq i<j\leq n\\ t^n_{n-1,i}(a)\circ^{n-1}_{j} t^n_{n-1,i}(b) \text{ if } 1\leq j<i\leq n\end{cases}$

Definition 3.8. Cubical ∞-magmas are cubical sets equipped with partial operations as above. A morphism between two cubical ∞-magmas is a morphism of their underlying cubical sets which respects partial operations $(\circ^n_j)_{n\geq 1, j\in [\![1,n]\!]}$. The category of cubical ∞-magmas is noted by ∞-$\mathbb{CM}$ag.

Definition 3.9. Cubical reflexive ∞-magmas are cubical reflexive set equipped a structure of ∞-magmas. A morphism between two cubical reflexive ∞-magmas is a morphism of their underlying cubical reflexive sets which respects partial operations $(\circ^n_j)_{n\geq 1, j\in [\![1,n]\!]}$. The category of cubical reflexive ∞-magmas is denoted by ∞-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$.

Now the forgetful functor:

$$\infty\text{-}\mathbb{CM}\mathrm{ag}_{\mathrm{r}} \xrightarrow{\quad V \quad} \mathbb{CS}\mathrm{ets},$$

is right adjoint and it induces the monad $\mathbb{M}=(M,\eta,\mu)$ of cubical reflexive ∞-magmas with its Kleisli category $\mathbb{K}\mathrm{l}(\mathbb{M})$. Denote by $\Theta_{\mathbb{M}}$ the full subcategory of $\mathbb{K}\mathrm{l}(\mathbb{M})$ which objects are objects of Θ_0. This small category $\Theta_{\mathbb{M}}$ equipped with the canonical inclusion functor: $\Theta_0 \xrightarrow{\ j\ } \Theta_{\mathbb{M}}$, is an important cubical theory because it is the basic data we need to build the coherator Θ^{∞}_{W} which models are cubical weak ∞-categories with connections.

Now let us define our second example of cubical theory: the cubical theory $\Theta_{\mathbb{M}^0}$ of cubical reflexive $(\infty,0)$-magmas[13]; first we use the notion of cubical $(\infty,0)$-sets, notion which underly a new sketch (see diagrams below) which we use to define a coherator which $\mathbb{S}$ets-models are cubical weak ∞-groupoids. Here we define cubical version of the formalism developed in [10] for globular $(\infty,0)$-sets. This formalism (defined in [15]) of this cubical world is very similar to its globular analogue. Consider a cubical set $\mathcal{C}=(C_n, s^n_{n-1,j}, t^n_{n-1,j})_{1\leq j\leq n}$. If $n\geq 1$ and $1\leq j\leq n$, then a (n,j)-reversor on it is given by a map $C_n \xrightarrow{\quad j^n_j \quad} C_n$ such that the following diagrams commute.

$$\begin{tikzcd} C_n \arrow[rr, "j^n_j"] \arrow[dr, "s^n_{n-1,j}"'] & & C_n \arrow[dl, "t^n_{n-1,j}"] \\ & C_{n-1} & \end{tikzcd} \qquad \begin{tikzcd} C_n \arrow[rr, "j^n_j"] \arrow[dr, "t^n_{n-1,j}"'] & & C_n \arrow[dl, "s^n_{n-1,j}"] \\ & C_{n-1} & \end{tikzcd}$$

[13] $(\infty,0)$-magmas are the *magmatic* incarnation of ∞-groupoids.

If for each $n > 0$ and for each $1 \leq j \leq n$, there are such (n,j)-reversor j^n_j on $\mathcal{C}$, then we say that $\mathcal{C}$ is a cubical $(\infty,0)$-set. The family of maps $(j^n_j)_{n>0,1\leq j\leq n}$ for all $(n \in \mathbb{N}^*)$ is called an $(\infty,0)$-structure, and in that case we shall say that $\mathcal{C}$ is equipped with the $(\infty,0)$-structure $(j^n_j)_{n>0,1\leq j\leq n}$. When we speak about such $(\infty,0)$-structure $(j^n_j)_{n>0,1\leq j\leq n}$ on $\mathcal{C}$, it means that it is for all integers $n \in \mathbb{N}^*$ where C_n must be non-empty. Seen as cubical $(\infty,0)$-set we denote it by

$$\mathcal{C} = ((C_n, s^n_{n-1,j}, t^n_{n-1,j})_{1\leq j\leq n}, (j^n_j)_{n>0,1\leq j\leq n}).$$

If

$$\mathcal{C}' = ((C'_n, s'^n_{n-1,j}, t'^n_{n-1,j})_{1\leq j\leq n}, (j'^n_j)_{n>0,1\leq j\leq n})$$

is another $(\infty,0)$-set, then a morphism of $(\infty,0)$-sets:

$$\mathcal{C} \xrightarrow{\quad f \quad} \mathcal{C}',$$

is given by a morphism of cubical sets such that for each $n > 0$ and for each $1 \leq j \leq n$ we have the following commutative diagrams.

$$\begin{tikzcd} C_n \arrow[r, "j^n_j"] \arrow[d, "f_n"'] & C_n \arrow[d, "f_n"] \\ C'_n \arrow[r, "j'^n_j"'] & C'_n \end{tikzcd}$$

The category of cubical $(\infty,0)$-sets is denoted by $(\infty,0)$-$\mathbb{CS}$ets. A cubical reflexive $(\infty,0)$-magma is an object of ∞-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$ such that its underlying cubical set is equipped with an $(\infty,0)$-structure. Morphisms between cubical reflexive $(\infty,0)$-magmas are those of ∞-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$ which are also morphisms of $(\infty,0)$-$\mathbb{CS}ets$, i.e they preserve the underlying $(\infty,0)$-structures. The category of cubical reflexive $(\infty,0)$-magmas is denoted by $(\infty,0)$-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$. Now the forgetful functor:

$$(\infty,0)\text{-}\mathbb{CM}\mathrm{ag}_{\mathrm{r}} \xrightarrow{\quad V \quad} \mathbb{CS}\mathrm{ets},$$

is right adjoint and it induces the monad $\mathbb{M}^0 = (M^0, \eta^0, \mu^0)$ of cubical reflexive $(\infty,0)$-magmas which Kleisli category is denoted by $\mathbb{K}\mathrm{l}(\mathbb{M}^0)$. Consider now $\Theta_{\mathbb{M}^0}$ the full subcategory of $\mathbb{K}\mathrm{l}(\mathbb{M}^0)$ which objects are objects of

Θ_0. This small category $\Theta_{\mathbb{M}^0}$ equipped with the canonical inclusion functor $\Theta_0 \xrightarrow{j^0} \Theta_{\mathbb{M}^0}$ is an important cubical theory because it is the basic data we need to build the coherator $\Theta^{\infty}_{W^0}$ which models are cubical weak ∞-groupoids with connections.

3.2.3 The Cubical Coherators Θ^{∞}_{W} and $\Theta^{\infty}_{W^0}$

Now we are ready to describe the cubical coherator Θ^{∞}_{W} which $\mathbb{S}$ets-models are cubical weak ∞-categories with connections, and the cubical coherator $\Theta^{\infty}_{W^0}$ of cubical weak ∞-groupoids with connections. More precisely the coherator Θ^{∞}_{W} is obtained as a colimit of a directed diagram $\Theta^{\bullet}_{M}$ in the category $\mathbb{C}$-$\mathbb{T}$h, where the diagram $\Theta^{\bullet}_{M}$ is built inductively starting with the cubical theory Θ_M described in 3.2.2. Also the coherator $\Theta^{\infty}_{W^0}$ is obtained as a colimit of a directed diagram $\Theta^{\bullet}_{M}$ in the category $\mathbb{C}$-$\mathbb{T}$h, where the diagram $\Theta^{\bullet}_{M^0}$ is built inductively starting with the cubical theory Θ_{M^0} described in 3.2.2. The main tools to gain these definitions are inspired by the following works:

1. The globular coherators as in 3.1 and in [8, 19];
2. The category Θ_0 of cubical pasting diagrams defined in [12];
3. The cubical contractions defined in [14, 16];
4. The cubical weak ∞-groupoids with connections defined in [15].

Consider an object $\mathbb{C} \xrightarrow{F} \Theta$ of the category $\mathbb{C}$-$\mathbb{T}$h of cubical theories and the unique functor $\Theta_0 \xrightarrow{\bar{F}} \Theta$. We put here $I^n = F(1(n))$; an I^n-arrow in Θ is one arrow of Θ with domain the object I^n. A pair (f, g) of I^n-arrows in Θ:

$$I^n \underset{g}{\overset{f}{\rightrightarrows}} X,$$

is called:

- admissible[14] if it doesn't belong to the image of $\bar{F}$;

[14] In the sequel we could also use *algebraic* instead of admissible. Grothendieck used the expression admissibility instead of algebraicity, this is the reason why we kept this termi-

- j-admissible (for a direction $j \in [\![1,n]\!]$) if it is admissible and verify $f \circ s^n_{n-1,j} = g \circ s^n_{n-1,j}$ and $f \circ t^n_{n-1,j} = g \circ t^n_{n-1,j}$,

$$\begin{tikzcd} I^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \\ I^{n-1} \arrow[u, "s^n_{n-1,j}"] \end{tikzcd} \qquad \begin{tikzcd} I^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \\ I^{n-1} \arrow[u, "t^n_{n-1,j}"] \end{tikzcd}$$

If a pair (f,g) of I^n-arrows: $I^n \overset{f}{\underset{g}{\rightrightarrows}} X$, is admissible, then we define its *j-lifting arrows* $[f,g]^n_{n+1,j}$ (for all $j \in [\![1,n+1]\!]$) or its j-lifting for short, as a map $[f,g]^n_{n+1,j}$ in Θ:

$$\begin{tikzcd} I^{n+1} \arrow[drr, dotted, "{[f,g]^n_{n+1,j}}"] & & \\ I^n \arrow[u, shift left, "s^{n+1}_{n,i}"] \arrow[u, shift right, "t^{n+1}_{n,i}"'] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & X \\ I^{n-1} \arrow[u, shift left, "s^n_{n-1,i}"] \arrow[u, shift right, "t^n_{n-1,i}"'] & & \end{tikzcd}$$

using an induction: we suppose that such operations $[-,-]^p_{p+1,k}$ ($p \leq n-1$ and $k \in [\![1,p+1]\!]$) exist for all faces of f and g; the definition of $[f,g]^n_{n+1,j}$ goes as follow:

- If $1 \leq i < j \leq n+1$, then

$$[f,g]^n_{n+1,j} \circ s^{n+1}_{n,i} = [f \circ s^n_{n-1,i}, g \circ s^n_{n-1,i}]^{n-1}_{n,j-1},$$
$$\text{and } [f,g]^n_{n+1,j} \circ t^{n+1}_{n,i} = [f \circ t^n_{n-1,i}, g \circ t^n_{n-1,i}]^{n-1}_{n,j-1};$$

nology. The intuitive meaning of admissibility is just that arrows which are admissibles, are operations for cubical theory (thus they do not belong to arrows in Θ_0), i.e operations which arities are the $X \in \Theta_0$. Thus the terminology *algebraic* is also correct if we follow the usual terminology of universal algebra.

- if $1 \leq j < i \leq n+1$ then

$$[f,g]^n_{n+1,j} \circ s^{n+1}_{n,i} = [f \circ s^n_{n-1,i-1}, g \circ s^n_{n-1,i-1}]^{n-1}_{n,j},$$
$$\text{and } [f,g]^n_{n+1,j} \circ t^{n+1}_{n,i} = [f \circ t^n_{n-1,i-1}, g \circ t^n_{n-1,i-1}]^{n-1}_{n,j};$$

- if $i = j$ then

$$[f,g]^n_{n+1,j} \circ s^{n+1}_{n,i} = f \text{ and } [f,g]^n_{n+1,j} \circ t^{n+1}_{n,i} = g;$$

- $[f,f]^n_{n+1,j} = 1^n_{n+1,j}(f)$.

If a pair (f,g) of I^n-arrows: $I^n \underset{g}{\overset{f}{\rightrightarrows}} X$, is j-admissible (for a direction $j \in [\![1,n]\!]$), then we define its $(j,-)$-*lifting arrow* $[f,g]^{n,-}_{n+1,j}$ or its $(j,-)$-lifting for short, as a map $[f,g]^{n,-}_{n+1,j}$ in Θ:

$$\begin{tikzcd}
I^{n+1} \arrow[drr, dotted, "{[f,g]^{n,-}_{n+1,j}}"] & & \\
I^n \arrow[u, shift left=2, "s^{n+1}_{n,i}"] \arrow[u, shift right=2, "t^{n+1}_{n,i}"'] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & X \\
I^{n-1} \arrow[u, shift left=2, "s^n_{n-1,i}"] \arrow[u, shift right=2, "t^n_{n-1,i}"'] & &
\end{tikzcd}$$

using an induction: we suppose that such operations $[-,-]^{p,-}_{p+1,k}$ ($p \leq n-1$ and $k \in [\![1,p]\!]$) exist for all faces of f and g, but also (see the induction used just below) we have to suppose that the operations $[-,-]^{p}_{p+1,k}$ ($p \leq n-1$ and $k \in [\![1,p+1]\!]$) defined above exist for such faces; the definition of $[f,g]^{n,-}_{n+1,j}$ goes as follow:

- for $1 \leq j \leq n$ we have
  - $[f;g]^{n,-}_{n+1,j} \circ s^{n+1}_{n,j} = f$ and $[f;g]^{n,-}_{n+1,j} \circ s^{n+1}_{n,j+1} = [f;g]^{n,-}_{n+1,j} \circ t^{n+1}_{n,j+1} = g$,

– $[f;g]^{n,-}_{n+1,j} \circ t^{n+1}_{n,j} = [f;g]^{n,-}_{n+1,j} \circ t^{n+1}_{n,j+1} = [f \circ t^{n}_{n-1,j}; g \circ t^{n}_{n-1,j}]^{n-1}_{n,j}$;

- for $1 \leq i, j \leq n+1$ we have

  – $[f;g]^{n,-}_{n+1,j} \circ s^{n+1}_{n,i} = \begin{cases} [f \circ s^{n}_{n-1,i}; g \circ s^{n}_{n-1,i}]^{n-1,-}_{n,j-1} & \text{if } 1 \leq i < j \leq n, \\ [f \circ s^{n}_{n-1,i-1}; g \circ s^{n}_{n-1,i-1}]^{n-1,-}_{n,j} & \text{if } 2 \leq j+1 < i \leq n+1; \end{cases}$

  – $[f;g]^{n,-}_{n+1,j} \circ t^{n+1}_{n,i} = \begin{cases} [f \circ t^{n}_{n-1,i}; g \circ t^{n}_{n-1,i}]^{n-1,-}_{n,j-1} & \text{if } 1 \leq i < j \leq n, \\ [f \circ t^{n}_{n-1,i-1}; g \circ t^{n}_{n-1,i-1}]^{n-1,-}_{n,j} & \text{if } 2 \leq j+1 < i \leq n+1; \end{cases}$

- $[f,f]^{n,-}_{n+1,j} = 1^{n,-}_{n+1,j}(f)$.

If a pair (f,g) of I^n-arrows: $I^n \overset{f}{\underset{g}{\rightrightarrows}} X$, is j-admissible (for a direction $j \in [\![1,n]\!]$), then we define its $(j,+)$-*lifting arrow* $[f,g]^{n,+}_{n+1,j}$ or its $(j,+)$-lifting for short, as a map $[f,g]^{n,+}_{n+1,j}$ in Θ:

$$\begin{tikzcd}
I^{n+1} \arrow[drr, dotted, "{[f,g]^{n,+}_{n+1,j}}"] & & \\
I^{n} \arrow[u, shift left, "s^{n+1}_{n,i}"] \arrow[u, shift right, "t^{n+1}_{n,i}"'] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & X \\
I^{n-1} \arrow[u, shift left, "s^{n}_{n-1,i}"] \arrow[u, shift right, "t^{n}_{n-1,i}"'] & &
\end{tikzcd}$$

using an induction: we suppose that such operations $[-,-]^{p,+}_{p+1,k}$ ($p \leq n-1$ and $k \in [\![1,p]\!]$) exist for all faces of f and g, but also (see the induction used just below) we have to suppose that the operations $[-,-]^{p}_{p+1,k}$ ($p \leq n-1$ and $k \in [\![1,p+1]\!]$) defined above exist for such faces; the definition of $[f,g]^{n,+}_{n+1,j}$ goes as follow:

- for $1 \leq j \leq n$ we have

  – $[f;g]^{n,+}_{n+1,j} \circ t^{n+1}_{n,j} = f$,

  – $[f;g]^{n,+}_{n+1,j} \circ s^{n+1}_{n,j} = [f;g]^{n,+}_{n+1,j} \circ s^{n+1}_{n,j+1} = [f \circ s^{n}_{n-1,j}; g \circ s^{n}_{n-1,j}]^{n-1}_{n,j}$;

- for $1 \leq i, j \leq n+1$ we have

– $s^{n+1}_{n,i}([f;g]^{n,+}_{n+1,j}) = \begin{cases} [f\circ s^{n}_{n-1,i}; g\circ s^{n}_{n-1,i}]^{n-1,+}_{n,j-1} \text{ if } 1\leq i<j\leq n, \\ [f\circ s^{n}_{n-1,i-1}; g\circ s^{n}_{n-1,i-1}]^{n-1,+}_{n,j} \text{ if } 2\leq j+1<i\leq n+1; \end{cases}$

– $t^{n+1}_{n,i}([f;g]^{n,+}_{n+1,j}) = \begin{cases} [f\circ t^{n}_{n-1,i}; g\circ t^{n}_{n-1,i}]^{n-1,+}_{n,j-1} \text{ if } 1\leq i<j\leq n, \\ [f\circ t^{n}_{n-1,i-1}; g\circ t^{n}_{n-1,i-1}]^{n-1,+}_{n,j} \text{ if } 2\leq j+1<i\leq n+1; \end{cases}$

- $[f,f]^{n,+}_{n+1,j} = 1^{n,+}_{n+1,j}(f)$.

Definition 3.10. A cubical theory Θ is contractible if for all integer $n\geq 0$, all pairs (f,g) of I^n-arrows in it which are admissible, have a j-lifting for all $j\in [\![1,n]\!]$, and if for all integer $n\geq 1$, all pairs (f,g) of I^n-arrows in it which are j-admissible ($j\in [\![1,n]\!]$), have a $(j,-)$-lifting and a $(j,+)$-lifting.

Given a cubical theory: $\mathbb{C} \xrightarrow{F} \Theta$, we are going to associate to it, functorialy, another cubical theory: $\mathbb{G} \xrightarrow{F_\infty} \Theta_\infty$, which has the property to be contractible. More precisely we are going to build by induction a directed diagram in $\mathbb{C}$-$\mathbb{T}$h consisting of inclusions of cubical theories:

$$\Theta^0 \overset{i_1}{\hookrightarrow} \Theta^1 \overset{i_2}{\hookrightarrow} \cdots \overset{i_n}{\hookrightarrow} \Theta^n \hookrightarrow \cdots$$

that we denote by

$$(\mathbb{N},\leq) \xrightarrow{\Theta^\bullet} \mathbb{C}\text{-}\mathbb{T}\mathrm{h},$$

such that $\Theta^0=\Theta$ and such that the colimit: colim $\Theta^\bullet$ in $\mathbb{C}$-$\mathbb{T}$h, gives the required contractible cubical theory Θ_∞:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[drrrrrr, "F_\infty"'] & \Theta^0 \arrow[r, hook, "i_1"] \arrow[drrrrr] & \Theta^1 \arrow[r, hook, "i_2"] \arrow[drrrr] & \cdots \arrow[r, hook, "i_n"] & \Theta^n \arrow[r, hook, "i_{n+1}"] \arrow[drr] & \cdots & \\ & & & & & & \Theta_\infty \end{tikzcd}$$

This colimit Θ_∞ of theories is a *coherator* in the sense of Grothendieck ([19]), i.e it is a contractible theory obtained as a colimit of a diagram $\Theta^\bullet$ in $\mathbb{C}$-$\mathbb{T}$h $\subset$ $\mathbb{C}$at of cubical theories. This directed diagram $\Theta^\bullet$ is defined inductively as follow:

- we start the induction with $\Theta^0 = \Theta$;
- we denote by E_0 the set which is the union of all admissible pairs of I^n-arrows in Θ (for all $n \geq 0$), plus all j-admissible pairs of I^n-arrows in Θ (for all $n \geq 1$ and for all directions $j \in [\![1,n]\!]$); thus we obtain the pair (Θ^0, E_0);
- Θ^1 is obtained by formally (see just below a precise meaning of "formally") adding in Θ^0 all kinds of liftings defined above, of elements of E_0; thus we obtain the inclusion of theories:

$$\Theta^0 \hookrightarrow \Theta^1;$$

- denote by E_1 the set which is the union of: all admissible pairs of I^n-arrows (for all $n \geq 0$) in Θ^1 which are not in E_0, all j-admissible pairs of I^n-arrows in Θ^1 (for all $n \geq 1$) which are not in E_0; thus we obtain the pair (Θ^1, E_1) and the construction:

$$(\Theta^0, E_0) \longrightarrow (\Theta^1, E_1)$$

- Θ^2 is obtained by formally adding in Θ^1 all kinds of liftings of elements of E_1;
- we suppose that until the integer $m-1$, thus we suppose that the sequence:

$$(\Theta^0, E_0) \to (\Theta^1, E_1) \to \cdots \to (\Theta^{m-2}, E_{m-2}) \to (\Theta^{m-1}, E_{m-1})$$

  is well defined. Thus Θ^m is obtained by formally adding in Θ^{m-1} all liftings of elements of E_{m-1}; and we obtain the following diagrams of inclusions of theories:

$$\Theta^0 \hookrightarrow \Theta^1 \hookrightarrow \cdots \hookrightarrow \Theta^{m-1} \hookrightarrow \Theta^m$$

- we associate to Θ^m the set E_m which is the union of: all admissible pairs of I^n-arrows in Θ^m (for all $n \geq 0$) which are not in E_{m-1}, and all j-admissible pairs of I^n-arrows (for all $n \geq 1$) in Θ^m which are not in E_{m-1}. This gives the pair (Θ^m, E_m), and completes our induction.

The cubical theory: $\mathbb{G} \xrightarrow{F_{\infty}} \Theta_{\infty}$, is called the cubical coherator associated to the cubical theory: $\mathbb{C} \xrightarrow{F} \Theta$. It is straightforward to see that this construction of coherators associated to cubical theory is functorial, and we get the following endofunctor Φ:

$$\begin{array}{ccc} \mathbb{C}\text{-}\mathbb{T}\mathrm{h} & \xrightarrow{\quad\Phi\quad} & \mathbb{C}\text{-}\mathbb{T}\mathrm{h} \\ & & \\ (F,\Theta) & \longmapsto & (F_{\infty},\Theta_{\infty}) \end{array}$$

Now we are ready to define the cubical coherator Θ^{∞}_{W} which $\mathbb{S}$ets-models are cubical weak ∞-categories with connections: it is defined as the coherator $\Phi(\Theta_M)$ associated to the theory Θ_M which $\mathbb{S}$ets-models are cubical reflexive ∞-magmas. Thus Θ^{∞}_{W} is the colimit in $\mathbb{C}$-$\mathbb{T}$h of the directed diagram $\Theta^{\bullet}_{M}$:

$$\begin{tikzcd}
\Theta^0_M \arrow[r, hook] \arrow[drrrrr] & \Theta^1_M \arrow[r, hook] \arrow[drrrr] & \Theta^2_M \arrow[r, hook] \arrow[drrr] & \cdots \arrow[r, hook] & \Theta^n_M \arrow[r, hook] \arrow[dr] & \Theta^{n+1}_M \arrow[r, hook] \arrow[d, hook] & \cdots \\
 & & & & & \Theta^{\infty}_{W} &
\end{tikzcd}$$

and is called the *cubical coherator* of cubical weak ∞-categories with connections. Denote by $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{W})$ the category of $\mathbb{S}$ets-models of Θ^{∞}_{W}, this is a category of models of cubical weak ∞-categories with connections. As we wrote in the introduction, in [14] we proposed another algebraic approach of cubical weak ∞-categories with connections using the notion of *cubical categorical stretchings*[15], and in [16] we proposed an operadic[16] approach of cubical weak ∞-categories with connections. We believe that these three approaches give equivalent models of cubical weak ∞-categories with connections and we don't hesitate to write the following conjecture:

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[15]These models of cubical weak ∞-categories with connections are the cubical analogue of the Penon's globular weak ∞-categories [20].

[16]These models of cubical weak ∞-categories with connections are the cubical analogue of the Batanin's globular weak ∞-categories [3].

Conjecture 3.11 (Algebraic Models of Cubical Weak ∞-Categories with Connections)**.** The following category of models of Cubical Weak ∞-Categories with Connections are equivalents:

- The category $\mathbb{W}$-$\mathbb{A}$lg of $\mathbb{W}$-algebras for the monad $\mathbb{W}=(W,\eta,\nu)$ acting on $\mathbb{C}\mathbb{S}$ets defined in [14];
- The category $\mathbb{B}^0_C$-$\mathbb{A}$lg of $\mathbb{B}^0_C$-algebras for the cubical operad $\mathbb{B}^0_C$ acting on $\mathbb{C}\mathbb{S}$ets defined in [16];
- The category $\mathbb{M}$od(Θ^∞_W) of $\mathbb{S}$ets-models of the cubical coherator Θ^∞_W defined in this article.

Also the cubical coherator $\Theta^\infty_{W^0}:=\Phi(\Theta_{M^0})$ associated to the theory Θ_{M^0} which $\mathbb{S}$ets-models are cubical reflexive $(\infty,0)$-magmas is the *cubical coherator* of cubical weak ∞-groupoids with connections. This theory $\Theta^\infty_{W^0}$ is the cubical analogue of the globular coherator of Grothendieck ([19]), i.e it is obtained as a colimit in $\mathbb{C}$-$\mathbb{T}$h of the directed diagram $\Theta^\bullet_{M^0}$:

$$\begin{tikzcd}
\Theta^0_{M^0} \arrow[r, hook] \arrow[drrrrr, hook] & \Theta^1_{M^0} \arrow[r, hook] \arrow[drrrr, hook] & \Theta^2_{M^0} \arrow[r, hook] \arrow[drrr, hook] & \cdots \arrow[r, hook] & \Theta^n_{M^0} \arrow[r, hook] \arrow[dr, hook] & \Theta^{n+1}_{M^0} \arrow[r, hook] \arrow[d, hook] & \cdots \\
 & & & & & \Theta^\infty_{W^0} &
\end{tikzcd}$$

Denote by $\mathbb{M}$od$(\Theta^\infty_{W^0})$ the category of $\mathbb{S}$ets-models of $\Theta^\infty_{W^0}$, this is a category of models of cubical weak ∞-groupoids with connections. As we wrote in the introduction, in [15] we proposed another algebraic approach of cubical weak ∞-groupoids with connections using the notion of *cubical groupoidal stretchings*[17]. We believe that these two approaches give equivalent models of cubical weak ∞-groupoids with connections and we don't hesitate to write the following conjecture:

Conjecture 3.12 (Algebraic Models of Cubical Weak ∞-Groupoids with Connections)**.** The following category of models of Cubical Weak ∞-Groupoids with connections are equivalents:

[17]These models of cubical weak ∞-groupoids with connections are the cubical analogue of the globular weak ∞-groupoids defined in [10].

- The category $\mathbb{W}^0$-$\mathbb{A}$lg of $\mathbb{W}^0$-algebras for the monad $\mathbb{W}^0 = (W^0, \eta^0, \nu^0)$ acting on $\mathbb{CS}$ets defined in [15];

- The category $\mathbb{M}$od$(\Theta^{\infty}_{W^0})$ of $\mathbb{S}$ets-models of the cubical coherator $\Theta^{\infty}_{W^0}$ defined in this article.

4 Epilogue

The aim of this epilogue is to describe a wide picture of how to reach cubical weak ∞-topos with cubical coherators. Even if it looks too conjectural, the work which have been done in this article plus the work in [11, 16], show that the discussion and the conjectures below are rather precisely stated. We believe that the easy conjecture will be the precise description of the cubical operads $\mathbb{B}^n_C$ and the cubical coherators $\Theta^{\infty}_{W_n}$ for all $n \geq 1$, for cubical weak (n, ∞)-transformations, plus their induced cocubical objects, and the equivalence of these two approaches (see below). The hard conjecture should be the contractibility of the induced cubical operads of coendomorphisms $\mathbb{C}$OEND$(\mathbb{B}^{\bullet}_C)$ and $\mathbb{C}$OEND$(\Theta^{\infty}_{W_{\bullet}})$ (see below).

After having defined in the last Section 3.2.3 the coherator Θ^{∞}_{W} which $\mathbb{S}$ets-models are cubical weak ∞-categories with connections, it is interesting to know how to weakened cubical strict ∞-functors by coherators. In [15] we weakened cubical strict ∞-functors (with connections) and cubical strict ∞-natural transformations (with connections), using the notion of *cubical stretchings*. In [16] we described[18] a cocubical object of cubical operads in the category $\mathbb{M}$nd of monads:

[18] Actually in [16] the cubical operads $\mathbb{B}^n_C$ for $n \geq 1$ have only been predicted without their precise constructions, however the globular work [11] on globular operads plus the monads $\mathbb{W}^1$ and $\mathbb{W}^1$ described in [15] show clearly that such cocubical object of operads must exist, even though it must be described by us or other mathematicians in future work.

$$
\begin{tikzcd}[column sep=large]
\mathbb{B}^0_C \arrow[r, shift left=1, "s^0_1"] \arrow[r, shift right=1, "t^0_1"'] & \mathbb{B}^1_C \arrow[r, shift left=3, "s^2_{1,2}"] \arrow[r, shift left=1, "s^2_{1,1}"] \arrow[r, shift right=1, "t^2_{1,1}"'] \arrow[r, shift right=3, "t^2_{1,2}"'] & \mathbb{B}^2_C \arrow[r, shift left=5, "s^3_{2,3}"] \arrow[r, shift left=3, "s^3_{2,2}"] \arrow[r, shift left=1, "s^3_{2,1}"] \arrow[r, shift right=1, "t^3_{2,1}"'] \arrow[r, shift right=3, "t^3_{2,2}"'] \arrow[r, shift right=5, "t^3_{2,3}"'] & \mathbb{B}^3_C \cdots \mathbb{B}^{n-1}_C \arrow[r, shift left=9, "s^n_{n-1,1}"] \arrow[r, shift left=7, dotted] \arrow[r, shift left=5, "s^n_{n-1,i}"] \arrow[r, shift left=3, dotted] \arrow[r, shift left=1, "s^n_{n-1,n-1}"] \arrow[r, shift right=1, "t^n_{n-1,n-1}"'] \arrow[r, shift right=3, dotted] \arrow[r, shift right=5, "t^n_{n-1,i}"'] \arrow[r, shift right=7, dotted] \arrow[r, shift right=9, "t^n_{n-1,1}"'] & \mathbb{B}^n_C \cdots
\end{tikzcd}
$$

such that $\mathbb{B}^0_C$ is the $\mathbb{S}^0$-operad[19] which algebras are cubical weak ∞-categories, but also the $\mathbb{S}^1$-operad $\mathbb{B}^1_C$ which algebras are cubical weak ∞-functors, the $\mathbb{S}^2$-operad $\mathbb{B}^2_C$ which algebras are cubical weak ∞-natural transformations, etc. The coherator version of these constructions is now not difficult, however it deserves another article; but we can already sketch the construction of the coherator $\Theta^{\infty}_{W_1}$ such that the category $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{W_1})$ of $\mathbb{S}$ets-models of $\Theta^{\infty}_{W_1}$ are models of cubical weak ∞-functors (with connections). For that we need first to define the category Θ_1 of arities for cubical ∞-functors. Objects of Θ_1 are triple $(X, f(X), X')$ where X and X' are cubical pasting diagrams in the usual sense, except that for X' we need to replace all variables $1(q)$ inside each basic divisors by the variables $2(q)$, i.e the usual pasting diagrams X have the color 1, whereas here cubical pasting diagrams X' have color 2. We need to well distinguished variables in this formalism of ∞-functors. Also $f(X)$ is a kind of formal image of X: $f(X)$ is still a cubical pasting diagram such that if $Adx^i_{k_i}$ is a basic n-divisor of X, then $f_n(A)dx^i_{k_i}$ becomes a basic n-divisor of $f(X)$. All the notions: *cubical extensions*, *cubical theories*, *coherators*, have their counterparts for this theory of cubical ∞-functors. If we denote $\Theta^{\infty}_{W_0} := \Theta^{\infty}_{W}$, then we obtain canonically, a 1-truncated cubical object in $\mathbb{C}$at:

[19]Here $\mathbb{S}^0$ is the monad $\mathbb{S}$ describe in Section 2; $\mathbb{S}^1$ is the monad for cubical strict ∞-functors, $\mathbb{S}^2$ is the monad for cubical strict ∞-natural transformations, and so on.

$$\begin{tikzcd}
\Theta^{\infty}_{W_0} \arrow[r, shift left=2, "s^0_1"] \arrow[r, shift right=2, "t^0_1"'] & \Theta^{\infty}_{W_1}
\end{tikzcd}$$

This 1-truncated cubical object can be extended with the coherator $\Theta^{\infty}_{W_2}$ such that the category $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{W_2})$ of $\mathbb{S}\mathrm{ets}$-models of $\Theta^{\infty}_{W_2}$ is a category of models of cubical weak ∞-natural transformations; thus we obtain a 2-truncated cubical object in $\mathbb{C}\mathrm{at}$:

$$\begin{tikzcd}
\Theta^{\infty}_{W_0} \arrow[r, shift left=2, "s^0_1"] \arrow[r, shift right=2, "t^0_1"'] & \Theta^{\infty}_{W_1} \arrow[r, shift left=6, "s^2_{1,2}"] \arrow[r, shift left=2, "s^2_{1,1}"] \arrow[r, shift right=2, "t^2_{1,1}"'] \arrow[r, shift right=6, "t^2_{1,2}"'] & \Theta^{\infty}_{W_2}
\end{tikzcd}$$

In fact we can draw the cocubical shape of cubical coherators for all cubical weak higher transformations that we hope to describe more accurately in a future work.

$$\begin{tikzcd}
\Theta^{\infty}_{W_0} \arrow[r, shift left=2, "s^0_1"] \arrow[r, shift right=2, "t^0_1"'] & \Theta^{\infty}_{W_1} \arrow[r, shift left=6, "s^2_{1,2}"] \arrow[r, shift left=2, "s^2_{1,1}"] \arrow[r, shift right=2, "t^2_{1,1}"'] \arrow[r, shift right=6, "t^2_{1,2}"'] & \Theta^{\infty}_{W_2} \arrow[r, shift left=10, "s^3_{2,3}"] \arrow[r, shift left=6, "s^3_{2,2}"] \arrow[r, shift left=2, "s^3_{2,1}"] \arrow[r, shift right=2, "t^3_{2,1}"'] \arrow[r, shift right=6, "t^3_{2,2}"'] \arrow[r, shift right=10, "t^3_{2,3}"'] & \Theta^{\infty}_{W_3} \cdots \Theta^{\infty}_{W_{n-1}} \arrow[r, shift left=18, "s^n_{n-1,1}"] \arrow[r, shift left=14, dotted] \arrow[r, shift left=10, "s^n_{n-1,i}"] \arrow[r, shift left=6, dotted] \arrow[r, shift left=2, "s^n_{n-1,n-1}"] \arrow[r, shift right=2, "t^n_{n-1,n-1}"'] \arrow[r, shift right=6, dotted] \arrow[r, shift right=10, "t^n_{n-1,i}"'] \arrow[r, shift right=14, dotted] \arrow[r, shift right=18, "t^n_{n-1,1}"'] & \Theta^{\infty}_{W_n} \cdots
\end{tikzcd}$$

For example the category $\mathbb{M}\mathrm{od}(\Theta^{\infty}_{W_3})$ of $\mathbb{S}\mathrm{ets}$-models of $\Theta^{\infty}_{W_3}$ is a category of models of cubical weak ∞-modifications[20] (cubical weak ∞-modifications are kinds of homotopies between cubical weak ∞-natural transformations). As we wrote in the introduction, in [5] it is proved that Batanin and

[20] Its globular analogue has been described in [9].

Grothendieck approaches of globular weak ∞-categories are both equivalent, and we believe that such equivalences are also true for cubical higher category theory, and not only for the cubical $\mathbb{S}^0$-operad $\mathbb{B}^0_C$ and the coherator $\Theta^\infty_{W_0}$, but also for algebras of $\mathbb{B}^n_C$ and $\mathbb{S}$ets-models of $\Theta^\infty_{W_n}$ for all integers $n \geq 1$. Let us renamed ∞-categories by: $(0,\infty)$-transformations, ∞-functors by: $(1,\infty)$-transformations, ∞-natural transformations by: $(2,\infty)$-transformations, etc. The following conjecture shoul-dn't be too difficult according to the globular result in [5]:

Conjecture 4.1 ($\mathbb{B}^n_C$-$\mathbb{A}$lg $\simeq \mathbb{M}$od$(\Theta^\infty_{W_n})$ for all $n \in \mathbb{N}$)**.** We have the equivalences $\mathbb{B}^n_C$-$\mathbb{A}$lg $\simeq \mathbb{M}$od$(\Theta^\infty_{W_n})$ for all $n \in \mathbb{N}$, where objects in $\mathbb{B}^n_C$-$\mathbb{A}$lg are operadic models of cubical weak (n,∞)-transformations (with connections), and where objects in $\mathbb{M}$od$(\Theta^\infty_{W_n})$ are sketches models of cubical weak (n,∞)-transformations (with connections).

The cocubical object $\Theta^\infty_{W_\bullet}$ above is a cocubical object in the 2-topos $\mathbb{C}$at of small categories and thanks to the work [16], we get a cubical operad of coendomorphisms $\mathbb{C}$OEND$(\Theta^\infty_{W_\bullet})$. As in [16] we can state the following conjecture.

Conjecture 4.2 (Contractibility of $\mathbb{C}$OEND$(\Theta^\infty_{W_\bullet})$)**.** The cubical operad $\mathbb{C}$OEND$(\Theta^\infty_{W_\bullet})$ is equipped with a composition system and is contractible in the sense of [16].

If this conjecture is true then we get a unique morphism !:

$$\mathbb{B}^0_C \xrightarrow{\quad ! \quad} \mathbb{C}\mathrm{OEND}(\Theta^\infty_{W_\bullet})$$

of cubical operads, which reveals that the cocubical object $\Theta^\infty_{W_\bullet}$ is a $\mathbb{B}^0_C$-coalgebra. If we accept the conjecture above

$$\mathbb{B}^n_C\text{-}\mathbb{A}\mathrm{lg} \simeq \mathbb{M}\mathrm{od}(\Theta^\infty_{W_n}) \text{ for all } n \in \mathbb{N},$$

plus this coalgebricity of $\Theta^\infty_{W_\bullet}$, these show that the cubical weak ∞-category with connections of cubical weak ∞-categories with connections exists with the formalism of coherators. Thus these conjectures open the perspective of an accurate approach by coherators of cubical weak ∞-topos of Grothendieck with connections, and also the perspective to have an accurate approach by coherators of cubical weak ∞-stacks with connections.

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