Categories and
General Algebraic Structures
with Applications

In press.

Combinatorial approach of the category Θ_0 of cubical pasting diagrams

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Abstract. In globular higher category theory the small category Θ_0 of finite rooted trees plays an important role: for example the objects of Θ_0 are the arities of the operations inside the free globular ω-operad $\mathbb{B}^0$ of Batanin, which $\mathbb{B}^0$-algebras are models of globular weak ∞-categories; also this globular Θ_0 is an important tool to build the coherator $\Theta^{\infty}_{W^0}$ of Grothendieck which $\mathbb{S}$ets-models are globular weak ∞-groupoids. Cubical higher category needs similarly its Θ_0. In this work we describe, combinatorially, the small category Θ_0 which objects are cubical pasting diagrams and which morphisms are morphisms of cubical sets.

1 Introduction

In globular higher category theory the small category Θ_0 of globular pasting diagrams plays an important role: for example the objects of Θ_0 are the arities of the operations inside the free globular ω-operad $\mathbb{B}^0$ of Batanin, which $\mathbb{B}^0$-algebras are algebraic models of globular weak ∞-categories; also this small category Θ_0 is an important tool to build the *coherator* $\Theta^{\infty}_{W^0}$

Keywords: Pasting diagrams, pasting schemes, sketch theory, higher order terms.
Mathematics Subject Classification [2010]: 03B15, 03B38, 03G99.
Received: 14 September 2023, Accepted: 29 October 2023.
ISSN: Print 2345-5853 Online 2345-5861.

of Grothendieck which $\mathbb{S}$ets-models are globular weak ∞-groupoids. Cubical higher category needs similarly its Θ_0. In this work we describe, combinatorially, the small category Θ_0 which objects are cubical pasting diagrams and which morphisms are morphisms of cubical sets. The monad $\mathbb{R} = (R, i, m)$ acting on the category of cubical sets, which algebras are cubical sets equipped with degeneracies, exhibits the sorts $A \in R(1)(n)$ of operations of the cubical theory, and the cubical pasting diagrams play the role of arities for these operations. Cubical pasting diagrams of dimension $n \in \mathbb{N}^*$ are rectangular finite conglomerate of basic n-cubes[1], where basic n-cubes are just elements $A \in R(1)(n)$. In order to build such conglomerate we need to have a good control of its basic n-cubes and a canonical way to control these basic cubical shapes is to equipped each basic n-cubes with a coordinate $(k_1, \cdots, k_n)$ in the network $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times; $\mathbb{Z}$ is the set of integers) in order it to be well located. Thus a basic n-cube is now a formal expression: $A(k_1, \cdots, k_n)$ which means that the n-cell $A \in R(1)(n)$ is located in the coordinate $(k_1, \cdots, k_n)$. But $A \in R(1)(n)$ is in particular an n-cube and it has faces which are $(n-1)$-cubes $s^n_{n-1,j}(A)$, and we can ask then what are the coordinate of it? The first approximation is to say that it is located in the coordinate $(k_1, \cdots, \widehat{k_j}, \cdots, k_n)$, which means that we removed k_j and it has the coordinate $(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n)$ in the network $\mathbb{Z}^{n-1}$. Also $A \in R(1)(n)$ can be degenerate in the $(n+1)$-cube $1^n_{n+1,j}(A)$ (classical degeneracies) or in the $(n+1)$-cube $1^{n,-}_{n+1,j}(A)$ (connections), and here we are attempted to say, at first approximation, that both are located in the coordinate $(k_1, \cdots, k_{j-1}, 1, k_j, \cdots, k_n)$, which means that we added 1 and it has coordinate in the network $\mathbb{Z}^{n+1}$. Such remove and addition of coordinate is well-known for tensor calculus in differential geometry, under the names *contraction of a tensor* and *dilatation of a tensor*. Thus we have chosen to use this tensorial notation to describe coordinates attached to basic n-cubes: the expression $Adx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$ says that the n-cell $A \in R(1)(n)$ is located in the coordinate $(k_1, \cdots, k_n)$, and in our jargon the tensor $dx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$ is called a coordinate. We shall often use the abbreviation $dx^i_{k_i}$ for the coordinate $dx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$. Also the face $s^n_{n-1,j}(A) \in R(1)(n-1)$ is located in the j-contraction of $dx^i_{k_i}$,

[1]We shall consider also some basic 0-cubes as 0-dimensional cubical pasting diagram, but not amalgamation of it, just because it doesn't make sense to glue such 0-cubes.

thus we write: $s^n_{n-1,j}(A)dx^1_{k_1}\otimes\cdots\otimes\widehat{dx^j_{k_j}}\otimes\cdots dx^n_{k_n}$, where here this j-contraction means the tensor $dx^1_{k_1}\otimes\cdots\otimes dx^{j-1}_{k_{j-1}}\otimes dx^j_{k_{j+1}}\cdots dx^{n-1}_{k_n}$; also the coordinate of the basic $(n+1)$-cube $1^n_{n+1,j}(A)$ is given by the j-dilatation $dx^1_{k_1}\otimes\cdots\otimes dx^{j-1}_{k_{j-1}}\otimes dx^j_1\otimes dx^{j+1}_{k_j}\otimes\cdots dx^{n+1}_{k_n}$ of $dx^i_{k_i}$.

However we need to reinforce this tensorial formalism in order it feats perfectly with the basic datas needed for cubical higher category; as a matter of fact, if $A\in R(1)(n)$ has coordinate $dx^1_{k_1}\otimes\cdots\otimes dx^n_{k_n}$ then two different faces of A may be different but with the same coordinate; for example if A is the degenerate 2-cube $1^{1,-}_{2,1}(1(1))\in R(1)(2)$:

$$A=\begin{tikzcd} 1(0) \arrow[r, "1(1)"] \arrow[d, "1(1)"'] \arrow[dr, phantom, "1^{1,-}_{2,1}(1(1))"] & 1(0) \arrow[d, "1^0_1(1(0))"] \\ 1(0) \arrow[r, "1^0_1(1(0))"'] & 1(0) \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd}$$

with coordinate $dx^1_{-4}\otimes dx^2_1$ (which means that A has coordinate $(-4,1)$ in $\mathbb{Z}^2$), then its faces: $s^2_{1,1}(A)=1(1)$ and $t^2_{1,1}(A)=1^0_1(1(0))$ have both the same coordinate dx^1_1 (it is reindexed after contraction). This example shows that the tensorial formalism alone leads to a lack of control of our cells, because we need that each part of our cubical pastings to be located individually. In order to remove such pathologies we are going to enriched the tensorial formalism with a concept of *formal box* which feats better with the entire shapes of each basic n-cubes in $R(1)(n)$. These formal boxes are specific degenerate n-cubes $(\square^{dx^i_{k_i}}_{1(n)},\equiv_A)$ called *degenerate boxes* which are equipped with coordinates $dx^i_{k_i}$ and whose aim is to contain any n-cubes which have the same degeneracies as $A\in R(1)(n)$. As we wrote above, the expression $Adx^i_{k_i}$ says that the n-cell A ($A\in R(1)(n)$) is located in the coordinate $dx^i_{k_i}$, and the expression $B(\square^{dx^i_{k_i}}_{1(n)},\equiv_A)$ now means that the n-cube $B\in R(X)(n)$ (here X is any cubical set) has the same degeneracies as $A\in R(1)(n)$, and B is inside the degenerate box $(\square^{dx^i_{k_i}}_{1(n)},\equiv_A)$ which itself is located in the coordinate $dx^i_{k_i}$. When $B=A$ then $A(\square^{dx^i_{k_i}}_{1(n)},\equiv_A)$ is called a *basic divisor.*

Such basic divisors $A(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$ are written $Adx^i_{k_i}$ to simplify the notation after the Section 5. We take advantage of this formalism to describe the monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets with these basic divisors, and show that $\mathbb{R} = (R, i, m)$ is a cartesian monad (5).

The definition of the degenerate boxes $(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$ $(A \in R(1)(n))$ in 4 are preceded by the definition of the basic one $\square^{dx^i_{k_i}}_{1(n)}$ in 3 called the *basic boxes*, i.e they are boxes with no degeneracies. These formal boxes (degenerates or not) are congruences of terms of a language $\mathcal{L}'_n = (\mathbb{Z}^n, \{-, +\})$ containing tensors $dx^i_{k_i}$, contractions of these tensors, and elements in $\{-, +\}$, as basic datas. The terms that we consider for this language are called *links* because their role is to exhibit a link between such formal boxes with their faces. Basic n-divisors are terms for a language $\mathcal{L}_n = (\mathcal{L}'_n, R(1)(n))$, and they constitute the basic pieces for rectangular n-divisors. Rectangular n-divisors are defined inductively as terms of a language $\mathbb{L}_n = (\mathcal{L}_n, (\circ^n_j)_{j \in [\![1;n]\!]})$. This inductive approach was possible thanks to the good control of the different faces that have the basic n-divisors. Rectangular n-divisors are written

$$X = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i},$$

and are characterized by a *rectangular n-configuration* C_n, i.e a finite subset of $\mathbb{Z}^n$ of the form

$$C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!]$$

such that the set $\{A_1 dx^i_{k^1_i}, \cdots, A_l dx^i_{k^l_i}, \cdots, A_r dx^i_{k^r_i}\}$ are the basic n-divisors of X, where now X can be seen as its n-configuration C_n *weighted* by this set of basic n-divisors. These rectangular n-divisors are our models of cubical pasting diagrams, and as we have expected they behave very well: thanks to their rectangular shapes they have a notion of sources and targets which allow them to be composed, but they can also be degenerated, etc. Thus they produce a cubical strict ∞-category[2] $\bullet$-$\mathbb{R}$ectDiv (6), and also the small category Θ_0 of cubical pasting diagrams. The small category Θ_0 is defined as the full subcategory of the category $\mathbb{CS}$ets of cubical sets which objects

[2]In fact the free cubical strict ∞-category $S(1)$ on a terminal object of $\mathbb{CS}$ets, where S is the underlying endofunctor of the monad $\mathbb{S} = (S, \lambda, \mu)$ of cubical strict ∞-categories with connections described in [3, 5].

are cubical pasting diagrams. The category Θ_0 can be seen as a cubical variation of Lawvere theory. Also each rectangular n-divisor X has an internal notion of sources and targets, which lead to an inductive sketch $\mathcal{E}_X$. This inductive sketch is built with the help of the formal sketch $\mathcal{E}_{C_n}$ of its n-configuration, which itself comes canonically from the lexicographical order on C_n (6). Thus we see here another crucial roles of coordinates which is to exhibit in a canonical way (with their intrinsic lexicographical features) the sketches of rectangular divisors. Formally these inductive sketches behaves like rectangular divisors, thus they are cubical sets which can be degenerated, composed, etc. Thus these sketches lead to another presentation of Θ_0 and to another presentation of the cubical strict ∞-category of cubical pasting diagrams. In the end of 7 we show that, for each rectangular n-divisor X, its associated sketch $\mathcal{E}_X$ is canonically a $(n-1)$-cubical object in the category $\mathbb{S}$ketch of sketches.

We can summarize the main definitions here:

- $Adx^i_{k_i}$ means that the n-cube A is located at the coordinate $dx^i_{k_i}$, see 2; usually $A \in R(1)(n)$ where R is the underlying endofunctor of the monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets; see 5;
- Basic box $\square^{dx^i_{k_i}}_{1(n)}$ of a coordinate $dx^i_{k_i}$, see 3; and degenerate boxes $(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$, see 4; they are formal boxes whose aim is to contains n-cubes. These formal boxes provide a better control of the coordinates, than the tensors, of the faces of the cubes they contain: their formalism allows to have an inductive definition of cubical pasting diagrams (6);
- Basic divisor: $B(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$; see 5; this is an n-cube B inside the n-box $(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$;
- Rectangular n-divisors: formal sum $X = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i}$, of basic one with rectangular shape, see 6; they are our models of cubical pasting diagrams;
- for each rectangular divisor X we associate an inductive sketch $\mathcal{E}_X$, see 7.

Applications of this cubical $\Theta_0 = (1 \downarrow S(1))$ are done in [5], where the monad $\mathbb{S} = (S, \lambda, \mu)$ of cubical strict ∞-categories with connections is described with the objects of Θ_0, and it is shown that it is a cartesian monad, solving a conjecture in [6]. Also in [5], two cubical coherators are defined with our cubical Θ_0: the coherator $\Theta^\infty_{W^0}$ which $\mathbb{S}$ets-models are cubical weak ∞-groupoids with connections, and the coherator Θ^∞_W which $\mathbb{S}$ets-models are cubical weak ∞-categories with connections.

The author has done previous work on cubical higher categories, some of them are published, see [3, 4]; but others were archived in IHES and removed after five years, see [6, 6]. We hope to make again available the work in [6, 6] very soon.

This article may be seen as an improved version of some aspects of the Arxiv version [2]. In [2] some materials were described for the question of pastings objects with cubical shapes in full generality, not only for the simpler cases of rectangular pastings as in the present work. Even if main ideas of this arxived version remain correct (like the idea of using coordinates to control the gluings), this article focuses only on rectangular gluings, which not only simplify the story, but is also more relevant for our main goals, i.e to capture objects in Θ_0.

2 Tensorial notation

The reader may read the first section in [3] for reminders of the basic definitions in cubical higher category theory: definition of cubical sets, definition of cubical strict ∞-categories with connections, etc.

Here we introduce tensorial notation and shall see that contraction and dilatation of tensors provide interesting structure for cubical sets, though trivial. This study (and introduction of tensorial notations) reveals the intrinsic cubical nature of tensorial calculus.

For each $n \in \mathbb{N}$, the n-dimensional network $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers) is used as a coordinate system; the elements $(k_1, \cdots, k_n) \in \mathbb{Z}^n$ are *coordinates* and are preferably denoted instead with the tensorial notation $dx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$ in order to freely use the dilatations and contractions operations on it, where these operations are commonly used in tensor calculus. When no confusion occur we write $dx^i_{k_i} := dx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$. These coordinates $dx^i_{k_i} \in \mathbb{Z}^n$ are used to indexed n-cubes, in order to well located them and to build conglomerates of n-cubes. Thus if A is an n-

cube, the notation $Adx^1_{k_1}\otimes\cdots\otimes dx^n_{k_n}$ means that A has the coordinate $dx^1_{k_1}\otimes\cdots\otimes dx^n_{k_n}$, which means that A is located at the *depth* $k_j\in\mathbb{Z}$ for the direction $j\in\{1,\cdots,n\}$.

Remark 2.1. It is easy to see that two coordinates $dx^i_{k_i}, dx^i_{k'_i}\in\mathbb{Z}^n$ are linked by translations. For example any coordinates $dx^i_{k_i}\in\mathbb{Z}^n$ gives the coordinate $dx^i_1 := dx^1_1\otimes\cdots\otimes dx^j_1\otimes\cdots dx^n_1$ by translations along all directions $j\in[\![1,n]\!]$. Usually we shall work with finite subsets $C_n\subset\mathbb{Z}^n$ named *n-configurations* below 6, and these n-configurations must be thought up to their translations in the network $\mathbb{Z}^n$. For example we have the *rectangular n-configurations* 6 which are n-configurations with a specific shape. Any translation of a rectangular n-configuration is still rectangular, and in fact these translations give the same rectangular n-configuration but with different coordinates (see 6).

Two coordinates $dx^i_{k_i} = dx^1_{k_1}\otimes\cdots\otimes dx^n_{k_n}$ and $dx^i_{k'_i} = dx^1_{k'_1}\otimes\cdots\otimes dx^n_{k'_n}$ are *j-adjacent* if for all $i\in[\![1,n]\!]\backslash j$, $k_i = k'_i$, and if $k_j = k'_j+1$ or $k_j = k'_j-1$.

The *j-contraction* of the coordinate $dx^1_{k_1}\otimes\cdots\otimes dx^j_{k_j}\otimes\cdots dx^n_{k_n}$ is defined as the coordinate

$$dx^i_{k_i}\setminus j = dx^1_{k_1}\otimes\cdots\otimes \widehat{dx^j_{k_j}}\otimes\cdots dx^n_{k_n},$$

in $\mathbb{Z}^{n-1}$ defined by removing the direction j and re-indexing:

$$dx^i_{k_i}\setminus j := dx^1_{k_1}\otimes\cdots\otimes dx^{j-1}_{k_{j-1}}\otimes dx^j_{k_{j+1}}\otimes\cdots dx^{n-1}_{k_n};$$

sometimes we use also the notation $c_j(dx^i_{k_i})$ for $dx^i_{k_i}\setminus j$. If we apply these contractions p-times then we obtain the following coordinate in $\mathbb{Z}^{n-p}$:

$$dx^i_{k_i}\setminus(j_1,\cdots,j_p),$$

where the order of occurrences of the $j's$ in $(j_1,\cdots,j_p)$ is important just because if σ is an element of the permutation group S_p then the action

$$\sigma\cdot dx^i_{k_i}\setminus(j_1,\cdots,j_p) := dx^i_{k_i}\setminus(j_{\sigma(1)},\cdots,j_{\sigma(p)}),$$

does not imply the equality between $dx^i_{k_i}\backslash(j_1,\cdots,j_p)$ and $dx^i_{k_i}\backslash(j_{\sigma(1)},\cdots,j_{\sigma(p)})$.

The j-*dilatation* of the coordinate $dx^1_{k_1} \otimes \cdots \otimes dx^j_{k_j} \otimes \cdots dx^n_{k_n}$ is a coordinate in $\mathbb{Z}^{n+1}$ defined by adding dx^j_1 in the direction j and re-indexing:

$$dx^i_{k_i} + j := dx^1_{k_1} \otimes \cdots \otimes dx^{j-1}_{k_{j-1}} \otimes dx^j_{k_1} \otimes dx^{j+1}_{k_j} \cdots dx^{n+1}_{k_n};$$

sometimes we use the notation $d_j(dx^i_{k_i})$ for $dx^i_{k_i} + j$; and if we apply these dilatations p-times then we obtain the following coordinate in $\mathbb{Z}^{n+p}$:

$$dx^i_{k_i} + (j_1, \cdots, j_p),$$

where the order of occurrences of the $j's$ in $(j_1, \cdots, j_p)$ is important.

Remark 2.2. The previous dilatation:

$$dx^i_{k_i} + j := dx^1_{k_1} \otimes \cdots \otimes dx^{j-1}_{k_{j-1}} \otimes dx^j_{k_1} \otimes dx^{j+1}_{k_j} \cdots dx^{n+1}_{k_n},$$

built by adding dx^j_1 is a convention, but we could add instead dx^j_k $(k \in \mathbb{Z})$ if necessary.

Consider now the following diagrams of different network $\mathbb{Z}^n$ for all $n \in \mathbb{N}$, such that $\mathbb{Z}^0$ is the singleton set $\{*\}$:

$$\cdots \mathbb{Z}^n \begin{array}{c} \xrightarrow{\sigma^n_{n-1,n}} \\ \dashrightarrow \\ \xrightarrow{\sigma^n_{n-1,i}} \\ \dashrightarrow \\ \xrightarrow{\sigma^n_{n-1,1}} \\ \xrightarrow[\tau^n_{n-1,1}]{} \\ \dashrightarrow \\ \xrightarrow[\tau^n_{n-1,i}]{} \\ \dashrightarrow \\ \xrightarrow[\tau^n_{n-1,n}]{} \end{array} \mathbb{Z}^{n-1} \cdots \mathbb{Z}^4 \begin{array}{c} \xrightarrow{\sigma^4_{3,4}} \\ \xrightarrow{\sigma^4_{3,3}} \\ \xrightarrow{\sigma^4_{3,2}} \\ \xrightarrow{\sigma^4_{3,1}} \\ \xrightarrow[\tau^4_{3,1}]{} \\ \xrightarrow[\tau^4_{3,2}]{} \\ \xrightarrow[\tau^4_{3,3}]{} \\ \xrightarrow[\tau^4_{3,4}]{} \end{array} \mathbb{Z}^3 \begin{array}{c} \xrightarrow{\sigma^3_{2,3}} \\ \xrightarrow{\sigma^3_{2,2}} \\ \xrightarrow{\sigma^3_{2,1}} \\ \xrightarrow[\tau^3_{2,1}]{} \\ \xrightarrow[\tau^3_{2,2}]{} \\ \xrightarrow[\tau^3_{2,3}]{} \end{array} \mathbb{Z}^2 \begin{array}{c} \xrightarrow{\sigma^2_{1,2}} \\ \xrightarrow{\sigma^2_{1,1}} \\ \xrightarrow[\tau^2_{1,1}]{} \\ \xrightarrow[\tau^2_{1,2}]{} \end{array} \mathbb{Z}^1 \begin{array}{c} \xrightarrow{\sigma^1_0} \\ \xrightarrow[\tau^1_0]{} \end{array} \mathbb{Z}^0$$

where for all positive integers $n \geq 2$ and all direction $i \in [\![1;n]\!]$, $\sigma^n_{n-1,i} =$

$\tau^n_{n-1,i} = c_i$ (the i-contractions); we also have the diagram:

$$
\begin{tikzcd}[column sep=large]
\mathbb{Z}^0 \arrow[r, "1^0_1"] & \mathbb{Z} \arrow[r, shift left=1, "1^1_{2,1}"] \arrow[r, shift right=1, "1^1_{2,2}"'] & \mathbb{Z}^2 \arrow[r, shift left=2, "1^2_{3,1}"] \arrow[r, "1^2_{3,2}"'] \arrow[r, shift right=2, "1^2_{3,3}"'] & \mathbb{Z}^3 \arrow[r, shift left=3, "1^3_{4,1}"] \arrow[r, shift left=1, "1^3_{4,2}"] \arrow[r, shift right=1, "1^3_{4,3}"'] \arrow[r, shift right=3, "1^3_{4,4}"'] & \mathbb{Z}^4 \cdots
\end{tikzcd}
$$

where $1^0_1(*) = dx^1_1$, and for all $n \geq 1$ and all $i \in [\![1; n+1]\!]$, $1^n_{n+1,i} = d_i$ (the i-dilatations), and the diagram:

$$
\begin{tikzcd}[column sep=large]
\mathbb{Z} \arrow[r, shift left=1, "1^{1,-}_{2,1}"] \arrow[r, shift right=1, "1^{1,+}_{2,1}"'] & \mathbb{Z}^2 \arrow[r, shift left=3, "1^{2,-}_{3,2}"] \arrow[r, shift left=1, "1^{2,-}_{3,1}"] \arrow[r, shift right=1, "1^{2,+}_{3,1}"'] \arrow[r, shift right=3, "1^{2,+}_{3,2}"'] & \mathbb{Z}^3 \arrow[r, shift left=5, "1^{3,-}_{4,3}"] \arrow[r, shift left=3, "1^{3,-}_{4,2}"] \arrow[r, shift left=1, "1^{3,-}_{4,1}"] \arrow[r, shift right=1, "1^{3,+}_{4,1}"'] \arrow[r, shift right=3, "1^{3,+}_{4,2}"'] \arrow[r, shift right=5, "1^{3,+}_{4,3}"'] & \mathbb{Z}^4 \arrow[r, shift left=7, "1^{4,-}_{5,4}"] \arrow[r, shift left=5, "1^{4,-}_{5,3}"] \arrow[r, shift left=3, "1^{4,-}_{5,2}"] \arrow[r, shift left=1, "1^{4,-}_{5,1}"] \arrow[r, shift right=1, "1^{4,+}_{5,1}"'] \arrow[r, shift right=3, "1^{4,+}_{5,2}"'] \arrow[r, shift right=5, "1^{4,+}_{5,3}"'] \arrow[r, shift right=7, "1^{4,+}_{5,4}"'] & \mathbb{Z}^5 \cdots
\end{tikzcd}
$$

where here for all $n \geq 1$, and all $i \in [\![1; n]\!]$, $1^{n,\gamma}_{n+1,i} = d_i$ (the i-dilatations), then it is straightforward to see that contractions and dilatations put a structure of cubical reflexive set (see [3]) on the collection of networks $\mathbb{Z}^\bullet = (\mathbb{Z}^n)_{n\in\mathbb{N}}$.

3 The basic boxes $\square^{dx^i_{k_i}}_{1(n)}$ of coordinates $dx^i_{k_i}$

A crucial and straightforward fact from the previous section is that given a coordinate $dx^i_{k_i}$ in $\mathbb{Z}^n$, it has a *trivial structure* of n-cubical set[3] where sources and targets are defined by contractions:

- $s^n_{n-1,j}(dx^i_{k_i}) = t^n_{n-1,j}(dx^i_{k_i}) := dx^i_{k_i} \setminus j$,

[3] And also a trivial structure of reflexive n-cubical set.

- $s^{n-p}_{n-p-1,k}(dx^i_{k_i} \setminus (j_1, \cdots, j_p)) = t^{n-p}_{n-p-1,k}(dx^i_{k_i} \setminus (j_1, \cdots, j_p)) := dx^i_{k_i} \setminus (j_1, \cdots, j_p, k)$

thus different contractions of $dx^i_{k_i}$ are the faces of its underlying trivial n-cubical set.

However this structure of n-cube that $dx^i_{k_i}$ has is too trivial because it does not distinguished sources and targets with the same direction j. And this distinction is crucial because our idea is too label any n-cubical sets A with a coordinate $dx^i_{k_i}$ of $\mathbb{Z}^n$, such that faces of A must have new coordinates $dx^i_{k_i} \setminus (j_1, \cdots, j_p)$ build by contractions and weighted by a notion of sources and targets. In order to correct this default we are going to enriched the coordinates with a notion of *link*, which are roughly speaking *coordinates equipped with* the symbols $\{-, +\}$.

Thus for each coordinate $dx^i_{k_i}$ of the network $\mathbb{Z}^n$ we shall associate an other n-cubical set $\square^{dx^i_{k_i}}_{1(n)}$ called *the box* of $dx^i_{k_i}$ and which formalise better the notion of n-cubical set A labelled by $dx^i_{k_i}$, in the sense that sources and targets of A are then labelled with weighted coordinates, which give the right information of the location of faces of A. Without these weights any p-face of A which is a source in the direction j has the same coordinate (because the trivial structure collapse this source-target information) as the other p-face of A which is a target in the same direction j, and this is counterintuitive: the role of $\square^{dx^i_{k_i}}_{1(n)}$ is to distinguished well coordinates of any faces of any n-cubical set labelled with the coordinate $dx^i_{k_i}$. This section is devoted to the description of these boxes $\square^{dx^i_{k_i}}_{1(n)}$.

Given a coordinate $dx^i_{k_i}$ and the elementary n-cube $1(n)$ (which is the unique n-cell of the cubical site $\mathbb{C}$), we associate to it a formal *free box* [4] $\square^{dx^i_{k_i}}_{1(n)}$ which is a non-degenerate n-cubical set which faces are congruences of terms for a language, and these terms are called here *links*. The datas of this language are the different contractions of $dx^i_{k_i}$: $dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_p)$ plus two symbols $\{-, +\}$ which label these contractions. These symbols $\{-, +\}$ must be interpreted as sources and targets of the different contractions they equipped, and provide a good notion of sources and targets for $\square^{dx^i_{k_i}}_{1(n)}$. The

[4] Here *free* has not to be interpreted in the algebraic sense of "freeness", but instead it must be interpreted as a box not linked with a higher dimensional box.

terms of this language are built inductively (see below) and congruences on it use notions of zigzag build with the cubical identities of sources and targets (see below). This n-cubical set $\square^{dx^i_{k_i}}_{1(n)}$ is called *the basic box with coordinate* $dx^i_{k_i}$. This box $\square^{dx^i_{k_i}}_{1(n)}$ is not degenerate because it is build with $1(n)$ which is non degenerate, and its degeneracies are discussed in the next section where *degenerate boxes* will be defined.

Remark 3.1. An other possible description of faces of $\square^{dx^i_{k_i}}_{1(n)}$ is given in the remark below, which looks more natural (it uses the Reverse Polish Notation), but less intuitive for us. Perhaps in the future we would prefer these *RPN notations.*

In this section we will describe only the underlying cubical set of the box $\square^{dx^i_{k_i}}_{1(n)}$ and degeneracies of it shall be described only in the next section, because they are more subtile and involve *notions of dilated free boxes equipped congruences for degeneracies* (see below). As we wrote in the previous section the role of $\square^{dx^i_{k_i}}_{1(n)}$ can be summarized as follow: if an n-cubical set X is labelled by a coordinate $dx^i_{k_i}$ it means that it is contained in the box $\square^{dx^i_{k_i}}_{1(n)}$ which faces are congruences of links.

The box $\square^{dx^i_{k_i}}_{1(n)}$ and all faces of $\square^{dx^i_{k_i}}_{1(n)}$ have underlying free boxes (see below). But when we consider the box associated to a face of $\square^{dx^i_{k_i}}_{1(n)}$ we forget that it was "linked" to $\square^{dx^i_{k_i}}_{1(n)}$ and then this box is named "free".

In order to keep the *linked information* of the faces of $\square^{dx^i_{k_i}}_{1(n)}$ we write these links as finite sequences of the form:

$$X = (dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_r), \pm)).$$

We can define them by finite decreasing induction:

Definition 3.2. • For any direction $j \in [\![1, n]\!]$, the term $s^n_{n-1,j}(\square^{dx^i_{k_i}}_{1(n)}) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j, -))$ and the term $t^n_{n-1,j}(\square^{dx^i_{k_i}}_{1(n)}) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j, +))$

are 1-links which must be interpreted respectively as the j-source and the j-target of the box $\square^{dx^i_{k_i}}_{1(n)}$.

- If $X = (dx^i_{k_i}, (dx^i_{k_i}\backslash j_1, \pm), (dx^i_{k_i}\backslash(j_1, j_2), \pm), ..., (dx^i_{k_i}\backslash(j_1, j_2, \cdots, j_{n-r}), \pm))$ is an $(n-r)$-link of the box $\square^{dx^i_{k_i}}_{1(n)}$, then for any direction $j \in [\![1, r]\!]$, the terms:

$$s^r_{r-1,j}(X) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., \\ (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm), (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j), -))$$

$$t^r_{r-1,j}(X) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., \\ (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm), (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j), +))$$

are $(n - r - 1)$-links of $\square^{dx^i_{k_i}}_{1(n)}$.

- $(n - r)$-links of sources-targets of $\square^{dx^i_{k_i}}_{1(n)}$, or $(n - r)$-links of $\square^{dx^i_{k_i}}_{1(n)}$ for short, are given by such sequences:

$$(dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm)).$$

Some notations shall be useful:

$$s^n_{n_2,j^1} := s^{n_2+1}_{n_2,j^1_{n_2+1}} \circ s^{n_2+2}_{n_2+1,j^1_{n_2+2}} \cdots \circ s^{n-1}_{n-2,j^1_{n-1}} \circ s^n_{n-1,j^1_n},$$

where $j^1 = (j^1_n, \cdots, j^1_{n_2+1})$ and $j^1_n \in [\![1, n]\!], j^1_{n-1} \in [\![1, n-1]\!], \cdots, j^1_{n_2+1} \in [\![1, n_2 + 1]\!]$;

$$t^n_{n_2,j^1} := t^{n_2+1}_{n_2,j^1_{n_2+1}} \circ t^{n_2+2}_{n_2+1,j^1_{n_2+2}} \cdots \circ t^{n-1}_{n-2,j^1_{n-1}} \circ t^n_{n-1,j^1_n},$$

where $j^1 = (j^1_n, \cdots, j^1_{n_2+1})$ and $j^1_n \in [\![1, n]\!], j^1_{n-1} \in [\![1, n-1]\!], \cdots, j^1_{n_2+1} \in [\![1, n_2 + 1]\!]$.

Also for any partition $n_p < n_{p-1} < \cdots < n_k < \cdots < n_2 < n_1 = n$ with $(p - 1)$ intervals $[\![n_{k+1}, n_k]\!]$ we have 6 different zigzags of sources and targets:

- $s^n_{n_2,j} := s^{n_2+1}_{n_2,j_{n_2+1}} \circ s^{n_2+2}_{n_2+1,j_{n_2+2}} \cdots \circ s^{n-1}_{n-2,j_{n-1}} \circ s^n_{n-1,j_n}$ where $j=(j_n,\cdots,j_{n_2+1})$ and $j_n \in [\![1,n]\!], j_{n-1} \in [\![1,n-1]\!],\cdots,j_{n_2+1} \in [\![1,n_2+1]\!]$ called string of sources (or string of type s);
- $t^n_{n_2,j} := t^{n_2+1}_{n_2,j_{n_2+1}} \circ t^{n_2+2}_{n_2+1,j_{n_2+2}} \cdots \circ t^{n-1}_{n-2,j_{n-1}} \circ t^n_{n-1,j_n}$ where $j=(j_n,\cdots,j_{n_2+1})$ and $j_n \in [\![1,n]\!], j_{n-1} \in [\![1,n-1]\!],\cdots,j_{n_2+1} \in [\![1,n_2+1]\!]$, called string of targets (or string of type t);
- $s^{n_{p-1}}_{n_p,j^{p-1}} \circ t^{n_{p-2}}_{n_{p-1},j^p} \cdots t^{n_k}_{n_{k+1},j^k} \circ s^{n_{k-1}}_{n_k,j^{k-1}} \cdots t^{n_2}_{n_3,j^2} \circ s^n_{n_2,j^1}$, called zigzag of sources-targets of type (s,s);
- $s^{n_{p-1}}_{n_p,j^{p-1}} \circ t^{n_{p-2}}_{n_{p-1},j^p} \cdots t^{n_k}_{n_{k+1},j^k} \circ s^{n_{k-1}}_{n_k,j^{k-1}} \cdots s^{n_2}_{n_3,j^2} \circ t^n_{n_2,j^1}$, called zigzag of sources-targets of type (s,t);
- $t^{n_{p-1}}_{n_p,j^{p-1}} \circ s^{n_{p-2}}_{n_{p-1},j^p} \cdots t^{n_k}_{n_{k+1},j^k} \circ s^{n_{k-1}}_{n_k,j^{k-1}} \cdots s^{n_2}_{n_3,j^2} \circ t^n_{n_2,j^1}$, called zigzag of sources-targets of type (t,t);
- $t^{n_{p-1}}_{n_p,j^{p-1}} \circ s^{n_{p-2}}_{n_{p-1},j^p} \cdots t^{n_k}_{n_{k+1},j^k} \circ s^{n_{k-1}}_{n_k,j^{k-1}} \cdots t^{n_2}_{n_3,j^2} \circ s^n_{n_2,j^1}$, called zigzag of sources-targets of type (t,s).

The number of occurences of the s and of the t in a string or zigzag is called *the size* of the string or the size of the zigzag. If X is a r-link of $\square^{dx^i_{k_i}}_{1(n)}$:

$$X = (dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1,j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1,j_2,\cdots,j_r), \pm))$$

then it can be written:

$$X = z_X(\square^{dx^i_{k_i}}_{1(n)}),$$

where z_X denotes its underlying string or zigzag of sources-targets.

All these zigzags or strings build the $(n-n_p)$-faces of any n-cube. Thanks to the cubical identities two different zigzags or strings may be equal. And these equalities build congruences on the sequences defined below, such that equivalence relations of these sequences are the faces of the free box $\square^{dx^i_{k_i}}_{1(n)}$.

More precisely consider two $(n-r)$-links $X = (dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1,j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1,j_2,\cdots,j_{n-r}), \pm))$ and $X' = (dx^i_{k_i}, (dx^i_{k_i} \setminus j'_1, \pm), (dx^i_{k_i} \setminus (j'_1,j'_2), \pm), ..., (dx^i_{k_i} \setminus (j'_1,j'_2,\cdots,j'_{n-r}), \pm))$. Denote by z_X the string or zigzag of sources-targets which gives X, i.e $X = z_X(\square^{dx^i_{k_i}}_{1(n)})$, and $z_{X'}$ the string or zigzag of sources-targets which gives X', i.e $X' = z_{X'}(\square^{dx^i_{k_i}}_{1(n)})$.

Definition 3.3. With the above notations, the $(n-r)$-link X is congruent to the $(n-r)$-link X' if and only if $z_X = z_{X'}$; in this case it is trivial to see that z_X and $z_{X'}$ have the same size. Then we write $X \equiv X'$. Equivalence classes of $(n-r)$-links of the free box $\square^{dx^i_{k_i}}_{1(n)}$ are r-faces of $\square^{dx^i_{k_i}}_{1(n)}$.

In fact the *terminal element* of the $(n-r)$-link X:

$$(dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm),$$

gives the precise information of an r-face of $\square^{dx^i_{k_i}}_{1(n)}$ that it can be a source or a target, depending on the sign in $\{-,+\}$: " $-$ " means source and " $+$ " means target.

Lemma 3.4. *If two $(n-r)$-links of $\square^{dx^i_{k_i}}_{1(n)}$ are congruents then they have the same terminal element.*

Proof. The proof is easy and is made by finite decreasing induction:

- We start the induction by proving it with sources and targets of $(dx^i_{k_i}) = \square^{dx^i_{k_i}}_{1(n)}$ (using the whole cubical identities $ss = ss$, $st = ts$, etc.) and verify that indeed they give the same terminal coordinates: this step shows the magical role of the trivial cubical structure of the coordinates. See the section above;
- We suppose that this is true for two congruent $(n-r)$-links. When we apply sources and targets of these $(n-r)$-links then it is straightforward to see that they have the same terminal coordinates.

$\square$

A simple consequence is the following fact:

Proposition 3.5. *A face of $\square^{dx^i_{k_i}}_{1(n)}$ is thus an equivalent class of links of $\square^{dx^i_{k_i}}_{1(n)}$ with the same terminal element.*

We can have in mind also that $(dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm)$ is an r-face of $\square^{dx^i_{k_i}}_{1(n)}$ equipped with (or linked by) the link:

$$(dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm));$$

thus when there is no confusion about the prescribed link of a face $(dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm)$ of $\square^{dx^i_{k_i}}_{1(n)}$ we denote this r-face of $\square^{dx^i_{k_i}}_{1(n)}$ just by:

$$(dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm),$$

without referring its link in $\square^{dx^i_{k_i}}_{1(n)}$.

The previous lemma allows to build the *free boxes associate to any faces* of $\square^{dx^i_{k_i}}_{1(n)}$.

Definition 3.6. The *free box* $\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)} = (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}))$ of the link $(dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm$ $))$ which represent an r-face of $\square^{dx^i_{k_i}}_{1(n)}$, is the basic box with coordinate $dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})$ in $\mathcal{Z}_r$.

When working with this free box $\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)}$, we forget the previous information that it was linked to $\square^{dx^i_{k_i}}_{1(n)}$. Thus the link $(dx^i_{k_i}, (dx^i_{k_i} \setminus j_1, \pm), (dx^i_{k_i} \setminus (j_1, j_2), \pm), ..., (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r-1}), \pm))$ which represents a face of $\square^{dx^i_{k_i}}_{1(n)}$, represents also a face of the underlying free box $\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)}$, but with the simpler link $(dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r-1}), \pm$
$))$ when we see it as a face of the free box $\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)}$. But when we work with faces of $\square^{dx^i_{k_i}}_{1(n)}$ we have to not forgot their links in order to have a complete informations about their locations. Thus faces of $\square^{dx^i_{k_i}}_{1(n)}$ can be seen as free boxes equipped with their links.

Remark 3.7. We have others natural notations for links X of $\square^{dx^i_{k_i}}_{1(n)}$ (Reverse Polish Notation, RPN):

$$X = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), \pm, \cdots, \pm);$$

this presentation allows the following definition of sources and targets of links of $\square^{dx^i_{k_i}}_{1(n)}$ by using the underlying free boxes of it: $s^n_{n-1,j}(\square^{dx^i_{k_i}}_{1(n)}) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j, -))$, and

$$s^r_{r-1,l}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r-1}), s^r_{r-1,l}(\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)}), \pm, \cdots, \pm),$$

thus

$$s^r_{r-1,l}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j_{n-(r-1)} = l), -), \pm, \cdots, \pm),$$

that we rewrite when removing redondant occurrences of brackets

$$s^r_{r-1,j}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j_{n-(r-1)} = l), -, \pm, \cdots, \pm);$$

and for targets: $t^n_{n-1,j}(\square^{dx^i_{k_i}}_{1(n)}) = (dx^i_{k_i}, (dx^i_{k_i} \setminus j, +))$, and

$$t^r_{r-1,l}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r-1}), t^r_{r-1,l}(\square^{dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r})}_{1(r)}), \pm, \cdots, \pm),$$

thus

$$t^r_{r-1,l}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ (dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j_{n-(r-1)} = l), +), \pm, \cdots, \pm),$$

that we rewrite when removing redondant occurrences of brackets:

$$t^r_{r-1,j}(X) = (dx^i_{k_i}, dx^i_{k_i} \setminus j_1, dx^i_{k_i} \setminus (j_1, j_2), ..., \\ dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}), dx^i_{k_i} \setminus (j_1, j_2, \cdots, j_{n-r}, j_{n-(r-1)} = l), +, \pm, \cdots, \pm).$$

4 Degenerate boxes $(\square^{dx^i_{k_i}}_{1(n)}, \equiv_A)$

We know that the following forgetful functor

$$[\mathbb{C}^{op}_r, \mathbb{S}\text{ets}] \xrightarrow{\quad U \quad} [\mathbb{C}^{op}, \mathbb{S}\text{ets}] = \mathbb{CS}\text{ets},$$

which sends cubical sets equipped with degeneracies and connections [3] to cubical sets is right adjoint. Its induced monad $\mathbb{R}$ (described in 5) applied to the terminal object 1 of the category $[\mathbb{C}^{op}, \mathbb{S}\text{ets}]$ of cubical sets, gives all kinds of degenerates n-cells $A \in \mathbb{R}(1)(n)$ (for all integers $n \in \mathbb{N}$) we need for cubical pasting diagrams. In Section 5 we shall describe this monad accurately in order to see that it is a cartesian monad.

Now for each $A \in R(1)(n)$ we are going to define a box $\square^{dx^i_{k_i}}_{1(n)} / \equiv_A$ which faces are modeled with those of A. For that purpose we are going to define a notion of zigzag of degeneracies in order to capture *the depth* of a degenerate n-cell $A \in \mathbb{R}(1)(n)$, which is the greatest integer r such that r-faces of A are of the form $1(r)$, i.e are non degenerate. We begin with the notations:

- $1^{n_2}_{n,i^1} := 1^{n-1}_{n,i^1_1} \circ 1^{n-2}_{n-1,i^1_2} \circ \cdots \circ 1^{n-k}_{n-k+1,i^1_k} \circ \cdots 1^{n_2}_{n_2+1,i^1_{n-n_2}}$, where $i^1 = (i^1_1, \cdots, i^1_k, \cdots, i^1_{n-n_2})$, $k \in [\![1, n-n_2]\!]$ and $i^1_1 \in [\![1, n]\!], \cdots, i^1_k \in [\![1, n-k+1]\!], \cdots, i^1_{n-n_2} \in [\![1, n_2+1]\!]$.
- $1^{n_2,\gamma}_{n,j^1} := 1^{n-1,\gamma}_{n,j^1_1} \circ 1^{n-2,\gamma}_{n-1,j^1_2} \circ \cdots \circ 1^{n-k,\gamma}_{n-k+1,j^1_k} \circ \cdots 1^{n_2,\gamma}_{n_2+1,j^1_{n-n_2}}$, where $j^1 = (j^1_1, \cdots, j^1_k, \cdots, j^1_{n-n_2})$, $k \in [\![1, n-n_2]\!]$ and $j^1_1 \in [\![1, n-1]\!], \cdots, j^1_k \in [\![1, n-k]\!], \cdots, j^1_{n-n_2} \in [\![1, n_2]\!]$.

Also for any partition $n_p < n_{p-1} < \cdots < n_k < \cdots < n_2 < n_1 = n$ with $(p-1)$ intervals $[\![n_{k+1}, n_k]\!]$ we have 6 different zigzags of reflexivities and connections:

- $1^{n_2}_{n,i} := 1^{n-1}_{n,i_1} \circ 1^{n-2}_{n-1,i_2} \circ \cdots \circ 1^{n-k}_{n-k+1,i_k} \circ \cdots 1^{n_2}_{n_2+1,i_{n-n_2}}$ where $i = (i_1, \cdots, i_k, \cdots, i_{n-n_2})$, $k \in [\![1, n-n_2]\!]$ called strings of degeneracies of type 1.
- $1^{n_2,\gamma}_{n,j} := 1^{n-1,\gamma}_{n,j_1} \circ 1^{n-2,\gamma}_{n-1,j_2} \circ \cdots \circ 1^{n-k,\gamma}_{n-k+1,j_k} \circ \cdots 1^{n_2,\gamma}_{n_2+1,j_{n-n_2}}$ where $j = (j_1, \cdots, j_k, \cdots, j_{n-n_2})$, $k \in [\![1, n-n_2]\!]$ called strings of degeneracies of type γ.

- $1^{n_2}_{n,i^1} \circ 1^{n_3,\gamma}_{n_2,i^2} \circ \cdots \circ 1^{n_k,\gamma}_{n_{k-1},i^k} \circ 1^{n_{k+1}}_{n_k,i^{k+1}} \circ \cdots \circ 1^{n_{p-1},\gamma}_{n_{p-2},i^{p-2}} \circ 1^{n_p}_{n_{p-1},i^{p-1}}$ called zigzags of degeneracies of type $(1,1)$.
- $1^{n_2,\gamma}_{n,i^1} \circ 1^{n_3}_{n_2,i^2} \circ \cdots \circ 1^{n_k,\gamma}_{n_{k-1},i^k} \circ 1^{n_{k+1}}_{n_k,i^{k+1}} \circ \cdots \circ 1^{n_{p-1},\gamma}_{n_{p-2},i^{p-2}} \circ 1^{n_p}_{n_{p-1},i^{p-1}}$ called zigzags of degeneracies of type $(\gamma,1)$.
- $1^{n_2,\gamma}_{n,i^1} \circ 1^{n_3}_{n_2,i^2} \circ \cdots \circ 1^{n_k,\gamma}_{n_{k-1},i^k} \circ 1^{n_{k+1}}_{n_k,i^{k+1}} \circ \cdots \circ 1^{n_{p-1}}_{n_{p-2},i^{p-2}} \circ 1^{n_p,\gamma}_{n_{p-1},i^{p-1}}$ called zigzags of degeneracies of type (γ,γ).
- $1^{n_2}_{n,i^1} \circ 1^{n_3,\gamma}_{n_2,i^2} \circ \cdots \circ 1^{n_k,\gamma}_{n_{k-1},i^k} \circ 1^{n_{k+1}}_{n_k,i^{k+1}} \circ \cdots \circ 1^{n_{p-1}}_{n_{p-2},i^{p-2}} \circ 1^{n_p,\gamma}_{n_{p-1},i^{p-1}}$ called zigzags of degeneracies of type $(1,\gamma)$.

The number of occurrences of the operations $1^r_{r+1,i}$, $1^{r,\gamma}_{r+1,i}$ in such zigzags or such strings are respectively called *the size of a zigzag* or *the size of a string.*

Definition 4.1. Consider an n-cell $A \in \mathbb{R}(1)(n)$ which is not equal to $1(n)$. Thus it is a degenerate n-cell and is build with zigzag or string of degeneracies as described just above. The *depth* of A is the integer $p \in \mathbb{N}$ such that A is equal to a zigzag of size $n-p$ or a string of size $n-p$ of degeneracies of the p-cell $1(p)$ of the cubical site, i.e $A = d_A(1(p))$ where d_A denotes its underlying string or zigzag of degeneracies and d_A has size equal to $n-p$.

Remark 4.2. Thanks to the axioms of degeneracies, the degenerate n-cell A has zigzags or strings of degeneracies with different shapes, which may be equals.

Suppose A is a degenerate n-cell in $\mathbb{R}(1)(n)$ with depth $p<n$. Zigzags or strings of sources-targets of A with sizes which are less or equal to $(n-p)$ are the one which build a congruence $\equiv_A$ on faces of the basic n-box $\square^{dx^i_{k_i}}_{1(n)}$, and this congruence is defined as follow: if $p<q\leq n$, two q-faces x and y of $\square^{dx^i_{k_i}}_{1(n)}$ are A-congruent: $x \equiv_A y$, if and only if any strings or zigzags of sources-targets z_x of x (i.e z_x is the underlying string or the underlying zigzag of sources-targets of any link of $\square^{dx^i_{k_i}}_{1(n)}$ which gives the q-face x (any two such links are equivalent)) and any strings or zigzags of sources-targets z_y (i.e z_y is the underlying string or the underlying zigzag of sources-targets of any

link of $\square_{1(n)}^{dx^i_{k_i}}$ which gives the q-face y (any two such links are equivalent)) of y, equalize A i.e are such that $z_x(A) = z_y(A)$.

Definition 4.3. The quotient $\square_{1(n)}^{dx^i_{k_i}} / \equiv_A$ is called a *degenerate box* with coordinate $dx^i_{k_i}$. We denote it with the bracket notation $(\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)$.

Sources, targets and degeneracies of the box $(\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)$ agree with those of A, thus it has a canonical structure of n-cubical set with degeneracies inherited by A and defined as follow:

Definition 4.4.

- Sources and targets of degenerate boxes:

$$s^n_{n-1,j}((\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)) := (\square_{1(n-1)}^{dx^i_{k_i}\backslash j}, \equiv_{s^n_{n-1,j}(A)})$$

and

$$t^n_{n-1,j}((\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)) := (\square_{1(n-1)}^{dx^i_{k_i}\backslash j}, \equiv_{t^n_{n-1,j}(A)})$$

- Degeneracies of degenerate boxes:

$$1^n_{n+1,j}((\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)) := (\square_{1(n+1)}^{dx^i_{k_i}+j}, \equiv_{1^n_{n+1,j}(A)})$$

and

$$1^{n,\gamma}_{n+1,j}((\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)) := (\square_{1(n+1)}^{dx^i_{k_i}+j}, \equiv_{1^{n,\gamma}_{n+1,j}(A)})$$

5 Basic divisors

The forgetful functor

$$[\mathbb{C}_r^{op}, \mathbb{S}\mathrm{ets}] \xrightarrow{\quad U \quad} [\mathbb{C}^{op}, \mathbb{S}\mathrm{ets}]$$

which sends cubical sets equipped with degeneracies and connections [3] to cubical sets is right adjoint and its induced monad is written $\mathbb{R} = (R, i, m)$ where $1_{\mathbb{C}\mathbb{S}\mathrm{ets}} \xrightarrow{\quad i \quad} R$ is its unit and $R^2 \xrightarrow{\quad m \quad} R$ is its multiplication.

Definition 5.1. A *basic n-divisor* is the formal expression $Adx^i_{k_i}$ where $A \in R(1)(n)$. Its interpretation is just: the n-cell A is located in its degenerate box $(\square_{1(n)}^{dx^i_{k_i}}, \equiv_A)$.

Also we have the following simple fact.

Proposition 5.2. *Any basic divisor has an underlying structure of reflexive cubical set with connections.*

Proof. The definitions of sources, targets, degeneracies are as follow:

- $s^n_{n-1,j}(Adx^i_{k_i}) := s^n_{n-1,j}(A)dx^i_{k_i} \setminus j$
- $t^n_{n-1,j}(Adx^i_{k_i}) := t^n_{n-1,j}(A)dx^i_{k_i} \setminus j$
- $1^n_{n+1,j}(Adx^i_{k_i}) := 1^n_{n+1,j}(A)(dx^i_{k_i} + j)$
- $1^{n,\gamma}_{n+1,j}(Adx^i_{k_i}) := 1^{n,\gamma}_{n+1,j}(A)(dx^i_{k_i} + j)$

□

Definition 5.3. Two basic divisors $Adx^i_{k_i}$, $A'dx^i_{k'_i}$ located respectively in the coordinates $dx^i_{k_i} = dx^1_{k_1} \otimes \cdots \otimes dx^n_{k_n}$ and $dx^i_{k'_i} = dx^1_{k'_1} \otimes \cdots \otimes dx^n_{k'_n}$ are j-adjacent for a direction $j \in [\![1, n]\!]$ if their coordinates are j-adjacent and if $s^n_{n-1,j}(A) = t^n_{n-1,j}(A')$ if $k_j = k'_j + 1$ or $t^n_{n-1,j}(A) = s^n_{n-1,j}(A')$ if $k_j = k'_j - 1$.

The set of basic divisors is denoted by $\mathbb{B}\mathrm{Div}$ and by the previous proposition it is straightforward that it has an underlying structure of cubical set with connections where its n-cells (for all $n \in \mathbb{N}$) are the basic n-divisors.

Now we are going to use basic divisors to describe the monad $\mathbb{R} = (R, i, m)$ just above and show that it is a cartesian monad: consider the full subcategory $\Theta_{\mathbb{B}\mathrm{Div}} \subset \mathbb{C}\mathbb{S}\mathrm{ets}$ which objects are basic divisors. The Yoneda embedding[5]

$$\begin{array}{ccc} \Theta_{\mathbb{B}\mathrm{Div}} & \xrightarrow{\quad \mathrm{Y} \quad} & \mathbb{C}\mathbb{S}\mathrm{ets} \\ & & \\ X & \longmapsto & \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(X, -), \end{array}$$

provides the following description of $R(C)$ where $C \in \mathbb{C}\mathbb{S}\mathrm{ets}$ is a cubical set:

$$R(C) := \coprod_{X \in \mathbb{B}\mathrm{Div}} \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(\mathrm{Y}(X), C).$$

[5]Of course, this is not a Yoneda embedding, stricto sensu, but because objects of $\Theta_{\mathbb{B}\mathrm{Div}}$ are terms build with representables, we have permitted ourself this abuse of language.

The multiplication m of the monad $\mathbb{R}$ is very simple: it is obtained with the concatenation of two strings of degeneracies, or one string of degeneracies with one zigzag of degeneracies, or with two zigzags of degeneracies. The unit i of the monad $\mathbb{R}$ sends n-cells c to the decorated box $cdx^i_{k_i}$.

Let us be more precise: the multiplication $R^2(C) \xrightarrow{\quad m \quad} R(C)$ is defined as follow: the cubical set $R^2(C)$ is defined by the formula:

$$R^2(C) = \coprod_{X\in\mathbb{B}\mathrm{Div}} \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}\Big(Y(X), R(C) = \coprod_{X'\in\mathbb{B}\mathrm{Div}} \hom_{\mathbb{C}\mathbb{S}\mathrm{ets}}(Y(X'), C)\Big)$$

thus an n-cell x of $R^2(C)$ is an expression of the form: $z(z'(c))$ where c is a p-cell of C, $p \leq n$ (for the case $p = n$ it means that x is non-degenerate and equal to c), z' is a string or a zigzag of degeneracies which when apply to c gives a degenerate q-cell $z'(c)$ of $R(C)$ ($p < q \leq n$), and where z is a string or a zigzag of degeneracies which degenerates again $z'(c)$. The multiplication m sends $z(z'(c)) \in R^2(C)$ to $(z + z')(c) \in R(C)$ where here $z + z'$ is just the concatenation of z and z'.

Proposition 5.4. *The monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets with connections is cartesian.*

Proof. The definition of the endofunctor R shows that it preserves fiber products.

We are going to prove that the multiplication m is cartesian, i.e we are going to prove that if $C \in \mathbb{C}\mathbb{S}\mathrm{ets}$ is a cubical set then the commutative diagram

$$\begin{tikzcd} R^2(C) \arrow[r, "R^2(!)"] \arrow[d, "m(C)"'] & R^2(1) \arrow[d, "m(1)"] \\ R(C) \arrow[r, "R(!)"'] & R(1) \end{tikzcd}$$

is a cartesian square; consider the commutative diagram in $\mathbb{C}\mathbb{S}$ets.

$$
\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, "f"'] & R^2(1) \arrow[d, "m(1)"] \\
R(C) \arrow[r, "R(!)"'] & R(1)
\end{tikzcd}
$$

Thus if x is an n-cell of C' then $f(x) = z(c)$ where $c \in C(q)$ $(q \leq n)$ and $R(!)(f(x)) = R(!)(z(c)) = z(1(q))$, and $g(x) = z"(z'(1(p)))$, thus $m(1)(g(x)) = m(1)(z"(z'(1(p)))) = (z" + z')(1(p))$, thus the commutativity of the square gives $z = z" + z'$ and $p = q$.

$$
\begin{tikzcd}
C' \arrow[dr, dotted, "l"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& R^2(C) \arrow[r, "R^2(!)"] \arrow[d, "m(C)"'] & R^2(1) \arrow[d, "m(1)"] \\
& R(C) \arrow[r, "R(!)"'] & R(1)
\end{tikzcd}
$$

Thus the unique arrow l is defined as follow: $l(x) = z"(z'(c))$, and we can see that $m(C)(z"(z'(c))) = (z" + z')(c) = z(c) = f(x)$ and that $R^2(!)(z"(z'(c))) = z"(z'(1(p))) = g(x)$.

The cartesianity of the unit

$$C \xrightarrow{i} R(C),$$

is easier and goes as follow: we start with a commutative diagram in $\mathbb{C}\mathbb{S}$ets.

$$
\begin{tikzcd}
C' \arrow[r, "!"] \arrow[d, "f"'] & 1 \arrow[d, "i(1)"] \\
R(C) \arrow[r, "R(!)"'] & R(1)
\end{tikzcd}
$$

Let x be an n-cell of C', thus we have $f(x) = z(c)$, thus $R(!)(z(c)) = z(1(p))$ and the commutativity gives: $z(1(p)) = i(1)(1(n)) = 1(n)$; which shows that $z = \emptyset$ and $p = n$, thus $f(x) = c$.

It shows that there is a unique map l

$$\begin{tikzcd}
C' \arrow[dr, dotted, "l"] \arrow[drr, "!"] \arrow[ddr, "f"'] & & \\
& C \arrow[r, "!"] \arrow[d, "i(C)"'] & 1 \arrow[d, "i(1)"] \\
& R(C) \arrow[r, "R(!)"'] & R(1)
\end{tikzcd}$$

defined by $l(x) = f(x)$. □

6 Rectangular divisors

Definition 6.1. An *n-configuration* is a finite subset $C_n \subset \mathbb{Z}^n$. A *rectangular n-configuration* is an n-configuration of the form

$$C_n = [\![p_1; q_1]\!] \times \cdots \times [\![p_j; q_j]\!] \times \cdots \times [\![p_n; q_n]\!] \subset \mathbb{Z}^n.$$

An n-configuration C_n must be thought up to its translations in the network $\mathbb{Z}^n$; the *normalization* of the rectangular n-configuration C_n just above is

$$[\![1; q_1 - (p_1 - 1)]\!] \times \cdots \times [\![1; q_j - (p_j - 1)]\!] \times \cdots \times [\![1; q_n - (p_n - 1)]\!],$$

and usually we will work with rectangular n-configurations with normalized shapes, i.e with n-configurations written as follow

$$C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!].$$

Now we are going to describe the sketch $\mathcal{E}_{C_n}$ of any rectangular n-configuration C_n (all the time normalized for simplicity) as above. For that perspective we are going to highlight some canonical orders on rectangular

n-configurations, all inherited from the lexicographic orders. Consider the rectangular n-configuration $C_n = [\![1;m_1]\!] \times \cdots \times [\![1;m_j]\!] \times \cdots \times [\![1;m_n]\!]$ as just above; with it we get m_n rectangular n-configurations $C_n^{k_n}$:

$$C_n^{k_n} = [\![1;m_1]\!] \times \cdots \times [\![1;m_j]\!] \times \cdots \times [\![1;m_{n-1}]\!] \times \{k_n\},$$

where $k_n \in [\![1;m_n]\!]$; and all these rectangular n-configurations $C_n^{k_n}$ are themselves ordered as follow

$$C_n^1 <_n \cdots <_n C_n^{k_n} <_n \cdots <_n C_n^{m_n}$$

where the orders $<_n$ are induced by the lexicographic order on C_n; these orders $<_n$ indicates the formal composition $\circ_n$ of these $C_n^{k_n}$ ($k_n \in [\![1;m_n]\!]$) along the direction $j = n$, thus the n-configuration

$$C_n = [\![1;m_1]\!] \times \cdots \times [\![1;m_j]\!] \times \cdots \times [\![1;m_n]\!] = \bigcup_{k_n \in [\![1;m_n]\!]} C_n^{k_n}$$

is preferably denoted by

$$C_n = C_n^1 \circ_n \cdots \circ_n C_n^{k_n} \circ_n \cdots \circ_n C_n^{m_n}$$

in order to keep the information of this formal $\circ_n$-composition of C_n; this also underlies the following sketch:

$$\begin{tikzcd}
 & & & & C_n & & & & \\
C_n^1 \arrow[urrrr, dotted, hook] & & C_n^2 \arrow[urr, dotted, hook] & & \cdots & & C_n^{m_n-1} \arrow[ull, dotted, hook'] & & C_n^{m_n} \arrow[ullll, dotted, hook'] \\
 & \{\circ_n\} \arrow[ul] \arrow[ur] & & & & & & \{\circ_n\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Remark 6.2. The sketch above is obtained by substituting all occurences $C_n^{k_n} \circ_n C_n^{k_n+1}$ in $C_n^1 \circ_n \cdots \circ_n C_n^{k_n} \circ_n \cdots \circ_n C_n^{m_n}$ with the formal base:

$$\begin{tikzcd}
C_n^{k_n} & & C_n^{k_n+1} \\
 & \{\circ_n\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and its dotted cocone shows the expected result (here C_n) of its colimit.

Remark 6.3. All our sketches are written with solid bases and dotted (co)cones.

This sketch is called the $\circ_n$-sketch of C_n, and it shows how to glue the $C_n^{k_n}$ ($k_n \in [\![1; m_n]\!]$) together, and its gluing is just C_n; now with the rectangular n-configurations $C_n^{l_n}$ ($l_n \in [\![1; m_n]\!]$ is fixed) we get m_{n-1} rectangular n-configurations $C_n^{l_n,k_{n-1}}$:

$$C_n^{l_n,k_{n-1}} = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_{n-2}]\!] \times \{k_{m-1}\} \times \{l_n\},$$

where $k_{m-1} \in [\![1; m_{n-1}]\!]$; and all these rectangular n-configurations $C_n^{l_n,k_{n-1}}$ are themselves ordered as follow

$$C_n^{l_n,1} <_{n-1} \cdots <_{n-1} C_n^{l_n,k_{n-1}} <_{n-1} \cdots <_{n-1} C_n^{l_n,m_{n-1}}$$

where the orders $<_{n-1}$ are induced by the lexicographic order on C_n; these orders $<_{n-1}$ indicate the formal composition $\circ_{n-1}$ of these $C_n^{l_n,k_{n-1}}$ ($k_{n-1} \in [\![1; m_{n-1}]\!]$) along the direction $j = n-1$, thus the n-configuration

$$C_n^{l_n} = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_{n-1}]\!] \times \{l_n\} = \bigcup_{k_{n-1} \in [\![1; m_{n-1}]\!]} C_n^{l_n,k_{n-1}}$$

is preferably denoted by

$$C_n^{l_n} = C_n^{l_n,1} \circ_{n-1} \cdots \circ_{n-1} C_n^{l_n,k_{n-1}} \circ_{n-1} \cdots \circ_{n-1} C_n^{l_n,m_{n-1}}$$

in order to keep the information of this formal $\circ_n$-composition of C_n; they are m_n of such formal $\circ_{n-1}$-compositions for C_n; this also underlies the following sketch.

$$\begin{tikzcd}
 & & & C_n^{l_n} & & & \\
C_n^{l_n,1} \arrow[urrr, dotted] & & C_n^{l_n,2} \arrow[ur, dotted] & \cdots & C_n^{l_n,m_{n-1}-1} \arrow[ul, dotted] & & C_n^{l_n,m_{n-1}} \arrow[ulll, dotted] \\
 & \{\circ_{n-1}\} \arrow[ul] \arrow[ur] & & & & \{\circ_{n-1}\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This sketch is called a $\circ_{n-1}$-sketch of C_n, and it shows how to glue the $C_n^{l_n,k_{m-1}}$ ($k_{n-1} \in [\![1;m_{n-1}]\!]$) together, and its gluing is $C_n^{l_n}$; they are m_n of such $\circ_{n-1}$-sketches for C_n; we may iterate this process: consider the m_j rectangular n-configurations

$$C_n^{l_n,\cdots,l_{j+1},k_j} = [\![1;m_1]\!] \times \cdots \times [\![1;m_{j-1}]\!] \times \{k_j\} \times \{l_{j+1}\} \cdots \times \{l_n\},$$

where $k_j \in [\![1;m_j]\!]$ and where $(l_{j+1},\cdots,l_n) \in [\![1;m_{j+1}]\!] \times \cdots \times [\![1;m_n]\!]$ is fixed, then it is straightforward that these rectangular n-configurations are ordered as follow

$$C_n^{l_n,\cdots,l_{j+1},1} <_j \cdots <_j C_n^{l_n,\cdots,l_{j+1},k_j} <_j \cdots <_j C_n^{l_n,\cdots,l_{j+1},m_j},$$

where the orders $<_j$ are induced by the lexicographic order on C_n; these orders $<_j$ indicate the formal composition $\circ_j$ of these $C_n^{l_n,\cdots,l_{j+1},k_j}$ ($k_j \in [\![1;m_j]\!]$) along the direction j, thus the n-configuration

$$C_n^{l_n,\cdots,l_{j+1}} = [\![1;m_1]\!] \times \cdots \times [\![1;m_j]\!] \times \{l_{j+1}\} \cdots \times \{l_n\} = \bigcup_{k_j \in [\![1;m_j]\!]} C_n^{l_n,\cdots,l_{j+1},k_j}$$

is preferably denoted by

$$C_n^{l_n,\cdots,l_{j+1}} = C_n^{l_n,\cdots,l_{j+1},1} \circ_j \cdots \circ_j C_n^{l_n,\cdots,l_{j+1},k_j} \circ_j \cdots \circ_j C_n^{l_n,\cdots,l_{j+1},m_j}$$

in order to keep the information of this formal $\circ_j$-composition; they are $m_{j+1}\cdots m_n$ of such formal $\circ_j$-compositions for C_n; this also underlies the following sketch.

$$\begin{tikzcd}
 & & & & C_n^{l_n,\cdots,l_{j+1}} & & & & \\
C_n^{l_n,\cdots,l_{j+1},1} \arrow[urrrr, dotted] & & C_n^{l_n,\cdots,l_{j+1},2} \arrow[urr, dotted] & & \cdots & & C_n^{l_n,\cdots,l_{j+1},m_j-1} \arrow[ull, dotted] & & C_n^{l_n,\cdots,l_{j+1},m_j} \arrow[ullll, dotted] \\
 & \{\circ_j\} \arrow[ul] \arrow[ur] & & & & & & \{\circ_j\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This sketch is called a $\circ_j$-sketch of C_n, and it shows how to glue the $C_n^{l_n,\cdots,l_{j+1},k_j}$ ($k_j \in [\![1;m_j]\!]$) together, and its gluing is $C_n^{l_n,\cdots,l_{j+1}}$; they are

$m_{j+1}\cdots m_n$ of such $\circ_j$-sketches for C_n. This construction ends with the n-configurations

$$C_n^{l_n,\cdots,l_2,k_1} = \{k_1\} \times \{l_2\} \times \cdots \times \{l_j\} \times \cdots \times \{l_n\},$$

where $k_1 \in [\![1;m_1]\!]$ and $(l_2,\cdots,l_n) \in [\![1;m_2]\!] \times \cdots \times [\![1;m_n]\!]$ is fixed, and where these m_1 rectangular n-configurations are ordered as follow

$$C_n^{l_n,\cdots,l_2,1} <_1 \cdots <_1 C_n^{l_n,\cdots,l_2,k_1} <_1 \cdots <_1 C_n^{l_n,\cdots,l_2,m_1},$$

where the orders $<_1$ are induced by the lexicographic order on C_n. But all these n-configurations $C_n^{l_n,\cdots,l_2,k_1}$ (for $k_1 \in [\![1;m_1]\!]$) are singletons, which means that they are just coordinates in C_n, and the order $<_1$ indicates the formal composition $\circ_1$ of these $C_n^{l_n,\cdots,l_2,k_1}$ along the direction $j = 1$, thus the n-configuration

$$C_n^{l_n,\cdots,l_2} = [\![1;m_1]\!] \times \{l_2\} \times \cdots \times \{l_n\} = \bigcup_{k_1 \in [\![1;m_1]\!]} C_n^{l_n,\cdots,l_2,k_1},$$

is preferably denoted by

$$C_n^{l_n,\cdots,l_2} = C_n^{l_n,\cdots,l_2,1} \circ_1 \cdots \circ_1 C_n^{l_n,\cdots,l_2,k_1} \circ_1 \cdots \circ_1 C_n^{l_n,\cdots,l_2,m_1}$$

in order to keep the information of this formal $\circ_1$-composition; there are $m_2\cdots m_n$ such formal $\circ_1$-compositions of C_n; this also underlies the following sketch.

$$\begin{tikzcd}
 & & & C_n^{l_n,\cdots,l_2} & & & \\
C_n^{l_n,\cdots,l_2,1} \arrow[urrr, dotted] & & C_n^{l_n,\cdots,l_2,2} \arrow[ur, dotted] & \cdots & C_n^{l_n,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & C_n^{l_n,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & \{\circ_1\} \arrow[ul] \arrow[ur] & & & & \{\circ_1\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This sketch is called a $\circ_1$-sketch of C_n, and it shows how to glue the $C_n^{l_n,\cdots,l_2,k_1}$ ($k_1 \in [\![1;m_1]\!]$) together, and its gluing is $C_n^{l_n,\cdots,l_2}$; there are $m_2\cdots m_r$ such $\circ_1$-sketches of C_n.

What we have done so far was to show how to reconstruct the n-configuration C_n from its parts with respect to the directions $j \in [\![1;n]\!]$. Let us explain the importance of this construction for a cubical strict ∞-category C. We can coherently weight the n-configuration C_n with n-cells of C, i.e in each coordinates of C_n we substitue an n-cell of C such that if two such n-cells $x, y \in C(n)$ have j-adjacents coordinates then $s^n_{n-1,j}(x) = t^n_{n-1,j}(y)$ or $t^n_{n-1,j}(x) = s^n_{n-1,j}(y)$. Such weighted C_n is called a composable array of n-cubes in [1] (page 350), and our n-configuration C_n can be seen as a formalization of the multi-dimensional arrays in [1]. From this weighted C_n and with respect to the structure of C, we get a unique n-cell $z \in C(n)$ which is the result obtained by applying the operations $\circ_j$ ($j \in [\![1;n]\!]$) in this weighted n-configuration. However in [1] the authors did not describe a complete procedure to get such $z \in C(n)$[6], but the sketches described just above for C_n provide such procedure, but for that we need to care about the order to apply these sketches: we need to apply the $\circ_1$-sketches of C_n, then the $\circ_2$-sketches of C_n, and so on, until the unique $\circ_n$-sketch of C_n. These kinds of sketches which have a hierarchical organization of their (co)cones have been studied before in [7] under the name *Trames*. Thus a trame is a sketch equipped with an ordered stratification of its set of (co)cones. In fact this way to ordered the computation of $z \in C(n)$ is already indicated by the formalism of the sketches of C_n because, following the terminology of *Trames* in [7]:

- The $m_2 \cdots m_n$ $\circ_1$-sketches of C_n are the first floor sketches of C_n;
- The $m_{j+1} \cdots m_n$ $\circ_j$-sketches of C_n are the j-floor sketches of C_n;
- The unique $\circ_n$-sketch of C_n is the n-floor sketch of C_n.

Thus any realization of the sketch $\mathcal{E}_{C_n}$ of C_n must be thought in an inductive way, starting from the $\circ_1$-sketches of C_n until the $\circ_n$-sketch of C_n.

Remark 6.4. In [1] page 350, the authors defined composable array of n-cubes in cubical strict ∞-groupoids, but of course this can be done also in any cubical strict ∞-category.

[6]They just write that this $z \in C(n)$ is obtained by applying the operations $\circ_i, \circ_j$ in any well-formed fashion

Remark 6.5. The decomposition of C_n above by its $\circ_n$-sketch, $\circ_{n-1}$-sketches, $\cdots$, $\circ_1$-sketches, was deliberate for simplicity. Let us call this simplification the $(\circ_n, \circ_{n-1}, \cdots, \circ_1)$-*decomposition* of C_n. We can do similar $(\circ_{j_n}, \circ_{j_{n-1}}, \cdots, \circ_{j_1})$-decompostion of C_n, where $j_i \in [\![1;n]\!]$ and $j_i \neq j_k$ if $i \neq k$; in this case C_n has one $\circ_{j_n}$-sketch (the n-floor sketch), has $m_{j_{i+1}} m_{j_{i+2}} \cdots m_{j_n}$ $\circ_{j_i}$-sketches (the (i)-floor skteches), and has $m_{j_2} \cdots m_{j_n}$ $\circ_{j_1}$-sketches (the first floor sketches). All these decompositions give the same information about how to glue C_n.

Now we are going to define by induction the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv (with connections) of rectangular divisors whose underlying cubical set is written as follow.

$$\bullet\text{-}\mathbb{R}\mathrm{ectDiv} = \cdots\; n\text{-}\mathbb{R}\mathrm{ectDiv} \begin{array}{c} \xrightarrow{\sigma^n_{n-1,n}} \\ \vdots \\ \xrightarrow{\sigma^n_{n-1,j}} \\ \vdots \\ \xrightarrow{\sigma^n_{n-1,1}} \\ \xrightarrow[\tau^n_{n-1,1}]{} \\ \vdots \\ \xrightarrow[\tau^n_{n-1,j}]{} \\ \vdots \\ \xrightarrow[\tau^n_{n-1,n}]{} \end{array} (n-1)\text{-}\mathbb{R}\mathrm{ectDiv} \cdots\; 2\text{-}\mathbb{R}\mathrm{ectDiv} \begin{array}{c} \xrightarrow{\sigma^2_{1,2}} \\ \xrightarrow{\sigma^2_{1,1}} \\ \xrightarrow[\tau^2_{1,1}]{} \\ \xrightarrow[\tau^2_{1,2}]{} \end{array} 1\text{-}\mathbb{R}\mathrm{ectDiv} \begin{array}{c} \xrightarrow{\sigma^1_0} \\ \xrightarrow[\tau^1_0]{} \end{array} 0\text{-}\mathbb{R}\mathrm{ectDiv}$$

Thus we suppose that its underlying $(n-1)$-cubical set have already been defined; we start to build the set n-$\mathbb{R}$ectDiv plus the diagrams (for $j \in [\![1;n]\!]$):

$$(n-1)\text{-}\mathbb{R}\mathrm{ectDiv} \begin{array}{c} \xleftarrow{\quad\sigma^n_{n-1,j}\quad} \\ \xleftarrow[\quad\tau^n_{n-1,j}\quad]{} \end{array} n\text{-}\mathbb{R}\mathrm{ectDiv},$$

where n-$\mathbb{R}$ectDiv is a set of congruences of terms; the congruences of terms in n-$\mathbb{R}$ectDiv are specific n-cubes X named *rectangular n-divisors*, which faces are denoted by $\sigma^n_{n-1,j}(X)$ and $\tau^n_{n-1,j}(X)$; the induction goes as follow.

- If $X \in \mathbb{B}$Div is a basic n-divisor (see 5), then $X \in n$-$\mathbb{R}$ectDiv such that $\sigma^n_{n-1,j}(X) := s^n_{n-1,j}(X)$ and $\tau^n_{n-1,j}(X) := t^n_{n-1,j}(X)$ (see 5 for the definition of $s^n_{n-1,j}$ and $t^n_{n-1,j}$);
- If $X, X' \in n$-$\mathbb{R}$ectDiv are such that $\tau^n_{n-1,j}(X) = \sigma^n_{n-1,j}(X')$, then $X \circ^n_j X' \in n$-$\mathbb{R}$ectDiv such that

- $\sigma^n_{n-1,j}(X \circ^n_j X') = \sigma^n_{n-1,j}(X)$ and $\tau^n_{n-1,j}(X \circ_j X') = \tau^n_{n-1,j}(X)$ for $1 \leq j \leq n$;
- $\sigma^n_{n-1,i}(X \circ^n_j X') = \begin{cases} \sigma^n_{n-1,i}(X) \circ^{n-1}_{j-1} \sigma^n_{n-1,i}(X') \text{ if } 1 \leq i < j \leq n \\ \sigma^n_{n-1,i}(X) \circ^{n-1}_{j} \sigma^n_{n-1,i}(X') \text{ if } 1 \leq j < i \leq n; \end{cases}$
- $\tau^n_{n-1,i}(X \circ^n_j X') = \begin{cases} \tau^n_{n-1,i}(X) \circ^{n-1}_{j-1} \tau^n_{n-1,i}(X') \text{ if } 1 \leq i < j \leq n \\ \tau^n_{n-1,i}(X) \circ^{n-1}_{j} \tau^n_{n-1,i}(X') \text{ if } 1 \leq j < i \leq n. \end{cases}$

We equip n-$\mathbb{R}$ectDiv with the congruences specific to the associativities and to the interchange laws.

Axioms 1 (Associativities)**.** If $X, X', X" \in n$-$\mathbb{R}$ectDiv such that the term $(X \circ^n_j X') \circ^n_j X" \in n$-$\mathbb{R}$ectDiv is well defined, then

$$(X \circ^n_j X') \circ^n_j X" \equiv X \circ^n_j (X' \circ^n_j X").$$

Axioms 2 (Interchange Laws)**.** If $X, X', X", X^{(4)} \in n$-$\mathbb{R}$ectDiv such that the term $(X \circ^n_j X') \circ^n_i (X" \circ^n_j X^{(4)}) \in n$-$\mathbb{R}$ectDiv is well defined, then

$$(X \circ^n_j X') \circ^n_i (X" \circ^n_j X^{(4)}) \equiv (X \circ^n_i X") \circ^n_j (X' \circ^n_j X^{(4)}).$$

Remark 6.6. We could have postponed these congruences after the definition of the maps $\epsilon^{n-1}_{n,i}(X)$ ($i \in [\![1;n]\!]$), $\Gamma^{n-1,-}_{n,i}(X)$ and $\Gamma^{n-1,+}_{n,i}(X)$ ($i \in [\![1;n-1]\!]$) for reflexivities defined just below; in this case we would have been obliged to chose a definition of the first transport law and the second transport law among their different presentations thanks to the interchange laws. See below.

The elements in $(n-1)$-$\mathbb{R}$ectDiv which are congruences of terms, are defined by hypothesis: they are specific $(n-1)$-cubes X named rectangular $(n-1)$-divisors, which degeneracies (defined below) are denoted by $\epsilon^{n-1}_{n,i}(X)$ ($i \in [\![1;n]\!]$), $\Gamma^{n-1,-}_{n,i}(X)$ and $\Gamma^{n-1,+}_{n,i}(X)$ ($i \in [\![1;n-1]\!]$) which live in n-$\mathbb{R}$ectDiv.

$$\begin{tikzcd}(n-1)\text{-}\mathbb{R}\text{ectDiv} \arrow[r, shift left=6, "\Gamma^{n-1,-}_{n,i}"] \arrow[r, "\epsilon^{n-1}_{n,i}"] \arrow[r, shift right=6, "\Gamma^{n-1,+}_{n,i}"'] & n\text{-}\mathbb{R}\text{ectDiv}.\end{tikzcd}$$

If $X \in (n-1)$-$\mathbb{R}$ectDiv, then $\epsilon^{n-1}_{n,i}(X)$ is defined inductively as follow.

- If $X \in (n-1)$-$\mathbb{R}$ectDiv is a basic $(n-1)$-divisor, then we put $\epsilon^{n-1}_{n,i}(X) := 1^{n-1}_{n,i}(X)$ (if $X \in \mathbb{B}$Div, see 5 for the definition of $1^{n-1}_{n,i}(X)$);
- If $X, X' \in (n-1)$-$\mathbb{R}$ectDiv such that $X \circ^{n-1}_j X' \in (n-1)$-$\mathbb{R}$ectDiv is well defined, then
  - $\epsilon^{n-1}_{n,i}(X \circ^{n-1}_j X') = \epsilon^{n-1}_{n,i}(X) \circ^n_{j+1} \epsilon^{n-1}_{n,i}(X')$ if $1 \leq i \leq j \leq n-1$;
  - $\epsilon^{n-1}_{n,i}(X \circ^{n-1}_j X') = \epsilon^{n-1}_{n,i}(X) \circ^n_j \epsilon^{n-1}_{n,i}(X')$ if $1 \leq j < i \leq n$.

If $X \in (n-1)$-$\mathbb{R}$ectDiv, then $\Gamma^{n-1,\gamma}_{n,i}(X)$ ($\gamma \in \{-,+\}$) is defined inductively as follow.

- If $X \in (n-1)$-$\mathbb{R}$ectDiv is a basic $(n-1)$-divisor, then we put $\Gamma^{n-1,\gamma}_{n,i}(X) := 1^{n-1,\gamma}_{n,i}(X)$ (if $X \in \mathbb{B}$Div, see 5 for the definition of $1^{n-1,\gamma}_{n,i}(X)$);
- If $X, X' \in (n-1)$-$\mathbb{R}$ectDiv such that $X \circ^{n-1}_j X' \in (n-1)$-$\mathbb{R}$ectDiv, is well defined, then
  - $\Gamma^{n-1,\gamma}_{n,i}(X \circ^{n-1}_j X') = \Gamma^{n-1,\gamma}_{n,i}(X) \circ^n_{j+1} \Gamma^{n-1,\gamma}_{n,i}(X')$ if $1 \leq i < j \leq n-1$
    $\Gamma^{n-1,\gamma}_{n,i}(X \circ^{n-1}_j X') = \Gamma^{n-1,\gamma}_{n,i}(X) \circ^n_j \Gamma^{n-1,\gamma}_{n,i}(X')$ if $1 \leq j < i \leq n-1$
  - First transport laws: for $1 \leq j \leq n-1$

$$\Gamma^{n-1,+}_{n,j}(X \circ^{n-1}_j X') = \begin{bmatrix} \Gamma^{n-1,+}_{n,j}(X) & \epsilon^{n-1}_{n,j}(X) \\ \epsilon^{n-1}_{n,j+1}(X) & \Gamma^{n-1,+}_{n,j}(X') \end{bmatrix} \qquad \begin{array}{c} \xrightarrow{j} \\ j+1 \big\downarrow \end{array}$$

where this notation means:

$$\begin{aligned}\Gamma^{n-1,+}_{n,j}(X \circ^{n-1}_j X') &= (\Gamma^{n-1,+}_{n,j}(X) \circ^n_j \epsilon^{n-1}_{n,j}(X)) \circ^n_{j+1} (\epsilon^{n-1}_{n,j+1}(X) \circ^n_j \Gamma^{n-1,+}_{n,j}(X')) \\ &= (\Gamma^{n-1,+}_{n,j}(X) \circ^n_{j+1} \epsilon^{n-1}_{n,j+1}(X)) \circ^n_j (\epsilon^{n-1}_{n,j}(X) \circ^n_{j+1} \Gamma^{n-1,+}_{n,j}(X'))\end{aligned}$$

(interchange laws)

  - Second transport laws: for $1 \leq j \leq n-1$

$$\Gamma^{n-1,-}_{n,j}(X \circ^{n-1}_j X') = \begin{bmatrix} \Gamma^{n-1,-}_{n,j}(X) & \epsilon^{n-1}_{n,j+1}(X') \\ \epsilon^{n-1}_{n,j}(X') & \Gamma^{n-1,-}_{n,j}(X') \end{bmatrix} \qquad \begin{array}{c} \xrightarrow{j} \\ j+1 \big\downarrow \end{array}$$

where this notation means:

$$\begin{aligned}\Gamma^{n-1,-}_{n,j}(X\circ^{n-1}_j X') &= (\Gamma^{n-1,-}_{n,j}(X)\circ^n_j \epsilon^{n-1}_{n,j+1}(X'))\circ^n_{j+1}(\epsilon^{n-1}_{n,j}(X')\circ^n_j \Gamma^{n-1,-}_{n,j}(X'))\\ &= (\Gamma^{n-1,-}_{n,j}(X)\circ^n_{j+1} \epsilon^{n-1}_{n,j}(X'))\circ^n_{j}(\epsilon^{n-1}_{n,j+1}(X')\circ^n_{j+1} \Gamma^{n-1,-}_{n,j}(X'))\\ &\qquad\text{(interchange laws)}\end{aligned}$$

We finish the definition of n-$\mathbb{R}$ectDiv by equipping it with the congruences specific to the unities.

Axioms 3 (Unities)**.** If $X\in n$-$\mathbb{R}$ectDiv then

$$X\circ^n_j \epsilon^{n-1}_{n,j}(\tau^n_{n-1,j}(X))\equiv X, \text{ and } \epsilon^{n-1}_{n,j}(\sigma^n_{n-1,j}(X))\circ^n_j X\equiv X;$$

$\Gamma^{n,+}_{n+1,j}(X)\circ^{n+1}_j \Gamma^{n,-}_{n+1,j}(X)\equiv \epsilon^n_{n+1,j+1}(X)$, and $\Gamma^{n,+}_{n+1,j}(X)\circ^{n+1}_{j+1} \Gamma^{n,-}_{n+1,j}(X)\equiv$

$\epsilon^n_{n+1,j}(X)$.

It is easy to see that the maps $\epsilon^{n-1}_{n,i}$, $\Gamma^{n-1,-}_{n,i}$, $\Gamma^{n-1,+}_{n,i}$:

$$\begin{tikzcd}(n-1)\text{-}\mathbb{R}\text{ectDiv} \arrow[r, shift left=4, "\Gamma^{n-1,-}_{n,i}"] \arrow[r, "\epsilon^{n-1}_{n,i}"] \arrow[r, shift right=4, "\Gamma^{n-1,+}_{n,i}"'] & n\text{-}\mathbb{R}\text{ectDiv},\end{tikzcd}$$

are well defined because they respect congruences, i.e if $x\equiv y$ in $(n-1)$-$\mathbb{R}$ectDiv then $\epsilon^{n-1}_{n,i}(x)\equiv\epsilon^{n-1}_{n,i}(y)$ and $\Gamma^{n-1,\gamma}_{n,i}(x)\equiv\Gamma^{n-1,\gamma}_{n,i}(y)$ in n-$\mathbb{R}$ectDiv. Let us denote by $\bullet$-$\mathbb{R}$ectDiv this cubical strict ∞-category with connections, we have

Theorem 6.7. *$\bullet$-$\mathbb{R}$ectDiv is the free cubical strict ∞-category with connections on the terminal object $1\in\mathbb{CS}$ets.*

The cells in the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv are our models of cubical pasting diagrams. Also for each $n\in\mathbb{N}$, elements of the set n-$\mathbb{R}$ectDiv are n-cubical sets, thus we put

Definition 6.8. The full subcategory $\Theta_0\subset\mathbb{CS}$ets which objects are cells of the cubical strict ∞-category $\bullet$-$\mathbb{R}$ectDiv defined above is called the *cubical* Θ_0.

Remark 6.9. The axioms of unities describe above 3 by congruences of terms are expressible with commutative diagrams for a projective sketch, as the one described in [3] for the axioms of interchange laws. In [3] these axioms were expressed in the level of models, this is the reason why we used a projective sketch for such diagrammatical formulation. We can also use inductive sketches to express these axioms, for example the congruences $\epsilon^{n-1}_{n,j}(\sigma^n_{n-1,j}(X)) \circ^n_j X \equiv X$ are encoded by the following cocones.

$$\begin{tikzcd}
\sigma^n_{n-1,j}(X) \arrow[rr, dotted, "\sigma^n_{n-1,j}"] & & X & & \\
\epsilon^{n-1}_{n,j}(\sigma^n_{n-1,j}(X)) \arrow[u, dotted, "\epsilon^{n-1}_{n,j}"] \arrow[urr, dotted, "\sigma^n_{n-1,j}\circ\epsilon^{n-1}_{n,j}"] & & & & X \arrow[ull, dotted, "\mathrm{id}_X"'] \\
& & \sigma^n_{n-1,j}(X) \arrow[ull, "\tau^n_{n-1,j}"] \arrow[urr, "\sigma^n_{n-1,j}"'] & &
\end{tikzcd}$$

Objects of Θ_0 are built with inductive sketches (see 7) because Θ_0 has to be seen as the main category of arities (in the sense of logic) for the theory of cubical ∞-category theory. The slogan of sketch theory could be: the syntax (logic) is diagrammatically governed by inductive sketches, and the semantic (structures, models) is diagrammatically governed by the projective sketches.

The inductive definition of cubical pasting diagrams shows the crucial role of coordinates which is to be guides to build terms with cubical shapes: if X and X' are basic n-divisors such that $t^n_{n-1,j}(X) = s^n_{n-1,j}(X')$ then the term $X \circ^n_j X'$ means that X and X' are located in the network $\mathbb{Z}^n$ with coordinates which are j-adjacent. The induction above plus the congruences show that rectangular n-divisors are first of all just rectangular filling of the network $\mathbb{Z}^n$ by basic n-divisors. Thus each $X \in n\text{-}\mathbb{R}\mathrm{ectDiv}$ is characterized by an n-configuration C_n in which in each coordinates $dx^i_{k_i} \in C_n$ is located a basic n-divisor $Adx^i_{k_i}$, such that if two basic n-divisors are located in two coordinates which are j-adjacent ($j \in [\![1, n]\!]$), then these basic n-divisors must be j-adjacent (see 5.3), and furthermore we demand that in these datas some sub terms of X must be congruents (3). Let us write $X \in n$-$\mathbb{R}$ectDiv as follow

$$X = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i},$$

where this writing means that the $A_l dx^i_{k^l_i}$ (for $l \in [\![1; \#C_n]\!]$) are located in $dx^i_{k^l_i} \in C_n$, and where here the n-configuration C_n of X is written

$$C_n = dx^i_{k^1_i} + \cdots + dx^i_{k^l_i} + \cdots + dx^i_{k^r_i},$$

where here $r = \#C_n = m_1 \cdots m_n$ if $C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!]$. When we write

$$X = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i},$$

then the elements of the set $\{A_1 dx^i_{k^1_i}, \cdots, A_l dx^i_{k^l_i}, \cdots, A_r dx^i_{k^r_i}\}$ are the basic n-divisors of X, and X can be seen as its n-configuration C_n *weighted* by this set of basic n-divisors, where this set is also equipped with congruences coming from the axioms of unities. Thus X is given by the couple of sets $(\{A_1 dx^i_{k^1_i}, \cdots, A_r dx^i_{k^r_i}\}, C_n)$ where $\{A_1 dx^i_{k^1_i}, \cdots, A_r dx^i_{k^r_i}\}$ and C_n are in bijection, and this bijection comes from a morphism of n-cubical set (see also 6.10). When we say that $X' \subset X$ is a sub rectangular n-divisor of X, it means that we considered X' equipped with a sub n-configuration $C'_n \subset C_n$ weighted by basic n-divisors $A' dx^i_{k_i}$ such that $dx^i_{k_i} \in C'_n$. The sketch $\mathcal{E}_{C_n}$ of C_n described above shows remarkable subsets of C_n: for example we defined the n-configuration $C^{l_n,\cdots,l_{j+1}}_n \subset C_n$, as a formal gluing along the direction $j \in [\![1; n]\!]$ of the n-configurations $C^{l_n,\cdots,l_{j+1},k_j}_n \subset C_n$, where $k_j \in [\![1; m_j]\!]$ and where $(l_{j+1}, \cdots, l_n) \in [\![1; m_{j+1}]\!] \times \cdots \times [\![1; m_n]\!]$ was fixed, and we formally described the following $\circ_j$-cocone of $\mathcal{E}_{C_n}$.

$$\begin{tikzcd}
 & & C^{l_n,\cdots,l_{j+1}}_n & & \\
C^{l_n,\cdots,l_{j+1},1}_n \arrow[urr, dotted] & C^{l_n,\cdots,l_{j+1},2}_n \arrow[ur, dotted] & \cdots & C^{l_n,\cdots,l_{j+1},m_j-1}_n \arrow[ul, dotted] & C^{l_n,\cdots,l_{j+1},m_j}_n \arrow[ull, dotted] \\
\{\circ_j\} \arrow[u] \arrow[ur] & & & \{\circ_j\} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

When we weight these n-configurations with the basic n-divisors of X we obtain sub rectangular n-divisors of X corresponding to these sub n-configurations; thus $X^{l_n,\cdots,l_{j+1}} \subset X$ is the sub rectangular n-divisor of X corresponding to the n-configuration $C_n^{l_n,\cdots,l_{j+1}}$; and $X^{l_n,\cdots,l_{j+1},k_j}$ are the sub rectangular n-divisors of X corresponding to the n-configurations $C_n^{l_n,\cdots,l_{j+1},k_j}$ ($k_j \in [\![1;m_j]\!]$); and the $\circ_j$-cocone above of $\mathcal{E}_{C_n}$ gives the following $\circ_j$-cocone of $\mathcal{E}_X$ (more precision are provided in 7):

$$\begin{tikzcd}
 & & & X^{l_n,\cdots,l_{j+1}} & & & \\
X^{l_n,\cdots,l_{j+1},1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_{j+1},2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_{j+1},m_j-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_{j+1},m_j} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,j} = \sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] & & & & \tau^n_{n-1,j} = \sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] &
\end{tikzcd}$$

when $\mathcal{E}_{C_n}$ is weighted by the set of basic n-divisors of X; here $\tau^n_{n-1,j} = \sigma^n_{n-1,j}$ at the bottom left means $\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},1}) = \sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},2})$, and $\tau^n_{n-1,j} = \sigma^n_{n-1,j}$ at the bottom right means $\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j-1}) = \sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j})$.

Remark 6.10. If $X \in n$-$\mathbb{R}$ectDiv, then another possibility for its notation is to describe it as a morphism of n-cubical set:

$$C_n \xrightarrow{\quad X \quad} R(1)$$

where $R(1)$ is the free reflexive cubical set on a terminal object $1 \in \mathbb{C}\mathbb{S}$ets, and where R is the underlying endofunctor of the monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets with connections described in 5. We recall that the cubical structure on C_n is given by the contractions of coordinates. With this description, the underlying map of sets for n-cubes[7] is

$$C_n \xrightarrow{\quad X(n) \quad} R(1)(n)$$

and we see that it induces a bijection of C_n on the image of $X(n)$. This description of X as morphism of n-cubical sets is interesting but for applications (see [5]) we need more concrete notation as the one given above.

[7]Here C_n is seen as a set.

7 Cubical inductive sketches

Let $X = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i}$ be an n-divisor where $C_n = dx^i_{k^1_i} + \cdots + dx^i_{k^l_i} + \cdots + dx^i_{k^r_i}$ is its n-configuration given by

$$C_n = [\![1; m_1]\!] \times \cdots \times [\![1; m_j]\!] \times \cdots \times [\![1; m_n]\!].$$

The sketch[8] $\mathcal{E}_X$ associated to X is provided by several underlying sketches $\mathcal{E}_{C_n}$ associated to its n-configuration C_n (see 6) called the $(\circ_{j_n}, \circ_{j_{n-1}}, \cdots, \circ_{j_1})$-decompositions of C_n (see 6.5) where $j_i \in [\![1; n]\!]$ and $j_i \neq j_k$ if $i \neq k$. Thanks to the congruences in 1 and 2, all these sketches provide equivalent formulation of $\mathcal{E}_X$, and in order to simplify the theory we shall use the $(\circ_n, \circ_{n-1}, \cdots, \circ_1)$-decomposition of C_n which has been accurately described in the beginning of Section 6. The sketch $\mathcal{E}_X$ is thus given by the $(\circ_n, \circ_{n-1}, \cdots, \circ_1)$-decomposition of C_n weighted by the basic n-divisors of X. Thus $\mathcal{E}_X$ consists of

- The $m_2 \cdots m_n$ $\circ_1$-cocones $\mathcal{E}^{l_n,\cdots,l_2}_X$ (first floor cocones), where $(l_2, \cdots, l_n) \in [\![1; m_2]\!] \times \cdots \times [\![1; m_n]\!]$:

$$\begin{tikzcd}
 & & & X^{l_n,\cdots,l_2} & & & \\
X^{l_n,\cdots,l_2,1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_2,2} \arrow[ur, dotted] & \dots & X^{l_n,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,1} = \sigma^n_{n-1,1} \arrow[ul, "\tau^n_{n-1,1}"] \arrow[ur, "\sigma^n_{n-1,1}"'] & & & & \tau^n_{n-1,1} = \sigma^n_{n-1,1} \arrow[ul, "\tau^n_{n-1,1}"] \arrow[ur, "\sigma^n_{n-1,1}"'] &
\end{tikzcd}$$

here $\tau^n_{n-1,1} = \sigma^n_{n-1,1}$ at the bottom left means

$$\tau^n_{n-1,1}(X^{l_n,\cdots,l_2,1}) = \sigma^n_{n-1,1}(X^{l_n,\cdots,l_2,2}),$$

and $\tau^n_{n-1,1} = \sigma^n_{n-1,1}$ at the bottom right means

$$\tau^n_{n-1,1}(X^{l_n,\cdots,l_2,m_1-1}) = \sigma^n_{n-1,1}(X^{l_n,\cdots,l_2,m_1}).$$

[8]This kind of sketches are known under the name *Trames* in [7].

- The $m_{j+1}\cdots m_n$ $\circ_j$-cocones $\mathcal{E}_X^{l_n,\cdots,l_{j+1}}$ (j-floor cocones), where $(l_{j+1},\cdots,l_n)\in [\![1;m_{j+1}]\!]\times\cdots\times[\![1;m_n]\!]$:

$$\begin{tikzcd}
 & & & X^{l_n,\cdots,l_{j+1}} & & & \\
X^{l_n,\cdots,l_{j+1},1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_{j+1},2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_{j+1},m_j-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_{j+1},m_j} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,j}=\sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] & & & & \tau^n_{n-1,j}=\sigma^n_{n-1,j} \arrow[ul, "\tau^n_{n-1,j}"] \arrow[ur, "\sigma^n_{n-1,j}"'] &
\end{tikzcd}$$

here $\tau^n_{n-1,j}=\sigma^n_{n-1,j}$ at the bottom left means

$$\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},1})=\sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},2}),$$

and $\tau^n_{n-1,j}=\sigma^n_{n-1,j}$ at the bottom right means

$$\tau^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j-1})=\sigma^n_{n-1,j}(X^{l_n,\cdots,l_{j+1},m_j}).$$

- The unique $\circ_n$-cocone of X (n-floor cocone):

$$\begin{tikzcd}
 & & & X & & & \\
X^1 \arrow[urrr, dotted] & & X^2 \arrow[ur, dotted] & \cdots & X^{m_n-1} \arrow[ul, dotted] & & X^{m_n} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,n}=\sigma^n_{n-1,n} \arrow[ul, "\tau^n_{n-1,n}"] \arrow[ur, "\sigma^n_{n-1,n}"'] & & & & \tau^n_{n-1,n}=\sigma^n_{n-1,n} \arrow[ul, "\tau^n_{n-1,n}"] \arrow[ur, "\sigma^n_{n-1,n}"'] &
\end{tikzcd}$$

here $\tau^n_{n-1,n}=\sigma^n_{n-1,n}$ at the bottom left means

$$\tau^n_{n-1,n}(X^1)=\sigma^n_{n-1,n}(X^2),$$

and $\tau^n_{n-1,n}=\sigma^n_{n-1,n}$ at the bottom right means

$$\tau^n_{n-1,n}(X^{m_n-1})=\sigma^n_{n-1,n}(X^{m_n}).$$

As we saw in 6, the $m_2 \cdots m_n$ $\circ_1$-cocones $\mathcal{E}_X^{l_n,\cdots,l_2}$ (first floor cocones of X), where $(l_2,\cdots,l_n) \in [\![1;m_2]\!] \times \cdots \times [\![1;m_n]\!]$:

$$
\begin{tikzcd}[column sep=small]
 & & & X^{l_n,\cdots,l_2} & & & \\
X^{l_n,\cdots,l_2,1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_2,2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & \tau^n_{n-1,1} = \sigma^n_{n-1,1} \arrow[ul, "\tau^n_{n-1,1}"] \arrow[ur, "\sigma^n_{n-1,1}"'] & & & & \tau^n_{n-1,1} = \sigma^n_{n-1,1} \arrow[ul, "\tau^n_{n-1,1}"] \arrow[ur, "\sigma^n_{n-1,1}"'] &
\end{tikzcd}
$$

are such that the $m_1 \cdots m_n$ n-configurations $X^{l_n,\cdots,l_2,k_1}$ ($k_1 \in [\![1;m_1]\!]$) are singletons, and then, they are just the $m_1 \cdots m_n$ basic n-divisors of X, where their sources are by definition $\sigma^n_{n-1,1} := s^n_{n-1,1}$ and their targets are by definition $\tau^n_{n-1,1} := t^n_{n-1,1}$ (see 5). And thus we can improve the description of the first floor of $\mathcal{E}_X$ by writing its $\circ_1$-cocones $\mathcal{E}_X^{l_n,\cdots,l_2}$ with their specific sources and targets:

$$
\begin{tikzcd}[column sep=small]
 & & & X^{l_n,\cdots,l_2} & & & \\
X^{l_n,\cdots,l_2,1} \arrow[urrr, dotted] & & X^{l_n,\cdots,l_2,2} \arrow[ur, dotted] & \cdots & X^{l_n,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & X^{l_n,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & t^n_{n-1,1} = s^n_{n-1,1} \arrow[ul, "t^n_{n-1,1}"] \arrow[ur, "s^n_{n-1,1}"'] & & & & t^n_{n-1,1} = s^n_{n-1,1} \arrow[ul, "t^n_{n-1,1}"] \arrow[ur, "s^n_{n-1,1}"'] &
\end{tikzcd}
$$

By definition each rectangular n-divisor X is an n-cell of $\bullet$-$\mathbb{R}$ectDiv, it is therefore an n-cube; let us denote by $f_p(X)$ the finite set of p-faces ($0 \leq p \leq n-1$) of this n-cube X, thus $f_p(X) \subset p$-$\mathbb{R}$ectDiv is a finite subset of rectangular p-divisors, and X seen as an n-cube may be depicted diagrammatically as follow.

$$\{X\}\;\begin{array}{c}\xrightarrow{\sigma^n_{n-1,n}}\\ \cdots\cdots\\ \xrightarrow{\sigma^n_{n-1,j}}\\ \cdots\cdots\\ \xrightarrow{\sigma^n_{n-1,1}}\\ \xrightarrow[\tau^n_{n-1,1}]{}\\ \cdots\cdots\\ \xrightarrow[\tau^n_{n-1,j}]{}\\ \cdots\cdots\\ \xrightarrow[\tau^n_{n-1,n}]{}\end{array}\; f_{n-1}(X)\cdots f_4(X)\;\begin{array}{c}\xrightarrow{\sigma^4_{3,4}}\\ \xrightarrow{\sigma^4_{3,3}}\\ \xrightarrow{\sigma^4_{3,2}}\\ \xrightarrow{\sigma^4_{3,1}}\\ \xrightarrow[\tau^4_{3,1}]{}\\ \xrightarrow[\tau^4_{3,2}]{}\\ \xrightarrow[\tau^4_{3,3}]{}\\ \xrightarrow[\tau^4_{3,4}]{}\end{array}\; f_3(X)\;\begin{array}{c}\xrightarrow{\sigma^3_{2,3}}\\ \xrightarrow{\sigma^3_{2,2}}\\ \xrightarrow{\sigma^3_{2,1}}\\ \xrightarrow[\tau^3_{2,1}]{}\\ \xrightarrow[\tau^3_{2,2}]{}\\ \xrightarrow[\tau^3_{2,3}]{}\end{array}\; f_2(X)\;\begin{array}{c}\xrightarrow{\sigma^2_{1,2}}\\ \xrightarrow{\sigma^2_{1,1}}\\ \xrightarrow[\tau^2_{1,1}]{}\\ \xrightarrow[\tau^2_{1,2}]{}\end{array}\; f_1(X)\;\begin{array}{c}\xrightarrow{\sigma^1_0}\\ \xrightarrow[\tau^1_0]{}\end{array}\; f_0(X)$$

This n-cube structure on X transfers to an n-cube structure on its sketch $\mathcal{E}_X$; indeed if $x \in f_p(X)$ is a p-face of X thus it is a rectangular p-divisor which sketch is $\mathcal{E}_x$; therefore if we define the sets: $f_p(\mathcal{E}_X) := \{\mathcal{E}_x / x \in f_p(X)\}$ ($p \in [\![0; n-1]\!]$), it highlights the following n-cube in $\mathbb{S}$ets:

$$\{\mathcal{E}_X\}\;\begin{array}{c}\xrightarrow{\sigma^n_{n-1,n}}\\ \cdots\cdots\\ \xrightarrow{\sigma^n_{n-1,j}}\\ \cdots\cdots\\ \xrightarrow{\sigma^n_{n-1,1}}\\ \xrightarrow[\tau^n_{n-1,1}]{}\\ \cdots\cdots\\ \xrightarrow[\tau^n_{n-1,j}]{}\\ \cdots\cdots\\ \xrightarrow[\tau^n_{n-1,n}]{}\end{array}\; f_{n-1}(\mathcal{E}_X)\cdots f_4(\mathcal{E}_X)\;\begin{array}{c}\xrightarrow{\sigma^4_{3,4}}\\ \xrightarrow{\sigma^4_{3,3}}\\ \xrightarrow{\sigma^4_{3,2}}\\ \xrightarrow{\sigma^4_{3,1}}\\ \xrightarrow[\tau^4_{3,1}]{}\\ \xrightarrow[\tau^4_{3,2}]{}\\ \xrightarrow[\tau^4_{3,3}]{}\\ \xrightarrow[\tau^4_{3,4}]{}\end{array}\; f_3(\mathcal{E}_X)\;\begin{array}{c}\xrightarrow{\sigma^3_{2,3}}\\ \xrightarrow{\sigma^3_{2,2}}\\ \xrightarrow{\sigma^3_{2,1}}\\ \xrightarrow[\tau^3_{2,1}]{}\\ \xrightarrow[\tau^3_{2,2}]{}\\ \xrightarrow[\tau^3_{2,3}]{}\end{array}\; f_2(\mathcal{E}_X)\;\begin{array}{c}\xrightarrow{\sigma^2_{1,2}}\\ \xrightarrow{\sigma^2_{1,1}}\\ \xrightarrow[\tau^2_{1,1}]{}\\ \xrightarrow[\tau^2_{1,2}]{}\end{array}\; f_1(\mathcal{E}_X)\;\begin{array}{c}\xrightarrow{\sigma^1_0}\\ \xrightarrow[\tau^1_0]{}\end{array}\; f_0(\mathcal{E}_X)$$

where $\sigma^p_{p-1,k}(\mathcal{E}_x) := \mathcal{E}_{\sigma^p_{p-1,k}(x)}$ and $\tau^p_{p-1,k}(\mathcal{E}_x) := \mathcal{E}_{\tau^p_{p-1,k}(x)}$ if $\mathcal{E}_x \in f_p(\mathcal{E}_X)$; therefore the sources and targets of this n-cubical set $\mathcal{E}_X$ send sketches to sketches.

Consider now the sketches $\mathcal{E}_{f_p(X)} := \bigcup f_p(\mathcal{E}_X) = \bigcup\limits_{x \in f_p(X)} \mathcal{E}_x$ for all $p \in [\![0; n-1]\!]$, i.e we consider the sketch $\mathcal{E}_{f_p(X)}$ obtained as the union of all cocones inside all sketches $\mathcal{E}_x$ where $x \in f_p(X)$ are the p-faces of X; in this case it highlights the following $(n-1)$-cubical object in the category $\mathbb{S}$ketch of sketches.

$$\mathcal{E}_X \begin{array}{c} \xrightarrow{\sigma^{n-1}_{n-2,n-1}} \\ \vdots \\ \xrightarrow{\sigma^{n-1}_{n-2,j}} \\ \vdots \\ \xrightarrow{\sigma^{n-1}_{n-2,1}} \\ \xrightarrow[\tau^{n-1}_{n-2,1}]{} \\ \vdots \\ \xrightarrow[\tau^{n-1}_{n-2,j}]{} \\ \vdots \\ \xrightarrow[\tau^{n-1}_{n-2,n-1}]{} \end{array} \mathcal{E}_{f_{n-1}(X)} \cdots \mathcal{E}_{f_5(X)} \begin{array}{c} \xrightarrow{\sigma^4_{3,4}} \\ \xrightarrow{\sigma^4_{3,3}} \\ \xrightarrow{\sigma^4_{3,2}} \\ \xrightarrow{\sigma^4_{3,1}} \\ \xrightarrow[\tau^4_{3,1}]{} \\ \xrightarrow[\tau^4_{3,2}]{} \\ \xrightarrow[\tau^4_{3,3}]{} \\ \xrightarrow[\tau^4_{3,4}]{} \end{array} \mathcal{E}_{f_4(X)} \begin{array}{c} \xrightarrow{\sigma^3_{2,3}} \\ \xrightarrow{\sigma^3_{2,2}} \\ \xrightarrow{\sigma^3_{2,1}} \\ \xrightarrow[\tau^3_{2,1}]{} \\ \xrightarrow[\tau^3_{2,2}]{} \\ \xrightarrow[\tau^3_{2,3}]{} \end{array} \mathcal{E}_{f_3(X)} \begin{array}{c} \xrightarrow{\sigma^2_{1,2}} \\ \xrightarrow{\sigma^2_{1,1}} \\ \xrightarrow[\tau^2_{1,1}]{} \\ \xrightarrow[\tau^2_{1,2}]{} \end{array} \mathcal{E}_{f_2(X)} \begin{array}{c} \xrightarrow{\sigma^1_0} \\ \xrightarrow[\tau^1_0]{} \end{array} \mathcal{E}_{f_1(X)}$$

As a matter of fact the cocones in $\mathcal{E}_x \subset \mathcal{E}_{f_p(X)}$ can be seen as ways of gluing of rectangular p-divisors along some rectangular $(p-1)$-divisors, and thus the integer $(p-1)$ is the dimension the sketch $\mathcal{E}_{f_p(X)}$ must have if we want to identify a cubical object in $\mathbb{S}$ketch inherited from the rectangular n-divisor X.

In order to justify the existence of this cubical object in $\mathbb{S}$ketch we need to describe the sources $\sigma^{p-1}_{p-2,k}$ and the targets $\tau^{p-1}_{p-2,k}$, which now must be morphisms of sketches, i.e are maps which send cocone to cocone. A $(p-1)$-cell in $\mathcal{E}_{f_p(X)}$ is a cocone in some $\mathcal{E}_x$ where $x \in f_p(X)$, and cocones in $\mathcal{E}_x$ are the one which belong to the first floor cocones until the p-floor cocones of $\mathcal{E}_x$; in fact we will show that $\sigma^{p-1}_{p-2,k}$ and $\tau^{p-1}_{p-2,k}$ send cocones of the first floor of $\mathcal{E}_x$ to respectively cocones of the first floor of $\mathcal{E}_{\sigma^p_{p-1,k'}(x)} \subset \mathcal{E}_{f_{p-1}(X)}$ and to cocones of the first floor of $\mathcal{E}_{\tau^p_{p-1,k'}(x)} \subset \mathcal{E}_{f_{p-1}(X)}$, where k and k' can be different or sometimes can be equal (see below). For cocones of the other floors the actions of $\sigma^{p-1}_{p-2,k}$ and $\tau^{p-1}_{p-2,k}$ are described similarly.

If $x = A_1 dx^i_{k^1_i} + \cdots + A_l dx^i_{k^l_i} + \cdots + A_r dx^i_{k^r_i}$ is a rectangular p-divisor ($x \in f_p(X)$) equipped with the p-configuration $C_p = [\![1; m_1]\!] \times \cdots \times [\![1; m_p]\!]$, as we see above $\mathcal{E}_x$[9] has $m_2 \cdots m_p$ $\circ_1$-cocones $\mathcal{E}^{l_p,\cdots,l_2}_x$ (first floor cocones of

[9] We deliberately use similar notation for $\mathcal{E}_x$ as for $\mathcal{E}_X$.

x), where $(l_2, \cdots, l_p) \in [\![1; m_2]\!] \times \cdots \times [\![1; m_p]\!]$:

$$\begin{tikzcd}[column sep=small]
 & & & x^{l_p,\cdots,l_2} & & & \\
x^{l_p,\cdots,l_2,1} \arrow[urrr, dotted] & & x^{l_p,\cdots,l_2,2} \arrow[ur, dotted] & \cdots & x^{l_p,\cdots,l_2,m_1-1} \arrow[ul, dotted] & & x^{l_p,\cdots,l_2,m_1} \arrow[ulll, dotted] \\
 & t^p_{p-1,1} = t^p_{p-1,1} \arrow[ul, "t^p_{p-1,1}"] \arrow[ur, "s^p_{p-1,1}"'] & & & & t^p_{p-1,1} = t^p_{p-1,1} \arrow[ul, "t^p_{p-1,1}"] \arrow[ur, "s^p_{p-1,1}"'] &
\end{tikzcd}$$

The bases of these cocones are such that $\sigma^p_{p-1,1} = s^p_{p-1,1}$ and $\tau^p_{p-1,1} = t^p_{p-1,1}$ because for all $k_1 \in [\![1; m_1]\!]$, the rectangular p-divisors $x^{l_p,\cdots,l_2,k_1}$ are just basic p-divisors; a fragment d:

$$\begin{tikzcd}
x^{l_p,\cdots,l_2,k_1-1} & & x^{l_p,\cdots,l_2,k_1} \\
 & \tau^p_{p-1,1} = \sigma^p_{p-1,1} \arrow[ul, "t^p_{p-1,1}"] \arrow[ur, "s^p_{p-1,1}"'] &
\end{tikzcd}$$

of these cocones is called a 1-*gluing data* of x; more generally a j-gluing data d of x is a cocone of the following shape

$$\begin{tikzcd}
A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} \\
 & t^p_{p-1,j}(A_l dx^i_{k^l_i}) = s^p_{p-1,j}(A_{l'} dx^i_{k^{l'}_i}) \arrow[ul, "t^p_{p-1,j}"] \arrow[ur, "s^p_{p-1,j}"'] &
\end{tikzcd}$$

where $A_l dx^i_{k^l_i}$, $A_{l'} dx^i_{k^{l'}_i}$ belong to the set of basic p-divisors of x; we are going to describe the action of $\sigma^{p-1}_{p-2,k}$ and $\tau^{p-1}_{p-2,k}$ on these j-gluing datas of x, because this action is similar and extend easily on its $\circ_1$-cocones $\mathcal{E}^{l_p,\cdots,l_2}_x$, and we also deliberately treat the general case of $j \in [\![1; p]\!]$ (and not only the case $j = 1$ of the first floor above) because this general case cover all cocones in $\mathcal{E}_x$ (for all floors). When in $\mathcal{E}_x$ gluing of rectangular p-divisors are involved, we just replace s with σ and t with τ in the computations; here $\sigma^{p-1}_{p-2,k} = s^{p-1}_{p-2,k}$ and $\tau^{p-1}_{p-2,k} = t^{p-1}_{p-2,k}$ because only basic divisors are involved.

We describe these morphisms of sketches by defining cocones $s^{p-1}_{p-2,k}(d)$ and $t^{p-1}_{p-2,k}(d)$ as precomposition of the j-gluing data d just above

$$\begin{tikzcd} A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} \\ & A' dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,j}"] \arrow[ur, "s^p_{p-1,j}"'] & \end{tikzcd}$$

where we write $A' = t^p_{p-1,j}(A_l dx^i_{k^l_i}) = s^p_{p-1,j}(A_{l'} dx^i_{k^{l'}_i})$; consider now $A_s" = s^{p-1}_{p-2,k}(A')$, $A_t" = t^{p-1}_{p-2,k}(A')$, and the maps

$$A_s" dx^i_{k"^l_i} \setminus (j,k) \xrightarrow{s^{p-1}_{p-2,k}} A' dx^i_{k^l_i} \setminus j, \qquad A_t" dx^i_{k"^l_i} \setminus (j,k) \xrightarrow{t^{p-1}_{p-2,k}} A' dx^i_{k^l_i} \setminus j$$

The maps $s^{p-1}_{p-2,k}$, $t^{p-1}_{p-2,k}$ send each cocone d of $\mathcal{E}_x$ to cocones $s^{p-1}_{p-2,k}(d)$, $t^{p-1}_{p-2,k}(d)$ respectively in the sketches $\mathcal{E}_{s^p_{p-1,k'}(x)} \subset \mathcal{E}_{f_{p-1}(X)}$, $\mathcal{E}_{t^p_{p-1,k'}(x)} \subset \mathcal{E}_{f_{p-1}(X)}$, by the precompositions

$$\begin{tikzcd} A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} \\ & A' dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,j}"] \arrow[ur, "s^p_{p-1,j}"'] & \\ & A"_s dx^i_{k"^l_i} \setminus (j,k) \arrow[u, "s^{p-1}_{p-2,k}"] & \end{tikzcd}$$

$$\begin{tikzcd}
A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} \\
& A' dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,j}"'] \arrow[ur, "s^p_{p-1,j}"] & \\
& A"_t dx^i_{k"^l_i} \setminus (j,k) \arrow[u, "t^{p-1}_{p-2,k}"] &
\end{tikzcd}$$

For this description of the maps $s^{p-1}_{p-2,k}$, $t^{p-1}_{p-2,k}$, we just use cubical identities as describe in [3].

- When $j = k$ we obtain $s^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} & \\
A' dx^i_{k^l_i} \setminus j \arrow[ur, "s^p_{p-1,j+1}"] & & A' dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,j}"'] \arrow[ur, "s^p_{p-1,j}"] & & A' dx^i_{k^l_i} \setminus j \arrow[ul, "s^p_{p-1,j+1}"'] \\
& & A" dx^i_{k"^l_i} \setminus (j,k) \arrow[ull, "t^{p-1}_{p-2,j}"] \arrow[u, "s^{p-1}_{p-2,k=j}"] \arrow[urr, "s^{p-1}_{p-2,j}"'] & &
\end{tikzcd}$$

where we denote $A" dx^i_{k"^l_i} \setminus (j,k) = s^{p-1}_{p-2,k}(A' dx^i_{k^l_i} \setminus j)$.

Remark 7.1. Of course we have also

$$t^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j) = s^{p-1}_{p-2,k}(A' dx^i_{k^l_i} \setminus j) = s^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j),$$

but in: $t^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j) = t^{p-1}_{p-2,j}(A') dx^i_{k^l_i} \setminus (j,j)$, and in: $s^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j) = s^{p-1}_{p-2,j}(A') dx^i_{k^l_i} \setminus (j,j)$, the basic divisors $A"$, $t^{p-1}_{p-2,j}(A')$ and $s^{p-1}_{p-2,j}(A')$ are not necessarily equals.

And thus the morphism of sketches $s^{p-1}_{p-2,k}$ sends d to the following cocone $s^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{s^{p}_{p-1,j+1}(x)}$:

$$\begin{tikzcd}
A'dx^{i}_{k^{l}_{i}} \setminus j & & A'dx^{i}_{k^{l}_{i}} \setminus j \\
& A"dx^{i}_{k"^{l}_{i}} \setminus (j,j) \arrow[ul, "t^{p-1}_{p-2,j}"] \arrow[ur, "s^{p-1}_{p-2,j}"'] &
\end{tikzcd}$$

And we obtain $t^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_{l}dx^{i}_{k^{l}_{i}} & & A_{l'}dx^{i}_{k^{l'}_{i}} & \\
A'dx^{i}_{k^{l}_{i}} \setminus j \arrow[ur, "t^{p}_{p-1,j+1}"] & & A'dx^{i}_{k^{l}_{i}} \setminus j \arrow[ul, "t^{p}_{p-1,j}"'] \arrow[ur, "s^{p}_{p-1,j}"] & & A'dx^{i}_{k^{l}_{i}} \setminus j \arrow[ul, "t^{p}_{p-1,j+1}"'] \\
& & A"dx^{i}_{k"^{l}_{i}} \setminus (j,k) \arrow[ull, "t^{p-1}_{p-2,j}"] \arrow[u, "t^{p-1}_{p-2,k=j}"] \arrow[urr, "s^{p-1}_{p-2,j}"'] & &
\end{tikzcd}$$

where we denote $A"dx^{i}_{k"^{l}_{i}} \setminus (j,k) = t^{p-1}_{p-2,k}(A'dx^{i}_{k^{l}_{i}} \setminus j)$, and thus the morphism of sketches $t^{p-1}_{p-2,k}$ sends d to the following cocone $t^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{t^{p}_{p-1,j+1}(x)}$:

$$\begin{tikzcd}
A'dx^{i}_{k^{l}_{i}} \setminus j & & A'dx^{i}_{k^{l}_{i}} \setminus j \\
& A"dx^{i}_{k"^{l}_{i}} \setminus (j,j) \arrow[ul, "t^{p-1}_{p-2,j}"] \arrow[ur, "s^{p-1}_{p-2,j}"'] &
\end{tikzcd}$$

- When $k<j$ then we obtain $s^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_l dx^i_{k^l_i} & & A_{l'} dx^i_{k^{l'}_i} & \\
A'dx^i_{k^l_i}\setminus j \arrow[ur, "s^p_{p-1,k}"] & & A'dx^i_{k^l_i}\setminus j \arrow[ul, "t^p_{p-1,j}"'] \arrow[ur, "s^p_{p-1,j}"] & & A'dx^i_{k^l_i}\setminus j \arrow[ul, "s^p_{p-1,k}"'] \\
& & A"dx^i_{k"^l_i}\setminus(j,k) \arrow[ull, "t^{p-1}_{p-2,j-1}"] \arrow[u, "s^{p-1}_{p-2,k}"] \arrow[urr, "s^{p-1}_{p-2,j-1}"'] & &
\end{tikzcd}$$

where we denote $A"dx^i_{k"^l_i}\setminus(j,k)=s^{p-1}_{p-2,k}(A'dx^i_{k^l_i}\setminus j)$.

Remark 7.2. Of course we have also

$$t^{p-1}_{p-2,j-1}(A'dx^i_{k^l_i}\setminus j)=s^{p-1}_{p-2,k}(A'dx^i_{k^l_i}\setminus j)=s^{p-1}_{p-2,j-1}(A'dx^i_{k^l_i}\setminus j),$$

but in $t^{p-1}_{p-2,j-1}(A'dx^i_{k^l_i}\setminus j)=t^{p-1}_{p-2,j-1}(A')dx^i_{k^l_i}\setminus(j,j-1)$ and in $s^{p-1}_{p-2,j-1}(A'dx^i_{k^l_i}\setminus j)=s^{p-1}_{p-2,j-1}(A')dx^i_{k^l_i}\setminus(j,j-1)$ the basic divisors $A"$, $t^{p-1}_{p-2,j-1}(A')$ and $s^{p-1}_{p-2,j-1}(A')$ are not necessarily equals.

And thus the morphism of sketches $s^{p-1}_{p-2,k}$ sends d to the following cocone $s^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{s^p_{p-1,k}(x)}$:

$$\begin{tikzcd}
A'dx^i_{k^l_i}\setminus j & & A'dx^i_{k^l_i}\setminus j \\
& A"dx^i_{k"^l_i}\setminus(j,j-1) \arrow[ul, "t^{p-1}_{p-2,j-1}"] \arrow[ur, "s^{p-1}_{p-2,j-1}"'] &
\end{tikzcd}$$

And we obtain $t^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_l dx^i_{k^l_i} & & & & A_{l'} dx^i_{k^{l'}_i} & \\
A' dx^i_{k^l_i} \setminus j \arrow[ur, "t^p_{p-1,k}"] & & & A' dx^i_{k^l_i} \setminus j \arrow[ull, "t^p_{p-1,j}"'] \arrow[urr, "s^p_{p-1,j}"] & & & A' dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,k}"'] \\
& & & A" dx^i_{k"^l_i} \setminus (j,k) \arrow[ulll, "t^{p-1}_{p-2,j-1}"] \arrow[u, "t^{p-1}_{p-2,k}"] \arrow[urrr, "s^{p-1}_{p-2,j-1}"'] & & &
\end{tikzcd}$$

where we denote $A" dx^i_{k"^l_i} \setminus (j,k) = t^{p-1}_{p-2,k}(A' dx^i_{k^l_i} \setminus j)$, and thus the morphism of sketches $t^{p-1}_{p-2,k}$ sends d to the following cocone $t^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{t^p_{p-1,k}(x)}$:

$$\begin{tikzcd}
A' dx^i_{k^l_i} \setminus j & & A' dx^i_{k^l_i} \setminus j \\
& A" dx^i_{k"^l_i} \setminus (j, j-1) \arrow[ul, "t^{p-1}_{p-2,j-1}"] \arrow[ur, "s^{p-1}_{p-2,j-1}"'] &
\end{tikzcd}$$

- When $k > j$ then we obtain $s^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_l dx^i_{k^l_i} & & & & A_{l'} dx^i_{k^{l'}_i} & \\
A' dx^i_{k^l_i} \setminus j \arrow[ur, "s^p_{p-1,k+1}"] & & & A' dx^i_{k^l_i} \setminus j \arrow[ull, "t^p_{p-1,j}"'] \arrow[urr, "s^p_{p-1,j}"] & & & A' dx^i_{k^l_i} \setminus j \arrow[ul, "s^p_{p-1,k+1}"'] \\
& & & A" dx^i_{k"^l_i} \setminus (j,k) \arrow[ulll, "t^{p-1}_{p-2,j}"] \arrow[u, "s^{p-1}_{p-2,k}"] \arrow[urrr, "s^{p-1}_{p-2,j}"'] & & &
\end{tikzcd}$$

where we denote $A" dx^i_{k"^l_i} \setminus (j,k) = s^{p-1}_{p-2,k}(A' dx^i_{k^l_i} \setminus j)$.

Remark 7.3. Of course we have also

$$t^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j) = s^{p-1}_{p-2,k}(A' dx^i_{k^l_i} \setminus j) = s^{p-1}_{p-2,j}(A' dx^i_{k^l_i} \setminus j),$$

but in $t^{p-1}_{p-2,j}(A'dx^i_{k^l_i}\backslash j) = t^{p-1}_{p-2,j}(A')dx^i_{k^l_i}\backslash(j,j)$ and $s^{p-1}_{p-2,j}(A'dx^i_{k^l_i}\backslash j) = s^{p-1}_{p-2,j}(A')dx^i_{k^l_i} \setminus (j,j)$ the basic divisors A", $t^{p-1}_{p-2,j}(A')$ and $s^{p-1}_{p-2,j}(A')$ are not necessarily equals.

And thus the morphism of sketches $s^{p-1}_{p-2,k}$ sends d to the following cocone $s^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{s^p_{p-1,k+1}(x)}$:

$$\begin{tikzcd}
A'dx^i_{k^l_i} \setminus j & & A'dx^i_{k^l_i} \setminus j \\
& A"dx^i_{k"^l_i} \setminus (j,j) \arrow[ul, "t^{p-1}_{p-2,j}"] \arrow[ur, "s^{p-1}_{p-2,j}"'] &
\end{tikzcd}$$

And we obtain $t^{p-1}_{p-2,k}(d)$ by using the diagram

$$\begin{tikzcd}
& A_l dx^i_{k^l_i} & & & & A_{l'} dx^i_{k^{l'}_i} & \\
A'dx^i_{k^l_i} \setminus j \arrow[ur, "t^p_{p-1,k+1}"] & & & A'dx^i_{k^l_i} \setminus j \arrow[ull, "t^p_{p-1,j}"'] \arrow[urr, "s^p_{p-1,j}"] & & & A'dx^i_{k^l_i} \setminus j \arrow[ul, "t^p_{p-1,k+1}"'] \\
& & & A"dx^i_{k"^l_i} \setminus (j,k) \arrow[ulll, "t^{p-1}_{p-2,j}"] \arrow[u, "t^{p-1}_{p-2,k}"] \arrow[urrr, "s^{p-1}_{p-2,j}"'] & & &
\end{tikzcd}$$

where we denote $A"dx^i_{k"^l_i} \setminus (j,k) = t^{p-1}_{p-2,k}(A'dx^i_{k^l_i} \setminus j)$, and thus the morphism of sketches $t^{p-1}_{p-2,k}$ sends d to the following cocone $t^{p-1}_{p-2,k}(d)$ of the sketch $\mathcal{E}_{t^p_{p-1,k+1}(x)}$:

$$\begin{tikzcd}
A'dx^i_{k^l_i} \setminus j & & A'dx^i_{k^l_i} \setminus j \\
& A"dx^i_{k"^l_i} \setminus (j,j) \arrow[ul, "t^{p-1}_{p-2,j}"] \arrow[ur, "s^{p-1}_{p-2,j}"'] &
\end{tikzcd}$$

which finalize our description of the $(n-1)$-cubical object in the category $\mathbb{S}$ketch of sketches associated to the rectangular n-divisor X.

Remark 7.4. It is interesting to see that the 1-faces $x \in f^1(X)$ of X are all of the form

$$x = A_0 dx_0^1 + \cdots + A_l dx_l^1 \cdots + A_r dx_r^1,$$

where any basic divisor $A_l dx_l^1$ of x can be $1(1)dx_l^1$ or $1_1^0(1(0))dx_l^1$, and the sketch $\mathcal{E}_{f_1(X)}$ is a set of cocones of the form

$$\begin{tikzcd}
 & & & & x & & & & \\
A_0dx_0^1 \arrow[urrrr, dotted] & & A_1dx_1^1 \arrow[urr, dotted] & & \cdots & & A_{r-1}dx_{r-1}^1 \arrow[ull, dotted] & & A_rdx_r^1 \arrow[ullll, dotted] \\
 & A \arrow[ul, "t_0^1"] \arrow[ur, "s_0^1"'] & & & & & & A \arrow[ul, "t_0^1"] \arrow[ur, "s_0^1"'] &
\end{tikzcd}$$

where A denotes the unique 0-cell $1(0)$ of the cubical site $\mathbb{C}$.

Acknowledgement

I thank mathematicians of the team AGA (Arithmétique et Géomètrie Algèbrique) who kindly organized me a talk on homotopy types, the 27th November 2019, and especially the good ambience provided in the LMO, Paris-Saclay, especially I want to mention Olivier Schiffmann, Benjamin Hennion, François Charles, Valentin Hernandez, and Patrick Massot. I also thank Ross Street, Michael Batanin, Mark Weber, Ronald Brown, Richard Steiner, with whom I interacted during the preparation of this article. I also thank a lot the precious work of the anonymous referee who helps us to improve this work. Especially we thank her/him to have warned us that the stratified inductive sketches $\mathcal{E}_X$ describe in 7 are known under the name *Trames* and were invented by Christian Lair and widely studied by him

in [7] in order to understand better higher order logics through sketch theory. This article has been written in November 2019, and circulated through mathematicians who interacted with me listed just above.

I dedicate this work to my daughters, Leïli and Amina-Tassadit.

References

[1] Brown, R., Higgins, P.J., and Sivera, R., "Nonabelian Algebraic Topology", European Mathematical Society, Tracts in Mathematics Volume 15, 2011.

[2] Kachour, C., *Combinatorial approach to the category Θ_0 of cubical pasting diagrams.* `https://arxiv.org/pdf/2102.09787.pdf`

[3] Kachour, C., *Algebraic models of cubical weak ∞-categories with connections*, Categ. General Alg. Structures Appl. 16(1) (2022), 143-187.

[4] Kachour, C., *Algebraic models of cubical weak higher structures*, Categ. General Alg. Structures Appl. 16(1) (2022), 189-220.

[5] Kachour, C., *The coherator Θ^{∞}_{W} of cubical weak ∞-categories with connections*, To appears in Categ. General Alg. Structures Appl. (2023).

[6] Camell Kachour, *Introduction to Higher Cubical Operads.* Prépublication de l'IHES, republished in HAL: `https://hal.science/hal-04306455`

[7] Lair, C., *Trames et sémantiques catégoriques des systèmes de trames*, Journal Diagrammes, 18 (1987), 47 pages.

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