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# S-metrizability and the Wallman basis of a frame

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Dedicated to Themba Dube on the occasion of his $65^{th}$ birthday

**Abstract.** The Wallman basis of a frame and the corresponding induced compactification was first investigated by Baboolal [2]. In this paper, we provide an intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame. Particularly, we show that a connected, locally connected frame is S-metrizable if and only if it has a countable locally connected and uniformly connected Wallman basis.

## 1 Introduction and Preliminaries

In [7], García-Máynez utilised the Wallman basis to construct locally connected compactifications and characterise S-metrizable spaces. The purpose of this paper is to generalise García-Máynez's characterisation of S-metrizable spaces. Thus we present a study of the Wallman basis of a frame, which was introduced by Baboolal in [2], and the corresponding construction of the Wallman compactification of frame. We present an isomorphism theorem for the Wallman compactification of dense metric sublocales of a

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metric frame. This together with Baboolal's work on insular ideals of a Wallman compactification (see [2]), leads to obtaining a generalization of García-Máynez's intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame.

We will first recall relevant material which will be required. A *frame* $L$ is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s | s \in S\},$$

for all $x \in L, S \subseteq L$, where $\bigvee S$ denotes $\bigvee\{s \mid s \in S\}$. The top element of a frame $L$ is denoted by $1_L$ and the bottom element by $0_L$. If no ambiguity is caused then we simply use 0 and 1. A map $h : L \longrightarrow M$ between frames is called a *frame homomorphism*, if $h$ preserves all finite meets, including the top element, and all arbitrary joins, including the bottom element. $h$ is *dense* if whenever $h(x) = 0_M$ then $x = 0_L$. $h$ is an *onto* frame homomorphism if for every $y \in L$ there is an $x \in M$ such that $h(x) = y$, and $h$ is *one-to-one* if whenever $h(a) = h(b)$, then $a = b$ for $a, b \in L$. $h$ is a frame *isomorphism* if and only if $h$ is onto, one-to-one. $h$ has a *right adjoint* $h_* : M \longrightarrow L$ satisfying the property that for all $x \in M$ and for all $y \in L$, $x \leq h_*(y)$ iff $h(x) \leq y$.

Given a topological space $X$, $\mathcal{O}X = \{U \subseteq X | \ U \text{ is open}\}$ is a frame. For any continuous map $f : X \longrightarrow Y$, from the topological space $X$ to a topological space $Y$, we have a frame homomorphism,

$$\begin{aligned} \mathcal{O}(f) : \ \mathcal{O}(Y) &\longrightarrow \mathcal{O}(X), \\ U &\mapsto f^{-1}(U). \end{aligned}$$

$\mathcal{O}$ : **Top** $\longrightarrow$ **Frm** is a contravariant functor, where **Top** denotes the category of topological spaces and continuous maps, and **Frm** denotes the category of frames and frame homomorphisms. The contravariant functor is given by

$$\begin{aligned} \Sigma : \mathbf{Frm} &\longrightarrow \mathbf{Top}, \\ L &\mapsto \Sigma L. \end{aligned}$$

$\Sigma L$, called the spectrum of $L$, is the space of all frame homomorphisms $\psi : L \longrightarrow \underline{2}$, where $\underline{2}$ denotes the two element frame $\{0, 1\}$. $\Sigma L$ has open sets $\Sigma_a = \{\psi \in \Sigma L \mid \psi(a) = 1\}$, for $a \in L$, and $\{\Sigma_a | \; a \in L\}$ is a topology on $\Sigma L$. For any frame homomorphism $h : L \longrightarrow M$, we have $\Sigma h : \Sigma M \longrightarrow \Sigma L$ which is defined by composing a frame homomorphism from $\Sigma M$ with $h$, that is, $\Sigma h(\psi) = \psi \cdot h$, for $\psi \in \Sigma M$.

We now recall definitions of corresponding topological concepts for frames. The *pseudocomplement* of $a$ is denoted $a^*$ and is characterized by the following formula

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

For elements $a, b$ in a frame $L$, we say that $a$ is *rather below* $b$, written $a \prec b$, if there exists an element $c$ in $L$ such that $a \wedge c = 0$ and $b \vee c = 1$. A frame $L$ is said to be *regular* if

$$a = \bigvee \{x \in L \mid x \prec a\}, \text{ for every } a \text{ in } L.$$

A frame $L$ is *compact* if whenever $\bigvee S = 1$, then there exists a finite subset $F$ of $S$ such that $\bigvee F = 1$. An element $x$ in a frame $L$ is said to be *connected* if whenever $x = b \vee c$ with $b \wedge c = 0$ we have either $b = 0$ or $c = 0$. Furthermore, a frame $L$ is connected if its top element 1 is connected, and it is said to be *locally connected* provided each element in the frame can be written as the join of connected elements.

A *compactification* of a frame $M$ is a compact regular frame $L$ together with a dense onto homomorphism $h : L \longrightarrow M$, denoted by $(L, h)$. A compactification $(L, h)$ is said to be *perfect* with respect to an element $u \in M$, if

$$h_*(u \vee u^*) = h_*(u) \vee h_*(u^*),$$

where $h_* : M \longrightarrow L$ is the right adjoint of $h$. The compactification $(L, h)$ is said to be a *perfect compactification* of $M$, if it is perfect with respect to every element $u \in M$.

We recall the following from Banaschewski [4]. A *strong inclusion* on a frame $M$ is a binary relation $\blacktriangleleft$ on $M$ such that:

1. if $x \leq a \blacktriangleleft b \leq y$ then $x \blacktriangleleft y$,
2. $\blacktriangleleft$ is a sublattice of $M \times M$,

3. $a \blacktriangleleft b \Longrightarrow a \prec b$,
4. $a \blacktriangleleft b \Longrightarrow a \prec c \prec b$, for some $c \in M$,
5. $a \blacktriangleleft b \Longrightarrow b^* \blacktriangleleft a^*$,
6. for each $a \in M$, $a = \bigvee\{x \in M \mid x \blacktriangleleft a\}$.

Let $S(M)$ be the set of all strong inclusions on $M$. Let $K(M)$ be the set of all compactifications of $M$, partially ordered by $(L, h) \leq (N, f)$ if and only if there exists a frame homomorphism $g : L \longrightarrow N$ making the following diagram commute.

$$\begin{array}{ccc} L & \xrightarrow{\;g\;} & N \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ M & = \! = \! = & M \end{array}$$

Banaschewski [4] showed that $K(M)$ is isomorphic to $S(M)$ by defining maps $K(M) \longrightarrow S(M)$ and $S(M) \longrightarrow K(M)$, which are inverses of each other and are order preserving. For the map $S(M) \longrightarrow K(M)$, let $\blacktriangleleft$ be any strong inclusion on $M$. Let $\gamma M$ be the set of all strongly regular ideals of $M$ (That is, the ideals $J$ of $M$ such that $x \in J$ implies there exists $y \in J$ with $x \blacktriangleleft y$). Then the join map $\bigvee : \gamma M \longrightarrow M$ is dense and onto and $\gamma M$ is a regular subframe of the frame of ideals of $M$, $\mathcal{I}(M)$. Hence $\bigvee : \gamma M \longrightarrow M$ is a compactification of $M$ associated with the given $\blacktriangleleft$.
We will be concerned with metric frames [10], which are defined as follows: A *diameter* on a frame $L$ is a map $d : L \longrightarrow \mathbb{R}^+$ (the non-negative reals including $\infty$) such that:
(M1) $d(0) = 0$.
(M2) If $a \leq b$ then $d(a) \leq d(b)$.
(M3) If $a \wedge b \neq 0$ then $d(a \vee b) \leq d(a) + d(b)$.
(M4) For each $\varepsilon > 0$, $U^d_\varepsilon = \{u \in L | \; d(u) < \varepsilon\}$ is a cover.
A diameter $d$ is called *compatible* if
(M5) For each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_d a\}$, where $x \lhd_d a$ means there exists $U^d_\varepsilon$ such that
$$U^d_\varepsilon x = \bigvee\{u \in U^d_\varepsilon \mid u \wedge x \neq 0\} \leq a.$$
A diameter $d$ is called a *metric diameter* if
(M6) For each $a \in L$ with $d(a) < \infty$, and $\varepsilon > 0$ there exist $u, v \leq a$,

$d(u), d(v) < \varepsilon$ such that

$$d(a) - \varepsilon < d(u \vee v).$$

A frame $L$ with a specified compatible metric diameter $d$ is called a *metric frame* and is denoted by $(L, d)$. $(L, d)$ is said to be uniformly locally connected (abbreviated ulc) if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(a) < \delta$ then there exists a connected $c$, $a \leq c$ and $d(c) < \varepsilon$.

## 2 The Wallman compactification and dense sublocales of compact metric frames

Our first aim is to show that every compact metric frame is a Wallman compactification of each of its dense sublocales. In order to do so, we will generalise a result of Steiner [13]. The Wallman compactification for frames was first introduced by Johnstone [8]. We begin by defining the Wallman compactification of a frame $M$.

**Definition 2.1.** For any frame $M$, $B \subseteq M$ is called a *Wallman basis* of $M$ if:

1. The bottom and top elements of $M$ are in $B$, and $a, b \in B$ implies that $a \vee b \in B$ and $a \wedge b \in B$.
2. For every $a \in M$, $a = \bigvee\{b \in B \mid b \prec_B a\}$, where $b \prec_B a$ means that there exists $c \in B$ such that $b \wedge c = 0$ and $c \vee a = 1$.
3. For $a, b \in B$ such that $a \vee b = 1$, there exist $c, d \in B$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

**Proposition 2.2** ([2]). *Let $M$ be a regular frame and $B$ a Wallman basis for $M$. Define $a \blacktriangleleft_B b$ in $M$ by*

$$a \blacktriangleleft_B b \text{ iff there exists } c \in B \text{ such that } a \prec_B c \prec_B b.$$

*Then $\blacktriangleleft_B$ is a strong inclusion on $M$.*

From Proposition 2.2, the corresponding compactification associated with this Wallman basis $B$, denoted by $\gamma_B M$, is called the *Wallman compactification* of $M$. Here $\gamma_B M$ consists of all strongly regular ideals of $M$ associated with $\blacktriangleleft_B$ and we have the join map $\bigvee : \gamma_B M \longrightarrow M$.

Baboolal [2] also showed how using the Wallman basis of a frame, one could obtain a Wallman basis for the corresponding Wallman compactification, using the join map.

**Proposition 2.3** ([2])**.** *Let $B$ be a Wallman basis of $M$, then $k(B)$ is a basis for $\gamma_B M$ where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$.*

We now recall a result of Steiner [13], in spaces. Before generalising the result in frames, we also recall the statement of the Boolean Ultrafilter Theorem which is required in the next proof we present.

**Proposition 2.4** ([13])**.** *If $(X, d)$ is a compact metric space, then it has a base $\mathcal{B}$ of open regular sets which satisfies the following: $B_1, B_2 \in \mathcal{B}$ implies that $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$. We say that $\mathcal{B}$ is a ring consisting of regular open sets.*

**Definition 2.5.** An element $a$ of a frame $M$ is called *regular* if $a = a^{**}$.

**Remark 2.6.** We note the following:

1. If $X$ is a topological space, then an open set $U$ is said to be regular open if $U = int(\overline{U})$.
2. It can be shown that an open set $U \in \mathcal{O}X$ is regular open if and only if $U = U^{**}$, where $U^*$ refers to the pseudocomplement of $U$ in the frame $\mathcal{O}X$.
   Thus an open set $U$ is *regular open* if and only if $U \in \mathcal{O}X$ is a regular element.

**Definition 2.7.** Let $M$ be a frame and $B \subseteq M$. $B$ is called a *ring* in $M$, if $b_1, b_2 \in B$ implies that $b_1 \wedge b_2 \in B$ and $b_1 \vee b_2 \in B$.

**Theorem 2.8** ([5], **(Boolean ultrafilter theorem)**)**.** *Every non trivial Boolean algebra contains an ultrafilter (That is, a maximal proper filter).*

**Lemma 2.9** ([5])**.** *The following are equivalent:*

1. *Every non trivial Boolean algebra contains an ultrafilter.*
2. *Every compact regular frame $M$ is spatial.*

3. *$\Sigma M \neq \emptyset$, for every non-trivial, compact regular $M$.*

In the next proposition we provide a generalisation Steiner's result.

**Proposition 2.10.** *If $(M, d)$ is a compact metric frame, then $M$ has a base $B$ of regular elements, and $B$ is a ring.*

*Proof.* If $(M, d)$ is a compact metric frame then $(M, d)$ is compact regular, since every metric frame is regular. If we assume the Boolean ultrafilter theorem, then by Lemma 2.9, $M$ is spatial. Thus

$$\eta : M \longrightarrow \mathcal{O}\Sigma M, \text{ given by } \eta(a) = \Sigma_a = \{\psi : M \longrightarrow \underline{2} \mid \psi(a) = 1\},$$

for $a \in M$, is an isomorphism. From [6], $(\Sigma M, \rho)$ is a metric space with metric given by

$$\rho(\xi, \eta) = \inf\{d(a) \mid \xi(a) = 1 = \eta(a)\}, \text{ for } \xi, \eta \in \Sigma M,$$

and $\tau_\rho$ (the topology on $\Sigma M$ generated by $\rho$) is exactly $\mathcal{O}\Sigma M$. Furthermore, since $M$ is compact, $\mathcal{O}\Sigma M$ is compact and therefore $\Sigma M$ is compact. So $(\Sigma M, \rho)$ is a compact metric space and by Proposition 2.4, has a ring base $\mathcal{B}$ consisting of regular open sets of $\Sigma M$. Each $\Sigma_a \in \mathcal{B}$ is regular open in $\Sigma M$, so $\Sigma_a \in \mathcal{O}\Sigma M$ is a regular element of the frame $\mathcal{O}\Sigma M$ . Since $\eta$ is an isomorphism, $\eta^{-1}(\mathcal{B}) = B$ is a ring base for $M$ consisting of regular elements. We can assume that $0_M, 1_M$ is also in $B$, without loss of generality, since $B \cup \{0_M, 1_M\}$ is still a ring base for $M$. □

The existence of a ring basis $B$ of regular elements for a compact frame $L$, is now guaranteed by Proposition 2.10. Utilizing this, we can show that for any dense onto frame homomorphism $h : L \to M$ where $L$ is compact, the image of $B$ under $h$ is a Wallman basis.

**Proposition 2.11.** *Let $h : L \longrightarrow M$ be a dense onto frame homomorphism. Suppose that $L$ is compact and let $B$ be a ring basis of regular elements of $L$. Then $h(B)$ is a Wallman basis of $M$.*

*Proof.* (1): Take any $h(b_1), h(b_2) \in h(B)$, for $b_1, b_2 \in B$. Then $h(b_1) \wedge h(b_2) = h(b_1 \wedge b_2)$, and since $B$ is a ring, $h(b_1 \wedge b_2) \in h(B)$. Now $h(b_1) \vee h(b_2) = h(b_1 \vee b_2) \in h(B)$, since $B$ is a ring. Also, $0_M = h(0_L) \in h(B)$ and $1_M = h(1_L) \in h(B)$.

(2): Take any $w \in M$. We will show that $w = \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$. Now $w = h(a)$, for some $a \in L$ since $h$ is onto, and
$a = \bigvee\{b \mid b \in B, b \prec a\}$, since $L$ is regular and $B$ is a basis of $L$.
Claim 1: $b \prec a \Longleftrightarrow b \prec_B a$. (2.1)
For $b \prec a$, we have $b^* \vee a = 1_L$. Now $b^* = \bigvee\{c \mid c \in B, c \leq b^*\}$, so by the compactness of $L$, we have $c_1 \vee c_2 \vee ... \vee c_n \vee a = 1_L$, for suitable $c_i \leq b^*$ and $c_i \in B$ for $i = 1, ..., n$. Since $B$ is closed under finite joins, then $c = c_1 \vee c_2 \vee ... \vee c_n \in B$, and so $c \vee a = 1_L$ with $c \in B$ and $c \leq b^*$. Hence $c \wedge b = 0_L$. Thus for $b \prec a$, we have shown that there exists $c \in B$ such that $b \wedge c = 0_L$ and $c \vee a = 1_L$. Hence $b \prec_B a$.
Now $b \prec_B a$ implies $b \prec a$ is immediate, hence $b \prec a$ if and only if $b \prec_B a$.

We also note that $b \prec_B a$ implies $h(b) \prec_{h(B)} h(a)$, since for $c \in B$ such that $b \wedge c = 0_L$ and $c \vee a = 1_L$, we have $h(b) \wedge h(c) = 0_M$, $h(c) \vee h(a) = 1_M$ and $h(c) \in h(B)$. Thus

$$\begin{aligned} w = h(a) &= h(\bigvee\{b \in B \mid b \prec a\}) \\ &= h(\bigvee\{b \in B \mid b \prec_B a\}) \\ &= \bigvee\{h(b) \mid b \in B, b \prec_B a\} \\ &\leq \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} h(a)\} \\ &= \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\} \\ &\leq w. \end{aligned}$$

So $w = \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$, as required.

(3): Take any $h(a), h(b) \in h(B)$ with $a, b \in B$, such that $h(a) \vee h(b) = 1_M$. Then $h(a \vee b) = 1_M$. We have to show that there exist $h(c), h(d) \in h(B)$ such that $h(c) \wedge h(d) = 0_M$ and $h(c) \vee h(a) = 1_M = h(d) \vee h(b)$. Now $a \vee b \in B$, so $a \vee b$ is regular.
Claim 2: If $x \in L$ is regular and $h(x) = 1_M$, then $x = 1_L$. (2.2)
Assume that $h(x) = 1_M$ where $x$ is regular. Then,

$$\begin{aligned} & (h(x))^* = 0_M \\ \Longrightarrow\ & h(x^*) = 0_M \end{aligned}$$

$$\implies x^* = 0_L \quad (\text{since } h \text{ is dense})$$
$$\implies x^{**} = 1_L.$$

Since $x$ is regular, $x = 1_L$, as claimed.

Hence $h(a \vee b) = 1_M$ implies $a \vee b = 1_L$. Now $a = \bigvee\{x \mid x \in B, x \prec_B a\}$, and $b = \bigvee\{y \mid y \in B,\ y \prec_B b\}$, therefore

$$\bigvee\{x \mid x \in B,\ x \prec_B a\} \vee \bigvee\{y \mid y \in B,\ y \prec_B b\} = 1_L.$$

Since $M$ is compact, there exists $x \in B$ with $x \prec_B a$, and there exists $y \in B$ with $y \prec_B b$ such that $x \vee y = 1_L$. $x \prec_B a$ implies that there exists $c \in B$, such that $x \wedge c = 0_L$ and $c \vee a = 1_L$, and $y \prec_B b$ implies that there exists $d \in B$ such that $y \wedge d = 0_L$ and $d \vee b = 1_L$. Now, $c \wedge d = (c \wedge d) \wedge (x \vee y) = (c \wedge d \wedge x) \vee (c \wedge d \wedge y) = 0_L$. Hence $h(c) \wedge h(d) = h(c \wedge d) = 0_M$. Furthermore, $h(c) \vee h(a) = 1_M$, since $c \vee a = 1_L$ and $h(d) \vee h(b) = 1_M$, since $d \vee b = 1_L$. Hence condition (3) is satisfied.

We have shown that $h(B)$ is a Wallman basis of $M$. □

We briefly discuss an application of Proposition 2.11 to dense metric sublocales to guarantee the existence of a Wallman basis for all dense metric sublocales of compact frames. We recall the definition of a metric sublocale [9].

**Definition 2.12** ([9])**.** Let $(L, \rho)$ be a metric frame and $h : L \longrightarrow M$ be an onto frame homomorphism. For $a \in M$, let

$$d(a) = \inf\{\rho(x) \mid a \leq h(x), x \in L\},$$

then $d$ is a compatible metric diameter on $M$, and $(M, d)$ is called a *metric sublocale* of $(L, \rho)$. Additionally, if $h$ is a dense map, then we call $(M, d)$ a *dense metric sublocale* of $(L, \rho)$.

**Corollary 2.13.** *Let $(M, d)$ be a dense metric sublocale of $(L, \rho)$, with a dense onto homomorphism $h : L \longrightarrow M$. Suppose that $L$ is compact and let $B$ be a ring basis of regular elements of $L$. Then $h(B)$ is a Wallman basis of $M$.*

*Proof.* Follows immediately from Proposition 2.11. □

We now recall a result that follows directly from the work of Banaschewski in [4].

**Theorem 2.14** ([4]). *Let $M$ be a frame. Let $(L, h)$ be a compactification of $M$ associated with strong inclusion $\blacktriangleleft_1$, and let $(N, f)$ be a compactification of $M$ associated with strong inclusion $\blacktriangleleft_2$. If $\blacktriangleleft_1 = \blacktriangleleft_2$, then $L \cong N$.*

It is well-known in the literature that rather below relation, $\prec$, interpolates in a compact regular frame. We recall this fact below and then present an isomorphism theorem for the Wallman compactification of dense sublocales of a frame.

**Proposition 2.15** ([5]). *Let $L$ be a compact regular frame. Then for any $a, b \in L$, $a \prec b$ implies that there exists $c \in L$ such that $a \prec c \prec b$. We say that $\prec$ interpolates in a compact regular frame.*

**Theorem 2.16.** *With the conditions as in Proposition 2.13, the Wallman compactification $\gamma_{h(B)}M$ of $M$ is isomorphic to $L$ (as frames).*

*Proof.* By Proposition 2.2, $h(B)$ determines a strong inclusion on $M$ given by: $x \blacktriangleleft y$ for $x, y \in M$ if and only if there exists $h(b)$ for $b \in B$, such that $x \prec_{h(B)} h(b) \prec_{h(B)} y$. Thus, $\gamma_{h(B)}M = \{J \mid J \text{ is a strongly regular ideal}\}$, where $J$ is said to be strong regular if $x \in J$ implies there exists $y \in J$ such that $x \blacktriangleleft y$. $\gamma_{h(B)}M$ is a compact regular frame and the join map

$$\begin{aligned} \bigvee : \gamma_{h(B)}M &\longrightarrow M \\ J &\mapsto \bigvee J \end{aligned}$$

makes $\gamma_{h(B)}M$ a compactification of $M$. We will show that $\gamma_{h(B)}M \cong L$. Let $h_*$ be the right adjoint of $h$. We note that $h : L \longrightarrow M$ is a compactification of $M$ (since $L$ is a compact regular frame), and this induces a strong inclusion $\blacktriangleleft_1$ on $M$ given by

$$x \blacktriangleleft_1 y \iff h_*(x) \prec h_*(y).$$

It suffices to show that $\blacktriangleleft = \blacktriangleleft_1$, for then by Theorem 2.14, $\gamma_{h(B)}M \cong L$. So suppose that $x \blacktriangleleft_1 y$, for $x, y \in M$. Then $h_*(x) \prec h_*(y)$ and therefore there exists $z \in L$ such that $h_*(x) \prec z \prec h_*(y)$, since $\prec$ interpolates in compact regular frames by Proposition 2.15. Now $h_*(x) \prec z$ implies $h_*(x)^* \vee z = 1_L$, and so $h_*(x)^* \vee \bigvee\{b \in B \mid b \leq z\} = 1_L$. Since $L$ is compact and $B$ is closed under finite joins, it follows that $h_*(x)^* \vee b = 1_L$, for some $b \in B$ with $b \leq z$. Now,

$$\begin{aligned}
& h_*(x) \prec b \leq z \prec h_*(y) \\
\implies & h_*(x) \prec b \prec h_*(y) \quad (b \in B) \\
\implies & h_*(x) \prec_B b \prec_B h_*(y) \quad \text{(by equation (2.1))} \\
\implies & hh_*(x) \prec_{h(B)} h(b) \prec_{h(B)} hh_*(y) \\
\implies & x \prec_{h(B)} h(b) \prec_{h(B)} y \\
\implies & x \blacktriangleleft y.
\end{aligned}$$

Now suppose $x \blacktriangleleft y$, for $x, y \in M$. Then there exists $b_1 \in B$ such that

$$x \prec_{h(B)} h(b_1) \prec_{h(B)} y.$$

$x \prec_{h(B)} h(b_1)$ implies there exists $c_1 \in B$ such that $x \wedge h(c_1) = 0_M$ and $h(c_1) \vee h(b_1) = 1_M$. Now $h(h_*(x) \wedge c_1) = hh_*(x) \wedge h(c_1) = x \wedge h(c_1) = 0_M$. So, $h_*(x) \wedge c_1 = 0_L$, since $h$ is a dense map. Furthermore, $c_1 \vee b_1 \in B$ and is therefore regular, so by equation (5.2), since $h(c_1 \vee b_1) = h(c_1) \vee h(b_1) = 1_M$, we must have $c_1 \vee b_1 = 1_L$. Hence we have shown that $h_*(x) \prec b_1$. Now, we observe that

$$\begin{aligned}
& h(b_1) \leq y \\
\implies & b_1 \leq h_*(y) \\
\implies & h_*(x) \prec b_1 \prec h_*(y) \\
\implies & h_*(x) \prec h_*(y) \\
\implies & x \blacktriangleleft_1 y.
\end{aligned}$$

Hence, we have shown that $\gamma_{h(B)}M \cong L$. □

## 3 S-metrizability and the Wallman basis

The purpose of this section is to provide one of the main results of this paper. We present a characterisation of S-metrizability in terms of the Wallman basis of a frame. S-metrizabilty of a frame is defined in terms of a connectedness property, called *Property S*, which is attributed to Sierpinski [12].

**Definition 3.1.** Let $(L, d)$ be a metric frame. $L$ is said to have *Property S* if, given any $\varepsilon > 0$, there exist $a_1, a_2, ..., a_n$ such that $\bigvee_{i=1}^{n} a_i = 1$, where $a_i$ is connected and $d(a_i) < \varepsilon$ for each $i$.

**Definition 3.2.** Let $(L, d)$ be a metric frame. Then $(L, d)$ is *S-metrizable* if $L$ admits a metric diameter that has Property S.

In what remains, we will let $M$ be a locally connected frame. We briefly state required theory from [2].

**Definition 3.3.** An element $0 \neq c \in M$ is a *component* of an element $u \in M$ if:

1. $c$ is connected and $c \leq u$,
2. $c$ is maximally connected in $u$ (that is, whenever $c \leq x \leq u$ and $x$ is connected in $M$, then $c = x$).

**Remark 3.4.** We note that if $c_\alpha$ and $c_\beta$ are components of $u \in M$, and $c_\alpha \neq c_\beta$, then $c_\alpha \wedge c_\beta = 0$

**Definition 3.5.** Let $B \subseteq M$ be a Wallman basis. Then $B$ is *locally connected* if each component of each element of $B$ is also in $B$.

**Definition 3.6.** A basis $B$ of $M$ is *uniformly connected* if whenever $A$ is finite, $\bigvee A = 1$ and $A \subseteq B$, then there exists finite cover $C \subseteq B$, such that every $c \in C$ is connected and $C$ is a refinement of $A$, denoted by $C \leq A$.

**Definition 3.7.** Let $\gamma_B M$ be the Wallman compactification associated with a Wallman basis $B$. An ideal $J \in \gamma_B M$ is said to be *insular* if whenever $x \in J$, there exists $y \in J$ having finitely many components, such that $y \in B$ and $x \blacktriangleleft y$.

In [2], Baboolal obtained the following characterisation for insular ideals of the Wallman compactification associated with a locally connect Wallman basis on a locally connected frame. This result plays an important role in the main result of this paper.

**Theorem 3.8** ([2]). *Let $B$ be a locally connected Wallman basis for the locally connected frame $M$. Then the following are equivalent*:

1. $\bigvee : \gamma_B M \longrightarrow M$ *is a perfect locally connected compactification of* $M$.
2. $B$ *is uniformly connected.*
3. *Every ideal $J$ in $\gamma_B M$ is insular.*

Although the following Lemma is known, it is difficult to find in the literature. We therefore, provide a proof for completeness.

**Lemma 3.9.** *Let $M$ be a locally connected frame and $c$ be a component of $v \in M$. Then $v \leq c \vee c^*$.*

*Proof.* By the local connectedness of $M$, $v = \bigvee_{\alpha \in I} c_\alpha$, where $c_\alpha$ are the components of $v$. Now $c = c_\alpha$, for some $\alpha \in I$. For $\beta \neq \alpha$, $c_\beta \wedge c_\alpha = 0_M$, so $c_\beta \leq c^*$. This implies that $\bigvee_{\beta \neq \alpha} c_\beta \leq c^*$, therefore $v = c \vee (\bigvee_{\beta \neq \alpha} c_\beta) \leq c \vee c^*$. □

Next we shall show that S-metrizability of a locally connected frame ensures the existence of a countable locally connected and uniformly connected Wallman basis. Before doing this, we need the following two propositions on *countability.*

**Proposition 3.10.** *Every compact metric frame has a countable base.*

*Proof.* Let $(M, d)$ be a compact metric frame. For each $n \in \mathbb{N}$, $U^d_{\frac{1}{n}} = \{x \in M \mid d(x) < \frac{1}{n}\}$ is a cover of $M$. So by compactness of $M$, there exists a finite cover $F_n \subseteq U^d_{\frac{1}{n}}$, of $M$.

Let $B = \bigcup_{n=1}^{\infty} F_n$. Then $B$ is countable. We shall show that $B$ is a base for $M$. Take any $a \in M$. Then $a = \bigvee\{x \in M \mid x \triangleleft_d a\}$. Now for any $x \triangleleft_d a$, there exists $\varepsilon > 0$, such that $U^d_\varepsilon x \leq a$. Take $n \in \mathbb{N}$, such that $\frac{1}{n} < \varepsilon$. Then $U^d_{\frac{1}{n}} x \leq a$. Since $F_n$ is a cover of $M$,

$$x = x \wedge \bigvee\{y \mid y \in F_n\} = \bigvee\{x \wedge y \mid y \in F_n, y \neq 0\}.$$

Now, $y \in F_n$ and $x \wedge y \neq 0$ imply that $y \leq a$ and therefore

$$x \leq \bigvee \{y \in F_n \mid x \wedge y \neq 0\} \leq a.$$

Since $a$ is a join of the $x$'s, it follows that $a$ is a join of elements that come from $B$, since each $y \in F_n$ is in $B$. So $B$ is a countable base. □

**Proposition 3.11.** *If $(M, d)$ is a compact locally connected metric frame, then each $u \in M$ has only countably many components.*

*Proof.* Since $M$ is locally connected, $u = \bigvee_{\alpha \in I} c_\alpha$, where $c_\alpha$ are the components of $u$. Let $B$ be a countable base of $M$. The existence of a countable base follows from Proposition 3.10. Each $c_\alpha$ is a join of elements from $B$, so we can choose any $b_\alpha \in B$ such that $b_\alpha \leq c_\alpha$. Whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$, then $c_\alpha \wedge c_\beta = 0$, therefore $b_\alpha \neq b_\beta$. Thus if $I$ were uncountable, then $\{b_\alpha\}_{\alpha \in I}$ would be uncountable. But $\{b_\alpha\}_{\alpha \in I} \subseteq B$, and $B$ is countable. Hence $\{b_\alpha\}_{\alpha \in I}$ is countable, which is a contradiction. Thus $I$ is countable. □

**Theorem 3.12** ([11])**.** *Let $(M, d)$ be a connected, locally connected metric frame. Then $(M, d)$ is S-metrizable if and only if $(M, d)$ has a perfect locally connected metrizable compactification.*

We are now ready to present the main result of this section:

**Proposition 3.13.** *Let $(M, d)$ be a connected metric frame. If $M$ is S-metrizable then $M$ has a countable, locally connected and uniformly connected Wallman basis.*

*Proof.* Assume that $(M, d)$ is S-metrizable. Then by Theorem 3.12, $(M, d)$ has a perfect locally connected metrizable compactification (just take the completion of $(M, d)$). Call it $(L, \rho)$ and let $h : (L, \rho) \longrightarrow (M, d)$ be a dense surjection where $\rho(a) = d(h(a))$, for all $a \in L$. We know by Propositions 2.10 and 3.10, that whenever $L$ is a compact metric frame, then $L$ has a countable ring basis, call it $B_0$, consisting of regular elements. Let

$$C_0 = \{c \in L \mid c \text{ is a component of some } b \in B_0\},$$

and let $B_1 = \langle B_0 \cup C_0 \rangle$, where $\langle B_0 \cup C_0 \rangle$ denotes the ring generated by $B_0$ and $C_0$. We will now show that $B_1$ is the smallest ring containing $B_0$ and $C_0$. Since $B_1 = \langle B_0 \cup C_0 \rangle$, we have

$$B_1 = \{x \in L \mid x \text{ is a finite join of elements } y,\ y = \bigwedge_{i=1}^{n} t_i,\ t_i \in B_0 \cup C_0\}.$$

Take any $x, y \in B_1$. Then $x = \bigvee_{i=1}^{n} x_i$, where $x_i = s_1^i \wedge ... \wedge s_{k_i}^i$, for $s_j^i \in B_0 \cup C_0$, and $y = \bigvee_{i=1}^{m} y_i$, where $y_i = t_1^i \wedge ... \wedge t_{q_i}^i$, for $t_{q_i}^i \in B_0 \cup C_0$. Thus $x \vee y = \bigvee_{i=1}^{n} x_i \vee \bigvee_{i=1}^{m} y_i$, with $x_i$ and $y_i$ as described above, so $x \vee y \in B_1$. Now, $x \wedge y = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i \wedge y_i)$, where $x_i \wedge y_i = s_1^i \wedge ... \wedge s_{k_i}^i \wedge t_1^i \wedge ... \wedge t_{q_i}^i$. So $x \wedge y \in B_1$. Hence $B_1$ is a ring containing $B_0$ and $C_0$, and $B_1$ is the smallest ring containing $B_0$ and $C_0$.

We now show that $B_1$ consists of regular elements. We first note that if $x$ and $y$ are regular then $x \wedge y$ is regular. For if $x = x^{**}$ and $y = y^{**}$, then $(x \wedge y)^{**} = x^{**} \wedge y^{**} = x \wedge y$ and so $x \wedge y$ is regular. If $c \in C_0$, then $c$ is a component of some $b \in B_0$. Now $c \leq b$ implies that $c^{**} \leq b^{**} = b$, so $c \leq c^{**} \leq b$. Now, $c$ is connected therefore $c^{**}$ is connected. Since $c$ is a component we must have $c = c^{**}$. Hence $c$ is regular. Thus $B_0 \cup C_0$ consists of regular elements and finite meets of elements from $B_0 \cup C_0$ is regular. Let

$$H_1 = \{x \in L \mid x \text{ is a finite meet of elements from } B_0 \cup C_0\}.$$

Then $H_1$ consists of regular elements. For each $m > 1$, let

$$H_m = \{x \in L \mid x \text{ is a join of at most } m \text{ elements from } H_1\}.$$

We prove by induction that each $H_m$ consists of regular elements. Let $m > 1$ and assume $H_{m-1}$ consists of regular elements. Let $x \in H_m$. Then there exist $h_1, h_2, ..., h_m \in H_1$ such that $x = h_1 \vee h_2 \vee ... \vee h_m$. Take any $h_k$ for $1 \leq k \leq m$. Now,

$$\begin{aligned} h_k &= b_1 \wedge ... \wedge b_t \wedge c_1 \wedge ... \wedge c_s \quad (\text{where } b_i \in B_0, c_j \in C_0) \\ &= b \wedge c_1 \wedge ... \wedge c_s, \end{aligned}$$

where $b = b_1 \wedge ... \wedge b_t \in B_0$, since $B_0$ is a ring. Each $c_i$ is a component of some $v_i \in B_0$, so

$$h_k = b \wedge c_1 \wedge ... \wedge c_s$$

$$\leq b \wedge v_1 \wedge ... \wedge v_s = d_k \in B_0.$$

Claim: $d_k \leq h_k \vee h_k^*$.
$h_k \vee h_k^* = (b \wedge c_1 \wedge ... \wedge c_s) \vee (b \wedge c_1 \wedge ... \wedge c_s)^*$. Now $h_k = b \wedge c_1 \wedge ... \wedge c_s \leq c_i$, for $i = 1, ..., s$. So $c_i^* \leq h_k^*$, for each $i$, and thus $c_1^* \vee ... \vee c_s^* \leq h_k^*$. Hence,

$$\begin{aligned}
h_k \vee h_k^* &\geq (b \wedge c_1 \wedge ... \wedge c_s) \vee (c_1^* \vee ... \vee c_s^*) \\
&= (b \vee (c_1^* \vee ... \vee c_s^*)) \wedge (c_1 \vee (c_1^* \vee ... \vee c_s^*)) \wedge \ ... \ \wedge (c_s \vee (c_1^* \vee ... \vee c_s^*)) \\
&\geq b \wedge (c_1 \vee c_1^* \vee ... \vee c_s^*) \wedge (c_2 \vee c_1^* \vee ... \vee c_s^*) \wedge \ ... \ \wedge (c_s \vee c_1^* \vee ... \vee c_s^*) \\
&\geq b \wedge (c_1 \vee c_1^*) \wedge (c_2 \vee c_2^*) \wedge \ ... \ \wedge (c_s \vee c_s^*) \quad \text{(By Lemma 3.9)} \\
&\geq b \wedge v_1 \wedge v_2 \wedge \ ... \ \wedge v_s = d_k.
\end{aligned}$$

Thus proving the claim that $d_k \leq h_k \vee h_k^*$.
We now show that $x$ is regular. Firstly, $x = h_1 \vee h_2 \vee ... \vee h_m \leq d_1 \vee d_2 \vee ... \vee d_m$. Hence $x^{**} \leq (d_1 \vee d_2 \vee ... \vee d_m)^{**} = d_1 \vee d_2 \vee ... \vee d_m$, since $d_i \in B_0$ and $B_0$ is a ring of regular elements. Fix any $i$, $1 \leq i \leq m$. Now $x = h_i \vee \bigvee_{j \neq i} h_j$, hence

$$\begin{aligned}
x \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \\
\implies \quad (x \wedge h_i^*)^{**} &\leq (\bigvee_{j \neq i} h_j)^{**} = \bigvee_{j \neq i} h_j \quad \text{(by the induction hypothesis)} \\
\implies \quad x^{**} \wedge h_i^{***} &\leq \bigvee_{j \neq i} h_j \\
\implies \quad x^{**} \wedge h_i^* &\leq \bigvee_{j \neq i} h_j
\end{aligned}$$

Hence for all $i$, we have $x^{**} \wedge h_i^* \leq \bigvee_{j \neq i} h_j$. Now,

$$\begin{aligned}
x^{**} &\leq d_1 \vee d_2 \vee ... \vee d_m \\
&\leq (h_1 \vee h_1^*) \vee (h_2 \vee h_2^*) \vee ... \vee (h_m \vee h_m^*) \\
&= (h_1 \vee ... \vee h_m) \vee (h_1^* \vee ... \vee h_m^*) \\
&= x \vee h_1^* \vee h_2^* ... \vee h_m^*.
\end{aligned}$$

Therefore,

$$x^{**} = x^{**} \wedge (x \vee h_1^* \vee h_2^* ... \vee h_m^*)$$

$$= (x^{**} \wedge x) \vee (x^{**} \wedge h_1^*) \vee (x^{**} \wedge h_2^*) \vee ... \vee (x^{**} \wedge h_m^*)$$
$$\leq x \vee \bigvee_{j \neq 1} h_j \vee \bigvee_{j \neq 2} h_j \vee ... \vee \bigvee_{j \neq m} h_j$$
$$\leq x.$$

Since $x \leq x^{**}$, we conclude that $x = x^{**}$, and so $x$ is regular.
Thus by induction on $m$, $H_m$ consists of regular elements for every $m > 1$. Thus $B_1 = \langle B_0 \cup C_0 \rangle$ consists of regular elements. Let $B_2 = \langle B_1 \cup C_1 \rangle$, where $C_1$ consists of components of elements from $B_1$. By a similar argument in which we showed that $B_1$ consists of regular elements, we can show that $B_2$ consists of regular elements. Thus $B = \bigcup_{n=0}^{\infty} B_n$, consists of regular elements. Also, B is a ring basis since $B_n \subseteq B_{n+1}$ and since each $B_n$ is a ring basis. Hence by Proposition 2.13, $h(B)$ is a Wallman basis for $(M, d)$.

Claim: $h(B)$ is countable.
$B_0$ is countable and by Proposition 3.11, since $(L, \rho)$ is compact and locally connected, it follows that $C_0$ is countable. Thus the ring generated by $B_0$ and $C_0$ is countable. So $B_1$ is countable. It follows that all $B_n$'s are countable. Hence $B = \bigcup_{n=0}^{\infty} B_n$ is countable. In addition, $h(B)$ would then be a countable base, as claimed.

We now show that $h(B)$ is a locally connected base. Take any $h(b) \in h(B)$, where $b \in B$. Let $w$ be a component of $h(b)$. We will show that $w \in h(B)$. Now, $b \in B_n$ for some $n$. We know that $b = \bigvee_\alpha \{c_\alpha \mid c_\alpha \text{ is a component of } b\}$, therefore
$$h(b) = \bigvee_\alpha \{h(c_\alpha) \mid c_\alpha \text{ is a component of } b\}.$$

Since $(L, \rho)$ is a perfect compactification, then each $h(c_\alpha)$ is connected in $M$. Now $w \leq h(b)$ implies $w \wedge h(c_\alpha) \neq 0_M$, for some component $c_\alpha$ of $b$. Therefore $w \leq w \vee h(c_\alpha) \leq h(b)$, with $w \vee h(c_\alpha)$ connected in $M$. Since $w$ is a component of $h(b)$, $h(c_\alpha) \leq w$. Also,

$$w = w \wedge h(b) = (w \wedge h(c_\alpha)) \vee \bigvee_{\beta \neq \alpha} (w \wedge h(c_\beta)).$$

Furthermore,

$$(w \wedge h(c_\alpha)) \wedge \bigvee_{\beta \neq \alpha} (w \wedge h(c_\beta)) = w \wedge (h(c_\alpha) \wedge \bigvee_{\beta \neq \alpha} h(c_\beta)) = 0_M.$$

Whenever $\beta \neq \alpha$, then $h(c_\alpha) \wedge h(c_\beta) = h(c_\alpha \wedge c_\beta) = h(0_L) = 0_M$. So since $w$ is connected and $w \wedge h(c_\alpha) \neq 0_M$, we must have $\bigvee_{\beta \neq \alpha}(w \wedge h(c_\beta)) = 0_M$. Hence $w = w \wedge h(c_\alpha) \leq h(c_\alpha)$, and therefore $w = h(c_\alpha)$. But $c_\alpha$ is a component of $b \in B_n$ for some $n$, so $c_\alpha \in B_{n+1} \subseteq B$. Thus $w = h(c_\alpha)$ with $c_\alpha \in B$, showing that $h(B)$ is a locally connected basis.
Lastly, we show that $h(B)$ is a uniformly connected base. We have $h : (L, \rho) \longrightarrow (M, d)$ is a perfect locally connected metrizable compactification of $M$, therefore by Proposition 2.16, the Wallman compactification $\gamma_{h(B)}M \cong L$, as frames. Thus $\gamma_{h(B)}M$ is a perfect locally connected compactification of $M$. By Theorem 3.8, $h(B)$ is uniformly connected. Thus $h(B)$ is a countable, locally connected and uniformly connected Wallman base for $M$. □

## 4 The Main Result

The following metrization theory from [9], is required for our main result:

**Definition 4.1.** A subset $X \subseteq M$ is said to be *locally finite* if there exists a cover $W$ of $M$ such that each $w \in W$ meets only finitely many elements from $X$.

**Definition 4.2.** A basis $B$ of $M$ is said to be *$\sigma-$locally finite* if $B = \bigcup_{n=1}^{\infty} B_n$ and each subset $B_n$ is locally finite.

**Theorem 4.3** ([9]). *Let $M$ be a regular frame. $M$ is metrizable if and only if $M$ has a $\sigma-$locally finite basis.*

We now establish our main result in this section, which is a generalisation of a result of García-Máynez [7].

**Theorem 4.4.** *Let $M$ be a connected and locally connected frame. The following are equivlent:*

1. *$M$ is S-metrizable.*
2. *$M$ has a countable locally connected and uniformly connected Wallman basis.*
3. *$M$ has a countable locally connected Wallman basis $B$ such that every ideal $J$ of $\gamma_B M$ is insular.*

*Proof.* $1 \Longrightarrow 2$: Follows from Proposition 3.13.

$2 \Longleftrightarrow 3$: Follows from Theorem 3.8.

$2 \Longrightarrow 1$: Suppose then that $M$ has a countable locally connected and uniformly connected Wallman basis $B$. By Theorem 3.8, $\bigvee : \gamma_B M \longrightarrow M$ is a perfect locally connected compactification of $M$. From Proposition 2.3, $k(B)$ is a basis for $\gamma_B M$, where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$. Since $B$ is countable, then $k(B)$ is countable. Thus $\gamma_B M$ has a countable basis and hence by Theorem 4.3 $\gamma_B M$ must be metrizable, since it is regular . So $M$ has a perfect locally connected metrizable compactification and hence by Theorem 3.12 is S-metrizable. □

**Remark 4.5.** It should be noted that in [7], García-Máynez does not assume connectedness nor local connectedness. However, it is not expected that local connectedness could be relaxed in the point-free context.

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