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Idempotent 2×2 matrices over linearly ordered abelian groups

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Abstract. In this paper we study the multiplicative semigroup of 2×2 matrices over a linearly ordered abelian group with an externally added bottom element. The multiplication of such a semigroup is defined by replacing addition and multiplication by join and addition in the usual formula defining matrix multiplication. We show that there are four types of idempotents in such a matrix semigroup and we determine which of them are 0-primitive. We also prove that the poset of idempotents of such a matrix semigroup with respect to the natural order is a lattice. It turns out that such a matrix semigroup is inverse or orthodox if and only if the abelian group is trivial.

1 Introduction

We start by recalling some terminology. A *semiring* is a set R with two binary algebraic operations $\oplus$ and $\odot$ such that (1) $(R, \oplus)$ is a commutative monoid with an identity element 0, (2) $(R, \odot)$ is a semigroup, (3) the distributivity laws hold, (4) $0 \odot r = 0 = r \odot 0$ for every $r \in R$. If $(R, \odot)$ is a

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monoid, then R is called a *semiring with identity.* A *semifield* is a semiring $(R, \oplus, \odot)$, where $(R \setminus \{0\}, \odot)$ is a group. Matrix rings over semirings can be defined in the same way as matrix rings over rings.

Let $\mathbf{A} = (A, +, \leq)$ be an ordered abelian group (see [3]). By adjoining an external bottom element $\bot$ and defining $a + \bot = \bot + a = \bot + \bot = \bot$ for all $a \in A$ we obtain a commutative pomonoid $\mathbf{A}^{\bot}$. Moreover, if $\mathbf{A}$ is a lattice-ordered abelian group, then $(A^{\bot}, \vee, +)$ is a semifield with multiplicative identity 0 and additive identity $\bot$ (cf. [7, Proposition 4.1]). Hence the set $M_2(\mathbf{A}^{\bot})$ of 2×2 matrices over $\mathbf{A}^{\bot}$ is a semiring with respect to componentwise joins of matrices and multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (a+e) \vee (b+g) & (a+f) \vee (b+h) \\ (c+e) \vee (d+g) & (c+f) \vee (d+h) \end{pmatrix},$$

where $I = \begin{pmatrix} 0 & \bot \\ \bot & 0 \end{pmatrix}$ is the multiplicative identity element (we call it the *identity matrix*) and $\Theta = \begin{pmatrix} \bot & \bot \\ \bot & \bot \end{pmatrix}$ is the zero element (we call it the *zero matrix*).

One of the important special cases is the linearly ordered abelian group $(\mathbb{R}, +, \leq)$. Usually the adjoined bottom element is denoted by $-\infty$ and the semiring $(\mathbb{R} \cup \{-\infty\}, \vee, +)$ is called the *tropical semiring.* Matrices over it are called *tropical matrices.* The study of such matrices is motivated by numerous applications.

In [6], Johnson and Kambites initiated the systematic study of the multiplicative semigroup of tropical matrices of order 2. Using methods of tropical geometry they described Green's relations and proved that this semigroup is regular. They also described all idempotents of this semigroup.

The aim of this paper is to continue that study, but in a more general situation — we will (mostly) consider 2×2 matrices over a linearly ordered abelian group with an adjoined bottom element. It is well known that there is a natural partial order on the set of idempotents of any semigroup. We will study that order on the set of idempotents of the semigroup $(M_2(\mathbf{A}^{\bot}), \cdot)$ where $\mathbf{A}$ is a linearly ordered abelian group. It will turn out that in this case the poset of idempotents is a lattice (see Theorem 4.1) which, in general, is not modular. We will also describe the 0-primitive idempotents of the semigroup $(M_2(\mathbf{A}^{\bot}), \cdot)$ in Proposition 3.5. In Section 5 we will examine regularity and some related properties of this matrix semigroup.

2 The description of idempotents

In [6, Theorem 4.1], Johnson and Kambites proved that the idempotent 2×2 matrices over the tropical semiring are of exactly four types. We can give an analogous description of idempotent elements of the multiplicative semigroup $M_2(\mathbf{A}^\perp)$, where $\mathbf{A} = (A, +, \leq)$ is a linearly ordered abelian group. Although our proof is similar to that in [6], we include it for the sake of completeness.

Theorem 2.1. *Let* $\mathbf{A}$ *be a lattice-ordered abelian group. Then the matrices from the set* $\{\Theta\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$*, where*

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \middle| \; x, y \in \mathbf{A}^\perp, x + y \leq 0 \right\},$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & x \\ y & x + y \end{pmatrix} \middle| \; x, y \in \mathbf{A}^\perp, x + y < 0 \right\},$$

$$\mathcal{C} = \left\{ \begin{pmatrix} x + y & x \\ y & 0 \end{pmatrix} \middle| \; x, y \in \mathbf{A}^\perp, x + y < 0 \right\},$$

are idempotents in the semigroup $(M_2(\mathbf{A}^\perp), \cdot)$*. If* $\mathbf{A}$ *is a linearly ordered abelian group, then every idempotent of the semigroup* $(M_2(\mathbf{A}^\perp), \cdot)$ *belongs to the set* $\{\Theta\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$*.*

Proof. It is easy to check that the matrices in the set $\{\Theta\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ are idempotent.

Conversely, suppose that $\mathbf{A}$ is linearly ordered and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A}^\perp).$$

Then the following equalities must hold:

$$(a + a) \vee (b + c) = a \quad (1) \qquad\qquad (a + b) \vee (b + d) = b \quad (2)$$

$$(a + c) \vee (c + d) = c \quad (3) \qquad\qquad (b + c) \vee (d + d) = d \quad (4).$$

From the equalities (1) and (4) it follows that $a + a \leq a$ and $d + d \leq d$. Since $\mathbf{A}$ is an ordered abelian group, we conclude that $a \leq 0$ and $d \leq 0$. For the element a we have two possibilities.

1) $a < 0$. Then $a + a < a$ and by (1) we must have $a = b + c$. If $d = 0$, then we have a matrix

$$\begin{pmatrix} b+c & b \\ c & 0 \end{pmatrix}, \text{ where } b, c \in \mathbf{A}^{\perp} \text{ and } b + c < 0.$$

On the other hand, if $d < 0$, then $d + d < d$ and by (4) we have $b + c = d$. If $b \in \mathbf{A}$, then $a + b, b + d < b$, which contradicts (2). If $c \in \mathbf{A}$, then $a + c, c + d < c$, which contradicts (3). Hence $b = c = \perp$. Now $a = d = b + c$ implies $a = d = \perp$. Therefore we have obtained the matrix Θ.

2) $a = 0$. From (1) we obtain $b + c \leq 0$ and (4) implies that either $d = 0$ or $d = b + c$. Therefore we have the matrices

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ c & b+c \end{pmatrix}, \text{ where } b, c \in \mathbf{A}^{\perp} \text{ and } b + c \leq 0.$$

The proof is complete. □

We say that e is an idempotent of *type* $\mathcal{A}$ (*type* $\mathcal{B}$, *type* $\mathcal{C}$) if $e \in \mathcal{A}$ (resp. $e \in \mathcal{B}$, $e \in \mathcal{C}$).

A semiring is called *simple*, if it has no nontrivial ideals. Precisely as in the case of matrix rings over fields one can prove the following result.

Proposition 2.2. *If R is a semifield and $n \in \mathbb{N}$, then the matrix semiring $M_n(R)$ is simple.*

An idempotent e of a semiring R is called *full*, if $R = ReR$, where

$$ReR = \left\{ \sum_{i=1}^{n} r_i e r_i' \mid n \in \mathbb{N}, r_i, r_i' \in R \right\}.$$

Similarly we call an idempotent e of a semigroup S *full* if $S = SeS$, where $SeS = \{ses' \mid s, s' \in S\}$.

Corollary 2.3. *If R is a semifield and $n \in \mathbb{N}$, then every non-zero idempotent of the semiring $M_n(R)$ is full.*

Proof. If $e \in M_n(R)$ is a non-zero idempotent, then $M_n(R) \cdot e \cdot M_n(R)$ is a non-zero ideal of the semiring $M_n(R)$. Simplicity implies that $M_n(R) \cdot e \cdot M_n(R) = M_n(R)$. □

In particular, the non-zero idempotents described in Theorem 2.1 are full in the semiring $M_2(\mathbf{A}^\perp)$. It is natural to ask if they are full also in the multiplicative semigroup $(M_2(\mathbf{A}^\perp), \cdot)$? It turns out that they need not be.

Example 2.4. Suppose that the idempotent matrix $X = \begin{pmatrix} 0 & -\infty \\ -\infty & -\infty \end{pmatrix}$ is a full idempotent in the semigroup $(M_2(\overline{\mathbb{R}}), \cdot)$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. Then there exist $a, b, c, d, e, f, g, h \in \overline{\mathbb{R}}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -\infty \\ -\infty & -\infty \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & -\infty \\ c & -\infty \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$
$$= \begin{pmatrix} a+e & a+f \\ c+e & c+f \end{pmatrix}.$$

Consequently,

$$\begin{cases} a+e &= 0 \\ a+f &= 0 \\ c+e &= 0 \\ c+f &= 1. \end{cases}$$

These equalities imply that a, c, e, f are real numbers. From $a + e = a + f$ we conclude that $e = f$. But then $1 = c + f = c + e = 0$, which is a contradiction, showing that this system of linear equations cannot have a solution. Thus X cannot be a full idempotent.

Idempotents (especially full idempotents) play an important role both in the Morita theory of semigroups and semirings. Due to [2, Proposition 4.14], if e is a full idempotent in a semiring R with an identity, then R is Morita equivalent to its local subsemiring eRe.

Our matrix semiring $M_2(\mathbf{A}^\perp)$ has the identity I. Hence we have the following result.

Corollary 2.5. *If* $\mathbf{A}$ *is a linearly ordered abelian group, then the semiring* $M_2(\mathbf{A}^\perp)$ *is Morita equivalent to all its non-zero local subsemirings.*

In general, it is not easy to compute those local subsemirings. For some idempotents, however, we can do this. We will give one such example.

Proposition 2.6. *If* $\mathbf{A}$ *is a linearly ordered abelian group, then the local subsemiring induced by the idempotent* $e = \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix}$ *in the semiring* $R = (M_2(\mathbf{A}^\perp), \cdot)$ *is*

$$eRe = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \leq a \leq b, c \leq d \leq b \right\}.$$

Proof. Let H denote the set of matrices on the right hand side of the last equality. First we prove that, for every matrix $X \in M_2(\mathbf{A}^\perp)$, the product eXe belongs to H. If $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(\mathbf{A}^\perp)$, then

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix} &= \begin{pmatrix} x \vee z & y \vee w \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix} \\ &= \begin{pmatrix} x \vee z & x \vee z \vee y \vee w \\ z & z \vee w \end{pmatrix}. \end{aligned}$$

From the definition of the least upper bound it follows that the last matrix is in H. Thus $eRe \subseteq H$.

Consider now a matrix $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. Using the inequalities $c \leq a \leq b$ and $c \leq d \leq b$ we see that

$$\begin{aligned} eY &= \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \vee c & b \vee d \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y, \\ Ye &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \perp & 0 \end{pmatrix} = \begin{pmatrix} a & a \vee b \\ c & c \vee d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y. \end{aligned}$$

Hence $Y = eYe \in eRe$. □

3 The poset of idempotents

If S is a semigroup, then its set of idempotents $E(S)$ is a poset with respect to the natural order relation $\leq$ which is defined by

$$f \leq e \iff ef = f = fe$$

(see [1, Section 1.8]). As usual, we write $f < e$ if $f \leq e$ and $f \neq e$. In this section we will investigate the structure of the poset $E(M_2(\mathbf{A}^\perp))$ where $\mathbf{A}$ is a linearly ordered abelian group. It is clear that Θ is the bottom element and I is the top element in this poset.

Lemma 3.1. *Let $\Theta \neq f < e$ for two idempotents $e, f \in E(M_2(\mathbf{A}^\perp))$. Then $e \in \mathcal{A}$.*

Proof. We know that $ef = f = fe$. Suppose that $e = \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \in \mathcal{B}$, so $a, b \in \mathbf{A}^\perp$ and $a + b < 0$. We will show that this leads to a contradiction. A similar proof gives a contradiction when $e \in \mathcal{C}$. We have three possibilities for the matrix f.

1) $f \in \mathcal{A}$. Let $f = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, where $x, y \in \mathbf{A}^\perp$ and $x + y \leq 0$. Then $f = fe$ means that

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} = \begin{pmatrix} 0 \vee (x+b) & a \vee (x+a+b) \\ y \vee b & (y+a) \vee (a+b) \end{pmatrix}.$$

Therefore $y = y \vee b$ and $0 = (y + a) \vee (a + b)$. These equalities imply $b \leq y$ and $y + a = 0$. On the other hand, $f = ef$ means that

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 \vee (y+a) & x \vee a \\ b \vee (y+a+b) & (x+b) \vee (a+b) \end{pmatrix}.$$

In particular, $y = b \vee (y + a + b) = b \vee (0 + b) = b$. It follows that $a + b = a + y = 0$. This contradicts the inequality $a + b < 0$.

2) $f \in \mathcal{B}$. Let $f = \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}$, where $x, y \in \mathbf{A}^\perp$ and $x + y < 0$. Then $f = fe$ means that

$$\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix} \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix}$$
$$= \begin{pmatrix} 0 \vee (x+b) & a \vee (x+a+b) \\ y \vee (x+y+b) & (y+a) \vee (x+y+a+b) \end{pmatrix}.$$

In particular, $x = a \vee (x + a + b)$. From $a + b < 0$ we conclude that $x + a + b < x$, so the definition of join implies $x = a$. On the other hand,

$f = ef$ means that

$$\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}$$
$$= \begin{pmatrix} 0 \vee (y+a) & x \vee (x+y+a) \\ b \vee (y+a+b) & (x+b) \vee (x+y+a+b) \end{pmatrix}.$$

In particular, $y = b \vee (y+a+b)$. Again $a+b < 0$ implies $y+a+b < y$, and so $y = b$. We have shown that $e = f$, contradicting the assumption $f < e$.

3) $f \in \mathcal{C}$. Let $f = \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix}$, where $x, y \in \mathbf{A}^\perp$ and $x + y < 0$. Then $f = fe$ means that

$$\begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix}$$
$$= \begin{pmatrix} (x+y) \vee (x+b) & (x+y+a) \vee (x+a+b) \\ y \vee b & (y+a) \vee (a+b) \end{pmatrix}.$$

Consequently, $0 = (y+a) \vee (a+b)$. Since $a+b < 0$, we have $0 = y+a$.

On the other hand, $f = ef$ means that

$$\begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (x+y) \vee (y+a) & x \vee a \\ (x+y+b) \vee (y+a+b) & (x+b) \vee (a+b) \end{pmatrix}.$$

Therefore $0 = (x+b) \vee (a+b)$, where $a+b < 0$. We conclude that $0 = x+b$. Hence $0 = y+a+x+b = x+y+a+b$. But the inequalities $x+y < 0$ and $a+b < 0$ imply $x+y+a+b < 0$. We have obtained a contradiction. □

Lemma 3.1 shows that there are no non-zero idempotents below elements of $\mathcal{B}$ and $\mathcal{C}$. So all idempotents of types $\mathcal{B}$ and $\mathcal{C}$ are atoms in the poset $E(M_2(\mathbf{A}^\perp))$.

The next lemma describes when two idempotents are in the relation $\leq$.

Lemma 3.2. *Let $e, f \in E(M_2(\mathbf{A}^\perp))$ be idempotents, where e is of type $\mathcal{A}$. Then*

(i)

$$\mathcal{A} \ni f = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \le \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = e \iff c \le a \ \ and \ \ d \le b,$$

(ii)

$$\mathcal{B} \ni f = \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \le \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = e \iff c \le a \ \ and \ \ d \le b,$$

(iii)

$$\mathcal{C} \ni f = \begin{pmatrix} a+b & a \\ b & 0 \end{pmatrix} \le \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = e \iff c \le a \ \ and \ \ d \le b.$$

Proof. (i) Since f, e are of type $\mathcal{A}$, $a + b \le 0$ and $c + d \le 0$.

Sufficiency. Assume that $c \le a$ and $d \le b$. Then $a + d \le a + b \le 0$, $b + c \le b + a \le 0$ and

$$ef = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 \vee (b+c) & a \vee c \\ b \vee d & (a+d) \vee 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = f,$$

$$fe = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 \vee (a+d) & a \vee c \\ b \vee d & (b+c) \vee 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = f.$$

Thus $f \le e$.

Necessity. If $f \le e$, then $a \vee c = a$ and $b \vee d = b$. Therefore $c \le a$ and $d \le b$.

(ii) Since f is of type $\mathcal{B}$ and e is of type $\mathcal{A}$, $a + b < 0$ and $c + d \le 0$.

Sufficiency. Assume that $c \le a$ and $d \le b$. Then $a + d \le a + b < 0$, $b + c \le b + a < 0$ and

$$\begin{aligned} ef &= \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} = \begin{pmatrix} 0 \vee (b+c) & a \vee (a+b+c) \\ b \vee d & (a+d) \vee (a+b) \end{pmatrix} \\ &= \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} = f, \\ fe &= \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 \vee (a+d) & a \vee c \\ b \vee (a+b+d) & (b+c) \vee (a+b) \end{pmatrix} \\ &= \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} = f. \end{aligned}$$

Thus $f \leq e$.

Necessity. If $f \leq e$, then $a \vee c = a$ and $b \vee d = b$. Therefore $c \leq a$ and $d \leq b$.

(iii) This case is similar to the case (ii). □

Recall that a non-zero idempotent e of a semigroup S with zero is called 0-*primitive* if, for every non-zero idempotent f, $f \leq e$ implies $e = f$.

Definition 3.3. We say that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A}^{\perp})$ is *balanced* if $a + d = b + c$.

Example 3.4. If $\mathbf{A} = (\mathbb{Z}, +, \leq)$, then some of the balanced matrices are

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \perp \\ 5 & \perp \end{pmatrix}, \begin{pmatrix} 4 & \perp \\ \perp & \perp \end{pmatrix}, \begin{pmatrix} \perp & \perp \\ \perp & \perp \end{pmatrix}.$$

Proposition 3.5. *Let* $\mathbf{A}$ *be a linearly ordered abelian group. A non-zero idempotent of the semigroup* $(M_2(\mathbf{A}^{\perp}), \cdot)$ *is* 0*-primitive if and only if it is balanced.*

Proof. **Necessity.** If an idempotent is non-balanced, then it is of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in \mathcal{A}$, where $a + b < 0$. Such idempotents are not 0-primitive, because

$$\Theta \neq \begin{pmatrix} 0 & a \\ b & a+b \end{pmatrix} < \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Sufficiency. From Theorem 2.1 we see that the set of non-zero balanced idempotents is

$$\mathcal{B} \cup \mathcal{C} \cup \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mid a \in \mathbf{A} \right\}.$$

Lemma 3.1 implies immediately that all idempotents in $\mathcal{B} \cup \mathcal{C}$ are 0-primitive. It remains to prove that each matrix $e = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $a \in \mathbf{A}$, is a 0-primitive idempotent. We will prove that $ef = f = fe$ implies $e = f$ for all matrices f of types $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

1) $f = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in \mathcal{A}$, where $x, y \in \mathbf{A}^{\perp}$ and $x + y \leq 0$. Then $f \leq e$ implies $a \leq x$ and $-a \leq y$, so $0 = a - a \leq x + y \leq 0$. We conclude that $x + y = 0$, so $a \leq x = -y \leq a$, which implies $a = x$. Now $a + y = 0$ gives $y = -a$, and we have proved that $e = f$.

2) $f = \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix} \in \mathcal{B}$, where $x, y \in \mathbf{A}^{\perp}$ and $x + y < 0$. Then $f \leq e$ implies $a \leq x$ and $-a \leq y$, so $0 = a - a \leq x + y \leq 0$. Similarly to the case 1) we obtain $e = f$.

3) $f \in \mathcal{C}$. This case is analogous to the case 2). □

4 The lattice of idempotents

It turns out that the poset of idempotent matrices has the structure of a lattice.

Theorem 4.1. *If* $\mathbf{A}$ *is a linearly ordered abelian group, then the poset of idempotents of the semigroup* $M_2(\mathbf{A}^{\perp})$ *is a lattice.*

Proof. We must prove that any two non-equal idempotents of the semigroup $M_2(\mathbf{A}^{\perp})$ have the least upper bound (supremum) and the greatest lower bound (infimum). If one of the idempotents is Θ or I, then it is clear what the supremum or infimum is. Therefore it suffices to consider the following pairwise different idempotents:

$$e_{\mathcal{A}} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad e_{\mathcal{B}} = \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}, \quad e_{\mathcal{C}} = \begin{pmatrix} z+w & z \\ w & 0 \end{pmatrix},$$

$$f_{\mathcal{A}} = \begin{pmatrix} 0 & a' \\ b' & 0 \end{pmatrix}, \quad f_{\mathcal{B}} = \begin{pmatrix} 0 & x' \\ y' & x'+y' \end{pmatrix}, \quad f_{\mathcal{C}} = \begin{pmatrix} z'+w' & z' \\ w' & 0 \end{pmatrix}.$$

We will prove that

1. $\sup(e_{\mathcal{A}}, f_{\mathcal{A}}) = \begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix}$, $\inf(e_{\mathcal{A}}, f_{\mathcal{A}}) = \begin{pmatrix} 0 & a \vee a' \\ b \vee b' & 0 \end{pmatrix}$ if $a \vee a' + b \vee b' \leq 0$ and $\inf(e_{\mathcal{A}}, f_{\mathcal{A}}) = \Theta$ if $a \vee a' + b \vee b' > 0$,
2. $\sup(e_{\mathcal{A}}, e_{\mathcal{B}}) = \begin{pmatrix} 0 & a \wedge x \\ b \wedge y & 0 \end{pmatrix}$, $\inf(e_{\mathcal{A}}, e_{\mathcal{B}}) = \begin{pmatrix} 0 & a \vee x \\ b \vee y & 0 \end{pmatrix}$ if $a \vee x + b \vee y \leq 0$ and $\inf(e_{\mathcal{A}}, e_{\mathcal{B}}) = \Theta$ if $a \vee x + b \vee y > 0$,

3. $\sup(e_{\mathcal{A}}, e_{\mathcal{C}}) = \begin{pmatrix} 0 & a \wedge z \\ b \wedge w & 0 \end{pmatrix}$, $\inf(e_{\mathcal{A}}, e_{\mathcal{C}}) = \begin{pmatrix} 0 & a \vee z \\ b \vee w & 0 \end{pmatrix}$ if $a \vee z + b \vee w \leq 0$ and $\inf(e_{\mathcal{A}}, e_{\mathcal{C}}) = \Theta$ if $a \vee z + b \vee w > 0$,

4. $\sup(e_{\mathcal{B}}, f_{\mathcal{B}}) = \begin{pmatrix} 0 & x \wedge x' \\ y \wedge y' & 0 \end{pmatrix}$, $\inf(e_{\mathcal{B}}, f_{\mathcal{B}}) = \Theta$,

5. $\sup(e_{\mathcal{B}}, e_{\mathcal{C}}) = \begin{pmatrix} 0 & x \wedge z \\ y \wedge w & 0 \end{pmatrix}$, $\inf(e_{\mathcal{B}}, e_{\mathcal{C}}) = \Theta$,

6. $\sup(e_{\mathcal{C}}, f_{\mathcal{C}}) = \begin{pmatrix} 0 & z \wedge z' \\ w \wedge w' & 0 \end{pmatrix}$, $\inf(e_{\mathcal{C}}, f_{\mathcal{C}}) = \Theta$.

We have to look at 6 cases about the supremum and 6 cases about the infimum. Let us start with the suprema.

1. We want to prove that $\sup(e_{\mathcal{A}}, f_{\mathcal{A}}) = \begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix}$. Since $a \wedge a' \leq a$ and $b \wedge b' \leq b$, by Lemma 3.2 we have

$$\begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix} \geq e_{\mathcal{A}} \quad \text{and} \quad \begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix} \geq f_{\mathcal{A}}.$$

Suppose that a matrix g is also an upper bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$. By Lemma 3.1, g must be of type $\mathcal{A}$, so let $g = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, where $u, v \in \mathbf{A}^{\perp}$ and $u + v \leq 0$. Now $e_{\mathcal{A}} \leq g$ implies $u \leq a$, $v \leq b$, and $f_{\mathcal{A}} \leq g$ implies $u \leq a'$, $v \leq b'$. The inequalities $u \leq a$ and $u \leq a'$ imply $u \leq a \wedge a'$, and, similarly, $v \leq b \wedge b'$. So $\begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix} \leq g$ and we have shown that $\begin{pmatrix} 0 & a \wedge a' \\ b \wedge b' & 0 \end{pmatrix}$ is the least upper bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$.

2. We want to prove that $\sup(e_{\mathcal{A}}, e_{\mathcal{B}}) = \begin{pmatrix} 0 & a \wedge x \\ b \wedge y & 0 \end{pmatrix}$. Again it follows from Lemma 3.2 that $\begin{pmatrix} 0 & a \wedge x \\ b \wedge y & 0 \end{pmatrix}$ is an upper bound of $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$. Suppose that a matrix g is also an upper bound of $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$. By Lemma 3.1, g must be of type $\mathcal{A}$, so let $g = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, where $u, v \in \mathbf{A}^{\perp}$ and $u + v \leq 0$. Now $e_{\mathcal{A}} \leq g$ implies $u \leq a$, $v \leq b$, and $e_{\mathcal{B}} \leq g$ implies $u \leq x$, $v \leq y$. The inequalities $u \leq a$ and $u \leq x$ imply $u \leq a \wedge x$, and, similarly,

$v \leq b \wedge y$. So $\begin{pmatrix} 0 & a \wedge x \\ b \wedge y & 0 \end{pmatrix} \leq g$ and we have shown that $\begin{pmatrix} 0 & a \wedge x \\ b \wedge y & 0 \end{pmatrix}$ is the least upper bound of $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$.

In the cases 3–6, the proofs are similar.

Next, we look at the infima.

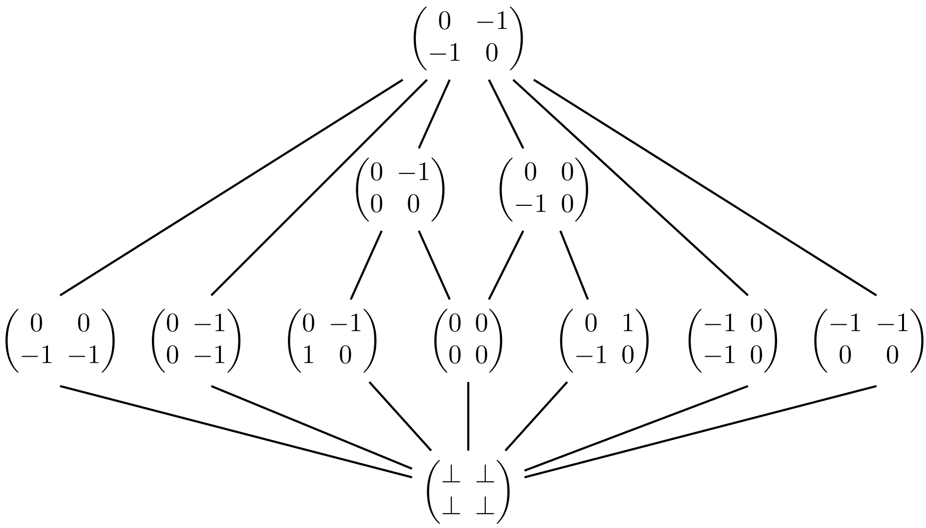
1. Consider first the case $a \vee a' + b \vee b' \leq 0$. Due to Lemma 3.2, the matrix $\begin{pmatrix} 0 & a \vee a' \\ b \vee b' & 0 \end{pmatrix} \in \mathcal{A}$ is a lower bound for $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$. Suppose that a matrix g is also a lower bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$. Then either $g = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, $g = \begin{pmatrix} u+v & u \\ v & 0 \end{pmatrix}$ or $g = \begin{pmatrix} 0 & u \\ v & u+v \end{pmatrix}$, where $u + v \leq 0$. In all three cases $g \leq e_{\mathcal{A}}$ implies $a \leq u$, $b \leq v$, and $g \leq f_{\mathcal{A}}$ implies $a' \leq u$, $b' \leq v$. The inequalities $a \leq u$ and $a' \leq u$ imply $a \vee a' \leq u$, and, similarly, $b \vee b' \leq v$. So, by Lemma 3.2 we have that $g \leq \begin{pmatrix} 0 & a \vee a' \\ b \vee b' & 0 \end{pmatrix}$, and we have shown that the last matrix is the greatest lower bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$.

Now suppose that $a \vee a' + b \vee b' > 0$. We will show that in this case Θ is the only lower bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$. Suppose that an idempotent $g = \begin{pmatrix} p & u \\ v & r \end{pmatrix} \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a lower bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$. As above, we conclude that $a \vee a' \leq u$ and $b \vee b' \leq v$. Hence $0 < a \vee a' + b \vee b' \leq u + v \leq 0$, a contradiction. Since Θ is the only lower bound of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$, it is the infimum of $e_{\mathcal{A}}$ and $f_{\mathcal{A}}$.

The cases 2 and 3 are similar to the case 1.

The cases 4–6 are clear, because the idempotents of type $\mathcal{B}$ and type $\mathcal{C}$ are 0-primitive. □

Example 4.2. The following figure, which depicts the lower cone of the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ in the lattice $E(M_2(\mathbb{Z}^{\perp}))$, shows immediately that the lattice of idempotents need not be modular. The lattice on this figure contains a pentagon sublattice.



5 On regularity and related properties

In [5, Theorem 4], Il'in gave necessary and sufficient conditions for regularity of the matrix semiring $M_n(R)$ over an arbitrary semiring R. In [6, Theorem 4.2], Johnson and Kambites proved that the multiplicative semigroup of 2×2 matrices over the tropical semiring is regular by showing that every $\mathcal{R}$-class contains an idempotent. This result was generalized by Gould, Johnson and Naz to linearly ordered abelian groups as follows.

Theorem 5.1 ([4])**.** *If* $\mathbf{A}$ *is a linearly ordered abelian group, then the matrix semigroup* $(M_2(\mathbf{A}^\bot), \cdot)$ *is regular.*

One could also consider the subsemigroup $(M_2(\mathbf{A}), \cdot)$ of $(M_2(\mathbf{A}^\bot), \cdot)$ and ask if it is regular. The answer is positive and this fact could be deduced by a closer examination of [4, Remark 5.6]. However, we prefer to give an explicit proof here.

Proposition 5.2. *If* $\mathbf{A}$ *is a linearly ordered abelian group, then the matrix semigroup* $(M_2(\mathbf{A}), \cdot)$ *is regular.*

Proof. Consider a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A})$. As the order is linear, we have either $a + d \geq b + c$ or $a + d < b + c$. We will show that in both cases there exists a matrix Y such that $X = XYX$.

1. If $a + d \geq b + c$, then

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -a & -a+b-d \\ -a+c-d & -d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 \vee (b-a+c-d) & (b-d) \vee (b-d) \\ (c-a) \vee (c-a) & (c-a+b-d) \vee 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & b-d \\ c-a & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

2. If $a + d < b + c$, then

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -b-c+d & -c \\ -b & a-b-c \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} (a-b-c+d) \vee 0 & (a-c) \vee (a-c) \\ (d-b) \vee (d-b) & 0 \vee (a-b-c+d) \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & a-c \\ d-b & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

□

If $n \geq 3$, then the semigroup $(M_n(\mathbf{A}^{\perp}), \cdot)$ is not regular. It is not even abundant (see [4, Remark 5.6]). We recall that a semigroup S is called *abundant* if every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class of S contains an idempotent, where $\mathcal{R}^*$ and $\mathcal{L}^*$ are certain generalized Green's relations whose definitions can be found, for example, in [4].

It is natural to ask if the semigroup $(M_2(\mathbf{A}), \cdot)$ has any other nice properties in addition to regularity.

Recall that a semigroup S is *inverse* if it is regular and its idempotents commute. A semigroup is *orthodox* if it is regular and the product of any two idempotents is an idempotent.

Proposition 5.3. *For a lattice-ordered abelian group* $\mathbf{A} = (A, +, \leq)$ *the following assertions are equivalent.*

(i) *The semigroup* $(M_2(\mathbf{A}), \cdot)$ *is inverse.*

(ii) *The semigroup* $(M_2(\mathbf{A}), \cdot)$ *is orthodox.*

(iii) $|A| = 1$.

Proof. Clearly (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii). We will prove the converses.

Suppose that $|A| > 1$. It is not difficult to see that then there exists $a \in A$ such that $a < 0$. Then also $2a = a + a < 0$. By Proposition 2.1, the matrices

$$e = \begin{pmatrix} 0 & a \\ a & 2a \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 2a & a \\ a & 0 \end{pmatrix}$$

are idempotent. If they would commute, then

$$\begin{pmatrix} 2a & a \\ 3a & 2a \end{pmatrix} = ef = fe = \begin{pmatrix} 2a & 3a \\ a & 2a \end{pmatrix},$$

in particular $a = 3a$, which implies $2a = 0$. This contradicts with $2a < 0$. So $(M_2(\mathbf{A}), \cdot)$ is not inverse. Note also that

$$(fe)^2 = \begin{pmatrix} 2a & 3a \\ a & 2a \end{pmatrix} \cdot \begin{pmatrix} 2a & 3a \\ a & 2a \end{pmatrix} = \begin{pmatrix} 4a & 5a \\ 3a & 4a \end{pmatrix} \neq \begin{pmatrix} 2a & 3a \\ a & 2a \end{pmatrix} = fe,$$

so $(M_2(\mathbf{A}), \cdot)$ is orthodox neither. □

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