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Morita equivalence of certain crossed products

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Abstract. We introduce an alternative criterion for Morita equivalence over graded tensor categories using equivariant centers and equivariantizations.

1 Introduction and Preliminaries

The concept of Morita equivalence in tensor categories shares similarities with its counterpart in ring theory. In ring theory, two rings are considered Morita equivalent if there exists an invertible bimodule category connecting them. Similarly, in tensor categories, the notion of Morita equivalence arises when an invertible bimodule category can be established between two categories. This relationship provides valuable insights into the structure of the categories themselves.

While Morita equivalence has been extensively studied in the context of fusion categories, primarily through the examination of their centers, re-

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cent advancements [7] have broadened its scope to encompass graded tensor categories. This paper presents a novel criterion for characterizing Morita equivalence in graded tensor categories by leveraging equivariant centers and equivariantizations. Notably, the identification of Morita equivalence can be expedited when the Brauer Picard groups are known, offering an efficient approach to establishing the equivalence relationship.

To generalize the properties of fusion categories to finite tensor categories, we utilize the concept of an exact module category, which was introduced by Etingof and Ostrik [5, Section 3]. Exact module categories offer an intermediary restriction between the semisimple module categories of a fusion category and more general cases that may not be semisimple or finite. This work contributes by providing comprehensive proofs for certain results that were previously established solely in the semisimple setting.

First, building upon the work of [8], we describe the center of a G-graded tensor category in terms of its trivial component. Consequently, the center can be understood in relation to a smaller category

$$Z(\mathcal{C}) \simeq Z_{\mathcal{C}_1}(\mathcal{C})^G.$$

Second, inspired by [9], we provide an explicit description of tensor functors between categorical duals. As a valuable application, we establish conditions under which two equivariantizations become equivalent. By combining these results, we demonstrate that two G-graded tensor categories, $\mathcal{C}$ and $\mathcal{D}$, possess equivalent centers if and only if certain conditions are met. Specifically, there must exist:

1. S a G-equivariant $Z_{\mathcal{C}_1}(\mathcal{C})$-module category,
2. a faithfull G^{op}-grading in $\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C}\rtimes G)}(S)$ such that

$$(\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C}\rtimes G)}(S))_1 \simeq \mathcal{D}.$$

Then, as a Corollary, we stablish an alternative criterion to prove Morita equivalence over graded tensor categories.

2 Center of a tensor category

In this section, we generalized and proved in detail some results of Gelaki, Naidu and Nikshych [8] about the center of a tensor category. In particular we show an equivalence of braided categories between the center of a category and an equivariantization of a certain relative center.

Definition 2.1. Let $\mathcal{C}$ be a finite tensor category and $\mathcal{D}$ be a full tensor subcategory of $\mathcal{C}$. The *relative center* $Z_{\mathcal{C}}(\mathcal{D})$ (or equivariant center following the notation of [7]) is the tensor category where

- objects: are the pairs (M, γ) where M is an object of $\mathcal{D}$ and $\{\gamma_X : X \otimes M \to M \otimes X\}_{X \in \mathcal{C}}$ is a natural family of isomorphism, called the *central structure* such that
$$(\gamma_X \otimes \mathrm{id}_Y)\alpha^{-1}_{X,M,Y}(\mathrm{id}_X \otimes \gamma_Y) = \alpha^{-1}_{M,X,Y}\gamma_{X\otimes Y}\alpha^{-1}_{X,Y,M}, \tag{2.1}$$
where α denotes the associativity constraints in $\mathcal{C}$.
- arrows from $(M, \gamma) \to (M', \gamma')$: are morphisms $f : M \to M'$ in $\mathcal{C}$ such that $(f \otimes \mathrm{id}_X)\gamma_X = \gamma'_X(\mathrm{id}_X \otimes f)$, for all $X \in \mathcal{C}$.
- tensor product for $(X, \gamma), (X', \gamma') \in Z_{\mathcal{D}}(\mathcal{C})$ given by
$$(X, \gamma)\otimes(X', \gamma') := (X \otimes X', \alpha^{-1}_{X,X'.-}(\mathrm{id}_X \otimes \gamma')\alpha_{X,-,X'}(\gamma \otimes \mathrm{id}_{X'})\alpha^{-1}_{-,X,X'}),$$
- unit object $(\mathbf{1}, \mathrm{id})$,
- duals $(X, \gamma)^* := (X^*, \gamma^*_{*(-)})$.

This construction is a special case of a more general construction considered by Majid in [12, Definition 3.2], and it is a generalization of *the categorical center* $Z(\mathcal{C}) := Z_{\mathcal{C}}(\mathcal{C})$, or Drinfeld center of $\mathcal{C}$. Making an abuse of notation, denoted each element (X, γ) in $Z_{\mathcal{D}}(\mathcal{C})$ by X.

Definition 2.2. Let G be a finite group and $\underline{G}$ be the monoidal category whose objects are elements of G, morphisms are identities and the tensor product is given by the multiplication in G. An *action of* G over $\mathcal{C}$ is a monoidal functor $(*, \mu) : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ with monoidal structure $\xi_{g,h} : (gh)_* \to g_*h_*$ for all $g, h \in G$; and the tensor structure of g_* is $\mu_g^{X,Y} : g_*(X \otimes Y) \to g_*(X) \otimes g_*(Y)$.

Example 2.3. [8, Section 3.1] Let $\mathcal{C}$ be a faithful G-graded tensor category and $\mathcal{D} = \mathcal{C}_1$. There is an action of G over $Z_{\mathcal{D}}(\mathcal{C})$:

Consider the faithful G-grading on $Z_{\mathcal{D}}(\mathcal{C}) = \bigoplus_{g \in G} Z_{\mathcal{D}}(\mathcal{C}_g)$ (follows from [13, Theorem 5.13, Proposition 5.15 and Theorem 5.19]). Since $Z(\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{D}\boxtimes\mathcal{D}^{rev}}(\mathcal{D}, \mathcal{D})$ [2, Prop 7.13.8], we obtain equivalences of $Z(\mathcal{D})$-bimodule categories [8, Proposition 3.1], where $\mathcal{D}' = \mathcal{D}\boxtimes\mathcal{D}^{rev}$

$$L_{g,h} : Z_{\mathcal{D}}(\mathcal{C}_h) \to \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}_g, \mathcal{C}_{hg}), \qquad R_{g,h} : Z_{\mathcal{D}}(\mathcal{C}_h) \to \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}_g, \mathcal{C}_{gh})$$
$$W \mapsto W \otimes - \qquad\qquad W \mapsto - \otimes W.$$

We need $* : \underline{G} \to \mathrm{Aut}_{\otimes}(Z_{\mathcal{D}}(\mathcal{C}))$ a monoidal functor. Since the center is graded, consider the following equivalence

$$Z_{\mathcal{D}}(\mathcal{C}_h) \xrightarrow{R_{g,h}} \mathrm{Fun}_{\mathcal{D}\boxtimes\mathcal{D}^{rev}}(\mathcal{C}_g, \mathcal{C}_{gh}) \xrightarrow{L^{-1}_{g,ghg^{-1}}} Z_{\mathcal{D}}(\mathcal{C}_{ghg^{-1}})$$
$$Y \mapsto - \otimes Y \simeq g_*(Y) \otimes - \mapsto g_*(Y),$$

in other words, for each $Y \in Z_{\mathcal{D}}(\mathcal{C}_h)$, in $\mathrm{Fun}_{\mathcal{D}\boxtimes\mathcal{D}^{rev}}(\mathcal{C}_g, \mathcal{C}_{gh})$ the functor $-\otimes Y$ as to be natural equivalent to some functor $Y' \otimes -$ for $Y' \in Z_{\mathcal{D}}(\mathcal{C}_{ghg^{-1}})$, then $Y' = g_*(Y)$. Define

$$g_* = \bigoplus_{h \in G} L^{-1}_{g,ghg-1} R_{g,h} : Z_{\mathcal{D}}(\mathcal{C}) \to Z_{\mathcal{D}}(\mathcal{C}).$$

In particular, there exists a natural family of isomorphisms in $\mathcal{C}$

$$c^g_{X,Y} : X \otimes Y \to g_*(Y) \otimes X, \quad X \in \mathcal{C}_g, Y \in Z_{\mathcal{D}}(\mathcal{C}), g \in G. \tag{2.2}$$

Consider the following natural isomorphism

$$g_*(X\otimes Y)\otimes Z \xrightarrow{(c^g_{Z,X\otimes Y})^{-1}} Z\otimes(X\otimes Y) \xrightarrow{\alpha^{-1}_{Z,X,Y}} (Z\otimes X)\otimes Y \xrightarrow{c^g_{Z,X}\otimes \mathrm{id}}$$
$$(g_*(X)\otimes Z)\otimes Y \xrightarrow{\alpha_{g_*(X),Z,Y}} g_*(X)\otimes(Z\otimes Y) \xrightarrow{\mathrm{id}\otimes c^g_{Z,Y}} g_*(X)\otimes(g_*(Y)\otimes Z)$$
$$\xrightarrow{\alpha^{-1}_{g_*(X),g_*(Y),Z}} (g_*(X)\otimes g_*(Y))\otimes Z,$$

for $Z \in \mathcal{C}_g$, $X, Y \in Z_{\mathcal{D}}(\mathcal{C})$, $g \in G$; so it induces $\mu_g^{X,Y} : g_*(X \otimes Y) \to g_*(X) \otimes g_*(Y)$ a natural isomorphism. Finally, if $g, h \in G$ and $(X, Y) \in \mathcal{C}_g \times \mathcal{C}_h$, then for all $Z \in Z_{\mathcal{D}}(\mathcal{C})$ the following natural isomorphism

$$(gh)_*(Z) \otimes (X \otimes Y) \xrightarrow{(c^{gh}_{X\otimes Y,Z})^{-1}} (X \otimes Y) \otimes Z \xrightarrow{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z) \xrightarrow{\mathrm{id} \otimes c^h_{Y,Z}}$$
$$X \otimes (h_*(Z) \otimes Y) \xrightarrow{\alpha^{-1}_{X,h_*(Z),Y}} (X \otimes h_*(Z)) \otimes Y \xrightarrow{c^g_{X,h_*(Z)} \otimes \mathrm{id}} (g_* h_*(Z) \otimes X) \otimes Y$$
$$\xrightarrow{\alpha_{g_*h_*(Z),X,Y}} g_* h_*(Z) \otimes (X \otimes Y),$$

induces $\xi_{g,h} : (gh)_* \to g_* h_*$ a natural isomorphism. Then we obtain an G-action over $Z_{\mathcal{D}}(\mathcal{C})$. $\square$

This action allows us to define new structures of tensor categories, has the equivariantization and crossed products.

Definition 2.4. Given an action of G over $\mathcal{C}$, the G-*equivariantization* of $\mathcal{C}$, denoted by $\mathcal{C}^G$, is a tensor category where

- objects: are the pairs $(X, u_g)_{g\in G}$ where $X \in \mathcal{C}$ and $u_g : g_*(X) \to X$ are isomorphism such that
$$u_g \circ g_*(u_h) = u_{gh}\xi_{g,h}, \quad \text{for all } g, h \in g.$$
- arrows from $(X, u_g) \to (Y, v_g)$: are morphisms $f : X \to Y$ in $\mathcal{C}$ such that $f u_g = v_g g_*(f)$;
- tensor product: is $(X, u_g) \otimes (Y, v_g) = (X \otimes Y, (u_g \otimes v_g)\mu_g^{X,Y})$, for all $g \in G$.

The next notion was introduced by Kirillov Jr. and Muger [11], [14].

Definition 2.5. A *braided G-crossed tensor category* $\mathcal{C}$ is a G-graded tensor category $\mathcal{C}$ equipped with the following structures for all $X \in \mathcal{C}_g, g, h \in G, Y, Z \in \mathcal{C}$:

- an action $(*, \xi)$ of G over $\mathcal{C}$ such that $g_*(\mathcal{C}_h) \subseteq \mathcal{C}_{ghg^{-1}}$,
- a natural collection of isomorphisms, called the G-braiding,
$$c^g_{X,Y} : X \otimes Y \to g_*(Y) \otimes X$$

such that, if $Y \in \mathcal{C}_h$,

$$(\xi^{-1}_{hgh^{-1},h}(Y) \otimes \mathrm{id}_{h_*(X)}) c^{hgh^{-1}}_{h_*(X),h_*(Y)} \mu^{X,Y}_h = (\xi^{-1}_{h,g}(Y) \otimes \mathrm{id}_{h_*(X)}) \mu^{g_*(Y),X}_h h_*(c^g_{X,Y}); \tag{2.3}$$

$$(\mathrm{id}_{g_*(Y)} \otimes c^g_{X,Z}) \alpha_{g_*(Y),X,Z} (c^g_{X,Y} \otimes \mathrm{id}_Z) = \alpha_{g_*(Y),g_*(Z),X} (\mu^{Y,Z}_g \mathrm{id}_X) c^g_{X,Y\otimes Z} \alpha_{X,Y,Z}; \tag{2.4}$$

$$(c^g_{x,h_*(Z)} \otimes \mathrm{id}_Y) \alpha^{-1}_{X,h_*(Z),Y} (\mathrm{id}_X \otimes c^h_{Y,Z}) \alpha_{X,Y,Z} = \alpha^{-1}_{g_* h_*(Z),X,Y} (\xi_{g,h}(Z) \otimes \mathrm{id}_{X\tilde{Y}}) c^{gh}_{X\otimes Y,Z}. \tag{2.5}$$

Example 2.6. [8, Theorem 3.3] Let $\mathcal{C}$ be a faithfull G-graded tensor category. The relative center $Z_{\mathcal{D}}(\mathcal{C})$ as a canonical braided G-crossed category structure:

We can assume that $\mathcal{C}$ is strict. Consider the action given in Example 2.3, then by the definition for all $g, h \in G$,

$$g_*(Z_{\mathcal{D}}(\mathcal{C}_h)) = L^{-1}_{g,ghg^{-1}} R_{g,h} Z_{\mathcal{D}}(\mathcal{C}_h) \subset Z_{\mathcal{D}}(\mathcal{C}_{ghg^{-1}}),$$

and the G-braiding given in Equation (2.2). We prove Equation (2.3):

Let $Z \in \mathcal{C}_h, X \in \mathcal{C}_{h^{-1}gh}, Y \in Z_{\mathcal{D}}(\mathcal{C})$. Then using the definition of ξ and μ we obtain

$$\begin{aligned}
((\xi_{hgh^{-1},h}(Y) \otimes \mathrm{id}_{h_*(X)}) (c_{h_*(X),h_*(Y)}) (\mu^{X,Y}_h) \otimes \mathrm{id}_Z \\
&= [c^{hg}_{h_*(X)\otimes Z,Y} (\mathrm{id}_{h_*(X)} \otimes (c^h_{Z,Y})^{-1}) (c^{hgh^{-1}}_{h_*(X),h_*(Y)} \otimes \mathrm{id}_Z)^{-1}] \\
&\quad (c^{hgh^{-1}}_{h_*(X),h_*(Y)} \otimes \mathrm{id}_Z)(\mu^{X,Y}_h \otimes \mathrm{id}_Z) \\
&= c^{hg}_{h_*(X)\otimes Z,Y} (\mathrm{id}_{h_*(X)} \otimes (c^h_{Z,Y})^{-1}) (\mu^{X,Y}_h \otimes \mathrm{id}_Z) \\
&= c^{hg}_{h_*(X)\otimes Z,Y} (\mathrm{id}_{h_*(X)} \otimes (c^h_{Z,Y})^{-1}) [(\mathrm{id}_{h_*(X)} \otimes c^h_{Z,Y}) \\
&\quad (c^h_{Z,X} \otimes \mathrm{id}_Y)(c^h_{XY,Z})^{-1}] \\
&= c^{hg}_{h_*(X)\otimes Z,Y} (c^h_{Z,X} \otimes \mathrm{id}_Y)(c^h_{XY,Z})^{-1},
\end{aligned}$$

and

$$\begin{aligned}
((\xi_{h,g}(Y) \otimes \mathrm{id}_{h_*(X)}) \mu^{g_*(Y),X}_h h_*(c^g_{X,Y})) \otimes \mathrm{id}_Z \\
&= c^{hg}_{h_*(X)\otimes Z,Y} (\mathrm{id}_{h_*(X)} \otimes c^g_{Z,Y})^{-1} (c^h_{h_*(X),g_*(Y)} \otimes \mathrm{id}_Z)^{-1} \\
&\quad \circ (\mathrm{id}_{h_* g_*(Y)} \otimes c^h_{Z,X})(c^h_{Z,g_*(Y)} \otimes \mathrm{id}_X)(c^h_{g_*(Y)X,Z})^{-1} (h_*(c^g_{X,Y}) \otimes \mathrm{id}_Z).
\end{aligned}$$

Since c^h and $c^h \otimes \mathrm{Id}$ are natural then

$$(c^h_{g_*(Y)X,Z})^{-1} (h_*(c^g_{X,Y}) \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c^g_{x,Y})(c^h_{XY,Z})^{-1},$$

$$(c^h_{h_*(X),g_*(Y)}\otimes \mathrm{id}_Z)^{-1}(\mathrm{id}_{h_*g_*(Y)}\otimes c^h_{Z,X})$$
$$(c^h_{Z,g_*(Y)}\otimes \mathrm{id}_X) = (\mathrm{id}_{h_*(X)}\otimes \tau^Y)c^h_{X,Zg_*(Y)} =$$
$$(\mathrm{id}_{h_*(X)}\otimes c^g_{Z,Y})(c^h_{Z,X}\otimes \mathrm{id}_Y)(\mathrm{id}_Z\otimes c^g_{X,Y})^{-1},$$

then we obtain Equation (2.3). In the same way, we prove Equation (2.4) and (2.5). □

This structure allows us to give a braided structure to an equivariantization. In particular, $(Z_{\mathcal{D}}(\mathcal{C}))^G$ is a braided tensor category.

Proposition 2.7. [8, Theorem 2.12] *If $\mathcal{C}$ is a braided G-crossed tensor product, then $\mathcal{C}^G$ is a braided tensor category.*

Proof. For $(X,u_g)_{g\in G}, (Y,v_g)_{g\in G}\in \mathcal{C}^G$ define

$$\hat{c}_{X,Y} = (\bigoplus_{g\in G} v_g\otimes \mathrm{id}_{X_g})(\bigoplus_{g\in G} c^g_{X_g,Y})$$

where

$$\hat{c}_{X,Y} : (X\otimes Y, (u_g\otimes v_g)\mu_g^{X,Y}) \to (Y\otimes X, (v_g\otimes u_g)\mu_g^{Y,X}), \tag{2.6}$$

explicitly

$$X\otimes Y = \bigoplus_{g\in G} X_g\otimes Y \xrightarrow{\bigoplus_{g\in G} c^g_{X_g,Y}} \bigoplus_{g\in G} g_*(Y)\otimes X_g \xrightarrow{\bigoplus_{g\in G} v_g\otimes \mathrm{id}_{X_g}}$$
$$\bigoplus_{g\in G} Y\otimes X_g = Y\otimes X.$$

This morphism is in $\mathcal{C}^G$:

$$\begin{aligned}
\hat{c}_{X,Y}(u_g\otimes v_g)\mu_g^{X,Y} &= (\bigoplus_{h\in G} v_h\otimes \mathrm{id}_{X_h})(\bigoplus_{h\in G} c^h_{X_h,Y})(u_g\otimes v_g)\mu_g^{X,Y}\\
&= \bigoplus_{h\in G}[(v_h\otimes \mathrm{id}_{X_h})(v_h^{-1}v_{gh}\xi_{ghg^{-1},g}\otimes u_g)c^{ghg^{-1}}_{g_*(X_h),g_*(Y)}\mu_g^{X,Y}]\\
&= \bigoplus_{h\in G}[(v_h\otimes \mathrm{id}_{X_h})(v_h^{-1}v_{gh}\xi^{-1}_{g,h}\otimes u_g)\mu^g_{h_*(Y),X}g_*(c^h_{X_h,Y})]\\
&= \bigoplus_{h\in G}[(v_g g_*(v_h)\otimes u_g)\mu^g_{h_*(Y),X}g_*(c^h_{X_h,Y})]
\end{aligned}$$

$$= \bigoplus_{h\in G}[(v_g\otimes u_g)\mu^g_{Y,X}g_*(v_h\otimes \mathrm{id}_{X_h})g_*(c^h_{X_h,Y})]$$
$$= (v_g\otimes u_g)\mu^g_{Y,X}g_*(\hat{c}_{X,Y}),$$

in the first equality we use the definition of $\hat{c}$, in the second that c is natural, in the third Equation (2.3), in the forth that (Y, v_g) is an equivariant object, in the fifth that μ^g is natural, in the sixth the definition of $\hat{c}$. Moreover, $\hat{c}$ is natural since c and v_g are natural transformations. Conditions over $\hat{c}$ be a braiding are equivalents to Equations (2.4) and (2.5). □

These constructions allow us to give another description of the Drinfeld's center of the category $\mathcal{C}$.

Theorem 2.8. [8, Theorem 3.5] *If $\mathcal{C}$ is a G-graded tensor category, $Z_{\mathcal{D}}(\mathcal{C})^G \simeq Z(\mathcal{C})$ as braided tensor categories, where $\mathcal{D} = \mathcal{C}_1$.*

Proof. We can assume that $\mathcal{C}$ is strict. Consider the following braided functor $F : Z_{\mathcal{D}}(\mathcal{C})^G \to Z(\mathcal{C})$ given by $F((X,\gamma),u) = (X,\tau)$ where $\tau_Y = (u_g\otimes \mathrm{Id}_Y)c^g_{Y,X}$ if $Y \in \mathcal{C}_g$.

We first check that F is well defined: Consider $B \in \mathcal{C}_g, A \in \mathcal{C}_h, X \in Z_{\mathcal{D}}(\mathcal{C})$, then

$$
\begin{aligned}
(\tau_A\otimes \mathrm{id}_B)(\mathrm{id}_A \otimes\tau_B)
&= (u_h\otimes \mathrm{id}_{AB})(c^h_{A,Y}\otimes \mathrm{id}_B)(\mathrm{id}_A \otimes u_g\otimes \mathrm{id}_B)(\mathrm{id}_A \otimes c^g_{B,X})\\
&= (u_h\otimes \mathrm{id}_{AB})(h_*(u_g)\otimes \mathrm{id}_{AB})(c^h_{A,g_*(X)}\otimes \mathrm{id}_B)(\mathrm{id}_A \otimes c^g_{B,X})\\
&= (u_h h_*(u_g)\otimes \mathrm{id}_{AB})(\xi_{h,g}\otimes \mathrm{id}_{AB})c^{hg}_{AB,X}\\
&= (u_{hg}\otimes \mathrm{id}_{AB})c^{hg}_{AB,X} = \tau_{AB},
\end{aligned}
$$

here we use in the first equality the definition of τ, in the second that $c^h_{A,-}\otimes \mathrm{id}_B$ is natural, in the third the definition of $\xi_{h,g}$, in the fourth that X is equivariant. Then $F((X,\gamma),u) \in Z(\mathcal{C})$. Consider f a morphism in $Z_{\mathcal{D}}(\mathcal{C})^G$ and define $Ff = f$, then f is a morphism in $Z(\mathcal{C})$, since $c^g_{Y,-}$ is natural and f is equivariant morphism.

The tensor structure of F is given by the identity $\mathrm{id}_{X\otimes Y}$, since

$$F(((X,\gamma),u_g)\otimes((Y,\tau),v_g)) =(X\otimes Y,((u_g\otimes v_g)\mu^{X,Y}_g\otimes \mathrm{id})c^g_{-,X\otimes Y})$$

$$F((X,\gamma),u_g)\otimes F((Y,\tau),v_g) = (X\otimes Y, (\mathrm{id}_X\otimes(v_g\otimes \mathrm{id})c^g_{-,Y})\\ ((u_g\otimes \mathrm{id})c^g_{-,X})\otimes \mathrm{id}_Y)$$

and for $Z\in\mathcal{C}_g$

$$\begin{aligned}((u_g\otimes v_g)\mu_g^{X,Y}\otimes \mathrm{id}_Z)c^g_{Z,X\otimes Y} &= (u_g\otimes v_g)(\mathrm{id}_{g_*(X)}\otimes c^g_{Z,Y})(c^g_{Z,X}\otimes \mathrm{id}_Y)\\ &=(\mathrm{id}_X\otimes v_g\otimes \mathrm{id}_Z)(u_g\otimes \mathrm{id}_{g_*(Y)\otimes Z})(\mathrm{id}_{g_*(X)}\otimes c^g_{Z,Y})(c^g_{Z,X}\otimes \mathrm{id}_Y)\\ &=(\mathrm{id}_X\otimes(v_g\otimes \mathrm{id})c^g_{-,Y})((u_g\otimes \mathrm{id})c^g_{-,X})\otimes \mathrm{id}_Y,\end{aligned}$$

here we use in the first equality the definition of μ_g, in the second and third that $(f\otimes f') = (\mathrm{id}\otimes f)(f'\otimes \mathrm{id})$, for any morphisms f, f'.

Let $U : Z(\mathcal{C}) \to Z_{\mathcal{D}}(\mathcal{C})$ be the forgetful functor where $U(Y, c_{-,Y}) = (Y, c_{-,Y}|_{\mathcal{D}})$, and consider the functor

$$G : Z(\mathcal{C}) \to Z_{\mathcal{D}}(\mathcal{C})^G$$
$$G(Y, c_{-,Y}) = (U(Y, c_{-,Y}), u_g)$$

where u_g is induced by the following natural isomorphism

$$c_{X,Y}(c^g_{X,Y})^{-1} : g_*(Y)\otimes X \to Y\otimes X, \text{ for } X\in\mathcal{C}_g$$

and c^g is defined in Equation (2.2). We check that G is well defined: we have to prove that $u_{gh}\xi^{-1}_{g,h} = u_g g_*(u_h)$ for all $g, h \in G$, this is equivalent, using the definition of u_g and ξ, to prove that for $X \in \mathcal{C}_h, Y \in \mathcal{C}_g$ and $(Z, c_{-,Z}) \in Z(\mathcal{C})$,

$$[c_{YX,Z}(c^{gh}_{YX,Z})^{-1}][(c^{gh}_{YX,Z})(\mathrm{id}_Y\otimes(c^h_{X,Z})^{-1})((c^g_{Y,h_*(Z)})^{-1}\otimes \mathrm{id}_X)][(\mathrm{id}\otimes c^g_{Y,X})^{-1}\mu^g_{h_*(Z),X}] = \tag{2.7}$$
$$[(c_{Y,Z}\otimes \mathrm{id}_X)((c^g_{Y,Z})^{-1}\otimes \mathrm{id}_X)][(\mathrm{id}\otimes(c^g_{Y,X})^{-1})(\mu^g_{Z,X}\ \mathrm{id}_Y)][(g_*(c_{X,Z})\otimes \mathrm{id}_Y)(g_*(c^h_{X,Z})^{-1}\otimes \mathrm{id}_Y)], \tag{2.8}$$

then

$$\begin{aligned}(2.7) &= c_{YX,Z}(\mathrm{id}_Y\otimes(c^h_{X,Z})^{-1})(c^g_{Y,h_*(Z)X})^{-1}\\ &= c_{YX,Z}(c^g_{Y,XZ})^{-1}(g_*(c^h_{X,Z})\otimes \mathrm{id}_Y)\\ &= (c_{Y,Z}\otimes \mathrm{id}_X)(\mathrm{id}_Y\otimes c_{X,Z})(c^g_{Y,XZ})^{-1}(g_*(c^h_{X,Z})^{-1}\otimes \mathrm{id}_Y)\\ &= (c_{Y,Z}\otimes \mathrm{id}_X)(c^g_{Y,ZX})^{-1}(g_*(c_{X,Z})\otimes \mathrm{id}_Y)(g_*(c^h_{X,Z})^{-1}\otimes \mathrm{id}_Y) = (2.8),\end{aligned}$$

in the first equality we use the definition of μ^g, in the second that c^g is natural, in the third that c is braided, in the forth that c^g is natural, in the fifth the definition of μ^g.

Finally, F and G are quasi-inverse of each other:

$$F(G(Y, c_{-,Y})) = F((Y, c_{-,Y}|_{\mathcal{D}})u_g) = (Y, (u_g \otimes \mathrm{id})c^g_{-,X}) = (Y, c_{-,Y}).$$

This implies that F is an equivalence of braided tensor categories. □

3 Equivalences between equivariantizations

In this section, we generalized some results of Galindo and Plavnik [9] about how to classify tensor equivalences between equivariantizations, when the tensor category is not a fusion category. The main change when we work over tensor categories not longer semisimple, is that the involved functors have to be exacts.

The first step is to prove an equivalence between two categories [9, Theorem 1.1]. Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, $\mathcal{M}$ be a $\mathcal{C}$-module category and $\mathcal{N}$ be a $\mathcal{D}$-module category. Consider the pairs $(\mathcal{S}, \tau)$ with $\mathcal{S}$ a $(\mathcal{C}, \mathcal{D})$-bimodule category and $\tau : \mathcal{S} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{M}$ an equivalence of $\mathcal{C}$-module categories, the structure of $\mathcal{C}$-module category of $\mathcal{S} \boxtimes_{\mathcal{D}} \mathcal{N}$ is given in [10, Proposition 3.13]. Over these pairs defined the following relation: $(\mathcal{S}, \tau) \sim (\mathcal{S}', \tau')$ if there exist $\phi : \mathcal{S} \to \mathcal{S}'$ a $(\mathcal{C}, \mathcal{D})$-bimodule equivalence and $a : \tau \to \tau'(\phi \boxtimes_{\mathcal{D}} \mathrm{Id}_{\mathcal{N}})$ a natural isomorphism of left $\mathcal{C}$-module functors.

Lemma 3.1. [9, Lemma 4.1] *The relation $\sim$ is an equivalence relation.*

Proof. Reflexive: $(\mathcal{S}, \tau) \sim (\mathcal{S}, \tau)$ since $\phi = \mathrm{Id}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule equivalence and $a = \mathrm{id}$ a natural isomorphism of left $\mathcal{C}$-module functors.

Symmetric: if $(\mathcal{S}, \tau) \sim (\mathcal{S}', \tau')$ we have $\phi : \mathcal{S} \to \mathcal{S}'$ is a $(\mathcal{C}, \mathcal{D})$-bimodule equivalence and $a : \tau \to \tau'(\phi \boxtimes_{\mathcal{D}} \mathrm{Id}_{\mathcal{N}})$ is a natural isomorphism, then there exist $\hat{\phi} : \mathcal{S}' \to \mathcal{S}$ a $(\mathcal{C}, \mathcal{D})$-bimodule equivalence and $\hat{a} : \tau'(\phi \boxtimes_{\mathcal{D}} \mathrm{Id}_{\mathcal{N}}) \to \tau$ a natural isomorphism. In particular there exists $\zeta : \mathrm{Id}_{\mathcal{S}} \simeq \hat{\phi} \circ \phi$ a natural isomorphism. Define $\overline{a} = \hat{a} \circ \tau'(\zeta \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}) : \tau' \to (\hat{\phi} \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}})$ a natural isomorphism of left $\mathcal{C}$-module functors. Then $(\mathcal{S}', \tau') \sim (\mathcal{S}, \tau)$, using $\hat{\phi}$ and $\overline{a}$.

Transitive: suppose that $(\mathcal{S}, \tau) \sim (\mathcal{S}', \tau')$ via ϕ, a and $(\mathcal{S}', \tau') \sim (\mathcal{S}'', \tau'')$ via ϕ', a', then $(\mathcal{S}, \tau) \sim (\mathcal{S}'', \tau'')$ via $\phi' \circ \phi, a' \circ a$ where

$$\tau \xrightarrow{a} \tau'(\phi \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}) \xrightarrow{a'} \tau''(\phi' \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}})(\phi \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}) = \tau''(\phi\phi' \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}).$$

□

Definition 3.2. Let FUNCT be the category with

- Objects: are the pairs $(\mathcal{C}, \mathcal{M})$ with $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an exact indecomposable left $\mathcal{C}$-module category,
- Arrows from $(\mathcal{C}, \mathcal{M}) \to (\mathcal{D}, \mathcal{N})$: are equivalence classes of tensor functors from $\mathcal{C}^*_{\mathcal{M}}$ to $\mathcal{D}^*_{\mathcal{N}}$, (notice that $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a finite tensor category if $\mathcal{M}$ is an indecomposable module category [5, Lemma 3.24]).
- Composition: is the equivalence class of usual composition of tensor functors.

Let COR be the category with

- Objects: are the pairs $(\mathcal{C}, \mathcal{M})$ with $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an exact indecomposable left $\mathcal{C}$-module category,
- Arrows from $(\mathcal{C}, \mathcal{M}) \to (\mathcal{D}, \mathcal{N})$: are equivalence classes of pairs $(\mathcal{S}, \tau)$ via the relation $\sim$ previously defined, denoted each equivalence class by $\overline{(\mathcal{S}, \tau)}$,
- Composition: $\overline{(\mathcal{P}, \beta)} \circ \overline{(\mathcal{S}, \tau)} := \overline{(\mathcal{S} \boxtimes_{\mathcal{D}} \mathcal{P}, \tau(\mathrm{id}_{\mathcal{S}} \boxtimes_{\mathcal{D}} \beta)\alpha_{\mathcal{S},\mathcal{P},\mathcal{N}})} : (\mathcal{C}, \mathcal{M}) \to (\mathcal{D}, \mathcal{N}) \to (\mathcal{E}, \mathcal{O})$, where α is the associativity constrain:

$$\alpha_{\mathcal{S},\mathcal{P},\mathcal{N}} : (\mathcal{S} \boxtimes_{\mathcal{D}} \mathcal{P}) \boxtimes_{\mathcal{E}} \mathcal{O} \to \mathcal{S} \boxtimes_{\mathcal{D}} (\mathcal{P} \boxtimes_{\mathcal{E}} \mathcal{O}).$$

Theorem 3.3. [9, Theorem 1.1] *The category FUNCT is equivalent to the category COR.*

We will construct a well defined full faithful and essentially surjective functor between both categories, in order to prove the equivalency.

Proof. Consider the following functor

$$\begin{aligned} K : FUNCT &\to COR \\ (\mathcal{C}, \mathcal{M}) &\mapsto (\mathcal{C}, \mathcal{M}) \\ (\overline{F} : \mathcal{D}^*_{\mathcal{N}} \to \mathcal{C}^*_{\mathcal{M}}) &\mapsto \overline{(\mathrm{Fun}_{\mathcal{D}^*_{\mathcal{N}}}(\mathcal{N}, \mathcal{M}^F), \epsilon)}, \end{aligned}$$

where ϵ is the evaluation functor. $\mathcal{N}$ is a left $\mathcal{D}^*_{\mathcal{N}}$-module category, by definition, with the action induced from $T \times N \mapsto T(N)$ for all $T \in \mathcal{D}^*_{\mathcal{N}}$ and $N \in \mathcal{N}$. $\mathcal{M}^F$ denoted the left $\mathcal{N}$-module category where as abelian category is $\mathcal{M}$ and the action is induced from $T \times M \mapsto (F(T))(M)$ for $M \in \mathcal{M}$. This implies that the Abelian category

$$S_F := \mathrm{Fun}_{\mathcal{D}^*_{\mathcal{N}}}(\mathcal{N}, \mathcal{M}^F) \tag{3.1}$$

is well defined, where F is a representative of the class $\overline{F}$.

Claim: S_F is a left $\mathcal{C}$-module category.

Consider the functor $\overline{\otimes} : \mathcal{C} \times S_F \to S_F$ where $(X \overline{\otimes} \gamma)(N) = X \otimes \gamma(N)$ for $X \in \mathcal{C}$, $\gamma \in S_F$, $N \in \mathcal{N}$. It is well defined: Consider $\phi \in \mathcal{D}^*_{\mathcal{N}}, \gamma \in S_F, N \in \mathcal{N}$ and $X \in \mathcal{C}$ then

$$\begin{aligned} \phi \otimes ((X \overline{\otimes} \gamma)(N)) &= F(\phi) \otimes (X \otimes \gamma(N)) \\ &= F(\phi)(X \otimes \gamma(N)) \\ &= X \otimes F(\phi)(\gamma(N)) \\ &= X \otimes (\phi \otimes \gamma(N)) \\ &= X \otimes \gamma(\phi \otimes N) \\ &= (X \overline{\otimes} \gamma)(\phi \otimes N) \\ &= (X \overline{\otimes} \gamma)(\phi(N)) \end{aligned}$$

where in the first equality we used the definition of the actions over $\mathcal{M}^F$ and $\overline{\otimes}$, in the second the definition of the action of $\mathcal{C}$ over $\mathcal{M}$, in the third that $F(\phi)$ is a $\mathcal{C}$-morphism, in the fourth the action over $\mathcal{M}^F$, in the fifth that $\gamma \in S_F$, in the sixth the definition of $\overline{\otimes}$, in the last the action of $\mathcal{D}^*_{\mathcal{N}}$ over $\mathcal{N}$. So, $X \overline{\otimes} \gamma$ is a morphism of $\mathcal{D}^*_{\mathcal{N}}$-module categories.

Moreover $\overline{\otimes}$ is an exact functor, since the action $\otimes$ of $\mathcal{C}$ over $\mathcal{M}$ and the following functor are exact: for each $A \in \mathcal{N}$,

$$\begin{aligned}\epsilon_A : S_F &\to \mathcal{M}\\ \gamma &\mapsto \gamma(A).\end{aligned}$$

Define for $X, Y \in \mathcal{C}$, $\gamma \in S_F$, $N \in \mathcal{N}$ the natural isomorphisms $\overline{\alpha}_{X,Y,\gamma}(N) := \alpha_{X,Y,\gamma(N)}$ and $\overline{l}_\gamma(N) := l_{\gamma(N)}$, where α, l are the associativity and unity constrains of $\mathcal{M}$ as a $\mathcal{C}$-module. Then $(S_F, \overline{\otimes}, \overline{\alpha}, \overline{l})$ is a left $\mathcal{C}$-module category. □

Claim: S_F is a right $\mathcal{D}$-module category.

Consider the functor $\overline{\otimes} : S_F \times \mathcal{D} \to S_F$ where $(\gamma\overline{\otimes}D)(N) = \gamma(D\otimes N)$ for $D \in \mathcal{D}$, $\gamma \in S_F$, $N \in \mathcal{N}$. It is well defined: Consider $\phi \in \mathcal{D}^*_\mathcal{N}, \gamma \in S_F, N \in \mathcal{N}$ and $D \in \mathcal{D}$ then

$$\begin{aligned}\phi\otimes(\gamma\overline{\otimes}D)(N)) &= \phi\otimes(\gamma(D\otimes N))\\ &= \gamma(\phi\otimes(D\otimes N))\\ &= \gamma(\phi(D\otimes N))\\ &= \gamma(D\otimes\phi(N))\\ &= \gamma(D\otimes(\phi\otimes N))\\ &= (\gamma\overline{\otimes}D)(\phi\otimes N)\\ &= (\gamma\overline{\otimes}D)(\phi(N))\end{aligned}$$

where in the first equality we used the definition of the action over $\mathcal{M}^F$, in the second that $\gamma \in S_F$, in the third the action of $\mathcal{D}^*_\mathcal{N}$ over $\mathcal{N}$, in the fourth that ϕ is a $\mathcal{D}$-morphism, in the fifth the action of $\mathcal{D}^*_\mathcal{N}$ over $\mathcal{N}$, in the sixth the definition of $\overline{\otimes}$, in the last the action of $\mathcal{D}^*_\mathcal{N}$ over $\mathcal{N}$. So, $X\overline{\otimes}\gamma$ is a morphism of $\mathcal{D}^*_\mathcal{N}$-module categories.

Moreover $\overline{\otimes}$ is an exact functor since it is the composition of exact functors. Define for $X, Y \in \mathcal{D}$, $\gamma \in S_F$, $N \in \mathcal{N}$ the natural isomorphisms $\overline{\alpha}_{X,Y,\gamma}(N) := \gamma(\alpha_{X,Y,N})$ and $\overline{r}_\gamma(N) := \gamma(r_N)$, where α, r are the associativity and unity constrains of $\mathcal{N}$ as a $\mathcal{D}$-module. Then $(S_F, \overline{\otimes}, \overline{\alpha}, \overline{r})$ is a right $\mathcal{D}$-module category. □

If we define for $X \in \mathcal{C}, Y \in \mathcal{D}$, $\gamma \in S_F$, $N \in \mathcal{N}$, $\beta_{X,\gamma,Y}(N) = \mathrm{id}_{X\otimes\gamma(Y\otimes N)} : ((X\overline{\otimes}\gamma)\overline{\otimes}Y)(N) \to (X\overline{\otimes}(\gamma\overline{\otimes}Y))(N)$, then S_F is a $(\mathcal{C}, \mathcal{D})$-bimodule category.

By [5, Theorem 3.31], there exists an equivalence of 2-categories between the 2-category of left $\mathcal{D}^*_{\mathcal{N}}$-module categories and the 2-category of right $\mathcal{D}$-module categories, since $\mathcal{N}$ is an exact $\mathcal{D}$-module category; this implies [9, Theorem 3.1] that the counit ϵ of this adjunction is a natural 2-transformation which is an equivalence of module categories, explicitly

$$\begin{aligned} \epsilon_{\mathcal{S}} : \mathrm{Fun}_{\mathcal{D}^*_{\mathcal{N}}}(\mathcal{N}, \mathcal{S}) \boxtimes_{\mathcal{D}} \mathcal{N} &\to \mathcal{S} \\ \gamma \boxtimes N &\mapsto \gamma(N), \end{aligned}$$

for all $\mathcal{S}$ a $\mathcal{D}^*_{\mathcal{N}}$-module category. In particular, $\epsilon := \epsilon_{\mathcal{M}^F} : S_F \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{M}^F$ is an equivalence of $\mathcal{D}^*_{\mathcal{N}}$-module categories. Moreover, it is clear that ϵ is also a morphism of $\mathcal{C}$-module categories. Then $\overline{(S_F, \epsilon)}$ is an object of COR.

Claim: The pair (S_F, ϵ) does not depend on the equivalence class of F.

Let $F, G : \mathcal{D}^*_{\mathcal{N}} \to \mathcal{M}$ be the tensor functors such that $\theta : F \to G$ is a tensor natural isomorphism. Then $\hat{\theta} := (\mathrm{Id}_{\mathcal{M}}, \theta \otimes \mathrm{id}) : \mathcal{M}^F \to \mathcal{M}^G$ is an equivalence of $\mathcal{D}^*_{\mathcal{N}}$-module categories. Since if $\phi \in \mathcal{D}^*_{\mathcal{N}}$, then for all $M \in \mathcal{M}$

$$\mathrm{Id}_{\mathcal{M}}(\phi \otimes^F M) = F(\phi) \otimes M \xrightarrow{\theta_\phi \otimes \mathrm{id}} G(\phi) \otimes M = \phi \otimes^G \mathrm{Id}(M).$$

$\hat{\theta}$ induced by $\overline{\theta} : \mathrm{Fun}_{\mathcal{D}^*_{\mathcal{N}}}(\mathcal{N}, \mathcal{M}^F) \to \mathrm{Fun}_{\mathcal{D}^*_{\mathcal{N}}}(\mathcal{N}, \mathcal{M}^G)$ is a $(\mathcal{C}, \mathcal{D})$-bimodule equivalence; then $S_F \simeq S_G$ as $(\mathcal{C}, \mathcal{D})$-bimodule categories, explicitly, $\overline{\theta}(\gamma) = \gamma$ for $\gamma \in S_F$. Then if $a := \mathrm{id} : \epsilon_F \to \epsilon_G(\overline{\theta} \boxtimes_{\mathcal{D}} \mathrm{id})$, we obtain $(S_F, \epsilon_F) \sim (S_G, \epsilon_G)$. □

By [9, Lemma 4.2], the functor K is contravariant. A continuation, we proved that K is a full functor, in other words, for any pairs $(\mathcal{C}, \mathcal{M}), (\mathcal{D}, \mathcal{N})$, the map

$$\begin{aligned} \mathrm{Hom}_{FUNCT}((\mathcal{D}, \mathcal{N}), (\mathcal{C}, \mathcal{M})) &\to \mathrm{Hom}_{COR}(F(\mathcal{D}, \mathcal{N}), F(\mathcal{C}, \mathcal{M})) \\ F &\mapsto \overline{(S_F, \epsilon)} \end{aligned}$$

is surjective. Consider the next map for each $\overline{(S, \tau)}$ an arrow in COR

$$\begin{aligned} L : \mathcal{D}^*_{\mathcal{N}} &\to \mathcal{M} \\ F &\mapsto \tau(\mathrm{Id}_S \boxtimes_D F)\tau^*, \end{aligned}$$

where $\tau^* : \mathcal{M} \to S\boxtimes_{\mathcal{D}}\mathcal{N}$ is the quasi-inverse of τ. By [9, Lemma 4.3], L does not depend on the election of the representing (S,τ). L is well defined since τ is an equivalence of $\mathcal{C}$-module categories and the left action of $\mathcal{C}$ over $S\boxtimes_{\mathcal{D}}\mathcal{N}$ is over the first component.

Claim: $\overline{L}$ is an arrow in FUNCT.

Consider $F, G \in \mathcal{D}^*_{\mathcal{N}}$ and remember that the tensor product in $\mathcal{D}^*_{\mathcal{N}}$ is the composition. Using τ^* is the quasi-inverse of τ, we obtain

$$\begin{aligned} L(F)\circ L(G) &= \tau(\mathrm{Id}_S \boxtimes_D F)\tau^*\tau(\mathrm{Id}_S \boxtimes_D G)\tau^* \\ &\simeq \tau(\mathrm{Id}_S \boxtimes_D FG)\tau^* \\ &= L(F\circ G). \end{aligned}$$

Since L is the composition of functors, L is a tensor functor. □

Claim: The assignment $\overline{(S,\tau)} \mapsto \overline{L} \mapsto \overline{(S_L,\epsilon)}$ is an identity.

Define $\phi : S \to S_L$, $\phi(s)(N) = \tau(s\boxtimes_{\mathcal{D}} N)$ for $s \in S, N \in \mathcal{N}$. ϕ is well defined since for $\gamma \in \mathcal{D}^*_{\mathcal{N}}$

$$\begin{aligned} \gamma\otimes\phi(s)(N) &= L(\gamma)(\phi(s)(N)) \\ &= \tau(\mathrm{id}\boxtimes\gamma)\tau^*(\tau(s\boxtimes N)) \\ &\simeq \tau(s\boxtimes\gamma(N)) \\ &= \phi(s)(\gamma\otimes N). \end{aligned}$$

It is clear that ϕ is an equivalence of left $\mathcal{C}$-module categories. If $Y \in \mathcal{D}$

$$\begin{aligned} \phi(s\otimes Y)(N) &= \tau((s\otimes Y)\boxtimes_{\mathcal{D}} N) \\ &= \tau(s\boxtimes_{\mathcal{D}}(Y\otimes N)) \\ &= \phi(s)(Y\otimes N) \\ &= (\phi(s)\otimes Y)(N), \end{aligned}$$

then ϕ is an equivalence of $(\mathcal{C},\mathcal{D})$-bimodules. Since $\epsilon(\phi\boxtimes_{\mathcal{D}}\mathrm{id}) = \tau$, take $a = \mathrm{id}$. This implies that $(S,\tau) \sim (S_L,\epsilon)$. □

Then, the map $F \mapsto \overline{S_F,\epsilon}$ is surjective and K is a full functor. A continuation we proved that K is faithful.

Claim: The assignment $\overline{F} \mapsto \overline{(S_F,\epsilon)} \mapsto \overline{L}$ is an identity.

By definition, for all $a \in \mathcal{D}^*_{\mathcal{N}}$, we have $L(a) = \epsilon(\mathrm{id} \boxtimes a)\epsilon^*$. For $\gamma \in S_F$, $N \in \mathcal{N}$ we have

$$\begin{aligned} \epsilon(\mathrm{id} \boxtimes a)(\gamma \boxtimes N) &= \gamma(a(N)) \\ &= a \boxtimes^F \gamma(N) \\ &= F(a)(\gamma(N)) \\ &= F(a)\epsilon(\gamma \boxtimes N), \end{aligned}$$

this implies that $L(a) = F(a)\epsilon\epsilon^* \simeq F(a)$, then $F \simeq L$ as tensor functors. □

K is essentially surjective since $K(\mathcal{C}, \mathcal{M}) = (\mathcal{C}, \mathcal{M})$, then K is an equivalence of categories. □

As a corollary, we can describe tensor functors between equivariantizations [9, Corollary 5.9]. Let $(*, \mu) : \underline{G} \to \underline{\mathrm{Aut}_{\otimes}(\mathcal{C})}$ be a monoidal functor. Denoted by $\psi : g_*(X \otimes Y) \simeq g_*(X) \otimes g_*(Y)$ the associated natural isomorphisms.

Consider in FUNCT the following objects $(\mathcal{D} \rtimes H, \mathcal{D})$ and $(\mathcal{C} \rtimes G, \mathcal{C})$ for $\mathcal{C}, \mathcal{D}$ finite tensor categories and G, H finite groups acting over $\mathcal{C}$ and $\mathcal{D}$, respectively:

$$\begin{aligned} \overline{\otimes} : (\mathcal{D} \rtimes H) \times \mathcal{D} &\to \mathcal{D}, & \overline{\otimes} : (\mathcal{C} \rtimes G) \times \mathcal{C} &\to \mathcal{C} \\ [A, h] \times B &\mapsto A \otimes h_*(B) & [C, g] \times D &\mapsto C \otimes g_*(D), \end{aligned}$$

where we are doing an abuse of notation calling $()_*$ two different actions over $\mathcal{C}$ and $\mathcal{D}$.

Given an arrow $\overline{F} : (\mathcal{C} \rtimes G, \mathcal{C}) \to (\mathcal{D} \rtimes H, \mathcal{D})$, there exists $F : (\mathcal{C} \rtimes G)^*_{\mathcal{C}} \to (\mathcal{D} \rtimes H)^*_{\mathcal{D}}$ a tensor functor; and $K(F)$ is an arrow in COR which has associated the following data:

- $S_F = \mathrm{Fun}_{(\mathcal{C} \rtimes G)^*_{\mathcal{C}}}(\mathcal{C}, \mathcal{D}^F)$ (from Equation (3.1)) a $(\mathcal{D} \rtimes H, \mathcal{C} \rtimes G)$-bimodule category, which is invertible by [9, Proposition 5.1]. Denoted by $S := (S_F)^{op}$ as left $\mathcal{C} \rtimes G$-module category where the action is given by $a \overline{\otimes} s = s \overline{\otimes} a^*$ for $a \in \mathcal{C} \rtimes G$, $s \in S$.
- $\epsilon : S_F \boxtimes_{\mathcal{C} \rtimes G} \mathcal{C} \to \mathcal{D}$ an equivalence of $\mathcal{D} \rtimes H$-module categories.

Then, as left $\mathcal{D} \rtimes H$-module categories we have

$$\begin{aligned}\mathcal{D} &\simeq S_F \boxtimes_{\mathcal{C}\rtimes G} \mathcal{C} \\ &\simeq \operatorname{Fun}^e_{\mathcal{C}\rtimes G}(S, \mathcal{C}) \\ &\simeq \operatorname{Fun}^e_{\mathcal{C}\rtimes G}(\mathcal{C}, S)^{op},\end{aligned}$$

when second equivalence is due to [10, Theorem 3.20] and $\operatorname{Fun}^e_{\mathcal{C}\rtimes G}(S, \mathcal{C})$ are the exact functors.

We want to describe strict $\mathcal{C} \rtimes G$-module categories in terms of strict $\mathcal{C}$-module categories plus some additional invariant G structure.

Definition 3.4. A *G-equivariant $\mathcal{C}$-module category* is a $\mathcal{C}$-module category $\mathcal{M}$ equipped with a G-graded monoidal functor $(*, \mu) : \underline{G} \to \underline{Aut^G_{\mathcal{C}}(\mathcal{M})}$, in other words, for each $g \in G$, there exists an $\mathcal{C}$-module automorphism $\mathcal{M} \simeq \mathcal{M}^{g*}$. A *$G$-equivariant $\mathcal{C}$-module functor* between two G-equivariant module categories $\mathcal{M}$ and $\mathcal{N}$ is a $\mathcal{C}$-module functor $(F, \alpha) : \mathcal{M} \to \mathcal{N}$ and (F, τ) is G-linear with natural isomorphisms $\tau_{g,M} : F(g * M) \to g * F(M)$ such that

$$(\mathrm{id}_g\, \alpha_{X,M})\tau_{g,XM} F(\psi_{X,M}) = \psi_{X,F(M)}(\mathrm{id}_{g_*X} \otimes \tau_{g,M})\alpha_{g_*X,g_*M}. \tag{3.2}$$

Here we make an abuse of notation using $*$ also for the action of G over $\mathcal{C}$. The next lemma, was firstly introduced in [17, Section 2].

Lemma 3.5. *There is a 2-equivalence between the 2-category of $\mathcal{C} \rtimes G$-module categories and the 2-category of G-equivariant $\mathcal{C}$-module categories.*

Proof. Over 0-cells, following [6, Proposition 5.12], there is a bijective correspondence between $\mathcal{C} \rtimes G$-module categories and G-equivariant $\mathcal{C}$-module categories, where

- given $(\mathcal{M}, \overline{\otimes})$ a structure of $\mathcal{C} \rtimes G$-module category, we defined

$$\begin{aligned}\otimes : \mathcal{C} \times \mathcal{M} &\to \mathcal{M} & \qquad * : G \times \mathcal{M} &\to \mathcal{M} \\ X \otimes M &= [X, 1] \overline{\otimes} M & g * M &= [1, g] \overline{\otimes} M,\end{aligned}$$

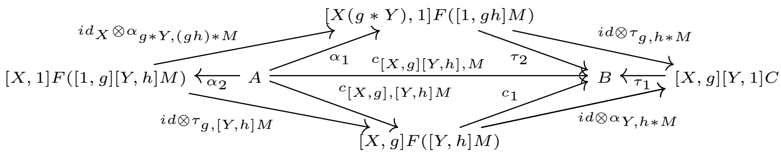
- given $(\mathcal{M}, \otimes, *)$ a structure of G-equivariant $\mathcal{C}$-module category, we defined

$$\begin{aligned} \overline{\otimes} : \mathcal{C} \times \mathcal{M} &\to \mathcal{M} \\ [X, g]\overline{\otimes}M &= X \otimes (g * M). \end{aligned}$$

Over 1-cells, consider $(F, c) : \mathcal{M} \to \mathcal{N}$ a $\mathcal{C} \rtimes G$-module functor and define the following natural isomorphisms $\alpha_{X,M} := c_{[X,1],M} : F(X \otimes M) \to X \otimes F(M)$ and $\tau_{g,M} := c_{[\mathbf{1},g],M} : F(g * M) \to g * F(M)$. Then c can be decompose as

$$c_{[X,g],M} = (\mathrm{id}_{[X,1]} \otimes \tau_{g,M})\alpha_{X,[\mathbf{1},g]\otimes M}. \tag{3.3}$$

Consider the following diagram, where
$A = F([X, g][Y, h]M), B = [X, g][Y, h]F(M), C = F([1, h]M)$



(3.4)

where

- the middle bottom triangle is equivalent to (F, c) is a $\mathcal{C} \rtimes G$-linear,
- left and right bottom, and middle up triangles are equivalent to Equation (3.3), with $\alpha_1 = \alpha_{X(g*Y),(gh)*M}$, $\alpha_2 = \alpha_{X,[g*Y,gh]M}$, $\tau_1 = id \otimes \tau_{h,M}, \tau_2 = id \otimes \tau_{gh,M}, c_1 = id \otimes c_{[Y,h],M}$,
- the left up triangle is equivalent to (F, α) is $\mathcal{C}$-linear,
- the right up triangle is equivalent to (F, τ) is G-linear,
- external diamond is Equation (3.2).

Therefore, (F, c) is a $\mathcal{C} \rtimes G$-module functor is equivalent to (F, α, τ) is a G-equivariant $\mathcal{C}$-module functor.

□

Then it is possible to consider the category of equivariant objects in S, denoted by S^G.

Denoted also by $\mu_{g,h} : (gh)_* \to g_* h_*$ the tensor structure of $*$ induced by the associativity constrain of the $\mathcal{C} \rtimes G$-action over S; and $c^g_{X,s} : g_*(X \otimes s) \to g_*(X) \otimes g_*(s)$ for $X \in \mathcal{C}, s \in S$ is induced by

$$[\mathbf{1}, g] \otimes X = [g_*(X), g] = g_*(X) \otimes [\mathbf{1}, g], \quad \text{for all } g \in G, X \in \mathcal{C}.$$

Lemma 3.6. [9, Remark 5.8.3] $\mathrm{Fun}^e_{\mathcal{C} \rtimes G}(\mathcal{C}, S) \simeq S^G$ *as categories.*

Proof. Defined

$$\begin{aligned} \varphi : \mathrm{Fun}^e_{\mathcal{C} \rtimes G}(\mathcal{C}, S) &\to S^G \\ \varphi(F, \zeta) &= (F(\mathbf{1}), \zeta^{-1}_{[\mathbf{1}, g], \mathbf{1}}) \\ \varphi(\eta) &= \eta_{\mathbf{1}}, \end{aligned}$$

where (F, ζ) is a functor of $\mathcal{C} \rtimes G$-modules with $\zeta_{[X,g],Y} : F([X, g] \otimes Y) \simeq [X, g] \otimes F(Y)$ is the structure of $\mathcal{C} \rtimes G$-module over F, for $X, Y \in \mathcal{C}$, $g \in G$; and $\eta : (F, \zeta) \to (F', \zeta')$ is a natural transformation. If for $g, h \in G$

$$\zeta^{-1}_{[\mathbf{1},g],\mathbf{1}}(\mathrm{id}_{[\mathbf{1},g]} \otimes \zeta^{-1}_{[\mathbf{1},h]\mathbf{1}}) = \zeta^{-1}_{[\mathbf{1},gh],\mathbf{1}}(m \otimes \mathrm{id}_{F(\mathbf{1})}) \alpha^{-1}_{[\mathbf{1},g],[\mathbf{1},h],F(\mathbf{1})},$$

then $\varphi(F, \zeta)$ is an equivariant object. We will denote $[\mathbf{1}, g] = g$ in $\mathcal{C} \rtimes G$. Since η is a natural transformation, $\zeta'_{g,\mathbf{1}} \eta_{\mathbf{1}} = (\mathrm{id}_g \otimes \eta_{\mathbf{1}}) \zeta_{g,\mathbf{1}}$ which is equivalent to $\eta_{\mathbf{1}}$ is a morphism in S^G. Then φ is well defined.

For $(s, u_g) \in S^G$, consider the functor of $\mathcal{C} \rtimes G$-modules $(- \otimes s, \zeta)$ with

$$\zeta_{[Y,g],Z} = (\mathrm{id}_Y \otimes (c^g_{Z,s})^{-1})(\mathrm{id}_{Y \otimes g_* Z} \otimes u_g^{-1}), \quad \text{for } Y, Z \in \mathcal{C}.$$

(F, ζ) is an exact functor since the action of $\mathcal{C}$ over S is an exact functor. Then φ is essentially surjective.

Consider $f \in \mathrm{Hom}_{S^G}(\varphi(F, \zeta), \varphi(F', \zeta'))$ and defined

$$\eta_X = \mathrm{id}_{[X,1]} \otimes f : [X, 1] \otimes F(\mathbf{1}) = F(X) \to F'(X) = [X, 1] \otimes F'(\mathbf{1}).$$

Consider $\beta : X \to Y$ in $\mathcal{C}$, it induces $\beta : [X, 1] \to [Y, 1]$ in $\mathcal{C} \rtimes G$ and we obtain

$$(\beta \otimes \mathrm{id}_{F'(\mathbf{1})})(\mathrm{id}_{[X,1]} \otimes f) = (\mathrm{id}_{[Y,1]} \otimes f)(\beta \otimes \mathrm{id}_{F(\mathbf{1})}),$$

this implies that η is a natural transformation. It is clear that the assignments $f \mapsto \eta = \mathrm{id} \otimes f \mapsto \varphi(\eta)$ and $\eta \mapsto \varphi(\eta) \mapsto \mathrm{id} \otimes \eta_{\mathbf{1}}$ are identities. This implies that φ is full and faithfull functor. Then φ is an equivalence of categories. □

Then, we induced over S^G a structure of right $\mathcal{D} \rtimes H$-module category, and we obtain

$$\mathcal{D} \simeq (S^G)^{op}$$

as left $\mathcal{D} \rtimes H$-module category. Moreover S^G results a $(\mathcal{C}^G, \mathrm{End}^e_{\mathcal{C} \rtimes G}(S))$-bimodule category [9, Remark 5.8(3)]. Since S is an invertible $(\mathcal{C} \rtimes G, \mathcal{D} \rtimes H)$-bimodule category we obtain

$$\mathrm{End}^e_{\mathcal{C} \rtimes G}(S) \simeq S^{op} \boxtimes_{\mathcal{C} \rtimes G} S \simeq \mathcal{D} \rtimes H,$$

then S^G is a $(\mathcal{C}^G, \mathcal{D} \rtimes H)$-bimodule category. Moreover S^G is invertible as $(\mathcal{C}^G, \mathcal{D} \rtimes H)$-bimodule category:

$$(S^G)^{op} \boxtimes_{\mathcal{C}^G} S^G \simeq \mathrm{End}^e_{\mathcal{C}^G}(S^G) \simeq \mathrm{End}^e_{\mathcal{C} \rtimes G}(S) \simeq \mathcal{D} \rtimes H,$$

since the module categories over $\mathcal{C} \rtimes G$ and $\mathcal{C}^G$ are in bijective correspondence [15, Proposition 3.2]. By [3, Proposition 4.2], the functor

$$R : (\mathcal{D} \rtimes H)^{op} \to \mathrm{End}^e_{\mathcal{C}^G}(S^G) \simeq \mathrm{End}^e_{\mathcal{C} \rtimes G}(S)$$
$$X \mapsto -\overline{\otimes} X$$

is an equivalence of tensor categories. Now, since $\mathcal{D} \rtimes H$ is an H-grading, then $\mathrm{End}^e_{\mathcal{C} \rtimes G}(S)$ as an H^{op}-grading, so as left $\mathcal{D} \rtimes H$-module categories $(S^G)^{op} \simeq \mathcal{D} \simeq (\mathcal{D} \rtimes H)_1$ and as right $\mathcal{D} \rtimes H$-module categories $S^G \simeq (\mathcal{D} \rtimes H)_1^{op} \simeq (\mathrm{End}^e_{\mathcal{C} \rtimes G}(S))_1$, then as left $(\mathrm{End}^e_{\mathcal{C} \rtimes G}(S))_1$-module categories

$$S^G \simeq (\mathrm{End}^e_{\mathcal{C} \rtimes G}(S))_1.$$

This proves the following Theorem.

Theorem 3.7. [9, Theorem 5.11] *$\mathcal{D}^H$ is tensor equivalent to $\mathcal{C}^G$ if, and only if, there exist a G-equivariant $\mathcal{C}$-module category S and a faithfull H^{op}-grading in $\mathrm{End}^e_{\mathcal{C}\rtimes G}(S)$ such that $(\mathrm{End}^e_{\mathcal{C}\rtimes G}(S))_1 \simeq \mathcal{D}$ and $\mathcal{D}^H \simeq (\mathrm{End}^e_{\mathcal{C}\rtimes G}(S))^H_1$.*

□

Remark 3.8. S has a structure of invertible $(\mathcal{C} \rtimes G, \mathcal{D} \rtimes H)$-bimodule category, then it is a simple [2, Exercise 4.3.11](and exact indecomposable) bimodule category. Regarding the left structure, S is also an exact left $\mathcal{C} \rtimes G$-module category: consider $P \in \mathcal{C}$ a projective object and $M \in S$ then $P \otimes M \simeq (P \otimes \mathbf{1}) \otimes M$ is projective.

Corollary 3.9. *$Z_{\mathcal{D}_1}(\mathcal{D})^H \simeq Z_{\mathcal{C}_1}(\mathcal{C})^G$ as tensor categories if, and only if, there exist*

- *S a G-equivariant $Z_{\mathcal{C}_1}(\mathcal{C})$-module category,*
- *a faithfull H^{op}-grading in $\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S)$ such that*

$$(\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S))_1 \simeq \mathcal{D}.$$

Remark 3.10. Since S is also an exact $Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G$-module category, by [2, Cor 7.10.5], there exists an algebra $B \in Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G$ such that S is equivalent to the B-modules in $Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G$. Then, by [2, Prop 7.11.1], we obtain

$$\mathcal{D} \simeq \mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S))_1 \simeq ({}_B(Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G)_B)_1$$
$$X \mapsto - \otimes X \mapsto B \otimes X.$$

Example 3.11. Consider $\mathcal{C}$ a G-graded finite tensor category and let A be an algebra in $\mathcal{C}$ such that ${}_A\mathcal{C}$, the category of left A-modules in $\mathcal{C}$, is an exact indecomposable $\mathcal{C}$-module category. Then the category of A-bimodules in $\mathcal{C}$, ${}_A\mathcal{C}_A$, is also G-graded and [2, Cor 7.16.2 and Rmk 7.12.5]

$$Z_{\mathcal{C}_1}(\mathcal{C})^G \simeq Z(\mathcal{C}) \simeq Z({}_A\mathcal{C}_A) \simeq Z_{{}_A(\mathcal{C}_1)_A}({}_A\mathcal{C}_A)^G,$$

then Corollary 3.9 implies that there exist

- S a G-equivariant $Z_{\mathcal{C}_1}(\mathcal{C})$-module (equivalently, $(Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G)$-module category (Lemma 3.5)),

- a graduation where $(\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S))_1 \simeq {}_A\mathcal{C}_A$.

Then

$$ {}_A\mathcal{C}_A \simeq (\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S))_1 \simeq ({}_B(Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G)_B)_1 $$

and the category of A-bimodules in $\mathcal{C}$ is a tensor subcategory of the category of B-bimodules in $Z_{\mathcal{C}_1}(\mathcal{C}) \rtimes G$.

By [7] two graded tensor categories are Morita equivalent if, and only if, their centers are braided crossed tensor equivalent; therefore Theorem 2.8 and Corollary 3.9 give us an extra condition to determined Morita equivalence:

Corollary 3.12. *There is a Morita equivalence between $\mathcal{C}$ and $\mathcal{D}$ if, and only if, there exist*

- *S a G-equivariant $Z_{\mathcal{C}_1}(\mathcal{C})$-module category,*
- *a faithfull H^{op}-grading in $\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S)$ such that*

$$ (\mathrm{End}^e_{Z_{\mathcal{C}_1}(\mathcal{C})\rtimes G}(S))_1 \simeq \mathcal{D}. $$

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