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On free acts over semigroups and their lattices of radical subacts

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Abstract. This study aims to investigate free objects in the category of acts over an arbitrary semigroup S. We consider two generalizations of free acts over arbitrary semigroups, namely acts with conditions (F1) and (F2), and give some new results about (minimal) prime subacts and radical subacts of any S-act with condition (F1). Furthermore, some lattice structures for some collections of radical subacts of free S-acts are introduced. We also obtain some results about the relationship between radical subacts of free S-acts and radical ideals of S. Moreover, for any prime ideal P of a semigroup S with a zero, we find a one-to-one correspondence between the collections of $P-$prime subacts of any two free S-acts. Also, it is shown that all free S-acts have isomorphic lattices of radical multiplication subacts.

1 Introduction

Let S be a semigroup. By a *right S-act* A_S (or *act* A for short), we mean a set A with an action $A \times S \longrightarrow A$ defined by $(a, s) \longmapsto as$ such that for any $a \in A$ and $s, t \in S$, $(as)t = a(st)$. Also, here the action of a semigroup

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on the empty set is allowed. If S is a monoid with identity element 1_S, this action can be considered *unitary* in the sense that, $a1_S = a$ for each $a \in A$. However, generally S-acts may not be unitary even if S is a monoid (see, for instance, [2, Chapter 11]). Here, unless stated otherwise, we study the more general case of all acts rather than unitary ones. As we will see, for non-unitary acts, some structures of acts can be developed in different ways (see, for instance, Definition 2.1). By an *S-homomorphism* (or a *homomorphism* for short) from an S-act A into an S-act B, we mean a mapping $f : A \longrightarrow B$ which preserves the actions in the sense that, $f(as) = f(a)s$, for any $a \in A$ and $s \in S$. For an arbitrary semigroup S, we use the notation **Act**-S to denote the category of all right S-acts in which the objects are all non-empty right S-acts and the morphisms are all S-homomorphism between right S-acts. Moreover, if we also consider the empty S-act as an object, the category of all S-acts is denoted by $\overline{\textbf{Act}}$-S. If a semigroup S contains a zero, 0, we consider any right S-act A to have a unique fixed element, denoted by θ, such that $a0 = \theta$ for any $a \in A$. Such an act is called a *centered act*. The category of all centered S-acts and all S-homomorphisms preserving the fixed element is denoted by $\mathbf{Act_0}$-S.

For any subact B of an S-act A, the set $\{s \in S | As \subseteq B\}$, denoted by $(B : A)$, is called *the colon of B in A*. It can easily be observed that $(B : A)$ is an ideal of S, whenever it is not empty. Moreover, for any subset T (possibly empty) of S, the subset $\{at \,|\, a \in A \,\text{and}\, t \in T\}$ of A is denoted by AT.

A proper subact B of A is called *prime*, if for any $a \in A$ and $s \in S$, $aSs \subseteq B$ implies that $a \in B$ or $s \in (B : A)$. The intersection of all prime subacts of A containing B is called the *prime radical of B*. Also, B is said to be a *radical subact* if it is equal to its radical. The set of all prime subacts of A containing B is known as the *prime variety of B*.

The notions of prime ideals of rings and prime submodules in the category of modules over rings (**Mod**-R) are remarkable subjects in the study of rings and modules, first proposed for **Mod**-R by Dauns in [3]. In the category of S-acts, [1] is known as the first paper on prime subacts in which the notion of prime acts is introduced as a generalization of prime modules. A great deal of work has been done on the concepts of prime ideals in semigroups and prime subacts in **Act**-S since then. For instances, [1, 4, 8, 10, 12, 13] studied different problems on prime ideals and prime subacts.

The notion of free act over a monoid S is defined as a free object in the category of unitary S-acts by different approaches (see, for example, [6, Section 1.5]). As an elementary approach, a unitary S-act F is free if it has a basis X for which every element $a \in F$ has a unique presentation in the form $a = xs$, for the unique $x \in X$ and $s \in S$. Equivalently, F is a free unitary S-act if it is a coproduct of a non-empty family of right S- acts each of which is isomorphic to S. These equivalent definitions for free S-acts satisfy the "universal property of free objects" in the category of all unitary S-acts ([6, Theorem 1.5.15]). By [6, Remark 2.3.7], a similar argument defines a free S-act in the category of unitary centered acts over a monoid S with a zero. In this case, for any free S-act F, there exists an index set I such that

$$F = (\dot{\bigcup}_{i \in I}(S_i \setminus \{\theta_i\}))\dot{\bigcup}\{\theta\},$$

where each S_i is isomorphic to S and θ is the fixed element of F. But, as we see in Remark 2.3, the above descriptions of an S-act F do not give a free object in the category of all (not only unitary) S-acts. Here, in Section 2 of this article, we consider an arbitrary semigroup S (possibly without identity and zero) and study the notion of free S-acts. This consideration gives different classes of objects in **Act**-S in Definition 2.1.

Similar to the prime subacts, some particular kinds of free acts are defined by using the translations of semigroups in acts. Hence in a sense, the study of prime subacts of these classes of objects in **Act**-S (and in $\overline{\textbf{Act}}$-S) can be more interesting than similar ones in **Mod**-R. In light of the examination of this aspect of the objects in $\overline{\textbf{Act}}$-S, a few results are given in Section 2 to characterize radical subacts of new defined acts.

Recently, there has been considerable interest in prime and radical ideals of semigroups (see, for instance, [10, 13]). However, studying these notions in **Act**-S is a new subject in literature. This study aimed to explore more about these issues in **Act**-S. In [8], some lattices of subacts of (free) acts over monoids were studied. We consider new generalizations of free acts, defined in Definition 2.1, and study generalizations of some results in [8] for the bigger classes of acts over arbitrary semigroups. In Section 3, the notion of (prime) radical subacts for a generalization of free acts is investigated. We consider lattice structures on different posets of radical subacts of this type of acts. In Theorem 3.7, it is proved that the algebraic connections between these structures are not trivial in the sense that non-isomorphic free acts may have isomorphic lattices of radical subacts.

For the preliminary results and definitions relating to semigroups and acts in this paper, we refer the readers to [2] and [6], respectively.

2 Free acts and some results on prime subacts of them

The concept of free acts can be defined as free objects in the category of acts over an arbitrary semigroup. However, the other approaches, defining freeness in the category of acts over monoids, may lead to different (non-equivalent) definitions for acts over arbitrary semigroups. We consider the following notions.

Definition 2.1. Let **Act**-S be the category of right acts over a semigroup S.

(1) In what follows, by a *free act on a non-empty set* X in **Act**-S, we mean a free object in **Act**-S which satisfies the *universal property* as follows.

There exists a mapping $i : X \longrightarrow F$ such that for any right S-act A in **Act**-S and any mapping $\phi : X \longrightarrow A$ there exists a unique homomorphism $\bar{\varphi} : F \longrightarrow A$, such that $\bar{\varphi}i = \varphi$ (when S contains a zero, homomorphisms of S-acts preserve zeros).

(2) A right S-act F in **Act**-S satisfies *Condition* (*F1*), if F has a generating set X such that every element $a \in F$, which is expressible in the form $a = xs$ for some $x \in X$ and $s \in S$, has a unique representation (when S contains a zero, 0, $xs = \theta$ if and only if $s = 0$).

(3) A right S-act F in **Act**-S satisfies *Condition* (*F2*), if F is a coproduct of a non-empty family of subacts each of which is isomorphic to S or S^1 (if S contains a zero, any coproduct of S-acts has a unique adjoined fixed element and is a subact of their products (see, [6, Proposition 2.1.15, Remark 2.1.16])).

The following theorem gives some relationships between these notions for different types of semigroups.

Theorem 2.2. *Let S be a semigroup.*

(1) *If S is a monoid (possibly with a zero) the above definitions are equivalent in the category of unitary right S-acts* **Act**-S ($\mathbf{Act_0}$-S).

(2) *Suppose that S has no identity element. There exists a free S-act on a non-empty set X in* **Act**-S*. Moreover, every free S-act satisfies Condition (F1). Furthermore, every act satisfying Condition (F1) also satisfies Condition (F2).*

(3) *If S has no left identity element, then any S-act F satisfies Condition (F1) if and only if F is free.*

Proof. Part (1) is a summary of [6, Theroems 1.5.13, 1.5.15 and Remark 2.3.7].

For any non-empty set X, consider the right S-act $X \times S^1$ defined by $(x, s)t = (x, st)$ for any $x \in X$, $s \in S^1$ and $t \in S$. Let $i : X \longrightarrow X \times S^1$ with $i(x) = (x, 1)$ and $f : X \longrightarrow A$ be a mapping for an arbitrary S-act A. Define $\bar{f} : X \times S^1 \longrightarrow A$ by $\bar{f}(x, s) = f(x)s$ and $\bar{f}(x, 1) = f(x)$, for any $s \in S$ and $x \in X$. Since $st \neq 1$, for any $s \in S^1$ and $t \in S$, $\bar{f}$ is a well-defined homomorphism. Moreover, since $(x, 1)$, $x \in X$, generate $X \times S^1$, it can be checked that $\bar{f}$ is unique with $\bar{f}(i(x)) = f(x)$. So, $X \times S^1$ is free on X.

To prove the second statement of (2), we show that if F is free on a set X with $i : X \longrightarrow F$ as in Definition 2.1 (1), then $i(X)$ is a generating set for F, satisfying Condition (F1). Consider mapping $i' : X \longrightarrow i(X)S \cup i(X)$, with $i'(x) = i(x)$, for each $x \in X$, and the inclusion map $j : i(X)S \cup i(X) \longrightarrow F$. Then there exists a unique homomorphism $\bar{\varphi} : F \longrightarrow i(X)S \cup i(X)$ extending i'. So $j\bar{\varphi} = id_F$, which implies that $a = j(\bar{\varphi}(a)) = \bar{\varphi}(a) \in i(X)S \cup i(X)$, for any $a \in F$. Thus $a = i(x)s$ or $a = i(x)$ generated by an element of $i(X)$. To prove the uniqueness of this expression for a, assume that there exist $x, x' \in X$ and $s, t \in S$ such that $a = i(x)s = i(x')t$. Consider the right S-act $X \times S^1$ as above and a mapping $f : X \longrightarrow i(X) \times S^1$, by $f(x) = (i(x), 1)$ and $f(x') = (i(x'), 1)$. Then for the unique homomorphism $\bar{f} : F \longrightarrow i(X) \times S^1$ we have,

$$(i(x), s) = (i(x), 1)s = f(x)s = \bar{f}(i(x))s = \bar{f}(i(x)s) = \bar{f}(a)$$

$$= \bar{f}(i(x')t) = \bar{f}(i(x'))t = f(x')t = (i(x'),1)t = (i(x'),t).$$

Therefore, the representation of $a = i(x)s \in F$ is unique.

To prove the last statement of (2), note that if F satisfies Condition (F1) with a generating set X, then every element of F is of the form x_i or has a unique expression $a = x_i s$, for some $x_i \in X$ and $s \in S$. For such x_i put $S_i = x_i S \cup \{x_i\}$. We have two cases. If $x_i = x_i t$, for some $t \in S$ then $f_i : S_i \longrightarrow S$ defined by $f_i(x_i s) = s$, for each $s \in S$ is a well-defined isomorphism of S-acts. In the second case, if $x_i \neq x_i t$, for each $t \in S$ then $g_i : S_i \longrightarrow S^1$ defined by $g_i(x_i) = 1$ and $g_i(x_i s) = s$, for each $s \in S$ is a well-defined isomorphism of S-acts. If $x_i s = x_j$, for some $s \in S$ and $x_i, x_j \in X$, then $x_i s^2 = x_j s$ which implies that $x_i = x_j$, by the assumption of Condition (F1). Hence for each $x_i, x_j \in X$ as above, $S_i \cap S_j = \emptyset$. Therefore, F is a coproduct of copies of S or S^1, i.e., F satisfies Condition (F2).

The sufficiency part in (3) follows from (2). For the proof of the necessity part in (3), suppose that F satisfies Condition (F1) with a generating set X as in Definition 2.1 (2). We show that F is free on X in **Act**-S. Consider the inclusion mapping $i : X \longrightarrow F$ and a mapping $f : X \longrightarrow A$ for an arbitrary S-act A. Since S has no left identity, Condition (F1) implies that $x \neq x's$, for any $x, x' \in X$ and $s \in S$. Thus, every element $a \in F$ is either an element of X or can be uniquely expressed in the form $a = xs$, for elements $x \in X$ and $s \in S$. Now, define $\bar{f} : F \longrightarrow A$ by $\bar{f}(xs) = f(x)s$ and $\bar{f}(x) = f(x)$, for any $x \in X$ and $s \in S$. Then by above argument, $\bar{f}$ is a well-defined homomorphism and uniquely extends f. □

Remark 2.3. (1) Note that Condition (F1) for an S-act F, may not lead to freeness of F in general. For example, take multiplicative subsemigroups $F = \{2,4\} \subseteq \mathbb{Z}_6$, $S = \{4,8\} \subseteq \mathbb{Z}_{12}$ and $A = \{0,4\} \subseteq \mathbb{Z}_8$ as S-acts. Consider the inclusion map $i : \{2\} \hookrightarrow F$ and the map $f : \{2\} \longrightarrow A$, defined by $f(2) = 4$. Then, there is no S-homomorphism $h : F \longrightarrow A$ with $hi = f$. However, $X = \{2\}$ generates F uniquely. Similar arguments for the inclusion map $\{4\} \hookrightarrow F$ shows that F is not a free S-act. Note that in this example, although S is a monoid, A is not a unitary S-act.

(2) A generating set for multiplicative semigroup $S = 4\mathbb{Z}$ as an S-act should contains 4 and 12. But $48 = 4 \times 12 = 12 \times 4$, has no unique presentation. So, Condition (F2) may not imply Condition (F1) in general.

There are different definitions in literature for the notion of primeness in **Act**-S by using elements, subacts or relations on acts (see, for instance, [1, 9, 12]). In the rest of this section, we shall investigate prime and radical subacts based on the following standard definitions (from [1, 8]) for an arbitrary semigroup S and acts over it.

Definition 2.4. (1) Let S be a semigroup, P a proper ideal of S and A an S-act. P is called a *prime ideal* if, for any $a, b \in S$, $aSb \subseteq P$ implies that $a \in P$ or $b \in P$. A proper subact B of A is called *prime*, if for any $a \in A$ and $t \in S$, $aSt \subseteq B$ implies that $a \in B$ or $At \subseteq B$ (i.e., $a \in B$ or $t \in (B : A)$). Clearly, if S is commutative, B is a prime subact of A if and only if $at \in B$ implies that $a \in B$ or $At \subseteq B$. Also, a centered S-act A is called a *prime act* if the one element subact $\{\theta\}$ is a prime subact of A. Obviously, the empty subact of any act is prime. By a $P-$*prime subact of* A, we mean a prime subact of A with colon P (possibly, $P = \emptyset$). The set of all $P-$prime subacts of A is called the $P-$*prime spectrum of* A and denoted by $\mathcal{S}pec_P(A)$. Moreover, $\mathcal{S}pec(A)$ denotes all prime subacts of A.

(2) For an arbitrary ideal I of S the *prime radical* of I is denoted by $\sqrt{I}$ and is defined to be the intersection of all prime ideals of S containing I. Also, for any subact B of an S-act A, the intersection of all prime subacts of A which contains B, is called the *prime radical* of B (*radical of* B for short) and is denoted by $rad_A(B)$ (or $rad(B)$ where the acts are delineated). We define $rad(B) = A$ when there is no prime subact in A containing B. Also, B is called a *radical subact* if $rad(B) = B$. The set of all radical subacts of A is denoted by $\mathcal{R}ad(A)$.

Clearly, every prime subact is a radical subact and $rad(rad(B)) = rad(B)$, for any subact B of A. We can express similar assertions for radicals of ideals in S.

The following proposition gives some of the useful properties of the prime radical in **Act**-S.

Proposition 2.5. ([8, Proposition 2.2]) *Let A and A' and A_γ, for any $\gamma \in \Gamma$ be S-acts. Assume that $\mathcal{S}ub(A)$ is the lattice of subacts of A and $B, C \in \mathcal{S}ub(A)$ and $B_\gamma \in \mathcal{S}ub(A_\gamma)$, for each $\gamma \in \Gamma$. The following assertions hold.*

(i) *For every prime subact P of A, $B \subseteq P$ if and only if $rad(B) \subseteq P$.*

(ii) *$rad(B) \subseteq rad(C)$ if and only if every prime subact P of A, which contains C, also contains B.*

(iii) *If $f : A \longrightarrow A'$ be a homomorphism then $f(rad(B)) \subseteq rad(f(B))$. In particular, if f is an isomorphism then $f(rad(B)) = rad(f(B))$. Moreover, for any $B' \in \mathcal{R}ad(A')$, $f^{-1}(B') \in \mathcal{R}ad(A)$.*

(iv) *$rad(\prod_{\gamma\in\Gamma} B_\gamma)$ in $\prod_{\gamma\in\Gamma} A_\gamma$ is a subset of $\prod_{\gamma\in\Gamma} rad(B_\gamma)$. In particular, if every prime subact of $\prod_{\gamma\in\Gamma} A_\gamma$ is also a product of subacts of $\{A_\gamma\}_{\gamma\in\Gamma}$, then the equality holds.*

Note that by Theorem 2.2, any result for acts with Condition (F1) can be trivially restated for free acts. So, in the following, some of the results are stated for acts with Condition (F1). Some of these results were studied in [8] for free acts over monoids. However, here we consider the bigger classes of acts over arbitrary semigroups. Note that, as we have seen in Remark 2.3 (1), even if S is a monoid then, an arbitrary S-act may not be unitary. Therefore, it is reasonable to recheck the mentioned results of [8], also for acts with Condition (F1).

Lemma 2.6. *Let A be an S-act, F be an S-act satisfying Condition (F1) and I, J and I_j for any $j \in \mathcal{J}$ be ideals of S. The following assertions hold.*

(i) *$FI \subseteq FJ$ if and only if $I \subseteq J$.*

(ii) *$(FI : F) = I$.*

(iii) *Let I be a proper ideal of S. Then, I is a prime ideal if and only if FI is a prime subact of F. Moreover, FS is a prime subact provided that $FS \neq F$.*

(iv) *$F(\bigcap_{j\in\mathcal{J}} I_j) = \bigcap_{j\in\mathcal{J}} FI_j$.*

(v) *Let B be a prime subact of A. If $(B : A)$ is a non-empty proper subset of S then it is a prime ideal of S.*

Proof. The proofs of (i), (ii), (iii), and (v) are routine. To prove (iv), suppose that X is a generating set with unique representations for elements of F. Let $x \in \bigcap_{j\in\mathcal{J}} FI_j$. Then, for every $k \in \mathcal{J}$, there exist $x_k \in X$ and $s_k \in I_k$ such that $x = x_k s_k$. Thus, for every $k, l \in \mathcal{J}$, $x_k s_k = x_l s_l$. Now, Condition (F1) for F implies that $s_k = s_l$ for every k, l. So, $s_k \in \bigcap_{j\in\mathcal{J}} I_j$, for each $k \in \mathcal{J}$. Hence, $x = x_k s_k \in F(\bigcap_{j\in\mathcal{J}} I_j)$. Also, $F(\bigcap_{j\in\mathcal{J}} I_j) \subseteq \bigcap_{j\in\mathcal{J}} FI_j$ is clear. So, the equality in (iv) holds. □

Lemma 2.7. ([8, Lemma 2.8]) *Let A be a right S-act and P a proper subact of A. Then, P is a prime subact of A if and only if for any (right) ideal I of S and any subact B of A, $BI \subseteq P$ implies that $I \subseteq (P : A)$ or $B \subseteq P$. Similarly, an ideal P of a semigroup S is prime if and only if for any ideals I and J of S, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.*

Remark 2.8. Let A be an S-act and B be a subact and P be a prime subact of A. Let I be an ideal of S. If $BI = A$, then $A = BI \subseteq B \subseteq A$ and $A = BI \subseteq AI \subseteq A$. Also, if $BI \neq A$, then by Lemma 2.7, $BI \subseteq P$ if and only if $I \subseteq (P : A)$ or $B \subseteq P$. So,

$$rad(BI) = rad(AI) \cap rad(B), \tag{2.1}$$

and therefore,

$$rad(rad(B)I) = rad(AI) \cap rad(rad(B)) = rad(AI) \cap rad(B) = rad(BI). \tag{2.2}$$

The following proposition is proved in [8, Proposition 2.13] for acts over monoids. We recall it for acts over arbitrary semigroups with more explanations in the proof.

Proposition 2.9. *Let S be a semigroup, $\{B_\gamma\}_{\gamma\in\Gamma}$ be a family of radical subacts of an S-act A, I be any ideal of S and $\{J_\lambda\}_{\lambda\in\Lambda}$ be a finite family of ideals of S such that $\bigcap_{\lambda\in\Lambda} J_\lambda \neq \emptyset$. The following assertions hold.*

(i) $rad(\bigcap_{\gamma\in\Gamma} B_\gamma) = \bigcap_{\gamma\in\Gamma} rad(B_\gamma)$.

(ii) $rad((\bigcap_{\gamma\in\Gamma} B_\gamma)I) = \bigcap_{\gamma\in\Gamma} rad(B_\gamma I)$.

(iii) $rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda)I) = rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda I))$.

(iv) $rad(rad(\bigcup_{\gamma\in\Gamma} B_\gamma)I) = rad(\bigcup_{\gamma\in\Gamma} rad(B_\gamma I))$.

Proof. The proof for (i) is clear.

(ii) Part (i) and equation (2.1) imply that:

$$\begin{aligned} rad((\bigcap_{\gamma\in\Gamma} B_\gamma)I) =& rad(AI) \cap rad(\bigcap_{\gamma\in\Gamma} B_\gamma) = rad(AI) \cap (\bigcap_{\gamma\in\Gamma} rad(B_\gamma)) \\ =& \bigcap_{\gamma\in\Gamma}(rad(AI) \cap rad(B_\gamma)) = \bigcap_{\gamma\in\Gamma} rad(B_\gamma I). \end{aligned}$$

(iii) Suppose that P is a prime subact of A containing $A(\bigcap_{\lambda\in\Lambda} J_\lambda)I$. Then, $(P : A)$ is not empty. So, by Lemma 2.6 (v), $(P : A) = S$ or it is a prime ideal of S. Hence,

$$(P : A) \supseteq (A(\bigcap_{\lambda\in\Lambda} J_\lambda)I : A) \supseteq (\bigcap_{\lambda\in\Lambda} J_\lambda)I.$$

Thus, putting $\Lambda = \{1, 2, \dots, n\}$, for some positive integer n, we get by Lemma 2.7 that $I \subseteq (P : A)$ or $\bigcap_{\lambda=1}^{n} J_\lambda \subseteq (P : A)$. In the latter case, $J_1 J_2 \dots J_n \subseteq \bigcap_{\lambda=1}^{n} J_\lambda$ implies by Lemma 2.7 that $J_\lambda \subseteq (P : A)$, for some $\lambda \in \Lambda$. So, $\bigcap_{\lambda\in\Lambda} J_\lambda I \subseteq (P : A)$, which implies that $A(\bigcap_{\lambda\in\Lambda} J_\lambda I) \subseteq P$. Therefore, by Proposition 2.5 (ii), $rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda)I) \supseteq rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda I))$. Also, clearly $rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda)I) \subseteq rad(A(\bigcap_{\lambda\in\Lambda} J_\lambda I))$. So, we get the result.

(iv) First, note that if P is a prime subact of A containing $\bigcup_{\gamma\in\Gamma} B_\gamma I$ then $P \supseteq rad(B_\gamma I)$, for each $\gamma \in \Gamma$. Hence, $P \supseteq \bigcup_{\gamma\in\Gamma} rad(B_\gamma I)$ which implies that:

$$rad(\bigcup_{\gamma\in\Gamma} rad(B_\gamma I)) \subseteq rad(\bigcup_{\gamma\in\Gamma} B_\gamma I).$$

Since for each $\gamma \in \Gamma$, $B_\gamma I \subseteq rad(B_\gamma I)$, the reverse inclusion is clear. So, we get that:

$$rad(\bigcup_{\gamma\in\Gamma} rad(B_\gamma I)) = rad(\bigcup_{\gamma\in\Gamma} B_\gamma I) = rad((\bigcup_{\gamma\in\Gamma} B_\gamma)I) = rad(rad(\bigcup_{\gamma\in\Gamma} B_\gamma)I),$$

where the last equality follows from equation (2.2). □

Let P be a prime subact of an S-act A and B any subact of A. Then, P is called a *minimal prime over* B if $B \subseteq P$ and there is no prime subact strictly between B and P. By a *non-trivial minimal prime in* A we mean a non-empty prime subact of A which is minimal in the set of all non-empty prime subacts of A, with respect to $\subseteq$. If A is a centered act, any minimal prime subact over $\{\theta\}$ is a minimal prime in A. The minimal prime ideals of S are defined analogously. In the rest of this section, we obtain some results on prime and radical subacts of acts with Condition (F1) as generalizations of the results in [8, Section 2].

Lemma 2.10. *Let I be a proper ideal of a semigroup S, F be an S-act satisfying Condition (F1) and G be a minimal prime subact of F (over FI). Then $(G : F) = S$ if and only if $G = FS$.*

Proof. Let G be a minimal prime subact (over FI) with $(G : F) = S$. Then, $FS \subseteq G$, and hence $FS \neq F$. By Lemma 2.6 (iii), FS is a prime subact of F and $FI \subseteq FS \subseteq G$. The minimality of G implies that $G = FS$. The converse is clear by Proposition 2.6 (ii). □

Lemma 2.11. *Let I be an ideal of a semigroup S and F be a right S-act satisfying Condition (F1). Then, every minimal prime subact G in F with $(G : F) \neq \emptyset$, is of the form $G = FS$ or $G = FP$ for a minimal prime ideal P in S. Moreover, every minimal prime subact G over FI is of the form $G = FS$ or $G = FP$ for a minimal prime ideal P over I.*

Proof. We prove the last statement. The first one is proved by a similar discussion for $\emptyset$ instead of I and FI.

By Lemma 2.10, if $G \neq FS$ then $(G : F) \neq S$. Let $(G : F) = P$. Then, $P \supseteq I$ is a prime ideal of S and $FI \subseteq FP \subseteq G$, by Lemma 2.6 (i), (ii), and (vi). Also, since FP is a prime subact, by Lemma 2.6 (iii), minimality of G over FI implies that $FP = G$. Moreover, if there exists a prime ideal P' with $I \subseteq P' \subset P$ then, by Lemma 2.6 (i), $FI \subseteq FP' \subset FP$, which is in contrast of the minimality of FP. So, P is a minimal prime ideal over I. □

The following theorem discusses an S-act with Condition (F1) which has a finite number of minimal prime subacts.

Theorem 2.12. *Let S be a semigroup and K be a proper ideal of S. Suppose that the product of any finite number of minimal prime ideals over K is finitely generated. If F is an S-act satisfying Condition (F1) then there are finitely many minimal prime subacts of F over FK.*

Proof. First note that, minimal primeness of the subact FS of F does not affect on the finiteness of the number of all prime subacts minimal over FK. So, we show there are finitely many minimal prime subacts $G \neq FS$ over FK in F. By Lemma 2.10, for every minimal prime subacts $G \neq FS$ of F over FK, $(G : F) \neq S$. Assume that F has a minimal prime subact over FK and put

$$\Pi = \{FI | I = P_1 P_2 \dots P_n, \text{ for some } n \in \mathbb{N}, \text{ and } P_i,\ 1 \leq i \leq n, \text{ are minimal prime ideals over } K\}.$$

Consider two possible cases as follows:

Case 1: There exists $FI \in \Pi$ such that $FI \subseteq FK$.

Suppose that $G \neq FS$ is an arbitrary minimal prime subact over FK and $I = P_1 P_2 \dots P_n$ for some $n \in \mathbb{N}$ and some minimal prime ideals P_i over K. We show that $G = FP_i$, for some $i = 1, 2, \dots n$. First, note that by the assumption of Case 1 and Lemma 2.11, $FI \subseteq FK \subseteq G = FP$ for a minimal prime ideal P over K. Hence, $P_1 P_2 \dots P_n = I \subseteq P$, by Lemma 2.6 (i). Thus, by Lemma 2.7, $P_i \subseteq P$, for some i, $1 \leq i \leq n$, and so $P = P_i$, by minimality of P over K. Hence, $G = FP_i$, for some i, $1 \leq i \leq n$. Therefore, the number of minimal prime subacts in F is finite in this case.

Case 2: For every $FI \in \Pi$, $FI \nsubseteq FK$.

We show that, case 2 leads to a contradiction. In this case the set

$$\Sigma = \{FJ \supseteq FK \,|\, J \text{ is an ideal of } S \text{ and } FI \nsubseteq FJ, \text{ for any } FI \in \Pi\}$$

is non-empty, for $FK \in \Sigma$. Now, we show that any chain $\{FJ_\lambda\}_{\lambda \in \Lambda}$ in $(\Sigma, \subseteq)$ has the upper bound, $\bigcup_{\lambda \in \Lambda} FJ_\lambda$ in Σ. On the contrary, let $\bigcup_{\lambda \in \Lambda} FJ_\lambda \supseteq FI$, for some $FI \in \Pi$. It can easily be checked that $\bigcup_{\lambda \in \Lambda} FJ_\lambda = F(\bigcup_{\lambda \in \Lambda} J_\lambda)$, which implies that $I \subseteq \bigcup_{\lambda \in \Lambda} J_\lambda$, by Lemma 2.6 (i). By assumption, I is finitely generated for each $FI \in \Pi$. Thus, the finite generating set for I should be contained in $\bigcup_{\lambda \in \Lambda'} J_\lambda$ for a finite subset Λ' of Λ. Since $\{FJ_\lambda\}_{\lambda \in \Lambda'}$ is a chain, by using Lemma 2.6 (i), $\{J_\lambda\}_{\lambda \in \Lambda'}$ is a chain of ideals in S. Thus, there exists $\alpha \in \Lambda'$ such that $I \subseteq J_\alpha$. This contradicts $FJ_\alpha \in \Sigma$. So $\bigcup_{\lambda \in \Lambda} FJ_\lambda \nsupseteq FI$. Hence, any chain $\{FJ_\lambda\}_{\lambda \in \Lambda}$ stops in $\bigcup_{\lambda \in \Lambda} FJ_\lambda = F(\bigcup_{\lambda \in \Lambda} J_\lambda) \in \Sigma$. Therefore, by Zorn's lemma Σ has a maximal element, say, $FQ \in \Sigma$. Now, we show that Q is a prime ideal of S. Let $MN \subseteq Q$

for some ideals M and N of S. If M and N are not contained in Q then $F(Q \cup M) \supset FQ$ and $F(Q \cup N) \supset FQ$. So, by maximality of FQ in Σ, there exist $FI, FI' \in \Pi$ such that $FI \subseteq F(Q \cup M)$ and $FI' \subseteq F(Q \cup N)$. By Lemma 2.6 (ii) and the assumption $MN \subseteq Q$,

$$F(II') \subseteq F((Q \cup M)(Q \cup N)) \subseteq FQ.$$

On the other hand, since $F(II') \in \Pi$ and $FQ \in \Sigma$, $F(II') \nsubseteq FQ$ which is a contradiction. So, $M \subseteq Q$ or $N \subseteq Q$. Therefore, by Lemma 2.7, Q is a prime ideal which contains K, by Lemma 2.6 (i). Now, we apply Zorn's lemma to show that Q contains a minimal prime ideal over K. Consider the set

$$\mathcal{P} = \{P | K \subseteq P \subseteq Q \text{ is a prime ideal of } S\},$$

partially ordered by $\supseteq$. Then, $Q \in \mathcal{P}$. Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a chain in $\mathcal{P}$, and $aSb \subseteq \bigcap_{\lambda\in\Lambda} P_\lambda$, for some $a \in S \setminus \bigcap_{\lambda\in\Lambda} P_\lambda$ and $b \in S$. So, there exists $\alpha \in \Lambda$ such that $a \notin P_\alpha$. Since P_α is prime, $aSb \subseteq P_\alpha$ implies that $b \in P_\alpha$ (and similarly, $b \in P_\beta$, for each $\beta \in \Lambda$ with $P_\beta \subseteq P_\alpha$). Moreover, clearly $b \in P_\beta$, for any $\beta \in \Lambda$ with $P_\alpha \subseteq P_\beta$. So, $b \in \bigcap_{\lambda\in\Lambda} P_\lambda$, which implies that, $\bigcap_{\lambda\in\Lambda} P_\lambda$ is a prime ideal in $\mathcal{P}$, larger than every P_λ, in the chain $\{P_\lambda\}_{\lambda\in\Lambda}$, with respect to relation $\supseteq$. So, by Zorn's lemma $\mathcal{P}$ has a maximal element, say P, with respect to $\supseteq$. Thus, $K \subseteq P \subseteq Q$ is minimal between prime ideals of S, with respect to $\subseteq$. So, $FP \in \Pi$ and $FQ \supseteq FP$, which contradicts $FQ \in \Sigma$. Hence, Case 2 is not true which completes the proof. □

Proposition 2.13. *Let I be an ideal of a semigroup S and F be a right S-act satisfying Condition (F1). Then, $F\sqrt{I} = rad(FI)$.*

Proof. If there is no prime subact of F containing FI then $FS = F$ and there is no prime ideal of S containing I. Thus, in this case $F\sqrt{I} = FS = F = rad(FI)$. Also, if there is no prime ideal of S containing I then $\sqrt{I} = S$ and $rad(FI) = FS$, and so we get the result. Hence, assume that there exist a prime subact P containing FI and a prime ideal J containing I. Then, $I \subseteq (FI : F) \subseteq (P : F)$. By Lemma 2.6 (v), $(P : F)$ is either S or a prime ideal of S. Thus, $\sqrt{I} \subseteq (P : F)$. Hence, for every prime subact P containing FI, we have $F\sqrt{I} \subseteq P$, which implies that, $F\sqrt{I} \subseteq rad(FI)$. By Lemma 2.6 (iii), $FJ \supseteq FI$ is a prime subact of F. Thus, by Lemma 2.6 (iv) we have,

$$rad(FI) = \bigcap_{FI \subseteq P \in Spec(F)} P \subseteq \bigcap_{I \subseteq J \in Spec(S)} FJ = F(\bigcap_{I \subseteq J \in Spec(S)} J) = F\sqrt{I}.$$

$\square$

Corollary 2.14. *Let I be an ideal of S and F be an S-act satisfying Condition (F1). Then, I is a radical ideal if and only if FI is a radical subact.*

Proof. The proof follows by Proposition 2.13 and Lemma 2.6 (i). $\square$

Recall from [2] that, a semigroup S is *regular*, if for any $s \in S$ there exists $t \in S$ such that $s = sts$.

Theorem 2.15. *Let S be a regular semigroup and F be an S-act satisfying Condition (F1). Then, for every ideal I of S, FI is a radical subact.*

Proof. We show that all ideals of S are radical. Let $x \in S \setminus I$. Put

$$\Sigma = \{K | K \text{ is an ideal of } S, K \supseteq I, x \notin K\}.$$

Then, $I \in \Sigma$. It can easily be checked by the Zorn's lemma that $(\Sigma, \subseteq)$ has a maximal element, say P. We show that P is a prime ideal. Let J and K be two ideals of S and $JK \subseteq P$. Let $s \in J \cap K$. Since S is regular, there exists $t \in S$ such that $s = sts$. Since $st \in J$ and $s \in K$, $s = sts \in JK$. Thus, $J \cap K \subseteq JK \subseteq P$. So, $P = P \cup (J \cap K) = (P \cup J) \cap (P \cup K)$. If $J \nsubseteq P$ and $K \nsubseteq P$ then $P \cup J$ and $P \cup K$ are two ideals properly containing P (and I). Hence, $x \in (P \cup J) \cap (P \cup K) = P$. But this contradicts $P \in \Sigma$. Therefore, $J \subseteq P$ or $K \subseteq P$, which implies by Lemma 2.7 that, P is a prime ideal of S. So, there exists a prime ideal P containing I and $x \notin P$, i.e., $x \notin \bigcap_{P \in V(I)} P = \sqrt{I}$. Therefore, $I = \sqrt{I}$. Now, the proof is completed by Corollary 2.14. $\square$

3 On lattices of radical subacts of free acts

In this section, we will study lattice structures of the collection of radical subacts of an S-act with Condition (F1) (in particular free acts). Some of the results in this section are given in [8] for free acts over monoids. Here, we generalize some results for acts with Condition (F1) over arbitrary semigroups. As in [11], a complete lattice $(Q, \vee, \wedge)$ is called a *quantale* if there is an associative binary operation $\cdot : Q \times Q \longrightarrow Q$ such that for any family $\{q_i\}_{i\in I}$ of elements of Q and for any $q \in Q$,

$$q \cdot \bigvee_{i\in I} q_i = \bigvee_{i\in I}(q \cdot q_i) \text{ and } (\bigvee_{i\in I} q_i) \cdot q = \bigvee_{i\in I}(q_i \cdot q).$$

We start with the following definition which is introduced in [8].

Definition 3.1. Suppose that S is a semigroup, $(\mathcal{I}(S), \cup, \cap)$ is the quantale of all ideals of S with the binary operation $IJ = \{ij | i \in I, j \in J\}$ for every $I, J \in \mathcal{I}(S)$, and $(\mathcal{L}, \vee, \wedge)$ is a lattice of subacts of an S-act F. Then, $\mathcal{L}$ is called a *right* $\mathcal{I}(S)$*−lattice* (S*-lattice*, for short) if $\mathcal{I}(S)$ acts on $\mathcal{L}$ by an operation denoted by "$*$" with the following properties. For any subacts G and H in $\mathcal{L}$ and any ideals I and J of S,

(1) $G * I \in \mathcal{L}$;

(2) $G * (IJ) = (G * I) * J$;

(3) $(G \vee H) * I = (G * I) \vee (H * I)$;

(4) $(G \wedge H) * I = (G * I) \wedge (H * I)$.

Moreover, if we have also

(5) $G * (I \cup J) = (G * I) \vee (G * J)$ and

(6) $G * (I \cap J) = (G * I) \wedge (G * J)$,

we say $\mathcal{L}$ is a *strong right* S*-lattice*.

Also, $\mathcal{L}$ is called a *complete right* S*-lattice* if for any arbitrary family $\{G_\gamma\}_{\gamma\in\Gamma}$ of S-acts in $\mathcal{L}$, meet and join exist in $\mathcal{L}$ and (3) and (4) hold, that is, (3′) $(\bigvee_{\gamma\in\Gamma} G_\gamma) * I = \bigvee_{\gamma\in\Gamma}(G_\gamma * I)$;

(4′) $(\bigwedge_{\gamma\in\Gamma} G_\gamma) * I = \bigwedge_{\gamma\in\Gamma}(G_\gamma * I)$.

By properties (1) and (2) in the definition, every S-lattice $\mathcal{L}$ is a right $\mathcal{I}(S)-$act which gives a representation of semigroup $\mathcal{I}(S)$ as well as of lattice $(\mathcal{I}(S), \cup, \cap)$. Note that a strong S-lattice of unitary S-acts is indeed an $\mathcal{I}(S)-$lattice in the notation of [7].

The set of all radical subacts of an S-act F, partially ordered by inclusion, forms an S-lattice and is denoted by $\mathcal{R}ad(F)$. The following theorem gives more details about $\mathcal{R}ad(F)$.

Theorem 3.2. ([8, Theorem 3.3]) *Suppose that S is a semigroup and F is an S-act. Then, the poset $(\mathcal{R}ad(F), \subseteq)$ of all radical subacts of F is a strong complete S-lattice, with the following operations. For every radical subacts $G, H \in \mathcal{R}ad(F)$ and every ideal I of S,*

$$G * I = rad(GI) \text{ and } G \wedge H = G \cap H \text{ and } G \vee H = rad(G \cup H).$$

By using Theorem 2.15, the following proposition gives an example of a strong sublattice of $\mathcal{R}ad(F)$ for an S-act F with Condition (F1).

Proposition 3.3. *Let S be a regular semigroup and F be an S-act satisfying Condition (F1). Then, $\mathcal{IS}ub(F) = \{FI | I \in \mathcal{I}(S)\}$ is a strong S-sublattice ($\mathcal{I}(S)-$subact) of $\mathcal{S}ub(F)$ and also of $\mathcal{R}ad(F)$.*

Proof. First, it can routinely be observed that $\mathcal{S}ub(F)$ with the natural operations (i.e., union and intersection and ideal multiplication) is an S-lattice if and only if for any subacts $G, H \in \mathcal{S}ub(F)$ and any ideal $I \in \mathcal{I}(S)$, $(G \cap H)I = GI \cap HI$. But this equality holds, for if $x \in GI \cap HI$, there exist $y \in G, z \in H$ and $r, s \in I$, such that $x = yr = zs$. Then, $x \in G \cap H$ and since S is regular, there exists $r' \in S$ such that $r = rr'r$. So $x = y(rr'r) = yr(r'r) = x(r'r) \in (G \cap H)I$. Thus, $GI \cap HI \subseteq (G \cap H)I$. The reverse inclusion is clear. Therefore, $\mathcal{S}ub(F)$ with the natural operations is an S-lattice.

By Theorem 2.15, for any ideals $I, J, K \in \mathcal{I}(S)$ we have $rad(FI) = FI$, which implies that $\mathcal{IS}ub(F) \subseteq \mathcal{R}ad(F)$ and $(FI) * J = rad((FI)J) = F(IJ) \in \mathcal{IS}ub(F)$.

Also, it is clear that $K(I \cup J) = KI \cup KJ$. Furthermore, since S is regular, any $x \in KI \cap KJ$ is of the form xyx for some $y \in S$. Also, $KI \cap KJ \subseteq I \cap J \cap K$. Thus, $x = x(yx) \in K(I \cap J)$. Hence, $K(I \cap J) = KI \cap KJ$. Now, it can easily be observed that $(FK) * (I \cup J) = ((FK) * I) \vee ((FK) * J)$ and $(FK) * (I \cap J) = ((FK) * I) \wedge ((FK) * J)$.

Moreover, by Lemma 2.6 (iv), $FI \cap FJ = F(I \cap J) \in \mathcal{ISub}(F)$. Also, clearly $FI \cup FJ = F(I \cup J) \in \mathcal{ISub}(F)$. Hence, in $\mathcal{R}ad(F)$,

$$FI \vee FJ = rad(FI \cup FJ) = rad(F(I \cup J)) = F(I \cup J) = FI \cup FJ,$$

that is, the join of subacts in $\mathcal{ISub}(F)$ coincides with the join of radical subacts in $\mathcal{R}ad(F)$. Furthermore,

$$(FI \vee FJ) * K = ((FI) * K) \vee ((FJ) * K)$$

and

$$(FI \wedge FJ) * K = ((FI) * K) \wedge ((FJ) * K).$$

Therefore, (1)-(6) in Definition 3.1 hold for $\mathcal{ISub}(F)$ as a subset of $\mathcal{R}ad(F)$ and $\mathcal{S}ub(F)$. □

In the following proposition we use multiplications of ideals of S by an S-act F with Condition (F1) to define a *lattice of radical (multiplication) subacts*, generalizing $\mathcal{ISub}(F)$ in Proposition 3.3 for arbitrary semigroups. In what follows, $\mathcal{I}(S) \cup \{\emptyset\}$ is denoted by $\overline{\mathcal{I}(S)}$.

Proposition 3.4. *Let F be an S-act with Condition (F1) and $(\mathcal{IR}ad(F), \subseteq)$ be the poset of all radical subacts of F, of the form $rad(FI)$, where $I \in \overline{\mathcal{I}(S)}$. Then, $\mathcal{IR}ad(F)$ is a bounded distributive strong S-lattice with the following operations and bounds. For any $I, J \in \overline{\mathcal{I}(S)}$ and any ideal K of S,*

$$rad(FI) * K = rad((FI)K) \text{ and}$$

$$rad(FI) \wedge rad(FJ) = rad(F(I \cap J)) \text{ and } rad(FI) \vee rad(FJ) = rad(F(I \cup J))$$

$$\text{and } \mathbf{1}_{\mathcal{IR}ad(F)} = rad(FS) = FS \text{ and } \mathbf{0}_{\mathcal{IR}ad(F)} = \emptyset.$$

Proof. The proof can routinely be obtained by noting that for any ideals I, J and K of S, $rad(F(I \cap J)K) = rad(F(IK \cap JK))$, by Proposition 2.9 (iii). □

Let $\mathcal{L}$ and $\mathcal{L}'$ be two S-lattices (resp. complete S-lattices) of S-acts. A mapping $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is called a homomorphism (resp. a complete homomorphism) if it preserves finite (resp. arbitrary) meets and joins and also operations $*$ of ideals of S on S-acts.

Proposition 3.5. *Every two isomorphic S-acts with Condition (F1) have the isomorphic lattices of radical subacts.*

Proof. The proof is routine and is omitted. □

Corollary 3.6. *Let S be a semigroup with a zero. Assume that F and F' are two S-acts satisfying Condition (F1) such that $\mathcal{R}ad(F)$ and $\mathcal{R}ad(F')$ are isomorphic S-lattices. Then for any arbitrary prime ideal P of S, there exists a one-to-one correspondence between $P-$prime subacts of F and F'.*

Proof. Let $f : \mathcal{R}ad(F) \longrightarrow \mathcal{R}ad(F')$ be a lattice isomorphism of $\mathcal{R}ad(F)$ and $\mathcal{R}ad(F')$ and π be the restriction of f to $\mathcal{S}pec_P(F)$, i.e., $\pi(B) = f(B)$ for each prime subact B of F with $(B : F) = P$. It suffices to show that π is a mapping onto $\mathcal{S}pec_P(F')$.

If $F' = \pi(B)$, for a $P-$prime subact B of F then,

$$F' = \pi(B) = f(B) = f(B \wedge F) = f(B) \wedge f(F) \subseteq f(F) \subseteq F',$$

i.e., $f(B) = f(F) = F'$. Hence, $B = F$, for f is an isomorphism. It contradicts the assumption that B is prime. Thus, for any $P-$prime subact B of F, $\pi(B)$ is a proper subact of F'. A similar argument shows that $f(F) = F'$.

Let $G'I \subseteq \pi(B) = f(B)$, for a subact G' of F' and an ideal I of S. Assume that $I \nsubseteq (\pi(B) : F')$. We claim that $G' \subseteq \pi(B)$. First, note that $rad(G'I) \subseteq rad(\pi(B)) = \pi(B)$. Thus, by (2.2) in Remark 2.8, $rad(rad(G')I) \subseteq \pi(B)$. Let $G \in \mathcal{R}ad(F)$ for which $f(G) = rad(G') \in \mathcal{R}ad(F')$. Therefore,

$$rad(rad(G')I) = rad(f(G)I) = f(G) * I = f(G * I) = f(rad(GI)).$$

So, $f(rad(GI)) = rad(rad(G')I) \subseteq \pi(B) = f(B)$. Since f preserves finite meets (and order of elements) in $\mathcal{R}ad(F)$, it can easily be checked that $GI \subseteq rad(GI) \subseteq B$. Hence, by Lemma 2.7, $G \subseteq B$ or $I \subseteq (B : F) = P$. The first case proves the claim, as one can easily observe that $G' \subseteq rad(G') = f(G) \subseteq f(B) = \pi(B)$. However, $I \subseteq (B : F) = P$ leads to a contradiction. Because it follows that $rad(FI) \subseteq B$ and $rad(FI) \wedge B = rad(FI)$. So, $f(rad(FI)) \wedge f(B) = f(rad(FI) \wedge B) = f(rad(FI))$. Also, $F' = f(F)$, as we explained in the first part of the proof. Thus,

$$F'I = f(F)I \subseteq rad(f(F)I) = f(F) * I = f(F * I) =$$

$$f(rad(FI)) = f(rad(FI)) \wedge f(B) \subseteq f(B) = \pi(B).$$

Therefore, contrary to the assumption of $I \nsubseteq (\pi(B) : F')$, we obtain $F'I \subseteq \pi(B)$. Thus, $\pi(B)$ is prime. Now, we show that $(\pi(B) : F') = P$, i.e., $\pi(B) \in \mathcal{S}pec_P(F')$. By assumption of Condition (F1) for F and Lemma 2.6 (iii) we have, $FP \subseteq B$ and $FP \in \mathcal{S}pec_P(F)$ and $F'P \in \mathcal{S}pec_P(F')$. So, $F'P = F' * P = f(F) * P = f(F * P) = f(FP) \subseteq f(B) = \pi(B)$, for f is a lattice isomorphism preserving order of $\mathcal{R}ad(F)$ (inclusion). Thus, $P \subseteq (\pi(B) : F')$. Moreover, since $f(B)$ is prime, $F' = f(F) \nsubseteq f(B)$. If $(\pi(B) : F') = J$, then $F'J \subseteq \pi(B) = f(B)$ and we have,

$$f(rad(FJ)) = f(F * J) = f(F) * J = F' * J = rad(F'J) \subseteq f(B).$$

So, $FJ \subseteq rad(FJ) \subseteq B$, for f is an isomorphism. Hence, $J \subseteq (B : F) = P$. Thus, $(\pi(B) : F') = P$.

Now, we show that π is onto. Let $G' \in \mathcal{S}pec_P(F')$. Since f is an isomorphism, there exists $F \neq G \in \mathcal{R}ad(F)$ for which $f(G) = G'$. We prove that $G \in \mathcal{S}pec_P(F)$, which implies that $\pi(G) = f(G) = G'$. The proof is similar to the previous part. First, since $F'P \subseteq G'$ and f preserves inclusion,

$$f(rad(FP)) = f(F * P) = f(F) * P = F' * P = rad(F'P) \subseteq G' = f(G).$$

So, it can easily be observed that, $rad(FP) \subseteq G$, and hence, $FP \subseteq G$, i.e., $P \subseteq (G : F)$. Let $HI \subseteq G$ for a subact H of F and an ideal I of S. If for any such ideal I, $I \subseteq P \subseteq (G : F)$, then by Lemma 2.7, G is prime. Assume that for such an ideal I, $I \nsubseteq P = (G' : F')$. We have $rad(HI) \subseteq rad(G) = G$. Then, (2.2) in Remark 2.8 implies that,

$$f(rad(H))I \subseteq f(rad(H)) * I = f(rad(H) * I) = f(rad(rad(H)I))$$

$$= f(rad(HI)) \subseteq f(G) = G'.$$

Thus, $f(rad(H))I \subseteq f(G) = G'$. But, $G' \in \mathcal{S}pec_P(F')$ and $I \not\subseteq P = (G' : F')$. So, by Lemma 2.7, $f(rad(H)) \subseteq G' = f(G)$. Thus, $H \subseteq rad(H) \subseteq G$. Hence, G is also prime in this case. Furthermore,

$$F'(G : F) \subseteq rad(F'(G : F)) = F' * (G : F) = f(F) * (G : F) =$$

$$f(F * (G : F)) = f(rad(F(G : F))) \subseteq f(rad(G)) = f(G) = G'.$$

Hence, $F'(G : F) \subseteq G'$. Since F' satisfies Condition (F1), by Lemma 2.6 (ii) we have, $(G : F) = (F'(G : F) : F') \subseteq (G' : F') = P$. So, $G \in \mathcal{S}pec_P(F)$, and the proof is completed. □

The following result shows that non-isomorphic acts with Condition (F1) (in particular free acts) may have the isomorphic lattices of radical subacts.

Theorem 3.7. *For any two (possibly non-isomorphic) S-acts F and F' with Condition (F1), $\mathcal{IR}ad(F)$ and $\mathcal{IR}ad(F')$ are isomorphic S-lattices. In particular, if S is a commutative semigroup, $\mathcal{IR}ad(F) \cong \mathcal{R}ad(S)$, for any S-act F with Condition (F1).*

Proof. Define $f : \mathcal{IR}ad(F) \longrightarrow \mathcal{IR}ad(F')$ by $f(rad(FI)) = rad(F'I)$, where $I \in \overline{\mathcal{I}(S)}$. By Proposition 2.13 and Lemma 2.6, f is a well-defined one-to-one mapping. Also, clearly f is onto and for any ideal J of S,

$$f(rad(FI) * J) = f(rad(FIJ)) = rad(F'IJ) = rad(F'I) * J$$

and

$$f(rad(FI) \wedge rad(FJ)) = f(rad(F(I \cap J))) = rad(F'(I \cap J))$$

$$= rad(F'I) \wedge rad(F'J) = f(rad(FI)) \wedge f(rad(FJ)).$$

Similar assertion shows that f preserves joins and so f is an isomorphism of S-lattices. The last statement can be verified by a similar argument to that of above, by noting that for any ideal I of a commutative semigroup S, $rad(SI) = rad(I) = \sqrt{I}$. □

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