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General Algebraic Structures
with Applications

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Cancel Culture

The Search for Universally Cancellable Exponents of Posets

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Abstract. Let A, B, C, and D be posets. Assume C and D are finite with a greatest element. Also assume that $A^C \cong B^D$.

Then there exist posets E, X, Y, and Z such that $A \cong E^X$, $B \cong E^Y$, $C \cong Y \times Z$, and $D \cong X \times Z$. If $C \cong D$, then $A \cong B$.

This generalizes a theorem of Jónsson and McKenzie, who proved it when A and B were meet-semilattices.

1 Background

Bergman, McKenzie, and Nagy [1], building on the work of Jónsson, found the first known class of universally cancellable exponents—non-empty chains—meaning that if A and B are posets and C a non-empty chain, then $A^C \cong B^C$ implies $A \cong B$. Here, E^X (E, X posets) is the set of order-preserving maps from X to E, partially ordered pointwise: If $f, g \in E^X$, then $f \leq g$ means $f(x) \leq g(x)$ for all $x \in X$. See Figures 1 and 2 [7, page 54].

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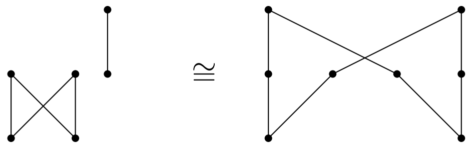

Figure 1.1. E^X *where* E *is the 4-element crown and* X *the 2-element chain*

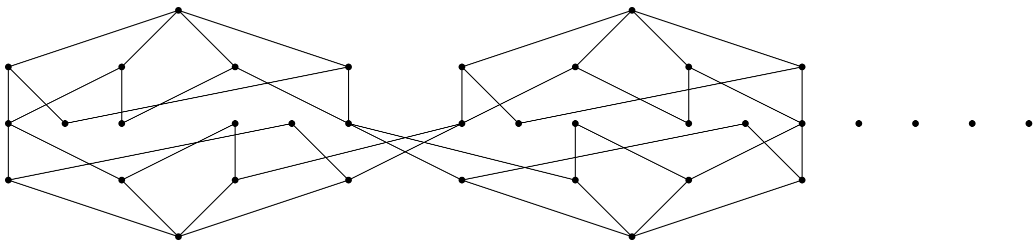

Figure 1.2. P^P *where* P *is the 4-element crown*

Professor Garrett Birkhoff conjectured—and McKenzie proved—that if A, B, and C are non-empty *finite* posets, then $A^C \cong B^C$ implies $A \cong B$, but this is not a *universal* cancellation result [2, p. 300], [13], [14]. Besides non-empty chains, no other universally cancellable exponent has been found—until now.

Every finite poset with a least or greatest element is universally cancellable as an exponent (Theorem 5 of Section 5, below).

Not every poset is universally cancellable: $\mathbf{2}^{\aleph_0} \cong (\mathbf{2}^{\aleph_0})^{\aleph_0}$ (since $E^{X\times Q} \cong (E^X)^Q$ [4, Exercise 1.26]), but $\mathbf{2} \ncong \mathbf{2}^{\aleph_0}$.

What is amusing is that our proof (inspired by Krebs and van der Zypen [11]) is more or less the same as Jónsson and McKenzie's proof of the following result: If A and B are $\wedge$-semilattices and C a finite poset with a greatest element, then $A^C \cong B^C$ implies $A \cong B$ (part of [10, Theorem 5.4]). They used an idea of Duffus, Jónsson, and Rival [8, Theorem (ii)], and an idea of Dilworth and Freese, to take repeatedly the poset of filters of a $\wedge$-semilattice and then use a limiting construction [5, page 264]. What we mean above by "same" is that the same idea works to prove our universal cancellation result, although we use ideas about algebraic posets of Erné [9].

2 Definition, notation, and terminology

Notation and definitions can be found in [4]. We denote the covering relation in a poset by "$\lessdot$."

Let A, B, P and Q be posets; P^∂ is the dual of P; P^Q was defined in Section 1; for $p \in P$, $\langle p \rangle$ is the constant map, $q \mapsto p$ (for all $q \in Q$).

Given $f \in B^A$, define a map f^Q from A^Q to B^Q as follows: for $g \in A^Q$, $f^Q(g) = f \circ g$.

If $Q \subseteq P$, ${\downarrow}Q$ is $\{p \in P \mid p \leq q \text{ for some } q \in Q\}$. Also, $Q \subseteq P$ is a *down-set* if ${\downarrow}Q = Q$. Dually, we define ${\uparrow}Q$; $\bigcap\{{\uparrow}q \mid q \in Q\}$ is denoted Q^u.

A subset $D \subseteq P$ is *directed* if $D \neq \emptyset$ and for all $d, d' \in D$, there exists $d'' \in D$ such that $d, d' \leq d''$. An *ideal* is a directed down-set. The poset of ideals of P ordered by inclusion is denoted P^σ.

If a directed subset D has a supremum $\bigvee D$, we sometimes denote it $\bigsqcup D$; the point is that the symbol $\bigsqcup$ in front of D indicates that D is directed. A function $f : P \to Q$ is *Scott-continuous* if whenever a directed subset $D \subseteq P$ has a supremum, so does $f[D]$ and $f(\bigsqcup D) = \bigvee f[D]$.

An element $k \in P$ is *compact* if, whenever $D \subseteq P$ is directed with supremum $\bigsqcup D$, $k \leq \bigsqcup D$ implies $k \leq d$ for some $d \in D$. The poset of compact elements of P is denoted $\kappa(P)$.

A poset is *algebraic* if every directed subset has a supremum and every element is the supremum of a directed subset of compact elements. An archetypal example is P^σ. In this example, $\kappa(P^\sigma) = \{{\downarrow}p \mid p \in P\} \cong P$ and the supremum of a directed subset of ideals of P is its union. Indeed, this is up to isomorphism the only example of an algebraic poset [9, Proposition 3 and Corollary 2].

Let **Poset** be the category of posets together with order-preserving maps. Let $(C_i)_{i \in \mathbb{N}_0}$ be a family of posets, where $\mathbb{N}_0$ is the set of non-negative integers, and let

$$(f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \le i}}$$

be a family of order-preserving maps with the following properties:

(1) $f_{ii} = \mathrm{id}_{C_i}$ for all $i \in \mathbb{N}_0$;

(2) $f_{ij} \circ f_{jk} = f_{ik}$ for all $i, j, k \in \mathbb{N}_0$ such that $k \le j \le i$.

Then

$$\mathcal{S} = \big((C_i)_{i \in \mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \le i}}\big)$$

is a *filtered system* in **Poset**.

Let $\mathcal{S}^Q$ be

$$\mathcal{S}^Q = \big((C_i^Q)_{i \in \mathbb{N}_0}, (f_{ij}^Q : C_j^Q \to C_i^Q)_{\substack{i,j \in \mathbb{N}_0 \\ j \le i}}\big).$$

Assume C is a poset and $(f_i : C_i \to C)_{i \in \mathbb{N}_0}$ is a family of order-preserving maps such that, for $i, j \in \mathbb{N}_0$, $j \le i$ implies $f_i \circ f_{ij} = f_j$. Then

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

is *compatible* with $\mathcal{S}$. It is a *filtered limit* of $\mathcal{S}$ if we assume further that, whenever $\big(C', (f_i' : C_i \to C')_{i \in \mathbb{N}_0}\big)$ is compatible with $\mathcal{S}$, there is a unique order-preserving map $f : C \to C'$ such that $f \circ f_i = f_i'$ for all $i \in \mathbb{N}_0$. As usual, the object C in a filtered limit is unique up to order-isomorphism. A reference for directed sets and limits is [18].

The following notation comes from [10, Section 3]. Let A, B, C, D be posets such that $C, D \ne \emptyset$. Let $\phi : A^C \cong B^D$.

$$\Delta(\phi) = \{f \in A^C \mid \phi(f) \text{ is a constant map}\}$$

$$R(\phi) = \{x \in A \mid \langle x \rangle \in \Delta(\phi)\}$$

A relation $\le_\phi$ is defined on $R(\phi)$ as follows: $x \le_\phi y$ if and only if

(1) $x, y \in R(\phi)$;

(2) $x \le y$ in A;
(3) if $f \in A^C$ and $f[C] \subseteq \{x, y\}$, then $f \in \Delta(\phi)$

Define a map $\mathring{\phi}$ from $R(\phi)$ to $R(\phi^{-1})$ as follows: for $x \in R(\phi)$, $\mathring{\phi}(x)$ is the element y such that $\phi(\langle x\rangle) = \langle y\rangle$.

By "$(\phi, 1)$," we mean: $\le_\phi$ is a partial ordering on $R(\phi)$.

By "$(\phi, 2)$," we mean: $\mathring{\phi}$ is a relation-preserving bijection from

$$\big(R(\phi), \le_\phi\big)$$

to

$$\big(R(\phi^{-1}), \le_{\phi^{-1}}\big)$$

and its inverse is relation-preserving.

By "$(\phi, 3)$," we mean that $\Delta(\phi)$ is the set of all relation-preserving maps from $(C, \le)$ to $\big(R(\phi), \le_\phi\big)$.

By "$(\phi, 4)$," we mean that $\le_\phi = \{(x, y) \in R(\phi) \times R(\phi) \mid x \le y\}$.

If $\alpha : A \cong A'$ and $\beta : B \cong B'$ (A', B' posets), then

$$\phi'(f') = \beta \circ \phi(\alpha^{-1} \circ f')$$

gives a map $\phi' : {A'}^C \to {B'}^D$. For $i \in \{1, 2, 3, 4\}$, (ϕ, i) implies (ϕ', i).

Below are results from [10] that we use. (Jónsson and McKenzie proved more than what we list.)

Theorem 2.1. (From [10, Theorem 3.2].) *If $C = D$ and $(\phi, 1)$, $(\phi, 2)$, $(\phi, 3)$, and $(\phi^{-1}, 3)$, then $A \cong B$.*

Theorem 2.2. (From [10, Theorem 3.3].) *If $(\phi, 1)$, $(\phi, 2)$, $(\phi, 3)$, $(\phi^{-1}, 3)$, and $(\phi, 4)$, then there exists a poset E such that $A \cong E^D$ and $B \cong E^C$.*

Lemma 2.3. (From [10, Lemma 4.1].) *If D has a top or bottom element, then $(\phi, 1)$.*

We do not need to know what Jónsson and McKenzie mean by "Property (a)" [10, Definition 4.2]; we only need the following:

Lemma 2.4. (From [10, Corollary 4.3].) *If C has a top element, then A^C has Property* (a)*.*

Lemma 2.5. (From [10, Lemma 4.5].) *If A^C and B^D have Property* (a)*, then $(\phi, 2)$.*

Jónsson and McKenzie call a poset *atomic* if, whenever $p < q$ in P, there exists $r \in P$ such that $p \lessdot r \leq q$ [10, page 92]. We call a poset *dually atomic* if its dual is atomic. A poset P is *directly irreducible* if $|P| \neq 1$ and whenever $P \cong E \times X$ we have $|E| = 1$ or $|X| = 1$.

Lemma 2.6. (From [10, Lemma 4.10].) *If A and B are atomic posets and C and D are finite directly irreducible posets with a greatest element, then $(\phi, 3)$.*

Lemma 2.7. (From [10, Lemma 4.8].) *If A is an atomic poset, if C and D have a top element and are directly irreducible, if B^D has Property* (a)*, and if $C \not\cong D$, then $(\phi, 4)$.*

Lemma 2.8. (From [10, Lemma 5.3].) *Let A' and B' be posets. Assume $A \subseteq A'$ and $B \subseteq B'$. Assume $\phi : A^C \cong B^D$ and $\psi : A'^C \cong B'^D$. Assume $\phi \subseteq \psi$.*

Then for $i \in \{1, 2, 3, 4\}$, (ψ, i) implies (ϕ, i).

3 Powers of algebraic posets and algebraic powers of posets

The main result of this section is that $(P^Q)^\sigma \cong (P^\sigma)^Q$ when P and Q are posets and Q is finite (Corollary 3). Duffus, Jónsson, and Rival had proven a special case of this theorem for P a lattice (so an "ideal" is the usual lattice ideal) [8, Theorem (ii)], with Jónsson and McKenzie noting that (a dual version of) this result held for semilattices [10, page 103].

We start with a familiar result (see [9, page 74], [19, Theorem 2.8], and [4, Exercise 9.6]):

Lemma 3.1. *Let A be an algebraic poset. Let Q be a poset such that every directed subset has a supremum. Let $f \in Q^{\kappa(A)}$.*

Then there exists a unique Scott-continuous map $F : A \to Q$ such that $F \restriction_{\kappa(A)} = f$. It is order-preserving.

Proof. Let $a \in A$. Let $D \subseteq \kappa(A)$ be any directed subset such that $\bigsqcup D = a$. Note that since f is order-preserving, $f[D]$ is directed, so $\bigsqcup f[D]$ exists by assumption. Define $F(a)$ to be $\bigsqcup f[D]$. (We are forced to make this definition, proving uniqueness.) We show that F is well-defined: If $D_1, D_2 \subseteq \kappa(A)$ are directed and $\bigsqcup D_1 = \bigsqcup D_2$, then ${\downarrow}D_1 = {\downarrow}D_2$. Thus for any $d_1 \in D_1$, there exists $d_2 \in D_2$ such that $d_1 \leq d_2$, so $f(d_1) \leq f(d_2)$ and hence $f(d_1) \leq \bigsqcup f[D_2]$. Therefore $\bigsqcup f[D_1] \leq \bigsqcup f[D_2]$ and by symmetry $\bigsqcup f[D_1] = \bigsqcup f[D_2]$.

Clearly $F(k) = f(k)$ for $k \in \kappa(A)$: let $D = \{k\}$. Note that F is order-preserving.

We show that F is Scott-continuous. Assume $C \subseteq A$ is directed. Then $F[C]$ is directed, so $\bigsqcup F[C]$ exists.

For all $c \in C$, let $D_c \subseteq \kappa(A)$ be a directed subset such that $\bigsqcup D_c = c$.

Claim. The set $D := \bigcup_{c \in C} D_c$ *is directed.*

Proof of claim. Let $c, c' \in C$ and let $d \in D_c$ and let $d' \in D_{c'}$. Then there exists $c'' \in C$ such that $c, c' \leq c''$. Since $d \leq \bigsqcup D_{c''}$, there exists $e \in D_{c''}$ such that $d \leq e$. Similarly, there exists $e' \in D_{c''}$ such that $d' \leq e'$.

Let $e'' \in D_{c''}$ be such that $e, e' \leq e''$. Then $d, d' \leq e'' \in D$.

Hence $\bigsqcup D = \bigvee_{c \in C}(\bigsqcup D_c) = \bigvee C$, so $F(\bigvee C) = \bigsqcup f[D] = \bigvee_{c \in C}(\bigvee f[D_c]) = \bigvee_{c \in C} F(c)$.

Thus F is Scott-continuous. □

Proposition 3.2. *Let A and A' be algebraic posets. Let $\phi : \kappa(A) \to \kappa(A')$ be an order-isomorphism. Then there exists a unique Scott-continuous map $\Phi : A \to A'$ such that, for all $k \in \kappa(A)$, $\Phi(k) = \phi(k)$. The map Φ is an order-isomorphism.*

Proof. Let Φ be the map of Lemma 3.1. Let $\Phi' : A' \to A$ be the map of Lemma 3.1 given by ϕ^{-1}. Then, for all $k \in \kappa(A)$, $(\Phi' \circ \Phi)(k) = k$, so, by Lemma 3.1, $\Phi' \circ \Phi = \mathrm{id}_A$. By symmetry, $\Phi \circ \Phi' = \mathrm{id}_{A'}$. □

Corollary 3.3. *Let P and Q be posets; assume Q is finite. For $f \in P^Q$ define a map $f^\sigma : Q \to P^\sigma$ as follows: for all $q \in Q$, $f^\sigma(q) = {\downarrow}f(q)$.*

Then there is a unique order-isomorphism from $(P^Q)^\sigma$ to $(P^\sigma)^Q$ sending $\downarrow f$ to f^σ for all $f \in P^Q$. In particular, $(P^\sigma)^Q$ is an algebraic poset and P^Q is order-isomorphic to $\kappa[(P^\sigma)^Q]$.

Proof. First, we show that directed subsets have suprema in $(P^\sigma)^Q$. Let $D \subseteq (P^\sigma)^Q$ be directed. For all $q \in Q$, let $D_q = \{d(q) \mid d \in D\}$; it is a directed subset of P^σ. Thus $\bigsqcup D_q$ exists in P^σ; we define $e(q)$ to be $\bigsqcup D_q$. The map $e \in (P^\sigma)^Q$ and $e \in D^u$. If $f \in D^u$, then $e \leq f$. Hence $e = \bigsqcup D$.

Claim 1. Let $f \in P^Q$. Then $f^\sigma \in \kappa[(P^\sigma)^Q]$.

Proof of claim. Let $D \subseteq (P^\sigma)^Q$ be directed and assume $f^\sigma \leq \bigsqcup D$. Then, for all $q \in Q$, $f^\sigma(q) = \downarrow f(q) \subseteq \bigsqcup D_q$, so there is a $d_q \in D$ such that $\downarrow f(q) \subseteq d_q(q)$. Since Q is finite, there exists a $g \in \{d_q \mid q \in Q\}^u \cap D$. Thus $f^\sigma \leq g$.

Let $G \in (P^\sigma)^Q$. Consider

$$\mathcal{H} = \{g^\sigma \mid g \in P^Q \text{ and } g(q) \in G(q) \text{ for all } q \in Q\}.$$

Claim 2. The set $\mathcal{H}$ is directed. Indeed, for all $q \in Q$, let $d_q \in G(q)$. There exists $g \in P^Q$ such that $d_q \leq g(q)$ for all $q \in Q$ and $g^\sigma \in \mathcal{H}$.

Proof of claim. For $r \in Q$, we define $g(r)$ by induction on $|\downarrow r|$. If $|\downarrow r| = 1$, let $g(r)$ be d_r. Now assume that $|\downarrow r| > 1$ and that $g(s)$ has been defined for all $s < r$ so that $g(s) \in G(s)$ and $d_s \leq g(s)$. As $G(r) \supseteq G(s)$ for all $s < r$, we know $g(s) \in G(r)$ for all $s < r$. Thus there exists $g(r) \in G(r)$ such that $g(s) \leq g(r)$ for all $s < r$ and $d_r \leq g(r)$. We conclude that $g \in P^Q$, and $g^\sigma \in \mathcal{H}$, and $d_q \leq g(q)$ for all $q \in Q$. Also, $\mathcal{H} \neq \emptyset$.

Now let $h, k \in P^Q$ be such that $h^\sigma, k^\sigma \in \mathcal{H}$. For $q \in Q$, pick $d_q \in G(q) \cap \{h(q), k(q)\}^u$. By the previous paragraph, there exists $g \in P^Q$ such that $g^\sigma \in \mathcal{H}$ and $d_q \leq g(q)$ for all $q \in Q$, so $h^\sigma, k^\sigma \leq g^\sigma$.

Clearly $\bigsqcup \mathcal{H} \leq G$, but, in fact, for all $q \in Q$

$$(\bigsqcup \mathcal{H})(q) = \bigvee_{\substack{g \in P^Q \\ g^\sigma \in \mathcal{H}}} \downarrow g(q) = \bigcup_{\substack{g \in P^Q \\ g^\sigma \in \mathcal{H}}} \downarrow g(q).$$

By Claim 2, for all $q \in Q$, $G(q) \subseteq (\bigsqcup \mathcal{H})(q)$, and thus $G \leq \bigsqcup \mathcal{H}$. Hence $\bigsqcup \mathcal{H} = G$.

This proves that $(P^\sigma)^Q$ is an algebraic poset and that $\kappa[(P^\sigma)^Q] = \{g^\sigma \mid g \in P^Q\}$. □

One does not simply dispense with the condition Q is finite, as the next two examples show.

Example 3.4. Let $P = \mathbb{N}_0 = Q$ with the natural order. Then there is no order-isomorphism from $(P^Q)^\sigma$ to $(P^\sigma)^Q$ sending, for each $f \in P^Q$, $\downarrow f$ to f^σ.

Consider $\mathrm{id}_{\mathbb{N}_0} \in P^Q$. Then $(\mathrm{id}_{\mathbb{N}_0})^\sigma$ should be compact.

For all $n \in \mathbb{N}_0$, let $g_n : Q \to P$ be the map $g_n(m) = \min\{m, n\}$ ($m \in \mathbb{N}_0$); then $g_n \in P^Q$ and $g_0^\sigma \leq g_1^\sigma \leq g_2^\sigma \leq \cdots$.

Claim. We have that $\bigsqcup_{n \in \mathbb{N}_0} g_n^\sigma = (\mathrm{id}_{\mathbb{N}_0})^\sigma$.

Proof of claim. Let $m, n \in \mathbb{N}_0$. If $m \leq n$, then $g_n^\sigma(m) = \downarrow m = (\mathrm{id}_{\mathbb{N}_0})^\sigma(m)$. If $n < m$, then $g_n^\sigma(m) = \downarrow n \subseteq \downarrow m = (\mathrm{id}_{\mathbb{N}_0})^\sigma(m)$.

Let $h \in (P^\sigma)^Q$ be such that $h \in \{g_n^\sigma \mid n \in \mathbb{N}_0\}^u$ in $(P^\sigma)^Q$. Then for all $n \in \mathbb{N}_0$, $g_n^\sigma(n) = \downarrow n \subseteq h(n)$, so $(\mathrm{id}_{\mathbb{N}_0})^\sigma(n) \subseteq h(n)$.

But if $(\mathrm{id}_{\mathbb{N}_0})^\sigma$ were compact, it would equal g_n^σ for some $n \in \mathbb{N}_0$.

Example 3.5. Let D be a directed set and Q a poset such that D^Q is not directed. (See [12], an example of someone with the username Emil Jeřábek.) Then $(D^\sigma)^Q \not\cong (D^Q)^\sigma$.

Indeed, $(D^\sigma)^Q$ has a top element.

Claim. The poset $(D^Q)^\sigma$ *does not have a top element.*

Proof of claim. Let $f, g \in D^Q$ be such that $\{f, g\}^u = \emptyset$ in D^Q. Then $\downarrow f, \downarrow g \in (D^Q)^\sigma$. Assume for a contradiction that $(D^Q)^\sigma$ has a top element K. Then $\downarrow f, \downarrow g \subseteq K$. Hence $f, g \in K$. Therefore there is $k \in K$ such that $f, g \leq k$, a contradiction.

4 Filtered limits of powers of posets of ideals

In order to get our universal cancellation result, we are going to use a result of Jónsson and McKenzie for atomic posets. We get dually atomic posets by taking a poset P, successively forming posets of ideals P^σ, $P^{\sigma\sigma}$, $P^{\sigma\sigma\sigma}$, $\ldots$, and taking a limit. If we start with a power poset P^Q, we would like the limit of P^Q, $(P^\sigma)^Q$, $(P^{\sigma\sigma})^Q$, $(P^{\sigma\sigma\sigma})^Q$, $\ldots$ to be the limit of P^Q, $(P^Q)^\sigma$, $(P^Q)^{\sigma\sigma}$, $(P^Q)^{\sigma\sigma\sigma}$, $\ldots$.

It is possible that a theorem from category theory (e.g., in a place like [3, Section 2.13] or [15]) implies our results, but the author would ask any reader who knows of such a theorem to confirm that verifying the theorem applies would take considerably less work than the proof we present below.

We give a concrete construction of a filtered limit and then show that any filtered limit would have the same properties.

Proposition 4.1. *Let*

$$\mathcal{S} = \big((C_i)_{i\in\mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j\in\mathbb{N}_0\\ j\leq i}}\big)$$

be a filtered system in **Poset***. Define a relation* $\leq$ *on* $\bigcup_{i\in\mathbb{N}_0}(C_i \times \{i\})$ *as follows: Let* $i, j \in \mathbb{N}_0$*. Let* $c_i \in C_i, c_j \in C_j$*. We say* $(c_i, i) \leq (c_j, j)$ *if there exists* $h \in \mathbb{N}_0$ *such that* $i, j \leq h$ *and* $f_{hi}(c_i) \leq f_{hj}(c_j)$ *in* C_h*.*

Then $\leq$ *is a preorder. Denote the equivalence class of* (c_i, i) *by* $[(c_i, i)]$ *and let* C *be the quotient poset.*

Let $h, i \in \mathbb{N}_0$ *be such that* $i \leq h$*. Let* $c_i \in C_i$*. Then* $[(c_i, i)] = [(f_{hi}(c_i), h)]$*.*

For all $i \in \mathbb{N}_0$*, define* $f_i : C_i \to C$ *by* $c_i \mapsto [(c_i, i)]$ $(c_i \in C_i)$*; then* f_i *is order-preserving.*

Finally,

$$\big(C, (f_i : C_i \to C)_{i\in\mathbb{N}_0}\big)$$

is a filtered limit of $\mathcal{S}$*.*

Proof. Let $i \in \mathbb{N}_0$, $c_i \in C_i$. Then $i \leq i$ and $f_{ii}(c_i) = c_i$, so $(c_i, i) \leq (c_i, i)$. Now let $j, k \in \mathbb{N}_0$, $c_j \in C_j$, $c_k \in C_k$. Assume $(c_i, i) \leq (c_j, j)$ and $(c_j, j) \leq (c_k, k)$. Then there exist $h', h'' \in \mathbb{N}_0$ such that $i, j \leq h'$ and $j, k \leq h''$ and $f_{h'i}(c_i) \leq f_{h'j}(c_j)$ and $f_{h''j}(c_j) \leq f_{h''k}(c_k)$. Let $h \in \mathbb{N}_0$ be such that $h', h'' \leq h$. Then $f_{hi}(c_i) = f_{hh'}\big(f_{h'i}(c_i)\big) \leq f_{hh'}\big(f_{h'j}(c_j)\big) = f_{hj}(c_j) = f_{hh''}\big(f_{h''j}(c_j)\big) \leq f_{hh''}\big(f_{h''k}(c_k)\big) = f_{hk}(c_k)$ so $(c_i, i) \leq (c_k, k)$.

Thus $\leq$ is a preorder.

Let $h, i \in \mathbb{N}_0$ be such that $i \leq h$. Let $c_i \in C_i$. Then $f_{hi}(c_i) = f_{hh}\big(f_{hi}(c_i)\big)$, so $[(c_i, i)] = [\big(f_{hi}(c_i), h\big)]$. Thus $f_i(c_i) = f_h\big(f_{hi}(c_i)\big)$.

Now let $i \in \mathbb{N}_0$ and let $c_i, c_i' \in C_i$ be such that $c_i \leq c_i'$. Then $(c_i, i) \leq (c_i', i)$ since $f_{ii}(c_i) = c_i \leq c_i' = f_{ii}(c_i')$, so $f_i(c_i) = [(c_i, i)] \leq [(c_i', i)] = f_i(c_i')$ in C. Thus f_i is order-preserving. We conclude that

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

is compatible with $\mathcal{S}$.

Let

$$\big(C', (f_i' : C_i \to C')_{i \in \mathbb{N}_0}\big)$$

be compatible with $\mathcal{S}$. Define $f : C \to C'$ as follows: Given $i \in \mathbb{N}_0$, $c_i \in C_i$, let $f\big([(c_i, i)]\big) = f_i'(c_i)$. We are forced into this definition if we want $f \circ f_i = f_i'$.

We show f is well-defined. If $i, j \in \mathbb{N}_0$, $c_i \in C_i$, $c_j \in C_j$, and $[(c_i, i)] = [c_j, j)]$, then there exist $h', h'' \in \mathbb{N}_0$ such that $i, j \leq h', h''$ and $f_{h'i}(c_i) \leq f_{h'j}(c_j)$ and $f_{h''j}(c_j) \leq f_{h''i}(c_i)$. Let $h \in \mathbb{N}_0$ be such that $h', h'' \leq h$. Then $f_{hi}(c_i) = f_{hh'}\big(f_{h'i}(c_i)\big) \leq f_{hh'}\big(f_{h'j}(c_j)\big) = f_{hj}(c_j)$ and $f_{hj}(c_j) = f_{hh''}\big(f_{h''j}(c_j)\big) \leq f_{hh''}\big(f_{h''i}(c_i)\big) = f_{hi}(c_i)$ so $f_{hi}(c_i) = f_{hj}(c_j)$.

Now $f_j'(c_j) = (f_h' \circ f_{hj})(c_j) = (f_h' \circ f_{hi})(c_i) = f_i'(c_i)$. Hence f is well-defined.

Now suppose $i, j \in \mathbb{N}_0$, $c_i \in C_i$, $c_j \in C_j$ and $[(c_i, i)] \leq [(c_j, j)]$. Then there exists $h \in \mathbb{N}_0$ such that $i, j \leq h$ and $f_{hi}(c_i) \leq f_{hj}(c_j)$. As $[(c_i, i)] = [\big(f_{hi}(c_i), h\big)]$ and $[(c_j, j)] = [\big(f_{hj}(c_j), h\big)]$, $f\big([(c_i, i)]\big) = f\Bigg([\big(f_{hi}(c_i), h\big)]\Bigg) = f_h'\big(f_{hi}(c_i)\big) \leq f_h'\big(f_{hj}(c_j)\big) = f\Bigg([\big(f_{hj}(c_j), h\big)]\Bigg) = f\big([(c_j, j)]\big)$, so f is order-preserving. □

Proposition 4.2. *Let*

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

be the filtered limit of the filtered system

$$\mathcal{S} = \big((C_i)_{i \in \mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \leq i}}\big).$$

Then for all $c \in C$*, there exist* $i \in \mathbb{N}_0$ *and* $c_i \in C_i$ *such that* $f_i(c_i) = c$.

For $c, d \in C$*,* $c \le d$ *is equivalent to each of the following:*

(1) *if* $i, j \in \mathbb{N}_0$*,* $c_i \in C_i$*,* $c_j \in C_j$*, and* $f_i(c_i) = c$ *and* $f_j(c_j) = d$*, then there exists* $h \in \mathbb{N}_0$ *such that* $i, j \le h$ *and* $f_{hi}(c_i) \le f_{hj}(c_j)$*;*

(2) *there exist* $h, i, j \in \mathbb{N}_0$*,* $c_i \in C_i$*,* $c_j \in C_j$ *such that* $i, j \le h$*,* $f_i(c_i) = c$*,* $f_j(c_j) = d$*, and* $f_{hi}(c_i) \le f_{hj}(c_j)$*.*

For (1) *or* (2)*, any* $h' \in \mathbb{N}_0$ *such that* $h \le h'$ *also works.*

Proof. Let

$$\big(C', (f'_i : C_i \to C')_{i \in \mathbb{N}_0}\big)$$

be the filtered limit for $\mathcal{S}$ constructed in Proposition 1. There exists an order-isomorphism $f : C \to C'$ such that $f \circ f_i = f'_i$ and $f_i = f^{-1} \circ f'_i$ for all $i \in \mathbb{N}_0$.

Since f^{-1} is onto, for every $c \in C$, there exists $c' \in C'$ such that $f^{-1}(c') = c$, but there exist $i \in \mathbb{N}_0$, $c_i \in C_i$ such that $c' = [(c_i, i)] = f'_i(c_i)$ so $c = f^{-1}\big(f'_i(c_i)\big) = f_i(c_i)$.

Now let $c, d \in C$. Let $c' = f(c)$, $d' = f(d)$. Then $c \le d$ if and only if $c' \le d'$. If $c' \le d'$, then there exist $h, i, j \in \mathbb{N}_0$ such that $i, j \le h$, $c_i \in C_i$, $c_j \in C_j$, $f'_i(c_i) = c'$, $f'_j(c_j) = d'$ and $f_{hi}(c_i) \le f_{hj}(c_j)$. Hence $f_i(c_i) = (f^{-1} \circ f'_i)(c_i) = f^{-1}(c') = c$, $f_j(c_j) = (f^{-1} \circ f'_j)(c_j) = f^{-1}(d') = d$. We get (2).

Now assume (2) holds and assume $\tilde{i}, \tilde{j} \in \mathbb{N}_0$, $\tilde{c}_{\tilde{i}} \in C_{\tilde{i}}$, $\tilde{c}_{\tilde{j}} \in C_{\tilde{j}}$, and $f_{\tilde{i}}(\tilde{c}_{\tilde{i}}) = c$ and $f_{\tilde{j}}(\tilde{c}_{\tilde{j}}) = d$. Then $f'_{\tilde{i}}(\tilde{c}_{\tilde{i}}) = c'$ and $f'_{\tilde{j}}(\tilde{c}_{\tilde{j}}) = d'$. That is, $[(c_i, i)] = [(\tilde{c}_{\tilde{i}}, \tilde{i})]$ and $[(c_j, j)] = [(\tilde{c}_{\tilde{j}}, \tilde{j})]$. Hence, as $[(c_i, i)] \le [(c_j, j)]$ by (2), (1) follows.

Now assume (1). Since there exist i and j and $c_i \in C_i$, $c_j \in C_j$ such that $f_i(c_i) = c$ and $f_j(c_j) = d$, the h exists, and $c = f_i(c_i) = f_h\big(f_{hi}(c_i)\big) \le f_h\big(f_{hj}(c_j)\big) = f_j(c_j) = d$.

Let the situation be as in (1) or (2). Let $h' \in \mathbb{N}_0$ be such that $h' \ge h$. Then $h' \ge i, j$ and $f_{h'i}(c_i) = f_{h'h}\big(f_{hi}(c_i)\big) \le f_{h'h}\big(f_{hj}(c_j)\big) = f_{h'j}(c_j)$. $\square$

Lemma 4.3. *Let*

$$\mathcal{S} = \big((C_i)_{i\in\mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j\in\mathbb{N}_0\\ j\le i}}\big)$$

be a filtered system in **Poset** *with filtered limit*

$$\big(C, (f_i : C_i \to C)_{i\in\mathbb{N}_0}\big).$$

Let

$$\big(C', (f'_i : C_i \to C')_{i\in\mathbb{N}_0}\big)$$

be compatible with $\mathcal{S}$. Let $f : C \to C'$ be an order-isomorphism such that $f \circ f_i = f'_i$ for all $i \in \mathbb{N}_0$. Then

$$\big(C', (f'_i : C_i \to C')_{i\in\mathbb{N}_0}\big)$$

is a filtered limit of $\mathcal{S}$.

Proof. Let

$$\big(C'', (f''_i : C_i \to C'')_{i\in\mathbb{N}_0}\big)$$

be compatible with $\mathcal{S}$. Then there exists a unique order-preserving map $g : C \to C''$ such that $g \circ f_i = f''_i$ for all $i \in \mathbb{N}_0$. Then $g \circ f^{-1} : C' \to C''$ is an order-preserving map such that, for all $i \in \mathbb{N}_0$, $(g\circ f^{-1}) \circ f'_i = g \circ f_i = f''_i$. If $h : C' \to C''$ is an order-preserving map such that $h \circ f'_i = f''_i$ for all $i \in \mathbb{N}_0$, then $h \circ f \circ f_i = f''_i$ for all $i \in \mathbb{N}_0$, so $h \circ f = g$ and thus $h = g \circ f^{-1}$. □

Lemma 4.4. *Let*

$$\big(C, (f_i : C_i \to C)_{i\in\mathbb{N}_0}\big)$$

be a filtered limit of the filtered system

$$\mathcal{S} = \big((C_i)_{i\in\mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j\in\mathbb{N}_0\\ j\le i}}\big).$$

(1) *If $f_{i+1,i}$ is an order-embedding for all $i \in \mathbb{N}_0$, then f_i is an order-embedding for all $i \in \mathbb{N}_0$.*

(2) *Let*

$$\big(C', (f'_i : C_i \to C')_{i\in\mathbb{N}_0}\big)$$

be compatible with $\mathcal{S}$. Let $f : C \to C'$ be the order-preserving map such that $f \circ f_i = f'_i$ for all $i \in \mathbb{N}_0$. If f'_i is an order-embedding for all $i \in \mathbb{N}_0$, then f is an order-embedding.

Remark: For something related, see [6].

Proof. (1) For $i, j \in \mathbb{N}_0$ such that $j \leq i$, then $f_{ij} = f_{i,i-1} \circ f_{i-1,i-2} \circ \cdots \circ f_{j+1,j}$, so f_{ij} is an order-embedding. Let $i \in \mathbb{N}_0$, $c_i, c_i' \in C_i$ be such that $f_i(c_i) \leq f_i(c_i')$. By Proposition 4.2(1), there exists $h \in \mathbb{N}_0$ such that $i \leq h$ and $f_{hi}(c_i) \leq f_{hi}(c_i')$, so $c_i \leq c_i'$. Thus f_i is an order-embedding.

(2) Let $i, j \in \mathbb{N}_0$, $c_i \in C_i$, $c_j \in C_j$ be such that $f\big(f_i(c_i)\big) \leq f\big(f_j(c_j)\big)$. Let $h \in \mathbb{N}_0$ be such that $i, j \leq h$. Then $f_i(c_i) = f_h\big(f_{hi}(c_i)\big)$ and $f_j(c_j) = f_h\big(f_{hj}(c_j)\big)$. Thus

$$f\Bigg(f_h\big(f_{hi}(c_i)\big)\Bigg) \leq f\Bigg(f_h\big(f_{hj}(c_j)\big)\Bigg)$$

so $f_h'\big(f_{hi}(c_i)\big) \leq f_h'\big(f_{hj}(c_j)\big)$ and $f_{hi}(c_i) \leq f_{hj}(c_j)$. Therefore $f_i(c_i) = f_h\big(f_{hi}(c_i)\big) \leq f_h\big(f_{hj}(c_j)\big) = f_j(c_j)$.

Hence f is an order-embedding. □

Lemma 4.5. *Let C_i be a poset for all $i \in \mathbb{N}_0$. Let $f_{i+1,i} : C_i \to C_{i+1}$ be an order-preserving map for all $i \in \mathbb{N}_0$. For all $i, j \in \mathbb{N}_0$ such that $j < i$, define $f_{ij} : C_j \to C_i$ as $f_{i,i-1} \circ f_{i-1,i-2} \circ \cdots \circ f_{j+2,j+1} \circ f_{j+1,j}$ and let $f_{jj} : C_j \to C_j$ be id_{C_j}.*

Then

$$\mathcal{S} = \big((C_i)_{i \in \mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \leq i}}\big).$$

is a filtered system in **Poset***.*

Let C be a poset. Let $f_i : C_i \to C$ be an order-preserving map for all $i \in \mathbb{N}_0$. Then

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

is compatible with $\mathcal{S}$ if and only if $f_{i+1} \circ f_{i+1,i} = f_i$ for all $i \in \mathbb{N}_0$.

Lemma 4.6. *Let A, B, and Q be posets. Then*

(1) *$f^Q \in (B^Q)^{(A^Q)}$ for all $f \in B^A$;*
(2) *for all $f_1, f_2 \in B^A$, $f_1 \leq f_2$ implies $f_1^Q \leq f_2^Q$;*
(3) *if $f \in B^A$ is an order-embedding, so is f^Q.*

Proof. (1) Let $g_1, g_2 \in A^Q$ be such that $g_1 \leq g_2$. Let $q \in Q$. Then $\big(f^Q(g_1)\big)(q) = f\big(g_1(q)\big) \leq f\big(g_2(q)\big) = \big(f^Q(g_2)\big)(q)$, so $f^Q(g_1) \leq f^Q(g_2)$.

(2) Let $g \in A^Q$, $q \in Q$. Then $[(f_1^Q)(g)](q) = (f_1 \circ g)(q) = f_1\big(g(q)\big) \leq f_2\big(g(q)\big) = (f_2 \circ g)(q) = [(f_2^Q)(g)](q)$.

Thus $f_1^Q(g) \leq f_2^Q(g)$ and hence $f_1^Q \leq f_2^Q$.

(3) Let $g_1, g_2 \in A^Q$ be such that $f^Q(g_1) \leq f^Q(g_2)$. Thus for all $q \in Q$, $[f^Q(g_1)](q) \leq [f^Q(g_2)](q)$ or $f\big(g_1(q)\big) = (f \circ g_1)(q) \leq (f \circ g_2)(q) = f\big(g_2(q)\big)$, so $g_1(q) \leq g_2(q)$.

Hence $g_1 \leq g_2$. □

Proposition 4.7. *Let*

$$\mathcal{S} = \big((C_i)_{i \in \mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \leq i}}\big).$$

be a filtered system in **Poset** *with filtered limit*

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

Let Q be a finite poset. Then $\mathcal{S}^Q$ is a filtered system with filtered limit

$$\big(C^Q, (f_i^Q : C_i^Q \to C^Q)_{i \in \mathbb{N}_0}\big).$$

Proof. Let $i, j, k \in \mathbb{N}_0$ be such that $k \leq j \leq i$. Then for all $g \in C_k^Q$

$$(f_{ij}^Q \circ f_{jk}^Q)(g) = f_{ij} \circ f_{jk} \circ g = f_{ik} \circ g = (f_{ik}^Q)(g),$$

so $f_{ij}^Q \circ f_{jk}^Q = f_{ik}^Q$.

For all $g \in C_i^Q$, $f_{ii}^Q(g) = f_{ii} \circ g = \mathrm{id}_{C_i} \circ g = g$, so $f_{ii}^Q = \mathrm{id}_{C_i^Q}$.

Hence $\mathcal{S}^Q$ is a filtered system. Observe that

$$\big(C^Q, (f_i^Q : C_i^Q \to C^Q)_{i \in \mathbb{N}_0}\big)$$

is compatible with $\mathcal{S}^Q$: for $i, j \in \mathbb{N}_0$ such that $j \leq i$ and $g \in C_j^Q$, $(f_i^Q \circ f_{ij}^Q)(g) = f_i \circ f_{ij} \circ g = f_j \circ g = f_j^Q(g)$ so $f_i^Q \circ f_{ij}^Q = f_j^Q$.

Let $\big(D, (g_i : C_i^Q \to D)_{i\in\mathbb{N}_0}\big)$ be a filtered limit of $\mathcal{S}^Q$.

Claim 1. For all $i, j \in \mathbb{N}_0$, $\gamma_i \in C_i^Q$, $\gamma_j \in C_j^Q$, $g_i(\gamma_i) \le g_j(\gamma_j)$ if and only if there exists $h \in \mathbb{N}_0$ such that $i, j \le h$ and, for all $q \in Q$, $f_{hi}\big(\gamma_i(q)\big) \le f_{hj}\big(\gamma_j(q)\big)$.

Proof of claim. ($\Rightarrow$) By Proposition 4.2(1), there exists $h \in \mathbb{N}_0$ such that $i, j \le h$ and $f_{hi}^Q(\gamma_i) \le f_{hj}^Q(\gamma_j)$—that is, $f_{hi} \circ \gamma_i \le f_{hj} \circ \gamma_j$. So for all $q \in Q$, $f_{hi}\big(\gamma_i(q)\big) \le f_{hj}\big(\gamma_j(q)\big)$.

($\Leftarrow$) We have $f_{hi} \circ \gamma_i \le f_{hj} \circ \gamma_j$ so $f_{hi}^Q(\gamma_i) \le f_{hj}^Q(\gamma_j)$. By Proposition 4.2(2), $g_i(\gamma_i) \le g_j(\gamma_j)$.

Define a map $\Psi : D \to C^Q$ as follows: for all $i \in \mathbb{N}_0$, $\gamma_i \in C_i^Q$, and $q \in Q$,

$$[\Psi\big(g_i(\gamma_i)\big)](q) = f_i\big(\gamma_i(q)\big).$$

Claim 2. The map Ψ is well-defined and order-preserving.

Proof of claim. Let $i, j \in \mathbb{N}_0$, $\gamma_i \in C_i^Q$, $\gamma_j \in C_j^Q$ be such that $g_i(\gamma_i) \le g_j(\gamma_j)$. By Claim 1 and Proposition 4.2(1), for all $q \in Q$, $f_i\big(\gamma_i(q)\big) \le f_j\big(\gamma_j(q)\big)$.

Claim 3. The map Ψ is an order-embedding.

Proof of claim. Let $i, j \in \mathbb{N}_0$, $\gamma_i \in C_i^Q$, $\gamma_j \in C_j^Q$ be such that $\Psi\big(g_i(\gamma_i)\big) \le \Psi\big(g_j(\gamma_j)\big)$. Hence, for all $q \in Q$, $f_i\big(\gamma_i(q)\big) \le f_j\big(\gamma_j(q)\big)$. Thus, by Proposition 4.2(1), there exists $h_q \in \mathbb{N}_0$ such that $i, j \le h_q$ and $f_{h_q i}\big(\gamma_i(q)\big) \le f_{h_q j}\big(\gamma_j(q)\big)$. Let $h \in \mathbb{N}_0$ be such that $h_q \le h$ for all $q \in Q$ and $i, j \le h$. (Here we use the finiteness of Q.) Then by Proposition 4.2, $f_{hi}\big(\gamma_i(q)\big) \le f_{hj}\big(\gamma_j(q)\big)$ for all $q \in Q$. By Claim 1, $g_i(\gamma_i) \le g_j(\gamma_j)$.

Claim 4. The map Ψ is onto.

Proof of claim. Start with an element of C^Q. For each $q \in Q$, let $i_q \in \mathbb{N}_0$, $c_{i_q} \in C_{i_q}$ be such that whenever $q, r \in Q$ and $q \le r$, we have $f_{i_q}(c_{i_q}) \le f_{i_r}(c_{i_r})$. Thus, whenever $q, r \in Q$ and $q \le r$, there exists (by Proposition 4.2(1)) $h_{q,r} \in \mathbb{N}_0$ such that $i_q, i_r \le h_{q,r}$ and $f_{h_{q,r} i_q}(c_{i_q}) \le f_{h_{q,r} i_r}(c_{i_r})$. Pick $h \in \mathbb{N}_0$ such that $h_{q,r} \le h$ for all $q, r \in Q$ with $q \le r$. (Here we use the finiteness of Q.) By Proposition 4.2, for all $q, r \in Q$ such that $q \le r$, $f_{h i_q}(c_{i_q}) \le f_{h i_r}(c_{i_r})$.

Define $\delta : Q \to C_h$ as follows: for all $q \in Q$, $\delta(q) = f_{hi_q}(c_{i_q})$. Then δ is order-preserving.

We will show that, for all $q \in Q$,

$$[\Psi\big(g_h(\delta)\big)](q) = f_{i_q}(c_{i_q}).$$

The left-hand side is $f_h\big(\delta(q)\big) = f_h\big(f_{hi_q}(c_{i_q})\big) = f_{i_q}(c_{i_q})$.

Let $i \in \mathbb{N}_0$, $\gamma_i \in C_i^Q$. Then, for all $q \in Q$,

$$\begin{aligned}\Big(\Psi\big(g_i(\gamma_i)\big)\Big)(q) &= f_i\big(\gamma_i(q)\big) \\ &= (f_i \circ \gamma_i)(q) \\ &= [f_i^Q(\gamma_i)](q),\end{aligned}$$

so $\Psi\big(g_i(\gamma_i)\big) = f_i^Q(\gamma_i)$.

By Lemma 4.3,

$$\big(C^Q, (f_i^Q : C_i^Q \to C^Q)_{i \in \mathbb{N}_0}\big)$$

is a filtered limit of $\mathcal{S}^Q$. □

Proposition 4.8. *Let*

$$\mathcal{S} = \big((C_i)_{i \in \mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \le i}}\big)$$

and

$$\mathcal{S}' = \big((C'_i)_{i \in \mathbb{N}_0}, (f'_{ij} : C'_j \to C'_i)_{\substack{i,j \in \mathbb{N}_0 \\ j \le i}}\big)$$

be filtered systems in **Poset** *with filtered limits*

$$\big(C, (f_i : C_i \to C)_{i \in \mathbb{N}_0}\big)$$

and

$$\big(C', (f'_i : C'_i \to C')_{i \in \mathbb{N}_0}\big),$$

respectively.

For all $i \in \mathbb{N}_0$, let $\phi_i : C_i \to C'_i$ be an order-isomorphism. Assume that, for all $i \in \mathbb{N}_0$, $f'_{i+1,i} \circ \phi_i = \phi_{i+1} \circ f_{i+1,i}$.

Then there is a unique order-isomorphism $f : C \to C'$ such that, for all $i \in \mathbb{N}_0$, $f \circ f_i = f'_i \circ \phi_i$.

Proof. We first show that

$$\big(C', (f'_i \circ \phi_i : C_i \to C')_{i \in \mathbb{N}_0}\big),$$

is compatible with $\mathcal{S}$, using Lemma 5: for $i \in \mathbb{N}_0$, $f'_{i+1} \circ \phi_{i+1} \circ f_{i+1,i} = f'_{i+1} \circ f'_{i+1,i} \circ \phi_i = f'_i \circ \phi_i$.

Thus there exists a unique order-preserving map $f : C \to C'$ such that, for all $i \in \mathbb{N}_0$, $f \circ f_i = f'_i \circ \phi_i$.

Now, for all $i \in \mathbb{N}_0$, $\phi_{i+1}^{-1} \circ f'_{i+1,i} = f_{i+1,i} \circ \phi_i^{-1}$, so by symmetry there is a unique order-preserving map $f' : C' \to C$ such that, for all $i \in \mathbb{N}_0$, $f' \circ f'_i = f_i \circ \phi_i^{-1}$; hence $f' \circ f'_i \circ \phi_i = f_i$ or $f' \circ f \circ f_i = f_i$. Of course, id_C is such that $\mathrm{id}_C \circ f_i = f_i$ for all $i \in \mathbb{N}_0$, so by uniqueness $\mathrm{id}_C = f' \circ f$. By symmetry $\mathrm{id}_{C'} = f \circ f'$. □

Corollary 4.9. *Let P and Q be posets, Q finite. For $i \in \mathbb{N}_0$, let $C_i = P^{\sigma \dots \sigma}$ and let $D_i = (P^Q)^{\sigma \dots \sigma}$ (i copies of "σ"). For $i \in \mathbb{N}_0$, let $f_{i+1,i} : C_i \to C_{i+1}$ be the order-embedding $x \mapsto \downarrow x$; let $g_{i+1,i} : D_i \to D_{i+1}$ be the order-embedding $y \mapsto \downarrow y$.*

Then for all $i \in \mathbb{N}_0$, there exists an order-isomorphism $\phi_i : C_i^Q \to D_i$ such that, for all $i \in \mathbb{N}_0$,

$$g_{i+1,i} \circ \phi_i = \phi_{i+1} \circ f^Q_{i+1,i}.$$

Proof. Let $\phi_0 = \mathrm{id}_{P^Q}$. Now assume $\phi_i : C_i^Q \to D_i$ is defined. By Proposition 3.2, there exists an order-isomorphism $\Phi_i : (C_i^Q)^\sigma \to D_i^\sigma$ such that, for all $f \in C_i^Q$, $\Phi_i(\downarrow f) = \downarrow [\phi_i(f)]$. By Corollary 3.3, there is an order-isomorphism $H_i : (C_i^\sigma)^Q \to (C_i^Q)^\sigma$ such that, for all $f \in C_i^Q$, $H_i(f^\sigma) = \downarrow f$.

Let $\phi_{i+1} = \Phi_i \circ H_i : C_{i+1}^Q \to D_{i+1}$. It is an order-isomorphism.

Let $f \in C_i^Q$. Note that, for all $q \in Q$,

$$[f^Q_{i+1,i}(f)](q) = (f_{i+1,i} \circ f)(q) = \downarrow [f(q)]$$

so $f^Q_{i+1,i}(f) = f^\sigma$.

Thus

$$\begin{aligned}\phi_{i+1}\big(f^Q_{i+1,i}(f)\big) &= \Phi_i\Big(H_i\big(f^Q_{i+1,i}(f)\big)\Big)\\ &= \Phi_i\big(H_i(f^\sigma)\big)\\ &= \Phi_i(\downarrow f)\\ &=\downarrow \phi_i(f).\end{aligned}$$

Also, $g_{i+1,i}\big(\phi_i(f)\big) =\downarrow [\phi_i(f)]$. Hence $g_{i+1,i} \circ \phi_i = \phi_{i+1} \circ f^Q_{i+1,i}$. □

Corollary 4.10. *Let all be as in Corollary* 4.9*. Let*

$$\mathcal{S} = \big((C_i)_{i\in\mathbb{N}_0}, (f_{ij} : C_j \to C_i)_{\substack{i,j\in\mathbb{N}_0\\ j\leq i}}\big)$$

and

$$\mathcal{T} = \big((D_i)_{i\in\mathbb{N}_0}, (g_{ij} : D_j \to D_i)_{\substack{i,j\in\mathbb{N}_0\\ j\leq i}}\big)$$

be filtered systems built as per Lemma 4.5. Let

$$\big(C, (f_i : C_i \to C)_{i\in\mathbb{N}_0}\big),$$

$$\big(C^Q, (f^Q_i : C^Q_i \to C^Q)_{i\in\mathbb{N}_0}\big),\ and$$

$$\big(D, (g_i : D_i \to D)_{i\in\mathbb{N}_0}\big)$$

be the filtered limits of $\mathcal{S}$, $\mathcal{S}^Q$, and $\mathcal{T}$, respectively (using Propositions 4.1 and 4.7).

Then there exists a unique order-isomorphism $f : C^Q \to D$ such that, for all $i \in \mathbb{N}_0$, $f \circ f^Q_i = g_i \circ \phi_i$.

Moreover, C is dually atomic.

Proof. We get f from Proposition 4.8.

We borrow from [5, page 264].

Claim 1. Let A be a poset. Let $a, b \in A$ be such that $a < b$. Then there exists $I \in A^\sigma$ such that $\downarrow a \subseteq I \lessdot \downarrow b$ in A^σ.

Proof of claim. Let $\mathcal{E}$ be the poset $\{J \in A^\sigma \mid {\downarrow}a \subseteq J \subsetneq {\downarrow}b\}$ ordered by set-inclusion. Then ${\downarrow}a \in \mathcal{E}$.

Now let $\mathcal{C}$ be a non-empty chain in $\mathcal{E}$. We see that $\bigcup \mathcal{C} \in \mathcal{E}$.

By Zorn's Lemma, $\mathcal{E}$ has a maximal member.

Claim 2. If $a \lessdot b$ in a poset A, then ${\downarrow}a \lessdot {\downarrow}b$ in A^σ.

Proof of claim. Assume for a contradiction that there exists $C \in A^\sigma$ such that ${\downarrow}a \subsetneq C \subsetneq {\downarrow}b$. Then $a \in C$ and there exists $c \in C$ such that $c \not\leq a$. Hence there exists $d \in C$ such that $a, c \leq d$. Therefore $a < d$.

As $d \leq b$, we conclude $d = b$. Thus $b \in C$, so ${\downarrow}b \subseteq C \subsetneq {\downarrow}b$, a contradiction.

Let $i, j \in \mathbb{N}_0$, $c_i \in C_i$, $c_j \in C_j$ be such that $f_i(c_i) < f_j(c_j)$. Then by Proposition 4.2, there exists $h \in \mathbb{N}_0$ such that $i, j \leq h$ and $f_{hi}(c_i) \leq f_{hj}(c_j)$. If $f_{hi}(c_i) = f_{hj}(c_j)$, then $f_i(c_i) = f_h\big(f_{hi}(c_i)\big) = f_h\big(f_{hj}(c_j)\big) = f_j(c_j)$, a contradiction. Thus $f_{hi}(c_i) < f_{hj}(c_j)$.

Now, there exists $c_{h+1} \in C_{h+1}$ such that $f_{h+1,h}\big(f_{hi}(c_i)\big) \leq c_{h+1} \lessdot f_{h+1,h}\big(f_{hj}(c_j)\big)$—that is, $f_{h+1,i}(c_i) \leq c_{h+1} \lessdot f_{h+1,j}(c_j)$.

Hence $f_i(c_i) = f_{h+1}\big(f_{h+1,i}(c_i)\big) \leq f_{h+1}(c_{h+1}) < f_{h+1}\big(f_{h+1,j}(c_j)\big) = f_j(c_j)$, using Lemma 4.4(1).

Assume for a contradiction that, for some $k \in \mathbb{N}_0$ and $c_k \in C_k$, $f_{h+1}(c_{h+1}) < f_k(c_k) < f_j(c_j)$. Let $\ell \in \mathbb{N}_0$ be such that $h+1, k \leq \ell$ and $f_{\ell,h+1}(c_{h+1}) \leq f_{\ell,k}(c_k)$ and let $m \in \mathbb{N}_0$ be such that $j, k \leq m$ and $f_{m,k}(c_k) \leq f_{m,j}(c_j)$. Pick $n \in \mathbb{N}_0$ such that $\ell, m \leq n$; by Proposition 4.2, we have

$$f_{n,h+1}(c_{h+1}) \leq f_{n,k}(c_k) \leq f_{n,j}(c_j) = f_{n,h+1}\big(f_{h+1,j}(c_j)\big).$$

By Claim 2, $f_{n,h+1}(c_{h+1}) \lessdot f_{n,h+1}\big(f_{h+1,j}(c_j)\big)$, so

$$f_{nk}(c_k) \in \{f_{n,h+1}(c_{h+1}), f_{nj}(c_j)\}$$

and thus

$$\begin{aligned} f_k(c_k) = f_n\big(f_{nk}(c_k)\big) &\in \{f_n\big(f_{n,h+1}(c_{h+1})\big), f_n\big(f_{nj}(c_j)\big)\} \\ &= \{f_{h+1}(c_{h+1}), f_j(c_j)\}, \end{aligned}$$

a contradiction.

Therefore $f_i(c_i) \leq f_{h+1}(c_{h+1}) \lessdot f_j(c_j)$. □

Lemma 4.11. *Let P, P', Q, Q' be posets such that Q and Q' are finite. Let $\mathcal{S}$ be as in Corollary 10 and similarly define $\mathcal{S}'$. Let $\Psi_0 : P^Q \to P'^{Q'}$ be an order-isomorphism.*

Then for all $i \in \mathbb{N}_0$, there exists an order-isomorphism $\Psi_n : C_i^Q \to C'^{Q'}_i$, so that, for all $i \in \mathbb{N}_0$,

$$f'^{Q'}_{i+1,i} \circ \Psi_i = \Psi_{i+1} \circ f^Q_{i+1,i}.$$

Proof. Define $\mathcal{T}$ as in Corollary 4.10, and similarly define $\mathcal{T}'$. By Corollary 4.9, $f^Q_{i+1,i}[C^Q_i] = \kappa[C^Q_{i+1}]$ and, by Lemma 4.6(3), $f^Q_{i+1,i}$ is an order-embedding. Also, C^Q_{i+1} is an algebraic poset (Corollary 3.3). By Proposition 3.2, there exists an order-isomorphism $\Psi_{i+1} : C^Q_{i+1} \to C'^{Q'}_{i+1}$ such that $\Psi_{i+1} \circ f^Q_{i+1,i} = f'^{Q'}_{i+1,i} \circ \Psi_i$. □

Proposition 4.12. *Let all be as in Lemma* 4.11*. Then for all $i \in \mathbb{N}_0$, f^Q_i and $f'^{Q'}_i$ are order-embeddings.*

Further, there exists a unique order-isomorphism $\Psi : C^Q \to C'^{Q'}$ such that, for all $i \in \mathbb{N}_0$, $\Psi \circ f^Q_i = f'^{Q'}_i \circ \Psi_i$.

Proof. Use Proposition 4.8, Lemma 4.4(1), and Lemma 4.6(3). □

5 Dually atomic posets and the refinement of powers and cancellation of exponents

In this section, we prove that finite posets C with a top element can be cancelled as exponents—$A^C \cong B^C$ implies $A \cong B$, even if A and B are infinite.

Jónsson and McKenzie's results are phrased in terms of atomic posets, whereas we were working earlier with dually atomic posets, so we show we can dualize the results of Jónsson and McKenzie.

Lemma 5.1. *Let A, B, C, D be posets such that $C, D \neq \emptyset$. Let $\phi : A^C \cong B^D$. Let $\phi^\partial : (A^\partial)^{(C^\partial)} \to (B^\partial)^{(D^\partial)}$ be given by $\phi^\partial(f) = \phi(f)$ for all $f \in (A^\partial)^{(C^\partial)}$ (so $\phi^\partial : (A^\partial)^{(C^\partial)} \cong (B^\partial)^{(D^\partial)}$).*

Then (ϕ, i) implies (ϕ^∂, i) for $i \in \{1, 2, 3, 4\}$.

Proof. Clearly $\Delta(\phi^\partial) = \Delta(\phi)$. Also $R(\phi^\partial) = \{x \in A \mid \langle x \rangle \in \Delta(\phi^\partial)\} = R(\phi)$.

Let $x, y \in A^\partial$. Then

$$x \leq_{\phi^\partial} y$$

if and only if

$x, y \in R(\phi^\partial)$, $x \leq y$ in A^∂, and for all $f \in (A^\partial)^{(C^\partial)}$ with $f[C] \subseteq \{x, y\}$, we have $f \in \Delta(\phi^\partial)$

if and only if

$x, y \in R(\phi)$, $x \geq y$ in A, and for all $f \in A^C$ with $f[C] \subseteq \{x, y\}$, we have $f \in \Delta(\phi)$

if and only if

$$x \geq_\phi y.$$

The map $\mathring{\phi}^\partial$ from $R(\phi^\partial)$ to $R(\phi^{\partial^{-1}})$ is such that, for $x \in R(\phi^\partial)$, $\mathring{\phi}^\partial(x)$ is the element y such that $\phi^\partial(\langle x \rangle) = \langle y \rangle$, so $\mathring{\phi}^\partial$ is $\mathring{\phi}$.

If $(\phi, 1)$, then $(\phi^\partial, 1)$.

If $(\phi, 2)$, then $(\phi^\partial, 2)$. (The set $R(\phi^{\partial^{-1}})$ is $R(\phi^{-1^\partial})$.)

Assume $(\phi, 3)$. Then

$$\begin{aligned} \Delta(\phi^\partial) &= \Delta(\phi) \\ &= \{f : C \to R(\phi) \mid \text{ for all } c, c' \in C,\ c \leq c' \text{ in } C \text{ implies } f(c) \leq_\phi f(c')\} \\ &= \{f : C \to R(\phi^\partial) \mid \text{ for all } c, c' \in C,\ c \geq c' \text{ in } C^\partial \text{ implies } f(c) \geq_{\phi^\partial} f(c')\} \end{aligned}$$

Thus $(\phi^\partial, 3)$ holds.

Assume $(\phi, 4)$ holds. Then

$$\begin{aligned} \geq_{\phi^\partial} &= \leq_\phi \\ &= \leq_A \cap \big(R(\phi) \times R(\phi)\big) \\ &= \geq_{A^\partial} \cap \big(R(\phi^\partial) \times R(\phi^\partial)\big), \end{aligned}$$

so $(\phi^\partial, 4)$ holds. □

Lemma 5.2. *Let A, B, C, D be posets. Assume A and B are dually atomic. Assume C and D are finite, directly irreducible posets with a bottom element. Assume $\phi : A^C \cong B^D$.*

Then $(\phi, 1)$, $(\phi, 2)$, $(\phi, 3)$, and $(\phi^{-1}, 3)$. If $C \ncong D$, then $(\phi, 4)$.

Proof. By Lemma 2.3, $(\phi, 1)$ holds. By Lemma 2.4, $(A^\partial)^{(C^\partial)}$ and $(B^\partial)^{(D^\partial)}$ have Property (a), so by Lemma 2.5, $(\phi^\partial, 2)$ holds, so by Lemma 5.1, $(\phi, 2)$ holds.

By Lemma 2.6, $(\phi^\partial, 3)$ holds, so by Lemma 5.1, $(\phi, 3)$ holds. By symmetry, $(\phi^{-1}, 3)$ holds.

Now assume $C \ncong D$. Then $(\phi^\partial, 4)$ by Lemma 2.4 and Lemma 2.7. By Lemma 5.1, $(\phi, 4)$ holds. □

Now we eliminate the hypothesis of being dually atomic.

Lemma 5.3. *Let A, B, C, D be posets such that C and D are finite, directly irreducible posets with a bottom element. Assume $\phi : A^C \to B^D$ is an order-isomorphism. Then $(\phi, 1)$, $(\phi, 2)$, $(\phi, 3)$, and $(\phi^{-1}, 3)$. If $C \ncong D$, then $(\phi, 4)$.*

Proof. By Lemma 4.4(1), Proposition 4.12 and Corollary 4.10, there exist dually atomic posets A_1 and B_1 and order-embeddings $f : A \to A_1$ and $g : B \to B_1$ and (by Lemma 4.6(3)) $f^C : A^C \to {A_1}^C$ and $g^D : B^D \to {B_1}^D$ and an order-isomorphism $\Psi : {A_1}^C \cong {B_1}^D$ such that $\Psi \circ f^C = g^D \circ \phi$.

Claim. If "Im" denotes the image of a map, $\operatorname{Im}(f^C) = (\operatorname{Im} f)^C$ *and* $\operatorname{Im}(g^D) = (\operatorname{Im} g)^D$.

Proof of claim. If $\alpha \in (\operatorname{Im} f)^C$, then for all $c \in C$, let $\alpha(c) = f(a_c)$. Define $\overline{\alpha} : C \to A$ by $c \mapsto a_c$ for all $c \in C$. This is order-preserving: if $c \leq c'$ in C, then $\alpha(c) \leq \alpha(c')$ so $a_c \leq a_{c'}$, since f is an order-embedding. That is, $\overline{\alpha} \in A^C$. For all $c \in C$, $[f^C(\overline{\alpha})](c) = (f \circ \overline{\alpha})(c) = f\big(\overline{\alpha}(c)\big) = f(a_c) = \alpha(c)$, so $f^C(\overline{\alpha}) = \alpha$.

Conversely, if $\alpha \in A^C$, then for all $c \in C$, $[f^C(\alpha)](c) = (f \circ \alpha)(c) = f\big(\alpha(c)\big) \in \operatorname{Im} f$.

By Lemma 5.2, $(\Psi, 1)$, $(\Psi, 2)$, $(\Psi, 3)$, and $(\Psi^{-1}, 3)$, and, if $C \not\cong D$, $(\Psi, 4)$.

By Proposition 4.12, the restriction of Ψ to $\operatorname{Im}(f^C)$ is essentially the same as ϕ; by the comment in Section 2 and Lemma 2.8, $(\phi, 1)$, $(\phi, 2)$, $(\phi, 3)$, $(\phi^{-1}, 3)$, and, if $C \not\cong D$, $(\phi, 4)$. □

Lemma 5.4. *Let all be as in Lemma* 5.3*. If $C \cong D$, then $A \cong B$. If $C \not\cong D$, then there is a poset E such that $A \cong E^D$ and $B \cong E^C$.*

Proof. This follows from Theorem 2.1 and Theorem 2.2. □

Theorem 5.5. *Let A, B, C, and D be posets such that C and D are finite and both have a least element (both have a greatest element). Assume $\phi : A^C \cong B^D$.*

Then there are posets E, X, Y, and Z such that $A \cong E^X$, $B \cong E^Y$, $C \cong Y \times Z$, and $D \cong X \times Z$.

If $C \cong D$, then $A \cong B$.

Proof. The proof is identical to that of [10, Theorem 5.2]. We will not even change the notation.

Let $C \cong C_1 \times C_2 \times \cdots \times C_m$ and $D \cong C_1 \times D_2 \times \cdots \times D_n$ where $m, n \in \mathbb{N}_0$, C_i is directly irreducible ($i = 1, \ldots, m$), and D_j is directly irreducible ($j = 1, \ldots, n$). Note that each C_i has a bottom element ($i = 1, \ldots, m$), as does each D_j ($j = 1, \ldots, n$). By Hashimoto's Refinement Theorem [16, Theorem 10.4.4], m and n are uniquely determined.

We proceed by induction on $m + n$.

If $m = 0$, let $E = B$, $X = D$, $Y = \mathbf{1}$, and $Z = \mathbf{1}$.

If $m = 1 = n$, use Lemma 4. (If $C \cong D$, let $E = A$, $X = Y = \mathbf{1}$, and $Z = C$.)

If $m + n \geq 3$, without loss of generality $m \geq 2$. Let $C' = C_1 \times \cdots \times C_{m-1}$, so $(A^{C_m})^{C'} \cong B^D$ and hence there exist posets E_1, X_1, Y_1, and Z_1 such that

$$A^{C_m} \cong E_1^{X_1},\ B \cong E_1^{Y_1},\ C' \cong Y_1 \times Z_1,\ D \cong X_1 \times Z_1.$$

By Hashimoto's Refinement Theorem, X_1 has at most n directly irreducible factors and C' has $m - 1$, so by induction there are posets E_2, X_2, Y_2, and Z_2 such that

$$A \cong E_2^{X_2},\ E_1 \cong E_2^{Y_2},\ C_m \cong Y_2 \times Z_2,\ X_1 \cong X_2 \times Z_2.$$

so let $E = E_2$, $X = X_2$, $Y = Y_1 \times Y_2$, and $Z = Z_1 \times Z_2$.

If $C \cong D$, we have $(A^{C'})^{C_m} \cong (B^{C'})^{C_m}$, so $A^{C'} \cong B^{C'}$ and by induction $A \cong B$.

If C and D have a greatest element, we have

$$(A^\partial)^{(C^\partial)} \cong (A^C)^\partial \cong (B^D)^\partial \cong (B^\partial)^{(D^\partial)},$$

where C^∂ and D^∂ have least elements. Thus there exist posets E, X, Y, and Z such that $A^\partial \cong (E^\partial)^{X^\partial}$, $B^\partial \cong (E^\partial)^{Y^\partial}$, $C^\partial \cong Y^\partial \times Z^\partial$, $D^\partial \cong X^\partial \times Z^\partial$, so $A \cong E^X$, $B \cong E^Y$, $C \cong Y \times Z$, and $D \cong X \times Z$.

Also, if $C \cong D$, then $C^\partial \cong D^\partial$, so $A^\partial \cong B^\partial$, and $A \cong B$. $\square$

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