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Internal Neighbourhood Structures II: Closure and closed morphisms

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Abstract. Internal preneighbourhood spaces inside any finitely complete category with finite coproducts and proper factorisation structure were first introduced in [49]. This paper proposes a closure operation on internal preneighbourhood spaces and investigates closed morphisms and its close allies. Consequently it introduces analogues of several well-known classes of topological spaces for preneighbourhood spaces. Some preliminary properties of these spaces are established in this paper. The results of this paper exhibit that preneighbourhood systems are more general than closure operators and conveniently allows identifying properties of classes of morphisms independent of *continuity* of morphisms with respect to induced closure operators.

1 Introduction

The notion of an internal preneighbourhood space was first considered in [49]. The present paper introduces a *closure operator* on an internal preneigh-

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bourhood space (see Definition 3.1). The *closure operator* entails in discussing *closed morphisms* (see Definition 4.1). The rest of the paper discusses notions closely aligned with *closed morphisms* — *dense morphisms* (see Definition 5.1), *proper morphisms* (see Definition 6.1), *separated morphisms* (see Definition 7.1) and *perfect morphisms* (see Definition 8.2). Alongside morphisms special classes of internal preneighbourhood spaces are introduced: *compact* spaces (see Definition 6.5), *Hausdorff spaces* (see Definition 7.5), *compact Hausdorff spaces* (see Definition 8.4(a)), *Tychonoff spaces* (see Definition 8.4(b)) and *absolutely closed spaces* (see Definition 8.4(c)). Detailed investigation on the special classes of internal preneighbourhood spaces shall be done in later papers. A quick perusal of Table 1 provides a glimpse of results achieved in this paper as well as helps to compare similar results in literature, e.g., in [32]. The table clearly exhibits the extent to which continuity of morphisms with respect to induced closure operations are essential in achieving these properties.

The paper is organised as follows:

(a) In §2 notions necessary for the paper are briefly introduced; in the process some seemingly new observations have been listed. In this paper a monotonic, extensional and grounded endomap on a poset is called a closure operator (see §2.1 for terminology). Given a complete lattice L, $\mathtt{EGM}(L)$ denotes the complete lattice of closure operators on L, $\mathtt{CBSMSL}(L)$ denotes the complete lattice of complete bounded sub-$\wedge$-semilattices of L. Proposition 2.1 shows $\mathtt{CBSMSL}(L)^{\mathtt{op}}$ is reflectively embedded in $\mathtt{EGM}(L)$ as the idempotent closure operations.

(b) §3 introduce *closure operation* on preneighbourhood spaces.

(i) Each preneighbourhood system μ on an object X, for each $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, partitions the subobjects of X into four subsets. The first partition consists of subobjects which are *far away* from p (see equation (3.1)); such subobjects are incompatible with p. The second partition consists of subobjects which are incompatible with p and not *far away* from p; the third partition consists of subobjects $x > p$ and the fourth of subobjects $x \leq p$. The first partition is a down-set of $\mathtt{Sub}_{\mathsf{M}}(X)$[1] (see Lemma 3.1 for details); in the

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[1] A subset $P \subseteq L$ of a lattice L is a *down-set* if it is non-empty and $x \leq y \in P \Rightarrow x \in P$. A down-set of the form $\downarrow p = \{x \in L : x \leq p\}$ is a *principal down-set* and is the smallest down-set containing p.

special instance when $\mathtt{Sub}_{\mathsf{M}}(X)$ is a frame and μ is a neighbourhood system (see Definition 2.7) the first partition is a principal down-set.

The set of subobjects in the fourth and second partitions should be the ones of concern in defining the *closure* $\mathrm{cl}_\mu p$ of p, see Definition 3.1; the fixed subobjects of cl_μ are *μ-closed subobjects.* The assignment $p \mapsto \mathrm{cl}_\mu p$ so defined is a closure operation, its initial properties discussed in Theorem 3.5. The closure cl_μ is not additive in general (unless when $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated and distributive, Theorem 3.5(d)), nor idempotent (unless $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated and μ, in particular, a neighbourhood system, see Theorem 3.5(e) for details). In case when $\mathtt{Sub}_{\mathsf{M}}(X)$ is pseudocomplemented then for a neighbourhood system μ on X, there is a Galois connection between the semilattices of closed subobjects and the open subobjects yielding a dual equivalence between regular closed and regular open subobjects (see Proposition 3.8 and Remark 3.10).

(ii) Continuing from §2.1 and specialising to the complete lattice $\mathtt{Sub}_{\mathsf{M}}(X)$, in §3.2 it is shown that the complete lattice of preneighbourhood systems on X dually contains a coreflective copy of closure operations on $\mathtt{Sub}_{\mathsf{M}}(X)$ (see Theorem 3.11). This exhibits the generality of the approach via preneighbourhood systems in comparison with closure operations.

(iii) In the absence of idempotence for cl_μ, there exists its *idempotent hull* $\widehat{\mathrm{cl}_\mu} \geq \mathrm{cl}_\mu$ (see Proposition 2.1 and Remark 2.2) having the same set of $\mathfrak{C}_\mu$ of closed subobjects. The notion of continuity with respect to cl_μ as well $\widehat{\mathrm{cl}_\mu}$ is discussed in §3.3. Proposition 3.15 shows continuity with respect to cl_μ implies continuity with respect to $\widehat{\mathrm{cl}_\mu}$; continuity with respect to $\widehat{\mathrm{cl}_\mu}$ (respectively, cl_μ) is called *μ-ϕ continuity* (respectively, *μ-ϕ continuity with respect to closures*) or *continuity* (respectively, *continuity with respect to closures*) in short. Theorem 3.18 shows every admissible monomorphism is continuous with respect to closures; equation (3.20) provides correspondence between closed subobjects of a closed subobject and closed subobjects of whole space (also Remark 3.21).

Note: for a morphism $X \xrightarrow{f} Y$ of the base category, its property of being *continuous with respect to preneighbourhood system μ on X and ϕ on Y* is precisely the definition of it being a preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ (see Definition 2.7(c)). On the other hand, each preneighbourhood system μ on X induces closure operations cl_μ, $\widehat{\mathrm{cl}_\mu}$, and *continuity* *with respect to these closure operations* is separate and is not affected by the presence or absence of *continuity with respect to preneighbourhood systems* in general.

(iv) A major obstruction to effecting continuity with respect to closures of a morphism is the inclusion of subobjects in the fourth partition in (i) while computing the join. An antidote to this occurs when $\mathtt{Sub}_{\mathsf{M}}(X)$ is *atom generated*: firstly, the closure has a simpler description (Remark 3.4), consequently, continuity with respect to closures for a large class of morphisms is obtained (Corollary 3.26(b)). However in general, continuity of every preneighbourhood morphism is ensured once every dense preneighbourhood morphism is continuous (Proposition 3.23 and Corollary 3.24).

(v) §3.5 illustrates notions in some specific contexts. Notable amidst them are closures with respect to functorial neighbourhood systems (see Theorem 3.38 and Definition 4.3, [49]) on locales, groups and commutative rings without identity. On locales it is known from [49] that the T-neighbourhood systems (see equation (2.19), [45, 46]) are functorial; it is shown here that the closure with respect to the T-neighbourhood system is precisely the usual closure of a sublocale (see §III.8, [58]); furthermore every localic map $X \xrightarrow{f} Y$ is continuous with respect to any preneighbourhood system μ on X, ϕ on Y if μ is larger than the T-neighbourhood system on X and ϕ is smaller than the T-neighbourhood system on Y (see §3.5). Example 2.21 shows the normal closure induces a functorial neighbourhood system ν_X on each group X. §3.5 shows $\mathrm{cl}_{\nu_X} A = \big\{x \in X : \text{ normal closure of } x \text{ meets } A\big\}$; furthermore every group homomorphism $X \xrightarrow{f} Y$ is continuous with respect to any preneighbourhood system $\mu \geq \nu_X$ on X and $\phi \leq \nu_Y$. Similarly, Example 2.22 shows the ideal closure of a subring induces a functorial neighbourhood system ι_X on a ring X. §3.5 shows $\mathrm{cl}_{\iota_X} A = \big\{x \in X : (\exists r \in X)(rx \in A)\big\}$; furthermore every ring homomorphism $X \xrightarrow{f} Y$ is continuous with respect to any preneighbourhood system $\mu \geq \iota_X$ on X and $\phi \leq \iota_Y$.

(c) §4.1 discusses *closed morphism*, i.e., given the preneighbourhood spaces (X, μ), (Y, ϕ), morphisms $X \xrightarrow{f} Y$ which preserve closed subobjects (see equation (4.1)). Theorem 4.2 provide properties of closed morphisms, while Theorem 4.4 provide sufficient examples of closed preneighbourhood morphisms.

(d) §5.1 introduces *dense morphisms* and their properties are discussed in Theorem 5.3. It is shown in Theorem 5.3(f) that if every preneighbourhood morphism is continuous then every preneighbourhood morphism factors as a dense preneighbourhood morphism followed by a closed embedding. This factorisation system is a proper factorisation system for the full subcategory of internal Hausdorff spaces (see Remark 5.5).

(e) §6.1 discusses stably closed morphisms, called *proper morphisms.* Theorem 6.2 discusses properties of proper morphisms. In §6.3 *compact preneighbourhood spaces* are introduced as preneighbourhood spaces (X, μ) for which the unique morphism $(X, \mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1}, \nabla_{\mathtt{1}})$ is proper. The full subcategory $\mathtt{K}[\mathtt{pNbd}[\mathbb{A}]]$ of compact preneighbourhood spaces is shown to be finitely productive, closed hereditary if every preneighbourhood morphism is continuous (Theorem 6.7(c)).

(f) §7.1 discusses *separated morphisms* (Definition 7.1), Theorem 7.4 discusses properties of separated morphisms. Internal *Hausdorff spaces* are introduced in Definition 7.5 as those preneighbourhood spaces (X, μ) for which $(X, \mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1}, \nabla_{\mathtt{1}})$ is a separated morphism, alternate characterisations are provided in Theorem 7.6, the full subcategory of internal Hausdorff spaces $\mathtt{Haus}[\mathtt{pNbd}[\mathbb{A}]]$ is shown to be finitely complete, closed under subobjects and images of preneighbourhood morphisms stably continuous and stably in E (Corollary 7.7).

(g) Finally in §8.1 *perfect morphisms* are discussed — they are preneighbourhood morphisms which are both proper and separated (Definition 8.2). The properties of perfect morphisms are discussed in Theorem 8.3. The paper concludes by introducing *compact Hausdorff spaces*, *Tychonoff spaces* and *absolutely closed spaces*, Definition 8.4.

The effort of internalising the notion of space has been pursued in different ways, at least in the references below as well as citations within them:

i. using closure operators as in [7–15, 19–26, 29–31, 33, 35–37, 39, 40, 42–44],

ii. using interior operators and neighbourhood operators (see [49], page 2, for definition) as in [16–18, 27, 28, 60–63, 67],

iii. using convergence structures as in [38, 47, 50, 56, 64, 68–76]

iv. using a set of axioms for closed morphisms as in [32]

v. using a set of axioms for proper morphisms as in [54].

In all of them the aspect of *continuity* of morphisms is built inside the axioms, or is an easy consequence of the axioms — for instance see §11.1, (F6) and its consequences in [32]. The approach in the present work is transversal: firstly a category with *nice* properties is shown to have a structure of categorical neighbourhood system. An object endowed with a neighbourhood system allows the formulation of a closure operation. In several convenient cases the closure operator possesses good familiar properties. The continuity of morphisms with respect to induced closure operation in general is not immediate and has to be checked. However, in the presence of continuity with respect to closure operations nicer properties are ensured as summarised by Table 1; however, several other properties do not need the presence of continuity with respect to the induced closure operation. Thus apart from the generalisation that the method allows it also reveals the extent to which the condition of continuity (with respect to closure operations) is required in obtaining familiar properties of well known classes of morphisms. It is essential to emphasise the generalisation obtained herein is conservative.

The notation and terminology adopted in this paper are largely in line with the usage in [57] or [3]. Finally, for any set I, the symbol 2^I (respectively, $2^I_{<\aleph_0}$) denotes the set of all subsets (respectively, finite subsets) of I and a set A is *small* if it is a member of some set.

2 Preliminaries

This section recalls facts relevant for this paper. In the process some observations are seemingly new.

2.1 This section establishes notations and terms with regards to posets as used in this paper.

Given a poset P with a smallest element $\mathsf{0}$ and a largest element $\mathsf{1}$, an order preserving endomap $P \xrightarrow{f} P$ is called *extensional* if $x \leq f(x)$ $(x \in P)$, *grounded* if $f(0) = 0$. A grounded and extensional order preserving endomap on P is called a *closure operation* on P and $\mathsf{EGM}(P)$ is the set of all closure operations on P. Evidently $\mathsf{EGM}(P)$ is ordered pointwise, i.e., $c \le d$ if $c(x) \le d(x)$, for each $x \in P$, where $c, d \in \mathsf{EGM}(P)$. The poset $\mathsf{EGM}(P)$ has smallest closure operation $\mathbf{1}_P$ and largest closure operation $P \xrightarrow{\lambda} P$, where $\lambda(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise} \end{cases}$. A closure operation $c \in \mathsf{EGM}(P)$ is *idempotent* if $c \circ c = c$. For each $c \in \mathsf{EGM}(P)$, $\mathtt{Fix}[c] = \{x \in P : c(x) = x\}$ is the set of *fixed points* of c.

Given a complete lattice L, for each $c \in \mathsf{EGM}(L)$, $0, 1 \in \mathtt{Fix}[c]$ and $\mathtt{Fix}[c]$ is closed under arbitrary meets, i.e., $\mathtt{Fix}[c]$ is a complete bounded sub-$\wedge$-semilattice of L. Define:

$$\hat{c}(y) = \bigwedge\{x \in \mathtt{Fix}[c] : y \le x\}, \qquad \text{for } y \in L. \tag{2.1}$$

Let $\mathtt{CBSMSL}(L)$ be the set of all complete bounded sub-$\wedge$-semilattices of the complete lattice L. Since an intersection of complete bounded sub-$\wedge$-semilattices is a complete bounded sub-$\wedge$-semilattice, the set $\mathtt{CBSMSL}(L)$ is a complete lattice with $\{0, 1\}$ as the smallest complete bounded sub-$\wedge$-semilattice of L, L the largest complete bounded sub-$\wedge$-semilattice of L and intersection of complete bounded sub-$\wedge$-semilattices as meet. Finally, for each $P \in \mathtt{CBSMSL}(L)$ define:

$$\nu_P(y) = \bigwedge\{x \in P : y \le x\}, \qquad \text{for } \in L. \tag{2.2}$$

Proposition 2.1. *For every complete lattice L there is an adjunction*

$$\begin{tikzcd} \mathtt{EGM}(L) \arrow[r, shift left, "\mathtt{Fix}"] \arrow[r, phantom, "\perp"] & \mathtt{CBSMSL(L)}^{\mathtt{op}} \arrow[l, shift left, "\nu"] \end{tikzcd}$$

of partially ordered sets and order preserving maps with $\mathtt{Fix} \circ \nu = \mathbf{1}_{\mathtt{CBSMSL(L)}^{\mathtt{op}}}$.

Furthermore, for any $c \in \mathtt{EGM}(L)$, $\hat{c} = \nu_{\mathtt{Fix}[c]}$ *and for any ordinal* α *if:*

$$c^\alpha = \begin{cases} \mathbf{1}_L, & \text{if } \alpha = 0 \\ c \circ c^\beta, & \text{if } \alpha = \beta + 1 \text{ is a non-limit ordinal}, \\ \bigvee_{\beta<\alpha} c^\beta, & \text{if } \alpha > 0 \text{ is a limit ordinal} \end{cases} \tag{2.3}$$

then $c^\alpha \in \mathsf{EGM}(L)$, $c \le c^\alpha \le \hat{c}$, $\hat{c} \circ c^\alpha = \hat{c} = c^\alpha \circ \hat{c}$ *and* $\widehat{c^\alpha} = \hat{c}$, *for all* $\alpha \ge 1$.

Proof. If $c, d \in \mathtt{EGM}(L)$, $P, Q \in \mathtt{CBSMSL}(L)$ then:

i. for any $y \in L$, since P is a complete sub-$\wedge$-semilattice of L, $\nu_P(y) \in P$, proving $\nu_P \in \mathtt{EGM}(L)$ is idempotent (i.e., $\nu_P \circ \nu_P = \nu_P$) and $\mathtt{Fix}[\nu_P] = P$;

ii. $c \leq d$ and $x \in \mathtt{Fix}[d]$ imply $c(x) \leq d(x) = x$, proving the map

$$\mathtt{EGM}(L) \xrightarrow{\mathtt{Fix}} \mathtt{CBSMSL}(L)^{\mathtt{op}}$$

is order preserving;

iii. if $P \subseteq Q$ then for each $x \in L$, then $\nu_Q(x) = \bigwedge\{t \in Q : x \leq t\} \leq \bigwedge\{t \in P : x \leq t\} = \nu_P(x)$, proving $\mathtt{CBSMSL}(L)^{\mathtt{op}} \xrightarrow{\nu} \mathtt{EGM}(L)$ is order preserving;

iv. $c \leq \nu_P \Leftrightarrow \Big(t \in P \Rightarrow \big(x \leq t \Rightarrow c(x) \leq t\big)\Big) \Leftrightarrow P \subseteq \mathtt{Fix}[c]$, proving $\mathtt{Fix} \dashv \nu$;

proving the first part of the statement. For the second part, since $c^0 = \mathbf{1}_L \leq c^1 = c \leq c^2 = c \circ c$, transfinite induction implies the first two conditions; hence for any $c \leq d \leq \hat{c}$, $\mathtt{Fix}[c] = \mathtt{Fix}[d] = \mathtt{Fix}[\hat{c}]$, $\hat{d} = \hat{c}$ and $\hat{c} \circ d = \hat{c} = d \circ \hat{c}$, completing the proof. □

Remark 2.2. The idempotent closure operation $\hat{c} = \nu_{\mathtt{Fix}[c]}$ is the smallest idempotent closure operation larger than c; it is called the *idempotent hull of* c (see §4.6, for more properties of $\hat{c}$, [41]).

Remark 2.3. In the context of (2.3), if there exists an ordinal α such that $c^\alpha = c^{\alpha+1}$, then c^α is idempotent and hence $c^\alpha = \hat{c}$ (see §4.6, [41]). Thus, if the underlying set of the lattice L is a small set then for each $c \in \mathsf{EGM}(L)$, $\hat{c} = c^\alpha$ for some ordinal α.

Remark 2.4. The adjunction provides a formula for join in $\mathtt{CBSMSL}(L)$:

$$\bigvee \mathcal{P} = \mathtt{Fix}[\textstyle\bigwedge_{P \in \mathcal{P}} \nu_P], \qquad \text{for all } \mathcal{P} \subseteq \mathtt{CBSMSL}(L). \tag{2.4}$$

Remark 2.5. The complete lattice $\mathtt{CBSMSL}(L)$ of all complete bounded sub-$\wedge$-semilattices of L is dually reflectively embedded inside the complete lattice $\mathtt{EGM}(L)$ of grounded closure operations on L as the idempotent closure operations.

In a lattice L for any $a \in L$ the following two formulae are equivalent:

$$a \neq \mathsf{0} \text{ and } (\forall x \in L)\big(x \leq a \Rightarrow x = \mathsf{0} \text{ or } x = a\big),$$
$$a \neq \mathsf{0} \text{ and } (\forall x \in L)\big(a \leq x \text{ or } a \wedge x = \mathsf{0}\big),$$

and the element a defined by such is an *atom* of L; the set of atoms of L is denoted by $\mathtt{atom}(L)$. A lattice L is *atomic* if for each $x \in L$, there exists $a \in \mathtt{atom}(L)$ with $a \leq x$, and *atom generated* if $x = \bigvee\big\{a \in \mathtt{atom}(L) : a \leq x\big\}$[2]. Evidently, every atom generated lattice is atomic, but the converse need not be true — e.g., consider the lattice of divisors of a positive natural number. In case of the lattice $\mathtt{Sub}_{\mathsf{M}}(X)$ (see §2.2), the symbol $\mathtt{atom}(X)$ abbreviate $\mathtt{atom}(\mathtt{Sub}_{\mathsf{M}}(X))$.

Recall from [52]: in a complete lattice L, a *pseudocomplement* of $a \in L$ is the element $a^* \in L$ such that $x \leq a^* \Leftrightarrow x \wedge a = \mathsf{0}$, i.e., $a^* = \max\big\{x \in L : a \wedge x = \mathsf{0}\big\}$; a complete lattice is *pseudocomplemented* if every element has a pseudocomplement. Evidently, $\mathsf{0}^* = \mathsf{1}$, $\mathsf{1}^* = \mathsf{0}$, $p \leq q \Rightarrow q^* \leq p^*$, the assignment $x \mapsto x^{**}$ is an idempotent closure operation on L, $(p \wedge q)^{**} = p^{**} \wedge q^{**}$ and $(\bigvee S)^* = \bigwedge_{s \in S} s^*$ $(S \subseteq L)$. Clearly, $p \parallel p^* \Leftrightarrow p, p^* \neq \mathsf{0}$, where $x \parallel y$ means x and y are incompatible.

An element $p \in L$ is said to be *implicative* if the order preserving endomap $L \xrightarrow{p\wedge-} L$ has a right adjoint $L \xrightarrow{(p \Longrightarrow -)} L$. Evidently, $p \in L$ is implicative if and only if $p \wedge -$ preserve arbitrary joins. Note: $p^* = (p \Longrightarrow \mathsf{0})$.

For a lattice L, a *down-set* is a subset $D \subseteq L$ such that $x \leq y \in D \Rightarrow x \in D$, and a *principal down-set* is a set of the form $\downarrow p = \big\{x \in L : x \leq p\big\}$ for some $p \in L$. Evidently, a principal down-set $\downarrow p$ is the smallest down-set containing p, and for every down-set D, $D = \bigcup_{p \in D}(\downarrow p)$. The set of down-sets of L is $\mathtt{Dn}(L)$ is a complete lattice with intersections being the meet and

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[2] Usages differ, e.g., in Chapter IV of [2], the terms *atomic* and *atom generated* are not distinguished, as in this paper, and the term *atomic* is used to mean atom generated.

unions as join, the smallest down-set being $\downarrow \mathtt{0} = \{\mathtt{0}\}$ and $\downarrow \mathtt{1} = L$ being the largest down-set.

An *up-set* of L is a down-set of L^{op}, and a *filter* in L is an up-set closed under finite meets.

2.2 Internal preneighbourhood spaces were considered in [49].

Let $\mathbb{A}$ be a finitely complete category with finite coproducts.

A morphism f of $\mathbb{A}$ is said to be *orthogonal* to a morphism g, written $f \perp g$ if there exists a unique morphism w such that $v = g \circ w$ and $u = w \circ f$ whenever $v \circ f = g \circ u$. For $\mathcal{H} \subseteq \mathbb{A}_1$ let ${}^{\perp}\mathcal{H} = \big\{ f \in \mathbb{A}_1 : h \in \mathcal{H} \Rightarrow f \perp h \big\}$, $\mathcal{H}^{\perp} = \big\{ f \in \mathbb{A}_1 : h \in \mathcal{H} \Rightarrow h \perp f \big\}$; then

$$\begin{tikzcd} 2^{\mathbb{A}_1} \arrow[r, shift left, "{{}^{\perp}\_}"] \arrow[r, phantom, "\perp"] & \left(2^{\mathbb{A}_1}\right)^{\mathsf{op}} \arrow[l, shift left, "{\_^{\perp}}"] \end{tikzcd}$$

, a pair $(\mathcal{A}, \mathcal{B})$ of subsets of $\mathbb{A}_1$ is called a *prefactorisation system* if $\mathcal{B} = \mathcal{A}^{\perp}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$; a prefactorisation system $(\mathcal{A}, \mathcal{B})$ is a *factorisation system* if every morphism factors as a $\mathcal{A}$-morphism followed by a $\mathcal{B}$-morphism and a factorisation system $(\mathcal{A}, \mathcal{B})$ is *proper* if $\mathcal{A} \subseteq \mathtt{Epi}(\mathbb{A})$, $\mathcal{B} \subseteq \mathtt{Mono}(\mathbb{A})$. The (possibly large) set of prefactorisation systems on $\mathbb{A}$ is a complete poset with $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$ if $\mathcal{A} \supseteq \mathcal{A}' \Leftrightarrow \mathcal{B} \subseteq \mathcal{B}'$ (for details see §2, [5]). Given a factorisation system $(\mathcal{A}, \mathcal{B})$, $\mathcal{A} \subseteq \mathtt{Epi}(\mathbb{A})$ if and only if for each object X the *diagonal* $d_X \in \mathcal{B}$, where $X \overset{d_X = (\mathbf{1}_X, \mathbf{1}_X)}{\rightarrowtail} X \times X$, (see Proposition 14.11, [1]). Hence a factorisation system $(\mathcal{A}, \mathcal{B})$ is proper if and only if for every object X, $d_X \in \mathcal{B}$ and the *codiagonal* $c_X \in \mathcal{A}$, where $X + X \xrightarrow{c_X} X$ is the unique morphism such that $c_X \circ \iota_1 = \mathbf{1}_X = c_X \circ \iota_2$, ι_1, ι_2 are the coproduct injections, and equivalently $\mathtt{ExtEpi}(\mathbb{A}) \subseteq \mathcal{A} \subseteq \mathtt{Epi}(\mathbb{A})$ and $\mathtt{ExtMon}(\mathbb{A}) \subseteq \mathcal{B} \subseteq \mathtt{Mono}(\mathbb{A})$.

Given a proper (E, M)-factorisation system, a M-subobject of an object X, also called an *admissible subobject* of X, is a $m \in \mathsf{M}$ with codomain X, any two equivalent admissible subobjects of X considered equal. The set of admissible subobjects of X is denoted by $\mathtt{Sub}_{\mathsf{M}}(X)$. In this paper the morphisms of E are depicted with arrows like $\twoheadrightarrow$ while the morphisms of M are depicted with arrows like $\rightarrowtail$. If $X \xrightarrow{f} Y$ be a morphism, then $X \overset{f^{\mathsf{E}}}{\twoheadrightarrow} \mathcal{I}_f \overset{f^{\mathsf{M}}}{\rightarrowtail} Y$ is the (E, M) factorisation of f; more generally, if $m \in \mathtt{Sub}_{\mathsf{M}}(X)$ (respectively, $n \in \mathtt{Sub}_{\mathsf{M}}(Y)$) then the *image* of m (respectively, *preimage* of n) under f is $\exists_f m$ (respectively, $f^{-\mathbf{1}}n$), where $f \circ m = (\exists_f m) \circ (f\big|_m)$ (respectively, $f \circ (f^{-\mathbf{1}}n) = n \circ f_n$) is the (E,M)-factorisation of $f \circ m$ (respectively, pullback of n along f), $(f\big|_m)$ is the *restriction of f on m* (respectively, f_n is the *corestriction* of f on n); obviously $f^{\mathsf{E}} = (f\big|_{\mathbf{1}_X})$ and $f^{\mathsf{M}} = \exists_f \mathbf{1}_X$. In presence of finite limits and finite coproducts, $\mathtt{Sub}_{\mathsf{M}}(X)$ is a lattice, the largest subobject is $\mathbf{1}_X$ and the smallest object is σ_X, where $\emptyset \overset{\mathtt{i}_{\emptyset_X}}{\twoheadrightarrow} \emptyset_X \overset{\sigma_X}{\rightarrowtail} X$ is the (E,M)-factorisation of the unique morphism $\mathtt{i}_X$ from the initial object to X. The image and preimage induce adjunction $\begin{tikzcd} \mathtt{Sub}_{\mathsf{M}}(X) \arrow[r, shift left, "\exists_f"] \arrow[r, phantom, "\perp"] & \mathtt{Sub}_{\mathsf{M}}(Y) \arrow[l, shift left, "f^{-\mathbf{1}}"] \end{tikzcd}$ for each morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$. A filter F on X is a filter in $\mathtt{Sub}_{\mathsf{M}}(X)$; the (possibly large) set of filters on X is $\mathtt{Fil}\,[X]$, which is a complete algebraic lattice, distributive if and only if $\mathtt{Sub}_{\mathsf{M}}(X)$ is distributive (see Theorem 1.2 [55] or Proposition 2.7, Corollary 2.8 [49]), with compact elements $\uparrow p = \big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \geq p\big\}$ $(p \in \mathtt{Sub}_{\mathsf{M}}(X))$. For each morphism $X \xrightarrow{f} Y$ the adjunction $\exists_f \dashv f^{-\mathbf{1}}$ induce adjunctions $\begin{tikzcd} \mathtt{Fil}\,[X] \arrow[r, shift left, "\overrightarrow{f}"] \arrow[r, phantom, "\top"] & \mathtt{Fil}\,[Y] \arrow[l, shift left, "\overleftarrow{f}"] \end{tikzcd}$, where:

$$\overrightarrow{f} A = \big\{y \in \mathtt{Sub}_{\mathsf{M}}(Y) : f^{-\mathbf{1}}y \in A\big\}, \qquad \text{for } A \in \mathtt{Fil}\,[X], \tag{2.5}$$

and

$$\overleftarrow{f} B = \big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : (\exists b \in B)(f^{-\mathbf{1}}b \leq x)\big\}, \qquad \text{for } B \in \mathtt{Fil}\,[Y]. \tag{2.6}$$

A connection between the smallest subobjects of objects need to be highlighted.

Proposition 2.6. *For every $Z \xrightarrow{g} Y$, there exists a unique $\emptyset_Z \overset{\omega_{Z,Y}}{\twoheadrightarrow} \emptyset_Y$ such that $g \circ \sigma_Z = \sigma_Y \circ \omega_{Z,Y}$ is the (E, M)-factorisation of $g \circ \sigma_Z$.*

In particular: $g \in \mathsf{M} \Rightarrow \omega_{Z,Y} \in \mathtt{Iso}(\mathbb{A})$, if $\mathbb{A}\,(Y, Z) \neq \emptyset$ then $\omega_{Y,Z} = {\omega_{Z,Y}}^{-1}$, and hence $\omega_{Y,Y} = \mathbf{1}_{\emptyset_Y}$.

Proof. Given a morphism $Z \xrightarrow{g} Y$, consider the diagram:

$$\begin{tikzcd}
& \emptyset \arrow[dl, two heads, "\mathtt{i}_{\emptyset_Z}"'] \arrow[dr, two heads, "\mathtt{i}_{\emptyset_Y}"] \arrow[ddl, bend right=60, "\mathtt{i}_Z"'] \arrow[ddr, bend left=60, "\mathtt{i}_Y"] & \\
\emptyset_Z \arrow[rr, dotted, two heads, "!\, w_g"] \arrow[d, tail, "\sigma_Z"'] & & \emptyset_Y \arrow[d, tail, "\sigma_Y"] \\
Z \arrow[rr, "g"'] & & Y
\end{tikzcd} \quad . \tag{2.7}$$

Since $g \circ \sigma_Z \circ \mathtt{i}_{\emptyset_Z} = g \circ \mathtt{i}_Z = \mathtt{i}_Y = \sigma_Y \circ \mathtt{i}_{\emptyset_Y}$, there exists the unique morphism w_g such that $w_g \circ \mathtt{i}_{\emptyset_Z} = \mathtt{i}_{\emptyset_Y}$ and $g \circ \sigma_Z = \sigma_Y \circ w_g$; evidently, $w_g \in \mathsf{E}$. If $Z \xrightarrow{g'} Y$ is any other morphism, then $w_{g'} \circ \mathtt{i}_{\emptyset Z} = \mathtt{i}_{\emptyset_Y} = w_g \circ \mathtt{i}_{\emptyset_Z}$ implies $w_g = w_{g'}$, i.e., w_g is independent of the choice of g; furthermore, it is evident that $w_g = (g\big|_{\sigma_Z})$ and the square represents the (E, M)-factorisation of $g \circ \sigma_Z$. Taking $\omega_{Y,Z} = w_g$ completes the proof. $\square$

Thus: $\emptyset_{\mathbf{1}}$ is an E-image of each $\emptyset_Y$ and if $\mathtt{i}_{\mathbf{1}}$ is an admissible monomorphism then each $\emptyset_Y \simeq \emptyset$ (see Theorem 2.11(f), Remark 2.12).

Preneighbourhood systems can now be defined.

Definition 2.7. (a) A *context* is $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$, where $\mathbb{A}$ is a finitely complete category with finite coproducts and a proper factorisation system (E, M) such that for each object X, $\mathtt{Sub}_{\mathsf{M}}(X)$ is a complete lattice.

(b) An order preserving map $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathsf{op}} \xrightarrow{\mu} \mathtt{Fil}\,[X]$ is a *preneighbourhood system* if $\mu(\sigma_X) = \mathtt{Sub}_{\mathsf{M}}(X)$ and $p \in \mu(m) \Rightarrow m \leq p$; if further, $p \in \mu(m) \Rightarrow (\exists q \in \mu(m))(p \in \mu(q))$ then μ is a *weak neighbourhood system*; moreover if $\mu(\bigvee S) = \bigcap_{s \in S} \mu(s)$ $(S \subseteq \mathtt{Sub}_{\mathsf{M}}(X))$ then μ is a *neighbourhood system.* A pair (X, μ), where X is an object of $\mathbb{A}$ and μ is a preneighbourhood system on X is called an *internal preneighbourhood* space. Likewise for *internal weak neighbourhood space* and *internal neighbourhood space.*

(c) If (X, μ) and (Y, ϕ) are internal preneighbourhood spaces then a morphism $X \xrightarrow{f} Y$ is a *preneighbourhood morphism* if $p \in \phi(u) \Rightarrow f^{-\mathbf{1}}p \in$

$\mu(f^{-1}u)$; if (X, μ) and (Y, ϕ) are internal neighbourhood spaces and f^{-1} preserve joins then it is a *neighbourhood morphism.* The category of internal preneighbourhood spaces and preneighbourhood morphisms is $\mathtt{pNbd}[\mathbb{A}]$; $\mathtt{wNbd}[\mathbb{A}]$ is the full subcategory of internal weak neighbourhood spaces and $\mathtt{Nbd}[\mathbb{A}]$ is the subcategory of internal neighbourhood spaces and neighbourhood morphisms.

(d) Given preneighbourhood system μ on X, a subobject $p \in \mathtt{Sub}_{\mathsf{M}}(X)$ is *μ-open* if $p \in \mu(p)$; the (possibly large) set of all μ-open sets is $\mathfrak{O}_\mu$.

(e) A neighbourhood system μ on X is a *topology* on X if $\mathfrak{O}_\mu$ is a frame in the partial order of $\mathtt{Sub}_{\mathsf{M}}(X)$. If μ is a topology on X then (X, μ) is an *internal topological space*, $\mathtt{Top}[\mathbb{A}]$ is the full subcategory of $\mathtt{Nbd}[\mathbb{A}]$ of all internal topological spaces.

(f) A morphism $X \xrightarrow{f} Y$ is *formally surjective* (or, also referred to in literature as *semistable*, e.g., in [66]) if for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$ there exists a $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ such that $y = \exists_f x$, or equivalently for every $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$ the corestriction f_y is in E.

(g) A morphism $X \xrightarrow{f} Y$ is a *Frobenius morphism* if for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ and $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $\exists_f(x \wedge f^{-1}y) = y \wedge \exists_f x$.

(h) A morphism $X \xrightarrow{f} Y$ is said to *reflect zero* if $f^{-1}\sigma_Y = \sigma_X$.

Remark 2.8. The condition (c) is often called a *continuity condition* with respect to preneighbourhood systems.

Remark 2.9. The set of preneighbourhood systems on X is $\mathtt{pnbd}[X]$; likewise $\mathtt{wnbd}[X]$, $\mathtt{nbd}[X]$, $\mathtt{top}[X]$ denote the set of weak neighbourhood systems, neighbourhood systems and topologies on X respectively. Each of $\mathtt{pnbd}[X]$, $\mathtt{wnbd}[X]$, $\mathtt{nbd}[X]$ are complete lattices (see Theorem 3.17 & Theorem 3.32, [49]), while $\mathtt{top}[X]$ is a complete sublattice of $\mathtt{nbd}[X]$ if and only if there exists a largest topology on X (see Theorem 3.36, [49]).

Remark 2.10. Given the preneighbourhood systems μ on X, ϕ on Y, a morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$ is a preneighbourhood morphism if and only if for any $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$ any one of the following three conditions is true: $\overleftarrow{f}\phi(y) \subseteq \mu(f^{-\mathbf{1}}y)$, $\phi(y) \subseteq \overrightarrow{f}\mu(f^{-\mathbf{1}}y)$, $\overleftarrow{f}\phi(\exists_f x) \subseteq \mu(x)$, see Theorem 3.40 [49][3]. The symbol $(X,\mu) \xrightarrow{f} (Y,\phi)$ is used to denote f is a preneighbourhood morphism.

Contexts abound — if $\mathbb{A}$ is finitely complete, finitely cocomplete and has *all* intersections then there is a $(\mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$-factorisation system on $\mathbb{A}$; in particular, every small complete, small cocomplete category $\mathbb{A}$, if well powered have a context $\mathcal{E} = (\mathbb{A}, \mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$, and if co-well powered have a context $\mathcal{M} = (\mathbb{A}, \mathtt{ExtEpi}(\mathbb{A}), \mathtt{Mono}(\mathbb{A}))$. As special cases are the contexts: $(\mathtt{FinSet}, \mathsf{Surjections}, \mathsf{Injections})$ Example 3.7 [49], $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$ Example 3.8 [49], $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$ Example 3.9 & Proposition 3.10 [49], $(\mathtt{Alg}[(\Omega,\Xi)], \mathsf{RegEpi}, \mathsf{Mon})$ Example 3.11 & Proposition 3.12 [49], $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMono})$ Example 3.13 [49], $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMono})$ Example 3.14 [49], every topos with its usual factorisation structure (page 5, (iii), [49]), every lextensive category [6] with a proper factorisation structure (see page, (v), [49]) (and this includes $\mathtt{Cat}$, $\mathtt{CRing}^{\mathtt{op}}$, $\mathtt{Sch}$, $\mathbb{A}^{\mathtt{op}}$ where $\mathbb{A}$ is a Zariski category, see Definition 1.2 [34]). Also given any context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ and any object X of $\mathbb{A}$, $(\mathcal{A} \downarrow X) = ((\mathbb{A} \downarrow X), (\mathsf{E} \downarrow X), (\mathsf{M} \downarrow X))$ is the context where

$$\begin{aligned} (\mathsf{E} \downarrow X) &= \big\{(X,x) \xrightarrow{e} (Y,y) : e \in \mathsf{E}\big\} \\ (\mathsf{M} \downarrow X) &= \big\{(X,x) \xrightarrow{M} (Y,y) : m \in \mathsf{M}\big\}, \end{aligned} \tag{2.8}$$

(see page 5 (iv), [49] and §2.10 [32] for details).

Given a preneighbourhood system μ on X, $x \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\mathtt{int}_\mu x = \bigvee\big\{p \in \mathfrak{O}_\mu : p \leq x\big\} \tag{2.9}$$

is the *μ-interior* of x. Evidently, $\sigma_X, \mathbf{1}_X \in \mathfrak{O}_\mu$, $\mathfrak{O}_\mu$ is closed under arbitrary joins if and only if each $\mathtt{int}_\mu x \in \mathfrak{O}_\mu$ (Theorem 3.20, [49]). Further, if each

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[3] The assignment $x \mapsto \overleftarrow{f}\phi(\exists_f x)$, $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ (respectively, $y \mapsto \overrightarrow{f}\mu(f^{-\mathbf{1}}y)$, $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$) is a preneighbourhood system [49], and is denoted by $\overleftarrow{f}\phi\exists_f$ (respectively, $\overrightarrow{f}\mu f^{-\mathbf{1}}$).

μ-interior is μ-open then for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ the following two conditions are equivalent:

$$\mu(x) = \bigcup\{\uparrow p : x \leq p \in \mathfrak{O}_\mu\} \tag{2.10}$$

$$p \in \mu(x) \Leftrightarrow x \leq \mathtt{int}_\mu p. \tag{2.11}$$

A preneighbourhood system μ is said to have *open interiors* if $\mathtt{int}_\mu x \in \mathfrak{O}_\mu$ for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ and is *open generated* if it has open interiors and satisfies (2.11). If μ have open interiors then:

$$\mathtt{int}_\mu(x \wedge y) = \mathtt{int}_\mu x \wedge \mathtt{int}_\mu y, \tag{2.12}$$

and $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathtt{int}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ is a meet preserving *intensional* (i.e., $\mathtt{int}_\mu x \leq x$ for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$) idempotent endomap with $\mathfrak{O}_\mu$ as its fixed set (Corollary 3.26 [49]). Finally, every neighbourhood system μ is open generated (Theorem 3.27, [49]).

The condition of a morphism reflecting zero shall be used in the paper. This section state some necessary facts about them.

Theorem 2.11. *Given the morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathbb{A}$, the following statements hold.*

(a) The following are equivalent:

  (i) f reflects zero, i.e., $f^{-\mathbf{1}}\sigma_Y = \sigma_X$.

  (ii) For any $x \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\exists_f x = \sigma_Y \Rightarrow x = \sigma_X. \tag{2.13}$$

  (iii) For all $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$:

$$y \wedge \exists_f x = \sigma_Y \Rightarrow x \wedge f^{-\mathbf{1}}y = \sigma_X.$$

(b) If $f^{-\mathbf{1}} \circ \exists_f = \mathbf{1}_{\mathtt{Sub}_{\mathsf{M}}(X)}$ then f reflects zero. In particular, every admissible morphism reflect zero.

(c) The set of morphisms reflecting zero is closed under compositions.

(d) If $g \circ f$ reflects zero then f reflects zero.

(e) For any morphism $X \xrightarrow{f} Y$ reflecting zero and $n \in \mathtt{Sub}_{\mathsf{M}}(Y)$, the corestriction f_n on N reflects zero.

(f) Let $\mathbb{A}$ be a category with pullbacks and initial object $\emptyset$. If every morphism of $\mathbb{A}$ reflect zero then $\emptyset$ is strict. Further if the unique morphism $\emptyset \xrightarrow{\mathtt{i}_1 = \mathtt{t}_\emptyset} \mathtt{1}$ is an admissible monomorphism and $\emptyset$ is strict then every morphism reflects zero.

Proof. Towards the proof of the equivalence in (a): the equivalence of (i) and (ii) follows from the adjunction $\exists_f \dashv f^{-1}$; since $\exists_f(x \wedge f^{-1}y) \leq y \wedge \exists_f x$, (ii) implies (iii), while taking $y = \sigma_Y$ and $x = \mathbf{1}_X$, (iii) implies (i). The statements in (b)-(d) follow from (a). Towards a proof of (e), in the pullback of $m \in \mathtt{Sub}_{\mathsf{M}}(Y)$ along f, if f reflect zero then

$$(f^{-1}m) \circ ({f_m}^{-1}\sigma_M) = f^{-1}(m \circ \sigma_M) = f^{-1}\sigma_Y = \sigma_X$$

implies ${f_m}^{-1}\sigma_M = \sigma_{f^{-1}M}$, since $f^{-1}m \in \mathsf{M}$, proving (e). Finally, the proof of (f) follows from Proposition 2.6. □

Remark 2.12. A finitely complete category with an initial object is *quasi-pointed* (see §1 [4], [51]) if the unique morphism $\emptyset \xrightarrow{\mathtt{i}_1} \mathtt{1}$ is a monomorphism. In many contexts, e.g., in $(\mathtt{FinSet}, \mathsf{Surjections}, \mathsf{Injections})$, $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$, $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMono})$ or $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMon})$ the unique morphism $\mathtt{i}_1$ is a regular monomorphism, and hence an admissible monomorphism. A context $\mathcal{A}$ is called *admissibly quasi-pointed* if its underlying category $\mathbb{A}$ has the unique morphism $\mathtt{i}_1$ an admissible monomorphism. Thus in an admissibly quasi-pointed context, the initial object is strict if and only if every morphism reflects zero.

Remark 2.13. Using Proposition 2.6 given the coterminating morphisms f and g consider the diagram in (▲) where the front vertical square is the pullback of f along g, the top horizontal square is the pullback of σ_Z along f_g; if both f and g reflect zero then the vertical right hand and base horizontal squares are pullback squares, enabling the existence of the unique morphism w to make the whole diagram commute; further all the squares are pullback squares; in particular, ${f_g}^{-\mathbf{1}}\sigma_Z = {g_f}^{-\mathbf{1}}\sigma_X$. Hence f_g reflects zero if and only if g_f reflects zero.

$$\begin{tikzcd}
 & {f_g}^{-\mathbf{1}}\emptyset_Z \arrow[rr, "(f_g)_{\sigma_Z}"] \arrow[dl, "{f_g}^{-\mathbf{1}}\sigma_Z"'] \arrow[dd, dotted, "!w"] & & \emptyset_Z \arrow[dl, "\sigma_Z"] \arrow[dd, twoheadrightarrow, "\omega_{Z,Y}"] \\
X \times_Y Z \arrow[rr, "f_g"] \arrow[dd, "g_f"'] & & Z \arrow[dd, "g"] & \\
 & \emptyset_X \arrow[rr, twoheadrightarrow, "\omega_{X,Y}"] \arrow[dl, "\sigma_X"] & & \emptyset_Y \arrow[dl, "\sigma_Y"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd} , \qquad (\blacktriangle)$$

Definition 2.14. A context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ called a *reflecting zero context* if all morphisms reflect zero.

This section exhibits a connection between formally surjective morphisms and Frobenius morphisms.

Proposition 2.15. *Every Frobenius* E*-morphism is formally surjective; if* $X \xrightarrow{f} Y$ *is formally surjective and* $\exists_f$ *is a homomorphism of meet semilattices then* f *is Frobenius.*

Proof. If f is a Frobenius morphism then for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$:

$$\exists_f f^{-\mathbf{1}}y = \exists_f(\mathbf{1}_X \wedge f^{-\mathbf{1}}y) = y \wedge \exists_f \mathbf{1}_X;$$

since $\exists_f \mathbf{1}_X = \mathbf{1}_Y \Leftrightarrow f \in \mathsf{E}$, every Frobenius E-morphism is formally surjective. The second part is trivial. $\square$

Remark 2.16. If $\mathtt{FS}[\mathbb{A}]$ (respectively, $\mathtt{FR}[\mathbb{A}]$) denote the (possibly large) set of formally surjective (respectively, Frobenius) morphisms of $\mathbb{A}$ then the following connections:

$$\begin{tikzcd}
\boxed{\mathsf{E}\text{ is pullback stable}} \arrow[r, Rightarrow] & \boxed{\mathsf{E} \subseteq \mathtt{FS}[\mathbb{A}]} \arrow[r, Rightarrow] & \boxed{\mathbb{A}_1 \subseteq \mathtt{FR}[\mathbb{A}]} \arrow[d, Rightarrow] \\
 & & \boxed{\mathsf{E} \subseteq \mathtt{FR}[\mathbb{A}]} \arrow[ul, Rightarrow]
\end{tikzcd}, \tag{2.14}$$

are well known (Theorem 5.13 [49] or Proposition 1.3 [30]).

The forgetful functor $\mathtt{pNbd}[\mathbb{A}] \xrightarrow{U} \mathbb{A}$ is a topological functor (Theorem 4.8(a) [49]). Consequently, each limit (respectively, colimit) object, unless mentioned to the contrary, is considered as an internal preneighbourhood space with the smallest (respectively, largest) preneighbourhood system which make each of the components of the limiting (respectively, colimiting) cone preneighbourhood morphisms. Thus, for instance:

i. The terminal object $\mathtt{1}$ being the empty product is always equipped with the smallest preneighbourhood system $\nabla_{\mathtt{1}}$ (see (2.18)). Note: the lattice $\mathtt{Sub}_{\mathsf{M}}(\mathtt{1})$ is not always trivial. For instance, in the context $(\mathtt{CRing}^{\mathtt{op}}, \mathsf{Epi}, \mathsf{RegMono})$ the terminal object is the commutative ring $\mathbb{Z}$ of integers, $\mathtt{Sub}_{\mathsf{RegMono}}(\mathbb{Z}) = \{n\mathbb{Z} : n \geq 0\}$, hence $\nabla_{\mathbb{Z}} <\uparrow_{\mathbb{Z}}$.

ii. Given an admissible monomorphism $M \overset{m}{\rightarrowtail} X$ and a preneighbourhood system μ on X, M is equipped with $(\mu\big|_m)$, where for any $a \in \mathtt{Sub}_{\mathsf{M}}(M)$:

$$\begin{aligned}(\mu\big|_m)(a) &= \big\{u \in \mathtt{Sub}_{\mathsf{M}}(M) : (\exists v \in \mu(m\circ a))(m^{-\mathbf{1}}v \leq u)\big\} \\ &= \big\{u \in \mathtt{Sub}_{\mathsf{M}}(X) : (\exists v \in \mu(m\circ a))(v \wedge m \leq m\circ u)\big\}.\end{aligned} \tag{2.15}$$

iii. The binary product $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ in $\mathbb{A}$ of the preneighbourhood spaces (X, μ), (Y, ϕ) is equipped with $\mu \times \phi$, where for any $(x, y) \in \mathtt{Sub}_{\mathsf{M}}(X \times Y)$:

$$(\mu \times \phi)(x, y) = \overleftarrow{p_1}\mu(x^{\mathsf{M}}) \vee \overleftarrow{p_2}\phi(y^{\mathsf{M}}). \tag{2.16}$$

iv. If $(X,\mu) \xrightarrow{f} (Z,\psi) \xleftarrow{g} (Y,\phi)$ and

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f_g"] \arrow[d, "g_f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is the pullback of f along g in $\mathbb{A}$ then $X \times_Z Y$ is equipped with $\mu \times_\psi \phi$, where for any $(x,y) \in \mathtt{Sub}_{\mathsf{M}}(X \times_Z Y)$:

$$(\mu \times_\psi \phi)(x,y) = \overleftarrow{g_f}\mu(x^{\mathsf{M}}) \vee \overleftarrow{f_g}\phi(y^{\mathsf{M}}). \tag{2.17}$$

In any context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ the (possibly large) set $\mathtt{pnbd}[X]$ of all preneighbourhood systems on X is a complete lattice (Theorem 3.17 [49]). The smallest is the *indiscrete* neighbourhood system $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathsf{op}} \xrightarrow{\nabla_X} \mathtt{Fil}\,[X]$ and the largest is the *discrete* neighbourhood system $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathsf{op}} \xrightarrow{\uparrow_X} \mathtt{Fil}\,[X]$, where

$$\nabla_X(x) = \begin{cases} \mathtt{Sub}_{\mathsf{M}}(X), & \text{if } x = \sigma_X \\ \{\mathbf{1}_X\}, & \text{if } x \neq \sigma_X \end{cases},$$

$$\text{and } \uparrow_X (x) = \big\{p \in \mathtt{Sub}_{\mathsf{M}}(X) : x \leq p\big\} \tag{2.18}$$

for any $x \in \mathtt{Sub}_{\mathsf{M}}(X)$.

Example 2.17. In the context $(\mathtt{FinSet}, \mathsf{Surjections}, \mathsf{Injections})$ the internal preneighbourhood systems are precisely extensional order preserving endomaps on the lattice 2^X of all subsets of X, the internal weak neighbourhood systems are the order preserving extensional idempotent endomaps on 2^X and the internal neighbourhood systems are the Kuratowski closure operations on 2^X (Example 3.7 [49]). Every neighbourhood system on a finite set precisely yield topologies (Corollary 2.13 & Figure 1 [49] for details).

Example 2.18. In the context $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$ the internal neighbourhood systems on X are precisely the topologies on X (Corollary 2.13, Example 3.8 & Figure 1 [49] for details).

Example 2.19. In the context $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMono})$, a preneighbourhood system is specified by a preneighbourhood system on the underlying set of the topological space; preneighbourhood morphisms are continuous functions which are preneighbourhood morphisms with respect to the involved preneighbourhood systems. In particular, neighbourhood systems on a topological space X is a second topology on the underlying set of the space X producing bitopological spaces (Example 3.13 [49]) and preneighbourhood morphisms are continuous functions which are also continuous with respect to the second topologies are neighbourhood morphisms.

Example 2.20. In the context $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMono})$ (Example 3.14 [49]) a special neighbourhood system shall be considered in this paper, namely the *T-neighbourhood system.* More precisely, given a locale X, the *T-neighbourhood system* on it is $\mathsf{Sub}_{\mathsf{RegMono}}(X)^{\mathsf{op}} \xrightarrow{\tau_X} \mathtt{Fil}\,[X]$:

$$\tau_X(S) = \big\{T \in \mathsf{Sub}_{\mathsf{RegMono}}(X) : (\exists a \in X)(S \subseteq \mathfrak{o}(a) \subseteq T)\big\}, \tag{2.19}$$

where $\mathfrak{o}(a) = \big\{\ (a \Longrightarrow x) : x \in L\big\}$ is the open sublocale for $a \in X$ (see §6.1.1 [58]), is an example of a functorial neighbourhood system on X (Theorem 3.38 & Definition 4.3 [49]). T-neighbourhood systems have been used extensively in [45, 46]. Since for a localic map $X \xrightarrow{f} Y$, the preimage $f^{-\mathbf{1}}$ does not preserve arbitrary joins, with X and Y empowered with T-neighbourhood systems, f is merely a preneighbourhood morphism and not a neighbourhood morphism. Furthermore, since $\mathsf{Sub}_{\mathsf{RegMono}}(X)$ is a coframe, and not a frame, neighbourhood systems on a locale is not an internal topology, internal topologies on a locale is not a reflective subcategory of neighbourhood spaces, (Theorem 4.8 [49]).

Example 2.21. In the context $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$, for a group X and a subgroup $A \subseteq X$, let $\mathsf{ncl}_X(A)$ denote the normal subgroup of X generated by A. The order preserving map $\mathsf{Sub}_{\mathsf{Mon}}(X)^{\mathsf{op}} \xrightarrow{\nu_X} \mathtt{Fil}\,[X]$ defined by:

$$\begin{aligned}\nu_X(A) &= \big\{U \in \mathsf{Sub}_{\mathsf{Mon}}(X) : \mathsf{ncl}_X(A) \subseteq U\big\}\\ &= \big\{U \in \mathsf{Sub}_{\mathsf{Mon}}(X) : (\exists N \lhd X)(A \subseteq N \subseteq U)\big\}\end{aligned} \tag{2.20}$$

is a preneighbourhood system on X. Since normal subgroups are closed under joins and intersections, ν_X is actually an internal neighbourhood system on the group X; moreover, every group homomorphism $X \xrightarrow{f} Y$ is a preneighbourhood morphism from the internal neighbourhood space (X, ν_X) to (Y, ν_Y), making ν_X a functorial preneighbourhood system, (Definition 4.3 [49]).

Example 2.22. In the context $(\mathtt{CRng}, \mathsf{RegEpi}, \mathsf{Mon})$, since $\mathtt{CRng}$ have objects commutative rings without identity, every ideal is a subring and feature as admissible monomorphisms. Given any ring X, a subring $A \subseteq X$, let $\mathsf{idl}_X(A)$ denote the ideal of X generated by A. The order preserving map $\mathtt{Sub}_{\mathsf{Mon}}(X)^{\mathsf{op}} \xrightarrow{\iota_X} \mathtt{Fil}\,[X]$ defined by:

$$\begin{aligned}\iota_X(A) &= \big\{U \in \mathtt{Sub}_{\mathsf{Mon}}(X) : \mathsf{idl}_X(A) \subseteq U\big\}\\ &= \big\{U \in \mathtt{Sub}_{\mathsf{Mon}}(X) : (\exists I \in \mathtt{Idl}\,[X])(A \subseteq I \subseteq U)\big\}\end{aligned} \tag{2.21}$$

is a preneighbourhood system on X. Since ideals are closed under joins and intersections, ν_X is actually an internal neighbourhood system on the group X; moreover, every ring homomorphism $X \xrightarrow{f} Y$ is a preneighbourhood morphism from the internal neighbourhood space (X, ι_X) to (Y, ι_Y), making ι_X a functorial preneighbourhood system (Definition 4.3 [49]).

3 Closure operations

This section introduce a *closure operation* on each preneighbourhood space and investigate its properties.

3.1 A preneighbourhood system μ on an object X induces the set:

$$\mathtt{Far}_{\mu}p = \big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : (\exists u \in \mu(x))(u \wedge p = \sigma_X)\big\}, \qquad p \in \mathtt{Sub}_{\mathsf{M}}(X) \tag{3.1}$$

of admissible subobjects of X which are *far away* from p, with respect to the preneighbourhood system μ.

Lemma 3.1. *For every object X of $\mathbb{A}$, $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathsf{op}} \times \mathtt{pnbd}[X] \xrightarrow{\mathtt{Far}} \mathtt{Dn}(X)$[4] is an order preserving function between complete lattices such that for any*

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[4] $\mathtt{Dn}(X)$ is abbreviation for $\mathtt{Dn}(\mathtt{Sub}_{\mathsf{M}}(X))$

set I, $\mu, \mu_i \in \texttt{pnbd}[X]$ $(i \in I)$, $p, q \in \texttt{Sub}_{\mathsf{M}}(X)$:

$$\texttt{Far}_{\mu}\sigma_X = \texttt{Sub}_{\mathsf{M}}(X) \textit{ and } \texttt{Far}_{\mu}\mathbf{1}_X = \{\sigma_X\}, \tag{3.2}$$

$$\texttt{Far}_{\nabla_X} p = \{\sigma_X\} \textit{ when } p \neq \sigma_X \textit{ and } \texttt{Far}_{\uparrow_X} p = \big\{x \in \texttt{Sub}_{\mathsf{M}}(X) : x \wedge p = \sigma_X\big\}, \tag{3.3}$$

$$x, p \neq \sigma_X, x \in \texttt{Far}_{\mu} p \Rightarrow x \parallel p, \tag{3.4}$$

$$\texttt{Far}_{\bigwedge_{i\in I}\mu_i} p = \bigcap_{i\in I} \texttt{Far}_{\mu_i} p, \tag{3.5}$$

if p is implicative,

$$\texttt{Far}_{\bigvee_{i\in I}\mu_i} p = \bigcup_{J \in 2^I_{<\aleph_0}} \texttt{Far}_{\mu_J} p, \tag{3.6}$$

where $\mu_J = \bigvee_{j\in J} \mu_j,\ J \in 2^I_{<\aleph_0}$,

$$\texttt{Far}_{\mu}(p \vee q) = \texttt{Far}_{\mu} p \cap \texttt{Far}_{\mu} q, \tag{3.7}$$

if $\texttt{Sub}_{\mathsf{M}}(X)$ *is distributive,*

$$(\forall i \in I)(x_i \in \texttt{Far}_{\mu} p) \Rightarrow \bigvee_{i\in I} x_i \in \texttt{Far}_{\mu} p, \tag{3.8}$$

if $\texttt{Sub}_{\mathsf{M}}(X)$ *is distributive,*
p is implicative
and μ is a neighbourhood system on X.

Proof. The first part of the statement as well as those in (3.2),(3.3) and (3.4) are simple verification. If $\big(\mu_i\big)_{i\in I}$ is a family of preneighbourhood systems on X, $\texttt{Far}_{\bigwedge_{i\in I}\mu_i} p \subseteq \bigcap_{i\in I} \texttt{Far}_{\mu_i} p$ $(p \in \texttt{Sub}_{\mathsf{M}}(X))$ since for any fixed p, $\texttt{Far}_{-} p$ is monotonic. On the other hand if $x \in \bigcap_{i\in I} \texttt{Far}_{\mu_i} p$ then for each $i \in I$ there exists a $u_i \in \mu_i(x)$ with $u_i \wedge p = \sigma_X$. If p is implicative then for

$u = \bigvee_{i \in I} u_i \in \bigcap_{i \in I} \mu_i(x)$, $u \wedge p = \bigvee_{i \in I}(u_i \wedge p) = \sigma_X$, proving (3.5). Since:

$$\begin{aligned} x \in \mathtt{Far}_{\bigvee_{i\in I}\mu_i} p &\Leftrightarrow (\exists u \in \bigvee_{i \in I} \mu_i(x))(u \wedge p = \sigma_X) \\ &\Leftrightarrow (\exists n \in \mathbb{N})(\exists i_0, i_1, \dots, i_{n-1} \in I) \\ &(\exists u_0 \in \mu_{i_0}(x), u_1 \in \mu_{i_1}(x), \dots, u_{n-1} \in \mu_{i_{n-1}}(x)) \\ &(u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge p = \sigma_X) \\ &\Leftrightarrow (\exists J \in 2^I_{<\aleph_0})(\exists u \in \mu_J(x))(u \wedge p = \sigma_X) \\ &\Leftrightarrow (\exists J \in 2^I_{<\aleph_0})(x \in \mathtt{Far}_{\mu_J} p) \\ &\Leftrightarrow x \in \bigcup_{J \in 2^I_{<\aleph_0}} \mathtt{Far}_{\mu_J} p, \end{aligned}$$

(3.6) is proved. From for any fixed $\mu \in \mathtt{pnbd}[X]$, $\mathtt{Far}_\mu$ is order reversing, $\mathtt{Far}_\mu(p \vee q) \subseteq \mathtt{Far}_\mu p \cap \mathtt{Far}_\mu q$. On the other hand, if $x \in \mathtt{Far}_\mu p \cap \mathtt{Far}_\mu q$ then there exist $u, v \in \mu(x)$ such that $u \wedge p = \sigma_X = v \wedge q$; if $\mathtt{Sub}_{\mathsf{M}}(X)$ is distributive then $(u \wedge v) \wedge (p \vee q) = (u \wedge v \wedge p) \vee (u \wedge v \wedge q) = \sigma_X$, proving (3.7). Finally, if μ is a neighbourhood system on X, for a family $\big(x_i\big)_{i \in I}$ of elements from $\mathtt{Far}_\mu p$, for each $i \in I$ there exists a $u_i \in \mu(x_i)$ such that $u_i \wedge p = \sigma_X$. Since $u = \bigvee_{i \in I} u_i \in \bigcap_{i \in I} \mu(x_i) = \mu(\bigvee_{i \in I} x_i)$, $u \wedge p = \bigvee_{i \in I}(u_i \wedge p) = \sigma_X$, whenever p is implicative, (3.8) stands proved. $\square$

Hence, given any $\mu \in \mathtt{pnbd}[X]$ and $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mathtt{Sub}_{\mathsf{M}}(X)$ is partitioned into four subsets: $\mathtt{Far}_\mu p$ which is a down-set (and a principal down-set in the special case when $\mathtt{Sub}_{\mathsf{M}}(X)$ is a frame and μ a neighbourhood system), the second is $\big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \parallel p\big\} \cap (\mathtt{Far}_\mu p)^c$, the third is $\big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x > p\big\}$ and the fourth is the principal down-set $\downarrow p$.

Definition 3.1. Given any internal preneighbourhood space (X, μ) define:

$$\mathrm{cl}_\mu p = \bigvee \big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : p \nless x \notin \mathtt{Far}_\mu p\big\} \tag{3.9}$$

the *μ-closure of p*, $\mathfrak{C}_\mu = \mathtt{Fix}[\mathrm{cl}_\mu] = \big\{p \in \mathtt{Sub}_{\mathsf{M}}(X) : p = \mathrm{cl}_\mu p\big\}$ is the (possibly large) set of *μ-closed admissible subobjects* of X.

Remark 3.2. Evidently $\mathrm{cl}_\mu p = p \vee \bigvee\big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \parallel p \text{ and } x \notin \mathtt{Far}_\mu p\big\}$; moreover:

$$\sigma_X \neq x < \mathrm{cl}_\mu p \Leftrightarrow p \nless x \notin \mathtt{Far}_\mu p;$$

noting the *strict inequality* on the left hand side above. Thus for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$ the statements:

(i) the set $\mathsf{N}_{\mu,p} = \{x \in \mathtt{Sub}_{\mathsf{M}}(X) : p \not< x \notin \mathtt{Far}_{\mu}p\}$ has a largest element;

(ii) $x \parallel p \Rightarrow x \in \mathtt{Far}_{\mu}p$ for any $x \in \mathtt{Sub}_{\mathsf{M}}(X)$;

(iii) the sets:

$$\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \leq p\}, \quad \{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x > p\} \quad \text{and} \quad \mathtt{Far}_{\mu}p$$

make a partition of $\mathtt{Sub}_{\mathsf{M}}(X)$;

(iv) $\mathrm{cl}_{\mu}p = p$;

are equivalent.

In particular:

$$p < \mathrm{cl}_{\mu}p \Leftrightarrow \Big(p \not< x \notin \mathtt{Far}_{\mu}p \Rightarrow (\exists y)(p \not< y \notin \mathtt{Far}_{\mu}p \text{ and } x < y)\Big). \quad (3.10)$$

Evidently $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_{\mu}} \mathtt{Sub}_{\mathsf{M}}(X)$ is a closure operation on $\mathtt{Sub}_{\mathsf{M}}(X)$, the idempotent hull of cl_{μ} is $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\widehat{\mathrm{cl}_{\mu}}} \mathtt{Sub}_{\mathsf{M}}(X)$, where:

$$\widehat{\mathrm{cl}_{\mu}}p = \bigwedge\{t \in \mathfrak{C}_{\mu} : p \leq t\},$$

both cl_{μ}, $\widehat{\mathrm{cl}_{\mu}}$ have the same fixed set $\mathfrak{C}_{\mu}$ and for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\widehat{\mathrm{cl}_{\mu}}p$ is the smallest μ-closed admissible subobject of X larger than p. In view of Remark 2.3, if $\mathtt{Sub}_{\mathsf{M}}(X)$ is a small set then $\widehat{\mathrm{cl}_{\mu}} = \bigvee_{\beta \leq \alpha} {\mathrm{cl}_{\mu}}^{\beta}$ for some ordinal α.

Remark 3.3. It shall turn out the condition $p \not< x$ in Definition 3.1 is an obstruction to many familiar properties of closure. One antidote, as shall be exhibited, is $\mathtt{Sub}_{\mathsf{M}}(X)$ being *atom generated.*

Remark 3.4. Obviously, $\mathrm{cl}_{\mu}p \geq \bigvee\{a \in \mathtt{atom}(X) : a \notin \mathtt{Far}_{\mu}p\}$; if $\mathtt{Sub}_{\mathsf{M}}(X)$ is atomic, then for each $\sigma_X \neq x < \mathrm{cl}_{\mu}p$ there exists an atom $a \leq x$ and hence $a < \mathrm{cl}_{\mu}p$ implying $a \notin \mathtt{Far}_{\mu}p$. If further $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated, $x = \bigvee\{a \in \mathtt{atom}(X) : a \leq x\}$ and hence $\mathrm{cl}_{\mu}p \geq \bigvee\{a \in \mathtt{atom}(X) : a \notin \mathtt{Far}_{\mu}p\} \geq x$, proving $\mathrm{cl}_{\mu}p = \bigvee\{a \in \mathtt{atom}(X) : a \notin \mathtt{Far}_{\mu}p\}$.

Thus, in *atom generated* $\mathtt{Sub}_{\mathsf{M}}(X)$, a convenient formula for computing the closure of a subobject is available.

Theorem 3.5. *The following statements are true.*

(a) If $p \in \mathtt{Sub}_{\mathsf{M}}(X)$ *is implicative then for any family* $\left(\mu_i\right)_{i\in I}$ *of preneighbourhood systems on* X*:*

$$\mathrm{cl}_{\bigwedge_{i\in I}\mu_i} p = \bigvee_{i\in I} \mathrm{cl}_{\mu_i} p. \tag{3.11}$$

(b) For any family $\left(\mu_i\right)_{i\in I}$ *of preneighbourhood systems on* X *and* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\mathrm{cl}_{\bigvee_{i\in I}\mu_i} p = \bigwedge_{J\in 2^I_{<\aleph_0}} \mathrm{cl}_{\mu_J} p, \qquad \textit{where } \mu_J = \bigvee_{j\in J} \mu_j. \tag{3.12}$$

(c) If $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented then for any preneighbourhood system* μ *on* X*:*

$$\sigma_X \neq x < \mathrm{cl}_{\mu} p \Leftrightarrow p \not< x \textit{ and } p^* \notin \mu(x). \tag{3.13}$$

Hence: $p \in \mathfrak{C}_\mu$ *(respectively,* $p \in \mathfrak{O}_\mu$*) if and only if* $p^* \in \mathfrak{O}_\mu$ *(respectively,* $p^* \in \mathfrak{C}_\mu$*).*

(d) If $\mathtt{Sub}_{\mathsf{M}}(X)$ *is distributive and atom generated then for any preneighbourhood space* (X, μ)*,* $p, q \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\mathrm{cl}_{\mu}(p \vee q) = \mathrm{cl}_{\mu} p \vee \mathrm{cl}_{\mu} q. \tag{3.14}$$

(e) If μ *is open generated and* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is atom generated then* cl_μ *is idempotent.*

Proof. Evidently from Lemma 3.1 and Definition 3.1, for $\mu, \psi \in \mathtt{pnbd}[X]$, if $\mu \leq \psi$ then $\mathrm{cl}_\psi \leq \mathrm{cl}_\mu$. Hence for any family $\left(\mu_i\right)_{i\in I}$ of preneighbourhood systems on X, $\mathrm{cl}_{\bigvee_{i\in I}\mu_i} \leq \mathrm{cl}_{\mu_i} \leq \mathrm{cl}_{\bigwedge_{i\in I}\mu_i}$, entailing $\mathrm{cl}_{\bigvee_{i\in I}\mu_i} p \leq \bigwedge_{i\in I} \mathrm{cl}_{\mu_i} p \leq$

$\bigvee_{i\in I}\mathrm{cl}_{\mu_i}p \leq \mathrm{cl}_{\bigwedge_{i\in I}\mu_i}p$, for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$. Since:

$$\begin{aligned} \sigma_X \neq x < \mathrm{cl}_{\bigwedge_{i\in I}\mu_i}p &\Leftrightarrow p \not< x \notin \mathtt{Far}_{\bigwedge_{i\in I}\mu_i}p \\ &\Leftrightarrow p \not< x \notin \bigcap_{i\in I}\mathtt{Far}_{\mu_i}p \qquad \text{(if } p \text{ is implicative, (3.5))} \\ &\Leftrightarrow (\exists i \in I)(p \not< x \notin \mathtt{Far}_{\mu_i}p) \\ &\Leftrightarrow (\exists i \in I)(\sigma_X \neq x < \mathrm{cl}_{\mu_i}p) \\ &\Rightarrow \sigma_X \neq x < \bigvee_{i\in I}\mathrm{cl}_{\mu_i}p, \end{aligned}$$

$\mathrm{cl}_{\bigwedge_{i\in I}\mu_i}p \leq \bigvee_{i\in I}\mathrm{cl}_{\mu_i}p$, proving (3.11). Since $\bigvee_{i\in I}\mu_i = \bigvee_{J\in 2^I_{<\aleph_0}}\mu_J$ with $\mu_J = \bigvee_{j\in J}\mu_j$ for $J \in 2^I_{<\aleph_0}$, $\mathrm{cl}_{\bigvee_{i\in I}\mu_i}p = \mathrm{cl}_{\bigvee_{J\in 2^I_{<\aleph_0}}\mu_J}p \leq \bigwedge_{J\in 2^I_{<\aleph_0}}\mathrm{cl}_{\mu_J}p$, for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$; further:

$$\begin{aligned} \sigma_X \neq x < \bigwedge_{J\in 2^I_{<\aleph_0}}\mathrm{cl}_{\mu_J}p &\Leftrightarrow (\forall J \in 2^I_{<\aleph_0})(\sigma_X \neq x < \mathrm{cl}_{\mu_J}p) \\ &\Leftrightarrow (\forall J \in 2^I_{<\aleph_0})(p \not< x \notin \mathtt{Far}_{\mu_J}p) \\ &\Leftrightarrow p \not< x \notin \mathtt{Far}_{\bigvee_{i\in I}\mu_i}p \qquad \text{(using (3.6))} \\ &\Leftrightarrow \sigma_X \neq x < \mathrm{cl}_{\bigvee_{i\in I}\mu_i}p, \end{aligned}$$

shows $\bigwedge_{J\in 2^I_{<\aleph_0}}\mathrm{cl}_{\mu_J}p \leq \mathrm{cl}_{\bigvee_{i\in I}\mu_i}p$, proving (3.12). In case when X is pseudo-complemented, $x \in \mathtt{Far}_{\mu}p \Leftrightarrow (\exists u \in \mu(x))(u \wedge p = \sigma_X) \Leftrightarrow (\exists u \in \mu(x))(u \leq p^*) \Leftrightarrow p^* \in \mu(x)$, proving (3.13). Clearly, $p \parallel p^* \Leftrightarrow p, p^* \neq \sigma_X$; if further:

i. $p \in \mathfrak{C}_\mu$, then $\sigma_X \neq p^* \not< p = \mathrm{cl}_\mu p \Leftrightarrow p < p^*$ or $p^* \in \mu(p^*) \Rightarrow p^* \in \mu(p^*)$ from (3.13) and assumption that $p \parallel p^*$. Hence $p^* \in \mathfrak{O}_\mu$.

ii. if $p \in \mathfrak{O}_\mu$ then $x \leq p \Leftrightarrow p \in \mu(x) \Rightarrow p^{**} \in \mu(x)$, so that (3.13) implies:

$$\sigma_X \neq x < \mathrm{cl}_\mu p^* \Rightarrow p^* \not< x \text{ and } x \not\leq p.$$

Since $p \parallel p^*$, $p \wedge \mathrm{cl}_\mu p^* = \sigma_X \Leftrightarrow \mathrm{cl}_\mu p^* \leq p^* \Leftrightarrow p^* \in \mathfrak{C}_\mu$.

Since $\{\sigma_X, \mathbf{1}_X\} \subseteq \mathfrak{C}_\mu \cap \mathfrak{O}_\mu$, the statements of the second part are trivially true if $p = \sigma_X$ or $p^* = \sigma_X$. This completes the proof of (c). In case $\mathtt{Sub}_{\mathsf{M}}(X)$

is distributive and atom generated then:

$$\mathrm{cl}_\mu(p \vee q) = \bigvee\big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu(p \vee q))^c\big)$$

$$= \bigvee\big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu p \cap \mathtt{Far}_\mu q)^c\big)$$

(using (3.7))

$$= \bigvee\Big(\big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu p)^c\big) \cup \big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu q)^c\big)\Big)$$

$$= \bigvee\big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu p)^c\big) \vee \bigvee\big(\mathtt{atom}(X) \cap (\mathtt{Far}_\mu q)^c\big)$$

$$= \mathrm{cl}_\mu p \vee \mathrm{cl}_\mu q$$

,

proving (d). Towards a proof of (e), if $\mathrm{cl}_\mu p$ is an atom then $p \in \mathfrak{C}_\mu$ or $p = \sigma_X$, and in either case $p = \mathrm{cl}_\mu p = \mathrm{cl}_\mu \mathrm{cl}_\mu p$. If $a \in \mathtt{atom}(X)$ and $a < \mathrm{cl}_\mu \mathrm{cl}_\mu p$, then since μ is open generated, $a \notin \mathtt{Far}_\mu \mathrm{cl}_\mu p$ implies $(\mathtt{int}_\mu u) \wedge \mathrm{cl}_\mu p \neq \sigma_X$ for every $u \in \mu(a)$. Choose and fix any $u \in \mu(a)$. Since $\mathtt{Sub}_{\mathsf{M}}(X)$ is atomic, for every atom $b \leq (\mathtt{int}_\mu u) \wedge \mathrm{cl}_\mu p$, $u \in \mu(b)$ and $(\mathtt{int}_\mu u) \wedge p \neq \sigma_X$ ($\because$, $b < \mathrm{cl}_\mu p$). Since this happens for each $u \in \mu(a)$, $a \notin \mathtt{Far}_\mu(p)$, and hence $a < \mathrm{cl}_\mu p$. Since $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated, (e) stands proved. □

Remark 3.6. A closure operation which preserves finite joins is called *additive* (see conditions (AD) and (GR) of §2.6, [41]). The condition (GR) of [41], however, is already embedded in the definition of a closure operation in this paper.

Remark 3.7. Theorem 3.5 shows if $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated then for every neighbourhood system μ on X, cl_μ is idempotent; furthermore, if $\mathtt{Sub}_{\mathsf{M}}(X)$ is distributive then for any neighbourhood system μ, cl_μ is additive (and hence a Kuratowski closure operation).

Proposition 3.8. *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented and the preneighbourhood system* μ *have open interiors then for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$(\widehat{\mathrm{cl}_\mu p})^* = \mathtt{int}_\mu p^*. \tag{3.15}$$

Proof. From Theorem 3.5(c) for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $p \leq \widehat{\mathrm{cl}_\mu} p \Rightarrow (\widehat{\mathrm{cl}_\mu} p)^* \leq p^* \Leftrightarrow (\widehat{\mathrm{cl}_\mu} p)^* \leq \mathtt{int}_\mu p^*$, since μ has interiors open. On the other hand, if $\mathrm{cl}_\mu p = \mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^* \Leftrightarrow p \leq \mathrm{cl}_\mu p \leq \mathtt{int}_\mu p^* \leq p^* \Leftrightarrow p = \sigma_X$. Hence for each $p \neq \sigma_X$, $\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^* < \mathrm{cl}_\mu p$ and hence $p \not\leq \mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^*$ and $p^* \notin \mu(\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^*)$, provided $\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^* \neq \sigma_X$ (using (3.13)). Since $\mathtt{int}_\mu p^*$ is μ-open, $\mathtt{int}_\mu p^* \in \mu(\mathtt{int}_\mu p^*) \subseteq \mu(\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^*) \Rightarrow p^* \in \mu(\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^*)$, forcing $\mathrm{cl}_\mu p \wedge \mathtt{int}_\mu p^* = \sigma_X$. Since μ has open interiors an use of Theorem 3.5(c) yields: $\mathrm{cl}_\mu p \leq (\mathtt{int}_\mu p^*)^* \Leftrightarrow \widehat{\mathrm{cl}_\mu} p \leq (\mathtt{int}_\mu p^*)^* \Rightarrow \mathtt{int}_\mu p^* \leq (\mathtt{int}_\mu p^*)^{**} \leq (\widehat{\mathrm{cl}_\mu} p)^*$. This completes the proof on observing trivial satisfaction of equality for $p = \sigma_X$. □

Corollary 3.9. *If* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented and atom generated then for any open generated preneighbourhood system* μ *on* X, $(\mathrm{cl}_\mu p)^* = \mathtt{int}_\mu p^*$.

Remark 3.10. Theorem 3.5(c) yields the adjunction on the top row of the diagram:

$$\begin{tikzcd}
\mathfrak{C}_\mu \arrow[r, shift left, "\star"] \arrow[r, phantom, "\perp"] & \mathfrak{O}_\mu{}^{\mathsf{op}} \arrow[l, shift left, "\star"] \\
\mathfrak{C}_\mu{}^* \arrow[u, tail] \arrow[r, shift left, "\star"] \arrow[r, phantom, "\wr|"] & (\mathfrak{O}_\mu{}^*)^{\mathsf{op}} \arrow[u, tail] \arrow[l, shift left, "\star"]
\end{tikzcd} \quad . \tag{3.16}$$

The adjunction restricts to an equivalence to the (possibly large) sets $\mathfrak{C}_\mu{}^\star = \{p \in \mathfrak{C}_\mu : p = p^{**}\}$ of *regular closed* subobjects and $\mathfrak{O}_\mu{}^\star = \{u \in \mathfrak{O}_\mu : u = u^{**}\}$ of *regular open* subobjects. Evidently, regular closed subobjects are closed under arbitrary meets and finite joins and hence regular open subobjects are open under arbitrary joins and finite meets (compare with Theorem 3.20 [49].

3.2 Given any object X, the symbols $\mathtt{EGM}(X)$, $\mathtt{CBSMSL}(X)$ abbreviate $\mathtt{EGM}(\mathtt{Sub}_{\mathsf{M}}(X))$, $\mathtt{CBSMSL}(\mathtt{Sub}_{\mathsf{M}}(X))$ respectively.

Theorem 3.11. *There is an adjunction* $\begin{tikzcd}\mathtt{EGM}(X) \arrow[r, shift left, "\Phi"] \arrow[r, phantom, "\perp"] & \mathtt{pnbd}[X]^{\mathsf{op}} \arrow[l, shift left, "\Psi"]\end{tikzcd}$ *with* $\Psi \circ \Phi = \mathbf{1}_{\mathtt{EGM}(X)}$, *where* $\Phi(c) = \uparrow_X \circ c^{\mathsf{op}}$ *(*$c \in \mathtt{EGM}(X)$*) and* $\Psi(\mu) = \bigwedge \mu$ *defined by* $(\bigwedge \mu)(x) = \bigwedge \mu(x)$ *(*$x \in \mathtt{Sub}_{\mathsf{M}}(X)$*).*

Proof. Evidently, both Φ, Ψ are order preserving and $\Psi\circ\Phi = \mathbf{1}_{\mathtt{EGM}(X)}$; furthermore, for any $c \in \mathtt{EGM}(X)$, $\mu \in \mathtt{pnbd}[X]$:

$$\begin{aligned}\Phi(c)\leq^{\mathrm{op}}\mu &\Leftrightarrow \mu \leq \Phi(c)\\ &\Leftrightarrow \big(x \in \mathtt{Sub}_{\mathsf{M}}(X) \Rightarrow (\mu(x) \subseteq \Phi(c)(x))\big)\\ &\Leftrightarrow \big(x \in \mathtt{Sub}_{\mathsf{M}}(X) \Rightarrow (p \in \mu(x) \Rightarrow c(x) \leq p)\big)\\ &\Leftrightarrow \big(x \in \mathtt{Sub}_{\mathsf{M}}(X) \Rightarrow (c(x) \leq \bigwedge \mu(x))\big)\\ &\Leftrightarrow c \leq \Psi(\mu),\end{aligned}$$

completing the proof. □

Proposition 3.12. *Using notation of Theorem 3.11, $c \in \mathtt{EGM}(X)$ is idempotent (respectively, idempotent and join preserving) if and only if $\Phi(c)$ is a weak neighbourhood system (respectively, neighbourhood system) on X.*

Proof. $\Phi(c)$ is a weak neighbourhood system if and only if for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\begin{array}{rrl} & u \in \Phi(c)(x) & \Leftrightarrow (\exists v \in \Phi(c)(x))(u \in \Phi(c)(v))\\ \Leftrightarrow & u \geq c(x) & \Leftrightarrow (\exists v \geq c(x))(u \geq c(v))\\ \Leftrightarrow & c^2(x) = c(x), & \end{array}$$

proving the idempotence of c. Further, $\Phi(c)$ is a neighbourhood system if and only if for every $A \subseteq \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\begin{array}{rrl} & \Phi(c)(\bigvee A) & \supseteq \bigcap_{x\in A}\Phi(c)(x)\\ \Leftrightarrow & \big(u \geq \bigvee_{x\in A} c(x) & \Rightarrow u \geq c(\bigvee A)\big)\\ \Leftrightarrow & \bigvee_{x\in A} c(x) & \geq c(\bigvee A)\end{array}$$

completing the proof. □

Remark 3.13. A closure operation is said to be *fully additive* if and only if it preserves arbitrary non-empty joins (see §2.6 condition (`FA`) [41]). Proposition 3.12 shows fully additive idempotent closure operators are special neighbourhood systems on X; evidently, every neighbourhood system is not of the form $\Phi(c)$ — for instance in the context $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$, the usual topology on the real line $\mathbb{R}$ is not of the form $\Phi(c)$, for any $c \in \mathtt{EGM}(\mathbb{R})$.

Remark 3.14. Alongside Proposition 2.1, Theorem 3.11 asserts: the idempotent closure operations on $\mathtt{Sub}_{\mathsf{M}}(X)$, identified as complete bounded sub-$\wedge$-semilattices of the lattice $\mathtt{Sub}_{\mathsf{M}}(X)$ are embedded reflectively inside the complete lattice of grounded closure operations on $\mathtt{Sub}_{\mathsf{M}}(X)$, which in turn are embedded coreflectively and dually inside the complete lattice of preneighbourhood systems on X as some special weak neighbourhood systems. Further, preneighbourhood systems induce closure operations $\mathtt{pnbd}[X]^{\mathrm{op}} \underset{\widehat{\mathrm{cl}}}{\overset{\mathrm{cl}}{\rightrightarrows}} \mathtt{EGM}(X)$ with $\mathrm{cl} \leq \widehat{\mathrm{cl}}$, $\Phi(\widehat{\mathrm{cl}_\mu}) \leq \Phi(\mathrm{cl}_\mu)$ and $\Phi(\widehat{\mathrm{cl}_\mu})$ a weak neighbourhood system on X. The diagram below summarise this.

$$\begin{tikzcd}
\mathtt{CBSMSL}(X)^{\mathrm{op}} \arrow[r, tail, shift right, "\nu"'] \arrow[r, phantom, "\perp"] & \mathtt{EGM}(X) \arrow[l, shift right, "\mathtt{Fix}"'] \arrow[r, tail, shift left, "\Phi"] \arrow[r, phantom, "\perp"] & \mathtt{pnbd}[X]^{\mathrm{op}} \arrow[l, shift left, "\Psi"] \arrow[l, bend right=50, "\mathrm{cl}"'] \arrow[l, bend right=80, "\widehat{\mathrm{cl}}"', ""{name=U}]
\end{tikzcd} \tag{3.17}$$

$\rotatebox{90}{$\leq$}$

In the context ($\mathtt{FinSet}$, $\mathsf{Surjections}$, $\mathsf{Injections}$) Φ is an isomorphism; the presence of non-discrete Hausdorff topological spaces ensure in the context ($\mathtt{Set}$, $\mathsf{Surjections}$, $\mathsf{Injections}$), Φ is not an isomorphism.

3.3 Given any preneighbourhood space (X, μ), there are two closure operations, cl_μ and $\widehat{\mathrm{cl}_\mu}$. The latter is idempotent, and both of them describe the same closed subobjects. The notion of *continuity* of morphisms with respect to cl_μ and $\widehat{\mathrm{cl}_\mu}$ needs mention.

Proposition 3.15. *Given the internal preneighbourhood spaces (X, μ), (Y, ϕ) and a morphism $X \xrightarrow{f} Y$, consider the statements:*

(a) For every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p$.

(b) For every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f \widehat{\mathrm{cl}_\mu} p \leq \widehat{\mathrm{cl}_\phi} \exists_f p$.

(c) For every $t \in \mathfrak{C}_\phi$, $f^{-1} t \in \mathfrak{C}_\mu$.

(d) For every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\widehat{\mathrm{cl}_\phi} \exists_f p = \widehat{\mathrm{cl}_\phi} \exists_f \mathrm{cl}_\mu p$.

Then: (a) implies (b); the statements (b), (d) and (c) are equivalent.

Proof. Assuming (b), for any $t \in \mathfrak{C}_\phi$ since $\exists_f \widehat{\mathrm{cl}_\mu} f^{-\mathbf{1}} t \leq \widehat{\mathrm{cl}_\phi} \exists_f f^{-\mathbf{1}} t \leq \widehat{\mathrm{cl}_\phi} t = t$, $f^{-\mathbf{1}} t \in \mathfrak{C}_\mu$, proving (c). Conversely, assuming (c):

$$\begin{aligned} f^{-\mathbf{1}} \widehat{\mathrm{cl}_\phi} \exists_f p &= f^{-\mathbf{1}} \bigwedge \{ t \in \mathfrak{C}_\phi : \exists_f p \leq t \} \\ &= \bigwedge \{ f^{-1} t : p \leq f^{-1} t, t \in \mathfrak{C}_\phi \} \quad (\because, \exists_f \dashv f^{-\mathbf{1}}) \\ &\geq \bigwedge \{ s \in \mathfrak{C}_\mu : p \leq s \} = \widehat{\mathrm{cl}_\mu} p, \end{aligned}$$

proving (b). Further, assuming the statement in (b):

$$\begin{array}{lrll} & \exists_f \widehat{\mathrm{cl}_\mu} p & \leq \widehat{\mathrm{cl}_\phi} \exists_f p & \text{(statement in (b))} \\ \Leftrightarrow & \widehat{\mathrm{cl}_\mu} p & \leq f^{-\mathbf{1}} \widehat{\mathrm{cl}_\phi} \exists_f p & \\ \Leftrightarrow & \mathrm{cl}_\mu p & \leq f^{-\mathbf{1}} \widehat{\mathrm{cl}_\phi} \exists_f p & \text{(since (b) implies (c))} \\ \Leftrightarrow & \exists_f \mathrm{cl}_\mu p & \leq \widehat{\mathrm{cl}_\phi} \exists_f p & \\ \Leftrightarrow & \big(t \in \mathfrak{C}_\phi \Rightarrow (\exists_f p \leq t & \Leftrightarrow \exists_f \mathrm{cl}_\mu p \leq t) \big) & \\ & & & \\ \Leftrightarrow & \widehat{\mathrm{cl}_\phi} \exists_f \mathrm{cl}_\mu p & \leq \widehat{\mathrm{cl}_\phi} \exists_f p & \\ \Leftrightarrow & \widehat{\mathrm{cl}_\phi} \exists_f p & = \widehat{\mathrm{cl}_\phi} \exists_f \mathrm{cl}_\mu p & (\because, p \leq \mathrm{cl}_\mu p) \end{array}$$

proves the equivalence of the statements (b), (d) and completing the proof of the equivalence of (b)-(c). Finally, assuming (a) for every $t \in \mathfrak{C}_\phi$ since $\exists_f \mathrm{cl}_\mu f^{-\mathbf{1}} t \leq \mathrm{cl}_\phi \exists_f f^{-\mathbf{1}} t \leq \mathrm{cl}_\phi t = t$, $f^{-\mathbf{1}} t \in \mathfrak{C}_\mu$, proving (c). □

Definition 3.16. Given the internal preneighbourhood spaces (X, μ) and (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is called μ-ϕ *continuous* or simply *continuous* if $f^{-\mathbf{1}}$ preserves closed subobjects, i.e., the statement Proposition 3.15(c) holds good; if f satisfies the statement Proposition 3.15(a), then f is called μ-ϕ *continuous with respect to closures* or simply *continuous with respect to closures.*

Remark 3.17. If $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{c_X} \mathtt{Sub}_{\mathsf{M}}(X)$ and $\mathtt{Sub}_{\mathsf{M}}(Y) \xrightarrow{c_Y} \mathtt{Sub}_{\mathsf{M}}(Y)$ are both monotonic then the adjunction $\exists_f \dashv f^{-\mathbf{1}}$ yields:

$$(\forall x \in \mathtt{Sub}_{\mathsf{M}}(X)) \left(\exists_f c_X(x) \le c_Y(\exists_f x)\right) \Leftrightarrow \\ (\forall y \in \mathtt{Sub}_{\mathsf{M}}(Y)) \left(c_X(f^{-\mathbf{1}}y) \le f^{-\mathbf{1}} c_Y(y)\right).$$

This provides alternative formulations of (a) and (b) respectively:

$$\mathrm{cl}_\mu f^{-\mathbf{1}} q \le f^{-\mathbf{1}} \mathrm{cl}_\phi q, \qquad \text{for all } q \in \mathtt{Sub}_{\mathsf{M}}(X),$$

and

$$\widehat{\mathrm{cl}_\mu} f^{-\mathbf{1}} q \le f^{-\mathbf{1}} \widehat{\mathrm{cl}_\phi} q, \qquad \text{for all } q \in \mathtt{Sub}_{\mathsf{M}}(X).$$

Theorem 3.18. *Given an internal preneighbourhood space* (X, μ) *and admissible monomorphisms* $A \overset{a}{\rightarrowtail} M \overset{m}{\rightarrowtail} X$ *:*

$$\mathrm{cl}_{(\mu|_m)} a \le m^{-\mathbf{1}} \mathrm{cl}_\mu (m \circ a). \tag{3.18}$$

Furthermore:

$$m \in \mathfrak{C}_\mu, a \in \mathfrak{C}_{(\mu|_m)} \Rightarrow m \circ a \in \mathfrak{C}_\mu \tag{3.19}$$

and

$$m \in \mathfrak{C}_\mu \Rightarrow \left(\widehat{\mathrm{cl}_{(\mu|_m)}} a = m^{-\mathbf{1}} \widehat{\mathrm{cl}_\mu} (m \circ a)\right). \tag{3.20}$$

Finally, if $\mathtt{Sub}_{\mathsf{M}}(X)$ *is atom generated or* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{m^{-\mathbf{1}}} \mathtt{Sub}_{\mathsf{M}}(M)$ *preserve joins then:*

$$\mathrm{cl}_{(\mu|_m)} a = m^{-\mathbf{1}} \mathrm{cl}_\mu (m \circ a). \tag{3.21}$$

Proof. Using the adjunction $\exists_m \dashv m^{-\mathbf{1}}$, for any $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $x \ge m \circ a \Leftrightarrow m^{-\mathbf{1}} x \ge a$, i.e., $x \not\ge m \circ a \Leftrightarrow m^{-\mathbf{1}} x \not\ge a$ $(\because, m \in \mathsf{M})$. Also, since $m^{-\mathbf{1}}(u \wedge m \circ a) =$

$m^{-\mathbf{1}}u \wedge a$, implying $m\circ(a \wedge m^{-\mathbf{1}}u) = m\circ\big(m^{-\mathbf{1}}(u \wedge m\circ a)\big) = m \wedge u \wedge m\circ a = u \wedge m\circ a$, the computations:

$$\begin{aligned}
x \in \mathtt{Far}_{\mu}(m\circ a) &\Leftrightarrow (\exists u \in \mu(x))(u \wedge m\circ a = \sigma_X) \\
&\Leftrightarrow (\exists u \in \mu(x))(m\circ(a \wedge m^{-\mathbf{1}}u) = \sigma_X) \\
&\Leftrightarrow (\exists u \in \mu(x)(a \wedge m^{-\mathbf{1}}u = \sigma_M)) && (\because, m \in \mathsf{M}) \\
&\Leftrightarrow (\exists v \in (\mu\big|_m)(m^{-\mathbf{1}}x))(a \wedge v = \sigma_M) && (\text{using } (2.15)) \\
&\Leftrightarrow m^{-\mathbf{1}}x \in \mathtt{Far}_{(\mu|_m)}(a),
\end{aligned}$$

show: $x \in \mathtt{Far}_{\mu}(m\circ a) \Leftrightarrow m^{-\mathbf{1}}x \in \mathtt{Far}_{(\mu|_m)}(a)$.

Therefore:

$$\begin{aligned}
\exists_m \mathrm{cl}_{(\mu|_m)} a &= \exists_m \bigvee\big\{x \in \mathtt{Sub}_{\mathsf{M}}(M) : a \not< x \notin \mathtt{Far}_{(\mu|_m)}(a)\big\} \\
&= \bigvee\big\{\exists_m x : x \in \mathtt{Sub}_{\mathsf{M}}(M), a \not< m^{-\mathbf{1}}(m\circ x) \notin \mathtt{Far}_{(\mu|_m)}(a)\big\} \\
&= \bigvee\big\{m\circ x : x \in \mathtt{Sub}_{\mathsf{M}}(M), m\circ a \not< m\circ x \notin \mathtt{Far}_{\mu}(m\circ a)\big\} \\
&\leq \bigvee\big\{y \in \mathtt{Sub}_{\mathsf{M}}(X) : m\circ a \not< y \notin \mathtt{Far}_{\mu}(m\circ a)\big\} \\
&= \mathrm{cl}_{\mu}(m\circ a),
\end{aligned}$$

proving $m\circ\mathrm{cl}_{(\mu|_m)} a \leq \mathrm{cl}_{\mu}(m\circ a) \Leftrightarrow \mathrm{cl}_{(\mu|_m)} a \leq m^{-\mathbf{1}}\mathrm{cl}_{\mu}(m\circ a)$.

The equations (3.19) & (3.20) are trivially true when a is an isomorphism; hence it is enough to prove them for a not an isomorphism.

For $m \in \mathfrak{C}_\mu$ and $\mathbf{1}_M \neq a \in \mathfrak{C}_{(\mu|_m)}$:

$$\begin{aligned}\mathrm{cl}_\mu(m \circ a) &= \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : m\circ a \not\leq x \notin \mathtt{Far}_\mu(m\circ a)\}\\ &= \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : a \not\leq m^{-\mathbf{1}}x \notin \mathtt{Far}_{(\mu|_m)}(a)\}\\ &= \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : m^{-\mathbf{1}}x \leq a\}\\ &\quad (\text{since } a \in \mathfrak{C}_{(\mu|_m)})\\ &= \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : \sigma_M \neq m^{-\mathbf{1}}x \leq a\} \vee\\ &\qquad \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : m^{-\mathbf{1}}x = \sigma_M\}\\ &= p \vee q,\end{aligned}$$

where $p = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : \sigma_M \neq m^{-\mathbf{1}}x \leq a\}$ and $q = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : m^{-\mathbf{1}}x = \sigma_M\}$. For $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ with $x < \mathrm{cl}_\mu(m\circ a)$ such that $m^{-\mathbf{1}}x = \sigma_M \Leftrightarrow x \wedge m = \sigma_X$, since $x \notin \mathtt{Far}_\mu(m\circ a)$, $x \notin \mathtt{Far}_\mu(m)$ ($\because, m\circ a < m$ and Lemma 3.1), and hence $x \geq m$ or $x \leq m$ ($\because, m \in \mathfrak{C}_\mu$). Since $x \not\geq m\circ a$, $x \not\geq m$ forces $x \leq m$. Hence there exists only one x contributing to the join for q, namely $x = \sigma_X$, i.e., $q = \sigma_X$. On the other hand, using Lemma 3.2 and $m \in \mathfrak{C}_\mu$, for each x contributing to the join for p, $x \leq m\circ a$. Thus:

$$\mathrm{cl}_\mu(m\circ a) = p = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \leq m\circ a\} = m\circ a$$

yielding $m\circ a \in \mathfrak{C}_\mu$; also:

$$\begin{aligned}m^{-\mathbf{1}}\widehat{\mathrm{cl}_\mu}(m\circ a) &= m^{-\mathbf{1}}\bigwedge\{s \in \mathfrak{C}_\mu : m\circ a \leq s\}\\ &= \bigwedge\{m^{-\mathbf{1}}s : a \leq m^{-\mathbf{1}}s, s \in \mathfrak{C}_\mu\}\\ &= \bigwedge\{t \in \mathfrak{C}_{(\mu|_m)} : a \leq t\} \qquad (\text{using (3.19)})\\ &= \widehat{\mathrm{cl}_{(\mu|_m)}}a,\end{aligned}$$

yielding (3.20).

If $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{m^{-\mathbf{1}}} \mathtt{Sub}_{\mathsf{M}}(M)$ preserve arbitrary joins then:

$$\begin{aligned} m^{-\mathbf{1}}\mathrm{cl}_{\mu}(m{\circ}a) &= m^{-\mathbf{1}}\bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : m{\circ}a \not< x \notin \mathtt{Far}_{\mu}(m{\circ}a)\} \\ &= \bigvee\{m^{-\mathbf{1}}x : x \in \mathtt{Sub}_{\mathsf{M}}(X), a \not< m^{-\mathbf{1}}x \notin \mathtt{Far}_{(\mu|_m)}(a)\} \\ &\leq \bigvee\{y \in \mathtt{Sub}_{\mathsf{M}}(M) : a \not< y \notin \mathtt{Far}_{(\mu|_m)}(a)\} = \mathrm{cl}_{(\mu|_m)}a, \end{aligned}$$

proving (3.21) in this case. On the other hand if $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated then for an atom b, $b \leq m \wedge \mathrm{cl}_{\mu}(m{\circ}a)$ if and only if $b \leq m$ and $b \notin \mathtt{Far}_{\mu}(m{\circ}a) \Leftrightarrow m^{-\mathbf{1}}b \notin \mathtt{Far}_{(\mu|_m)}(a)$; since $m^{-\mathbf{1}}b$ is also an atom, it implies $m^{-\mathbf{1}}b \leq \mathrm{cl}_{(\mu|_m)}a \Leftrightarrow m \wedge b = b \leq m{\circ}\mathrm{cl}_{(\mu|_m)}a$. Hence $m \wedge \mathrm{cl}_{\mu}(m{\circ}a) \leq m{\circ}\mathrm{cl}_{(\mu|_m)}a \Leftrightarrow m^{-\mathbf{1}}\mathrm{cl}_{\mu}(m{\circ}a) \leq \mathrm{cl}_{(\mu|_m)}a$, proving (3.21) in this case also. This completes the proof. □

Lemma 3.2. *Given admissible monomorphisms* $A \overset{a}{\rightarrowtail} M \overset{m}{\rightarrowtail} X$ *with* $a \neq \mathbf{1}_M$, *if* $\sigma_M \neq (x \wedge m) \leq m{\circ}a$ *then* $x < \mathrm{cl}_{\mu}m$.

Proof. If $x > m$ then $m = x \wedge m \leq m{\circ}a \leq m$ implies a is an isomorphism; hence if $a \neq \mathbf{1}_M$ then $x \not> m$. Also, $x \wedge m \leq m{\circ}a \Leftrightarrow m^{-\mathbf{1}}x \leq a \Rightarrow m^{-\mathbf{1}}x \notin \mathtt{Far}_{(\mu|_m)}(a) \Leftrightarrow x \notin \mathtt{Far}_{\mu}(m{\circ}a) \Rightarrow x \notin \mathtt{Far}_{\mu}(m)$. Thus, the conditions imply: $m \not< x \notin \mathtt{Far}_{\mu}(m) \Leftrightarrow x < \mathrm{cl}_{\mu}m$. □

Definition 3.19 ((see condition `(HE)`, §2.5 [41])). Given an internal preneighbourhood space (X, μ) and a $m \in \mathtt{Sub}_{\mathsf{M}}(X)$, cl_{μ} is *hereditary for* m if for all $a \in \mathtt{Sub}_{\mathsf{M}}(M)$:

$$\mathrm{cl}_{(\mu|_m)}a = m^{-\mathbf{1}}\mathrm{cl}_{\mu}(m{\circ}a). \tag{3.22}$$

Similarly, $\widehat{\mathrm{cl}_{\mu}}$ is *hereditary for* m if for all $a \in \mathtt{Sub}_{\mathsf{M}}(M)$:

$$\widehat{\mathrm{cl}_{(\mu|_m)}}a = m^{-\mathbf{1}}\widehat{\mathrm{cl}_{\mu}}(m{\circ}a). \tag{3.23}$$

For a subset $\mathfrak{a} \subseteq \mathtt{Sub}_{\mathsf{M}}(X)$, cl_{μ} (respectively, $\widehat{\mathrm{cl}_{\mu}}$) is $\mathfrak{a}$-*hereditary* if cl_{μ} (respectively, $\widehat{\mathrm{cl}_{\mu}}$) is hereditary for each $m \in \mathfrak{a}$; M-hereditary is shortened to *hereditary.*

Remark 3.20. In terms of Definition 3.19, equation (3.21) suggests in the special case when $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated, for every preneighbourhood system μ on X, cl_μ is hereditary; in general, if for an admissible subobject m, $m^{\mathbf{-1}}$ preserve joins then cl_μ is hereditary for m. For every admissible monomorphism $m \in \mathtt{Sub}_{\mathsf{M}}(X)$, $m^{\mathbf{-1}}$ preserve joins if and only if $\mathtt{Sub}_{\mathsf{M}}(X)$ is a frame ([48] or Theorem 2.11(a) [49]). Thus, cl_μ is hereditary for any preneighbourhood system μ on X if $\mathtt{Sub}_{\mathsf{M}}(X)$ is a frame.

However, in general from (3.20), for every preneighbourhood system μ on X, $\widehat{\mathrm{cl}_\mu}$ is $\mathfrak{C}_\mu$-hereditary.

Remark 3.21. Equations (3.18) and (3.19) together suggest:

$$m \in \mathfrak{C}_\mu \Rightarrow \big(m{\circ}a \in \mathfrak{C}_\mu \Leftrightarrow a \in \mathfrak{C}_{(\mu|_m)}\big),$$

i.e., the closed subobjects of a closed subobject M are precisely those which are closed in X.

Remark 3.22. Theorem 3.18 ensures each admissible monomorphism is continuous with respect to closures and satisfies the condition (see condition `(CC)`, §2.4 [41]).

3.4 Theorem 3.18 ensure every admissible monomorphism is continuous with respect to closures and hence continuous (Prop 3.15).

Proposition 3.23. *Given preneighbourhood morphism* $(X,\mu) \xrightarrow{f} (Y,\phi)$*, if* $f = m{\circ}h$ *where* $\widehat{\mathrm{cl}_\mu}$ *is hereditary for* m *and* h *is a preneighbourhood morphism then* f *is continuous if and only if* h *is continuous.*

Proof. If

$$\begin{tikzcd}(X,\mu) \arrow[r, "h"] \arrow[rr, bend right, "f"'] & \big(I,(\phi|_m)\big) \arrow[r, tail, "m"] & (Y,\phi)\end{tikzcd}$$

with $\widehat{\mathrm{cl}_\mu}$ hereditary for m then:

$$\begin{aligned}
& \exists_f \widehat{\mathrm{cl}_\mu} p && \leq \widehat{\mathrm{cl}_\phi} \exists_f p \\
\Leftrightarrow\quad & m{\circ}\exists_h \widehat{\mathrm{cl}_\mu} p && \leq \widehat{\mathrm{cl}_\phi}(m{\circ}\exists_h p) \\
\Leftrightarrow\quad & \exists_h \widehat{\mathrm{cl}_\mu} p && \leq m^{\mathbf{-1}} \widehat{\mathrm{cl}_\phi}(m{\circ}\exists_h p) \\
\Leftrightarrow\quad & \exists_h \widehat{\mathrm{cl}_\mu} p && \leq \widehat{\mathrm{cl}_{(\phi|_m)}} \exists_h p \qquad \text{(since } \widehat{\mathrm{cl}_\mu} \text{ is } m\text{-hereditary)},
\end{aligned}$$

completing the proof. □

Using Definition 5.1 and immediate observations:

Corollary 3.24. *In the context of Proposition 3.23, every preneighbourhood morphism is continuous if and only if every dense morphism is continuous.*

Lemma 3.3. *Consider internal preneighbourhood spaces (X, μ), (Y, ϕ), a morphism $X \xrightarrow{f} Y$, admissible subobjects $x, p \in \mathtt{Sub}_{\mathsf{M}}(X)$.*

(a) If f reflects zero and $\mu \supseteq \overleftarrow{f}\phi\exists_f$ then:

$$\exists_f x \in \mathtt{Far}_\phi(\exists_f p) \Rightarrow x \in \mathtt{Far}_\mu p. \tag{3.24}$$

(b) If f is a Frobenius morphism and $\mu \subseteq \overleftarrow{f}\phi\exists_f$ then:

$$x \in \mathtt{Far}_\mu p \Rightarrow \exists_f x \in \mathtt{Far}_\phi(\exists_f p). \tag{3.25}$$

Proof. If $\exists_f x \in \mathtt{Far}_\mu(\exists_f p)$ then there exists a $v \in \phi(\exists_f x)$ with $v \wedge \exists_f p = \sigma_Y$; if f reflects zero then $\sigma_X = f^{-1}(v \wedge \exists_f p) \geq p \wedge f^{-1}v \Rightarrow p \wedge f^{-1}v = \sigma_X$. Hence if $\mu \supseteq \overleftarrow{f}\phi\exists_f$ then $f^{-1}v \in \mu(x)$ proving (a). On the other hand, if $x \in \mathtt{Far}_\mu p$ then there exists a $u \in \mu(x)$ such that $u \wedge p = \sigma_X$. Since $\mu \subseteq \overleftarrow{f}\phi\exists_f$, there exists a $v \in \phi(\exists_f x)$ such that $f^{-1}v \leq u$ entailing $p \wedge f^{-1}v = \sigma_X$ for some $v \in \phi(\exists_f x)$. Since f is a Frobenius morphism, $\sigma_Y = \exists_f(p \wedge f^{-1}v) = v \wedge \exists_f p$, implying $\exists_f x \in \mathtt{Far}_\phi(\exists_f p)$. This completes the proof. □

Remark 3.25. Lemma 3.3 illustrates the obstruction in establishing continuity or continuity with respect to closures. Thus, in context of Lemma 3.3, if $t \in \mathfrak{C}_\phi$ and f is a zero reflecting preneighbourhood morphism then $x \notin \mathtt{Far}_\mu(f^{-1}t) \Rightarrow \exists_f x \notin \mathtt{Far}_\phi(\exists_f f^{-1}t)$ and hence $\exists_f x \notin \mathtt{Far}_\phi t$. Since $t \in \mathfrak{C}_\phi$, $\exists_f x \leq t \Leftrightarrow x \leq f^{-1}t$ or else $\exists_f x \geq t$. Consequently, $f^{-1}t$ would be closed if $\exists_f x \geq t \Rightarrow x \geq f^{-1}t$. With regards to continuity with respect to closures:

$$\begin{aligned} x < \mathrm{cl}_\mu p &\Leftrightarrow p \not< x \notin \mathtt{Far}_\mu p \\ &\Rightarrow p \not< x \text{ and } \exists_f x \notin \mathtt{Far}_\phi(\exists_f p) \\ &\quad (\text{if } \mu \supseteq \overleftarrow{f}\phi\exists_f \text{ and } f \text{ reflects zero}); \end{aligned}$$

hence if $\exists_f p < \exists_f x \Rightarrow p < x$ then $x < \mathrm{cl}_\mu p \Rightarrow \exists_f x < \mathrm{cl}_\phi \exists_f p$ entailing continuity with respect to closures.

Corollary 3.26. *Given* $(X,\mu) \xrightarrow{f} (Y,\phi)$*, the following statements hold.*

(a) If $f^{-\mathbf{1}} \circ \exists_f = \mathbf{1}_{\mathtt{Sub}_{\mathsf{M}}(X)}$ *then* f *is continuous with respect to closures.*

(b) If $\mathtt{Sub}_{\mathsf{M}}(X)$ *is atom generated,* f *reflects zero and preserve atoms then* f *is continuous with respect to closures.*

Remark 3.27. Every admissible monomorphism satisfy condition (a) of Corollary 3.26(a); however the property do not characterise admissible monomorphisms — in the context $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMon})$ injective continuous maps may not be extremal monomorphisms and yet satisfy the condition.

Remark 3.28. With regards to preservation of atoms, every Frobenius morphism preserve atoms: if $X \xrightarrow{f} Y$ is a Frobenius morphism and $a \in \mathtt{atom}(X)$, then for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$ either $a \leq f^{-\mathbf{1}}y \Leftrightarrow \exists_f a \leq y$ or else $a \wedge f^{-\mathbf{1}}y = \sigma_X \Rightarrow \sigma_Y = \exists_f(a \wedge f^{-\mathbf{1}}y) = y \wedge \exists_f a$, completing the proof. Hence, from Corollary 3.26(b), if $\mathtt{Sub}_{\mathsf{M}}(X)$ is atom generated then every reflecting zero Frobenius preneighbourhood morphism with (X,μ) as domain is continuous with respect to closures.

3.5 As observed *continuity* for morphisms with respect to induced closure operations is not automatic, even for preneighbourhood morphisms. This section illustrate its presence in many familiar contexts, as well as exhibit instances of several properties of closure operations from a preneighbourhood system discussed so far. However, before embarking on the examples, let for $\mu \in \mathtt{pnbd}[X]$ and $\phi \in \mathtt{pnbd}[Y]$:

$$\mathtt{CC}(\mu,\phi) = \left\{X \xrightarrow{f} Y : \mu \supseteq \overleftarrow{f}\,\phi\exists_f \text{ and } (\forall p \in \mathtt{Sub}_{\mathsf{M}}(X))(\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p)\right\}, \tag{3.26}$$

and

$$\mathtt{C}(\mu,\phi) = \left\{X \xrightarrow{f} Y : \mu \supseteq \overleftarrow{f}\,\phi\exists_f \text{ and } (\forall p \in \mathtt{Sub}_{\mathsf{M}}(X))(\exists_f \widehat{\mathrm{cl}_\mu} p \leq \widehat{\mathrm{cl}_\phi} \exists_f p)\right\}. \tag{3.27}$$

Evidently, $\mathtt{pnbd}[X] \times \mathtt{pnbd}[Y]^{\mathrm{op}} \underset{\mathtt{CC}}{\overset{\mathtt{C}}{\rightrightarrows}} 2^{\mathbb{A}\,(X,Y)}$ are both order preserving maps with $\mathtt{CC}(\mu,\phi) \subseteq \mathtt{C}(\mu,\phi)$.

Let $\mathcal{A}$ be any of the contexts: $(\mathtt{FinSet}, \mathsf{Surjections}, \mathsf{Injections})$, $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$, $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMon})$. In each of them for each object X, $\mathtt{Sub}_{\mathsf{M}}(X)$ is distributive, complemented, atom generated, morphisms preserve atoms and the contexts are admissibly quasi-pointed with strict initial object. Hence for every preneighbourhood system μ on X, cl_μ is additive (Theorem 3.5(d)), hereditary (Theorem 3.18, equation (3.21)), $p \in \mathfrak{C}_\mu \Rightarrow p^* \in \mathfrak{O}_\mu$ (respectively, $p \in \mathfrak{O}_\mu \Rightarrow p^* \in \mathfrak{C}_\mu$) (Theorem 3.5(c)) and each preneighbourhood morphism is continuous (Corollary 3.26(b), Theorem 2.11(f)); if further μ has open interiors then $(\widehat{\mathrm{cl}_\mu p})^* = \mathtt{int}_\mu p^*$ (Proposition 3.8). Moreover, if μ is open generated then cl_μ is idempotent (Theorem 3.5(e)). Thus for each neighbourhood system μ on X, cl_μ is a hereditary Kuratowski closure operation yielding internal topologies.

The context $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMon})$ has each of its $\mathtt{Sub}_{\mathsf{M}}(X)$ a coframe and the context is admissibly quasi-pointed with the initial object strict; however, unlike contexts in §3.5, preimages do not preserve arbitrary joins and images do not preserve atoms. Furthermore, each subobject (also called *sublocale*) $S \subseteq X$ is generated by *principal sublocales*: $S = \bigvee_{a \in S}[a]$, where $[a] = \big\{ (t \Longrightarrow a) : t \in X \big\}$ is the smallest sublocale containing a (see §III.10.2 [58]). Hence, for every preneighbourhood system μ on a locale X, cl_μ is hereditary (Theorem 3.18, equation (3.21)).

Consider the T-neighbourhood system for a locale X (see §2.20). The closed sublocale for $a \in X$ is $\uparrow a = \big\{x \in X : x \geq a\big\}$ is the complement of $\mathcal{O}a = \big\{ (a \Longrightarrow x) : x \in X \big\} = \big\{x \in X : x = (a \Longrightarrow x)\big\}$ in the lattice $\mathtt{Sub}_{\mathsf{M}}(X)$ of all sublocales of X (see Proposition 6.1.3 [58]). Since for any

$x \in X$: $1 \neq x \in \mathcal{O}a \Leftrightarrow x = (a \Longrightarrow x) \neq 1 \Leftrightarrow a \nleq x$, for any sublocale S:

$$\begin{aligned} x \in \mathrm{cl}_{\tau_X} S &\Leftrightarrow \big[x\big] \subseteq \mathrm{cl}_{\tau_X} S \\ &\Leftrightarrow \big(T \in \tau_X(\big[x\big]) \Rightarrow T \cap S \neq \{1\}\big) \\ &\Leftrightarrow \big(\big[x\big] \subseteq \mathcal{O}a \Rightarrow \mathcal{O}a \cap S \neq \{1\}\big) \\ &\Leftrightarrow \big(\mathcal{O}a \cap S = \{1\} \Rightarrow \big[x\big] \nsubseteq \mathcal{O}a\big) \\ &\Leftrightarrow (S \subseteq \uparrow a \Rightarrow x \geq a) \\ &\Leftrightarrow x \geq \bigwedge S. \end{aligned}$$

This proves $\mathrm{cl}_{\tau_X} S = \uparrow (\bigwedge S)$, the usual localic closure of S (see §III.8 [58]); In particular, cl_{τ_X} is additive and idempotent, i.e., a hereditary Kuratowski closure operation; since $\mathtt{Sub}_{\mathsf{M}}(X)$ is a coframe, this is not an internal topology in general. Further, for every localic map $X \xrightarrow{f} Y$, since $\exists_f \mathrm{cl}_{\tau_X} S \subseteq \mathrm{cl}_{\tau_Y} \exists_f S$ $(S \in \mathtt{Sub}_{\mathsf{RegMono}}(X))$ (see §III.8.4 [58]), the preneighbourhood morphism $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$ is continuous, and for preneighbourhoods $\mu \supseteq \tau_X$, $\tau_Y \supseteq \phi$, $(X, \mu) \xrightarrow{f} (Y, \phi)$ is continuous.

In the context $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$, for a group X consider the neighbourhood system ν_X of Example 2.21; in fact $\nu_X = \Phi(\mathtt{ncl}_X)$, where $\mathtt{ncl}_X$ is the normal closure of a subgroup G of X. Since $\mathtt{ncl}_X$ is an idempotent and join preserving closure operation, $\Phi(\mathtt{ncl}_X)$ is a neighbourhood system on X, $\mathrm{cl}_{\Phi(\mathtt{ncl}_X)}$ is idempotent, $\mathfrak{O}_{\Phi(\mathtt{ncl}_X)}$ is precisely the set of all normal subgroups of X, for any subgroup A:

$$\begin{aligned} x \in \mathrm{cl}_{\Phi(\mathtt{ncl}_X)} A &\Leftrightarrow \big[x\big] \in \mathrm{cl}_{\Phi(\mathtt{ncl}_X)} A \\ &\Leftrightarrow \mathtt{ncl}_X(x) \cap A \neq \{0\}, \end{aligned}$$

where $\big[x\big]$ is the cyclic group generated by x, $A \in \mathfrak{C}_{\Phi(\mathtt{ncl}_X)}$ if and only if for any $x \in X$:

$$\mathtt{ncl}_X(x) \cap A \neq \{0\} \Rightarrow x \in A,$$

$N \trianglelefteq X \Rightarrow \mathrm{cl}_{\Phi(\mathtt{ncl}_X)} N \trianglelefteq X$, $\mathtt{int}_{\Phi(\mathtt{ncl}_X)} A$ is the normal core of A (i.e., the largest normal subgroup contained in a subgroup), the preneighbourhood morphism $(X,\Phi(\mathtt{ncl}_X)) \xrightarrow{f} (Y,\Phi(\mathtt{ncl}_Y))$, for each group homomorphism $X \xrightarrow{f} Y$, is continuous and every $(X,\mu) \xrightarrow{f} (Y,\phi)$ where $\mu \supseteq \mathtt{ncl}_X$ and $\phi \subseteq \mathtt{ncl}_Y$ is continuous.

In the context $(\mathsf{CRng}, \mathsf{RegEpi}, \mathsf{Mon})$ for a ring X consider the neighbourhood system ι_X of Example 2.22; in fact $\iota_X = \Phi(\mathtt{idl}_X)$, where $\mathtt{idl}_X$ is the ideal closure of a subring of X. Since $\mathtt{idl}_X$ is an idempotent and join preserving closure operation, $\Phi(\mathtt{idl}_X)$ is a neighbourhood system, $\mathrm{cl}_{\Phi(\mathtt{idl}_X)}$ is idempotent, $\mathfrak{O}_{\Phi(\mathtt{idl}_X)} = \mathtt{Idl}\left[X\right]$ the set of all ideals of the ring X, for any subring A:

$$\begin{aligned} x \in \mathrm{cl}_{\Phi(\mathtt{idl}_X)}A &\Leftrightarrow \left[x\right] \in \mathrm{cl}_{\Phi(\mathtt{idl}_X)}A \\ &\Leftrightarrow \mathtt{idl}_X(x) \cap A \neq \{0\} \\ &\Leftrightarrow (\exists r \in X)(rx \in A), \end{aligned}$$

where $\left[x\right]$ is the subring generated by x, $A \in \mathfrak{C}_{\Phi(\mathtt{idl}_X)}$ if and only if for any $x \in X$:

$$(\exists r \in X)(rx \in A) \Rightarrow x \in A,$$

$I \in \mathtt{Idl}\left[X\right] \Rightarrow \mathrm{cl}_{\Phi(\mathtt{idl}_X)}I \in \mathtt{Idl}\left[X\right]$, $\mathtt{int}_{\Phi(\mathtt{idl}_X)}A$ is the *ideal core* of A (i.e., the largest ideal contained in a subring), the preneighbourhood morphism $(X,\Phi(\mathtt{idl}_X)) \xrightarrow{f} (Y,\Phi(\mathtt{idl}_Y))$, for each ring homomorphism $X \xrightarrow{f} Y$ is continuous and every $(X,\mu) \xrightarrow{f} (Y,\phi)$ where $\mu \supseteq \mathtt{idl}_X$ and $\phi \subseteq \mathtt{idl}_Y$ is continuous.

Given a context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$, for any $c \in \mathtt{EGM}(X)$:

$$\mathtt{Far}_{\Phi(c)}p = \big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : c(x) \wedge p = \sigma_X\big\}, \tag{3.28}$$

$$\mathrm{cl}_{\Phi(c)}p = \bigvee\big\{x \not> p : c(x) \wedge p \neq \sigma_X\big\}, \tag{3.29}$$

$$\mathfrak{C}_{\Phi(c)} = \Bigg\{p \in \mathtt{Sub}_{\mathsf{M}}(X) : x \in \mathtt{Sub}_{\mathsf{M}}(X) \Rightarrow \tag{3.30}$$

$$\Big((x > p) \text{ or } \big(c(x) \wedge p \neq \sigma_X \Rightarrow x \leq p\big)\Big)\Bigg\},$$

while $\mathfrak{O}_{\Phi(c)} = \mathtt{Fix}[c]$, $\mathtt{int}_{\Phi(c)}p = \bigvee\big\{x \in \mathtt{Fix}[c] : x \leq p\big\}$.

In particular, $\mathfrak{C}_{\nabla_X}$ is a chain and each element of $\mathfrak{C}_{\nabla_X}$ *cuts* the lattice $\mathtt{Sub}_{\mathsf{M}}(X)$ in two parts, one above and the other below. In presence of good conditions on $\mathtt{Sub}_{\mathsf{M}}(X)$ (e.g., if each element is complemented) $\mathrm{cl}_{\nabla_X}p$ is closed, i.e., cl_{∇_X} is idempotent. Further, since $\mathfrak{C}_{\nabla_X}$ is closed under joins, cl_{∇_X} is additive, irrespective of presence of distributivity. On the other hand, every atom is in $\mathfrak{C}_{\uparrow_X}$, the $p \in \mathfrak{C}_{\uparrow_X}$ is nearly like an atom: for every $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ either $x \wedge p = \sigma_X$, $p < x$ or $x \leq p$, the last condition being the exception to an atom.

The functions $\mathtt{Sub}_{\mathsf{M}}(X) \overset{C}{\underset{D}{\rightrightarrows}} \mathtt{Sub}_{\mathsf{M}}(X)$ provide competitors for a closure operation induced from a preneighbourhood system μ on X:

$$C(p) = \bigvee\big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \neq \mathbf{1}_X \text{ and } x \notin \mathtt{Far}_{\mu}p\big\}, \tag{3.31}$$

and

$$D(p) = \bigvee\big\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : \mu(x) \subsetneqq \mu(p) \text{ and } x \notin \mathtt{Far}_{\mu}p\big\} \tag{3.32}$$

Evidently, C, D are closure operations and $D \leq \mathrm{cl}_{\mu}p \leq C$ and hence $\mathtt{Fix}[C] \leq \mathfrak{C}_{\mathrm{cl}_{\mu}} \leq \mathtt{Fix}[D]$.

However, the function C is trivial, i.e., either $C(p) = \sigma_X$ or $C(p) = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \neq \mathbf{1}_X\}$ — to see this, let $C(p) \neq \sigma_X$ and choose a $x \neq \mathbf{1}_X$ with $x \notin \mathtt{Far}_\mu p$. Since $x \neq 1$ there exists a $y \neq \mathbf{1}_X$ such that $x \vee y \neq \mathbf{1}_X$ and for such a y, $x \vee y \notin \mathtt{Far}_\mu p$. Consequently:

$$C(p) = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \neq \mathbf{1}_X\} \leq \bigvee\{x \vee y : y \neq \mathbf{1}_X\} \leq C(p).$$

On the other hand, the function D uses a partial order relation on $\mathtt{Sub}_{\mathsf{M}}(X)$ larger than the usual partial order. Since the order structure on $\mathtt{Sub}_{\mathsf{M}}(X)$ make centre stage of this paper, the closure operation D is left for a future detailed investigation.

4 Closed morphisms

Having defined a closure operation induced from a preneighbourhood system, this section describe morphisms which preserve the closure operation.

4.1 Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

Definition 4.1. Given the internal preneighbourhood spaces (X, μ) and (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is μ-ϕ *closed*, or simply *closed* when the preneighbourhood systems are evident, if:

$$p \in \mathfrak{C}_\mu \Rightarrow \exists_f p \in \mathfrak{C}_\phi. \tag{4.1}$$

The (possibly large) set of closed morphisms is $\mathbb{A}_{\mathsf{cl}}$.

Theorem 4.2. *Given the internal preneighbourhood spaces (X, μ), (Y, ϕ), (Z, ψ) and the morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ the following statements are true.*

(a) The morphism f is a closed morphism if and only if for every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\widehat{\mathrm{cl}_\phi} \exists_f p \leq \exists_f \widehat{\mathrm{cl}_\mu} p. \tag{4.2}$$

(b) If f is continuous then f is closed if and only if for every $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f \widehat{\mathrm{cl}_\mu} p = \widehat{\mathrm{cl}_\phi} \exists_f p$. In particular, $m \in \mathtt{Sub}_{\mathsf{M}}(X)$ is a closed map if and only if $m \in \mathfrak{C}_\mu$.

(c) The set $\mathbb{A}_{\mathsf{cl}}$ contain all isomorphisms.

(d) The set $\mathbb{A}_{\mathsf{cl}}$ is closed under compositions.

(e) If $g \circ f$ is a closed morphism and f is formally surjective and continuous then g is a closed morphism.

(f) If f is a closed continuous morphism then for each $m \in \mathfrak{C}_\phi$ the corestriction f_m of f along m is closed and continuous.

Proof. The statement in (c) is immediate from definition, while (d) is immediate from $\exists_{g \circ f} = \exists_g \circ \exists_f$. If f is closed, then for any $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\widehat{\mathrm{cl}_\mu} p \in \mathfrak{C}_\mu$ implies $\exists_f \widehat{\mathrm{cl}_\mu} p \in \mathfrak{C}_\phi$ and hence $\widehat{\mathrm{cl}_\phi} \exists_f p \leq \exists_f \widehat{\mathrm{cl}_\mu} p$ ($\because$, $\exists_f p \leq \exists_f \widehat{\mathrm{cl}_\mu} p$); on the other hand, if (4.2) is true then for $p \in \mathfrak{C}_\mu$, $\widehat{\mathrm{cl}_\phi} \exists_f p \leq \exists_f p$ proving $\exists_f p \in \mathfrak{C}_\phi$. This proves (a). The first part of statement in (b) is immediate from (a) and continuity (Proposition 3.15(b)); the second part is immediate from definition and Remark 3.21. If $g \circ f$ is a closed morphism, f is formally surjective and continuous then for any $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$:

$$\begin{aligned}
\widehat{\mathrm{cl}_\psi} \exists_g y &= \widehat{\mathrm{cl}_\psi} \exists_g \exists_f f^{-1} y && \text{(since } f \text{ is formally surjective)}\\
&= \widehat{\mathrm{cl}_\psi} \exists_{g \circ f} f^{-1} y \\
&\leq \exists_{g \circ f} \widehat{\mathrm{cl}_\mu} f^{-1} y && \text{(since } g \circ f \text{ is closed)}\\
&= \exists_g \exists_f \widehat{\mathrm{cl}_\mu} f^{-1} y \\
&\leq \exists_g \widehat{\mathrm{cl}_\phi} \exists_f f^{-1} y && \text{(since } f \text{ is continuous)}\\
&= \exists_g \widehat{\mathrm{cl}_\phi} y && \text{(since } f \text{ is formally surjective),}
\end{aligned}$$

proving (e). Finally, given

$$\begin{tikzcd}
P \arrow[r, tail, "p"] & f^{-1}M \arrow[r, "f_m"] \arrow[d, tail, "f^{-1}m"'] & M \arrow[d, tail, "m"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

, where f is closed continuous, the square is the pullback of $m \in \mathfrak{C}_\phi$ along f, $f^{-1}m \in \mathfrak{C}_\mu$ (Proposition 3.15(c)), yielding:

$$\begin{aligned}
m\circ\widehat{\mathrm{cl}_{(\phi|_m)}}\exists_{fm}p &= \widehat{\mathrm{cl}_\phi}(m\circ(\exists_{fm}p)) \\
&\quad \text{(using (b), second part)} \\
&= \widehat{\mathrm{cl}_\phi}(\exists_f((f^{-\mathbf{1}}m)\circ p)) \\
&= \exists_f\widehat{\mathrm{cl}_\mu}((f^{-\mathbf{1}}m)\circ p) \\
&\quad \text{(since } f \text{ is closed continuous, (b))} \\
&= \exists_f(f^{-\mathbf{1}}m)\circ\widehat{\mathrm{cl}_{(\mu|_{f^{-\mathbf{1}}m})}}p \\
&\quad \text{(since } f^{-\mathbf{1}}m \in \mathfrak{C}_\mu\text{, (b), second part)} \\
&= m\circ\exists_{fm}\widehat{\mathrm{cl}_{(\mu|_{f^{-\mathbf{1}}m})}}p \\
\Rightarrow \widehat{\mathrm{cl}_{(\phi|_m)}}\exists_{fm}p &= \exists_{fm}\widehat{\mathrm{cl}_{(\mu|_{f^{-\mathbf{1}}m})}}p,
\end{aligned}$$

completing the proof of (f). □

Remark 4.3. In view of the part (*b*) of Theorem 4.2(b), for any internal preneighbourhood space (X,μ), $m \in \mathfrak{C}_\mu$ is called a *closed embedding* and $\mathbb{A}_{\mathsf{clemb}}$ is the (possibly large) set of closed embeddings.

4.2 In this section some examples of closed morphisms are provided.

Given the preneighbourhood spaces (X,μ) and (Y,ϕ) let:

$$\mathtt{Cl}(\mu,\phi) = \big\{X \xrightarrow{f} Y : \mu \supseteq \overleftarrow{f}\,\phi\exists_f \text{ and } (\forall p \in \mathtt{Sub}_{\mathsf{M}}(X))(\widehat{\mathrm{cl}_\phi}\exists_f p \leq \exists_f\widehat{\mathrm{cl}_\mu}p)\big\}.$$

Evidently, $\mathtt{pnbd}[X]^{\mathtt{op}} \times \mathtt{pnbd}[Y] \xrightarrow{\mathtt{Cl}} 2^{\mathbb{A}\,(X,Y)}$ is an order preserving map.

Theorem 4.4. *If* $X \xrightarrow{f} Y$ *is a reflecting zero Frobenius* E*-morphism and* $\phi \in \mathtt{pnbd}[Y]$ *then* $f \in \mathtt{Cl}(\overleftarrow{f}\,\phi\exists_f,\phi)$.

Proof. Since a Frobenius E-morphism is formally surjective (Proposition 2.15) for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $y = \exists_f f^{-\mathbf{1}}y$. Hence using Lemma 3.3, for any $p \in \mathfrak{C}_{\overleftarrow{f}\,\phi\exists_f}$, $y \notin \mathtt{Far}_\phi(\exists_f p) \Leftrightarrow f^{-\mathbf{1}}y \notin \mathtt{Far}_{\overleftarrow{f}\,\phi\exists_f}\,p \Rightarrow p \leq f^{-\mathbf{1}}y$ or $p \geq f^{-\mathbf{1}}y$; further $\exists_f p \not\leq y \Leftrightarrow p \not\leq f^{-\mathbf{1}}y$ (from $\exists_f \dashv f^{-\mathbf{1}}$). Hence if $y \notin \mathtt{Far}_\phi(\exists_f p)$ and $\exists_f p \not\leq y$, then $f^{-\mathbf{1}}y \leq p \Rightarrow y = \exists_f f^{-\mathbf{1}}y \leq \exists_f p$, proving $\exists_f p \in \mathfrak{C}_\phi$.

□

Regarding examples in some other specific contexts:

(i) In the context $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMon})$, if X and Y are equipped with their T- neighbourhood system then a localic map $X \xrightarrow{f} Y$ is a closed morphism if and only if f is a closed morphism in the usual localic sense.

(ii) In the context $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$ (respectively, $(\mathtt{CRng}, \mathsf{RegEpi}, \mathsf{Mon})$) if $X \xrightarrow{f} Y$ with $f \in \mathsf{RegEpi}$ then f is $\Phi(\mathtt{ncl}_X)$-$\Phi(\mathtt{ncl}_X)$ closed (respectively, $\Phi(\mathtt{idl}_X)$-$\Phi(\mathtt{idl}_X)$) closed.

5 Dense morphisms

In this section the notion of *dense morphisms* shall be introduced, the *dense*-(closed embedding) factorisation system exhibited.

5.1 Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

Definition 5.1. Given the internal preneighbourhood spaces (X, μ), (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is μ-ϕ *dense* morphism (or in short *dense* morphism, if μ and ϕ are evident) if $f = m \circ h$ for some $m \in \mathfrak{C}_\phi$ implies m is an isomorphism. The (possibly large) set of all dense morphisms is denoted by $\mathbb{A}_{\mathsf{d}}$.

Remark 5.2. Consider the commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "f^{\mathsf{E}}"'] & Y & \\
\mathcal{I}_f \arrow[ur, tail, "f^{\mathsf{M}}"] \arrow[r, tail, "j_f"'] \arrow[rr, bend right, "u_f"'] & \overline{\mathcal{I}_f} \arrow[u, tail, "\mathrm{cl}_\phi f^{\mathsf{M}}"] \arrow[r, tail, "k_f"'] & \widehat{\mathcal{I}_f} \arrow[ul, tail, "\widehat{\mathrm{cl}_\phi} f^{\mathsf{M}}"']
\end{tikzcd}$$

where $j_f, k_f, u_f = k_f \circ j_f$ are the comparisons between the respective admissible subobjects. Evidently, $f = (\widehat{\mathrm{cl}_\phi} f^{\mathsf{M}}) \circ u_f \circ f^{\mathsf{E}}$ is dense if and only if

$\widehat{\mathrm{cl}}_\phi f^{\mathsf{M}} = \mathbf{1}_Y$. For a general f, since $\big(u_f\circ f^{\mathsf{E}}\big)^{\mathsf{E}} = f^{\mathsf{E}}$, $\big(u_f\circ f^{\mathsf{E}}\big)^{\mathsf{M}} = u_f$, an use of (3.20) shows on taking $m = \widehat{\mathrm{cl}}_\phi f^{\mathsf{M}}$:

$$\widehat{\mathrm{cl}}_{(\phi|_m)}(u_f\circ f^{\mathsf{E}})^{\mathsf{M}} = \widehat{\mathrm{cl}}_{(\phi|_m)} u_f = m^{-\mathbf{1}}\widehat{\mathrm{cl}}_\phi(m\circ u_f) = m^{-\mathbf{1}}m = \mathbf{1}_{\widehat{\mathcal{I}_f}},$$

i.e., $u_f\circ f^{\mathsf{E}}$ is a dense morphism. In particular, $f = (\widehat{\mathrm{cl}}_\phi f^{\mathsf{M}})\circ(u_f\circ f^{\mathsf{E}})$ shows every morphism factor as a dense morphism followed by a closed embedding.

Theorem 5.3. *Given the internal preneighbourhood spaces (X,μ), (Y,ϕ), (Z,ψ) and the morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following statements hold.*

(a) The morphism f is a dense morphism if and only if $\widehat{\mathrm{cl}}_\phi f^{\mathsf{M}} = \mathbf{1}_Y$.

(b) $\mathsf{E} \subseteq \mathbb{A}_{\mathsf{d}}$ and $\mathbb{A}_{\mathsf{d}} \cap \mathbb{A}_{\mathsf{clemb}} = \mathtt{Iso}(\mathbb{A})$.

(c) If g is dense continuous and f is dense then $g\circ f$ is dense.

(d) If $g\circ f \in \mathbb{A}_{\mathsf{d}}$ then $g \in \mathbb{A}_{\mathsf{d}}$.

(e) ${\mathbb{A}_{\mathsf{d}}}^{\perp} \subseteq \mathbb{A}_{\mathsf{clemb}}$ and ${}^{\perp}\mathbb{A}_{\mathsf{clemb}} \subseteq \mathbb{A}_{\mathsf{d}}$.

(f) If every preneighbourhood morphism is continuous then $\mathbb{A}_{\mathsf{d}} \subseteq {}^{\perp}\mathbb{A}_{\mathsf{clemb}}$ and $(\mathbb{A}_{\mathsf{d}}, \mathbb{A}_{\mathsf{clemb}})$ is a factorisation system on $\mathbb{A}$.

(g) If all preneighbourhood morphisms are continuous then dense morphisms are pushout stable.

(h) If all preneighbourhood morphisms are continuous and $\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbb{A} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$ $\alpha_X \in \mathbb{A}_{\mathsf{d}}$ *($X \in \mathbb{X}_0$), $\varinjlim F$ and $\varinjlim G$ exists then $\varinjlim \alpha \in \mathbb{A}_{\mathsf{d}}$.*

Proof. Since $f = (\widehat{\mathrm{cl}}_\phi f^{\mathsf{M}})\circ u_f\circ f^{\mathsf{E}}$ (see paragraph before Theorem), the proof of (a) follows. If f is dense and $f \in \mathfrak{C}_\phi$ then $f = f^{\mathsf{M}} = \widehat{\mathrm{cl}}_\phi f^{\mathsf{M}}$, proving $\mathbb{A}_{\mathsf{clemb}} \cap \mathbb{A}_{\mathsf{d}} \subseteq \mathtt{Iso}(\mathbb{A})$; the other inclusion is obvious. Also, if $f \in \mathsf{E}$ and $f = m\circ h$ with $m \in \mathfrak{C}_\phi$ then $m \in \mathsf{E} \cap \mathsf{M} = \mathtt{Iso}(\mathbb{A})$. This proves (b). If g is dense continuous, f is dense and $g{\circ}f = m{\circ}h$ for some $m \in \mathfrak{C}_\psi$, then from the pullback of m along g:

$$\begin{tikzcd}
X \arrow[drr, bend left, "h"] \arrow[dr, dotted, "!\,u"] \arrow[ddr, bend right, "f"'] & & \\
& \cdot \arrow[r, "g_m"] \arrow[d, tail, "g^{-\mathbf{1}}m"'] & \cdot \arrow[d, tail, "m"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

$g^{-\mathbf{1}}m \in \mathfrak{C}_\phi$ (Proposition 3.15(c)), there exists a unique morphism u such that $f = (g^{-\mathbf{1}}m){\circ}u$; hence from density of f, $g^{-\mathbf{1}}m$ is an isomorphism. Hence $g = m{\circ}g_m{\circ}(g^{-\mathbf{1}}m)^{-1}$ forces m an isomorphism from density of g. This proves (c). If $g{\circ}f \in \mathbb{A}_{\mathsf{d}}$, $g = m{\circ}h$ for some $m \in \mathfrak{C}_\psi$, then $g{\circ}f = m{\circ}h{\circ}f$ forces from density of $g{\circ}f$, m is an isomorphism; hence $g \in \mathbb{A}_{\mathsf{d}}$, proving (d). If $f \in {\mathbb{A}_{\mathsf{d}}}^{\perp}$, then $(u_f{\circ}f^{\mathsf{E}}) \perp f$ forces f^{E} an isomorphism proving $f \in \mathsf{M}$. Consequently, $f = (\widehat{\mathrm{cl}_\phi f}){\circ}u_f$ and $u_f \perp f$ forces $\widehat{\mathrm{cl}_\phi} f \leq f$, i.e., $f \in \mathbb{A}_{\mathsf{clemb}}$, proving ${\mathbb{A}_{\mathsf{d}}}^{\perp} \subseteq \mathbb{A}_{\mathsf{clemb}}$; the inequality ${}^{\perp}\mathbb{A}_{\mathsf{clemb}} \subseteq \mathbb{A}_{\mathsf{d}}$ is trivial, proving (e). If every preneighbourhood morphism is continuous, f is dense and $v{\circ}f = m{\circ}u$ for some $m \in \mathbb{A}_{\mathsf{clemb}}$, then $v^{-\mathbf{1}}m$ is a closed embedding (Proposition 3.15(c)) and there exists a unique morphism w such that $f = (v^{-\mathbf{1}}m){\circ}w$, implying $v^{-\mathbf{1}}m$ an isomorphism; hence $v_m{\circ}(v^{-\mathbf{1}}m)^{-1}$ is the unique diagonal fill-in, i.e., $f \perp m$, proving $\mathbb{A}_{\mathsf{d}} \subseteq {}^{\perp}\mathbb{A}_{\mathsf{clemb}}$. Hence if every preneighbourhood morphism is continuous, $\mathbb{A}_{\mathsf{d}} = {}^{\perp}\mathbb{A}_{\mathsf{clemb}}$, ${\mathbb{A}_{\mathsf{d}}}^{\perp} = {{}^{\perp}\mathbb{A}_{\mathsf{clemb}}}^{\perp} \subseteq \mathbb{A}_{\mathsf{clemb}} \subseteq {{}^{\perp}\mathbb{A}_{\mathsf{clemb}}}^{\perp}$ proving ${\mathbb{A}_{\mathsf{d}}}^{\perp} = \mathbb{A}_{\mathsf{clemb}}$. This completes the proof of (f). The rest of the properties in (g), (h) follow from (f) and the properties of a prefactorisation system (see Proposition 2.2 [5]). □

Remark 5.4. If every preneighbourhood morphism is continuous then from part (*f*) of Theorem 5.3, $(\mathbb{A}_{\mathsf{d}} \cap \mathtt{pNbd}[\mathbb{A}]_1, \mathbb{A}_{\mathsf{clemb}} \cap \mathtt{pNbd}[\mathbb{A}]_1)$ is a factorisation structure for $\mathtt{pNbd}[\mathbb{A}]$. The dense-(closed embedding) factorisation of preneighbourhood morphisms is well known for topological spaces, i.e., for neighbourhood systems of the context $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$, as well as for locales with T-neighbourhood systems (see §XV.2.2 [58]). Theorem 5.3(f) generalise these results to larger subcategories of preneighbourhood spaces.

Remark 5.5. The factorisation system $(\mathbb{A}_{\mathsf{d}} \cap \mathtt{pNbd}[\mathbb{A}]_1, \mathbb{A}_{\mathsf{clemb}} \cap \mathtt{pNbd}[\mathbb{A}]_1)$ is proper if and only if for each internal neighbourhood space (X, μ), $(X, \mu) \overset{d_X}{\rightarrowtail} (X \times X, \mu \times \mu)$ is a closed embedding, i.e., (X, μ) is an internal *Hausdorff space* (see Definition 7.5 and Theorem 7.6) — note the assumption of *continuity of each preneighbourhood morphism* is already embedded by Theorem 5.3(f), relaxing the *proper* condition in Definition 7.5 (see also Theorem 6.2(c)).

5.2 This section exhibit examples of dense subobjects.

Example 5.6. In the context $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$ the dense morphisms between neighbourhood spaces are precisely the usual continuous maps with dense image.

In the context $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMon})$, the dense morphisms between neighbourhood spaces are precisely the bicontinuous functions between the bitopological spaces, which have dense image with respect to the second topologies.

Example 5.7. In the context $(\mathtt{Loc}, \mathsf{Epi}, \mathsf{RegMon})$, a localic map $X \xrightarrow{f} Y$ is a dense morphism with respect to the T-neighbourhood systems on X and Y, if and only if, $f(0) = 0$, i.e., is a dense localic map in the usual sense (see §8.2 [58]).

Example 5.8. In the context $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$ a group homomorphism $X \xrightarrow{f} Y$ is $\Phi(\mathtt{ncl}_X)$-$\Phi(\mathtt{ncl}_Y)$ dense if and only if the image $f(X)$ non-trivially meets every non-trivial normal subgroup of Y, i.e., $f(X)$ is *essential with respect to normal subgroups.*

Definition 5.9. An admissible subobject $m \in \mathtt{Sub}_{\mathsf{M}}(X)$ is *essential with respect to* $\mathfrak{a}$ or $\mathfrak{a}$-*essential* ($\mathfrak{a} \subseteq \mathtt{Sub}_{\mathsf{M}}(X)$) if $a \in \mathfrak{a} \Rightarrow a \wedge m \neq \sigma_X$; $\mathtt{Sub}_{\mathsf{M}}(X)$-essential is shortened to *essential.*[5]

Example 5.10. In the context $(\mathtt{CRng}, \mathsf{RegEpi}, \mathsf{Mon})$ a ring homomorphism $X \xrightarrow{f} Y$ is $\Phi(\mathtt{idl}_X)$-$\Phi(\mathtt{idl}_Y)$ dense if and only if the image $f(X)$ non-trivially meets every non-trivial ideal Y, i.e., the image $f(X)$ is essential with respect to ideals.

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[5] The idea of essential abelian groups can be obtained from (see page 19 [53]).

6 Stably closed morphisms

This section discuss pullback stable closed morphisms and *compact preneighbourhood* spaces as a special case.

6.1 Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

Definition 6.1. A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is said to be *proper* if it is stably in $\mathbb{A}_{\mathsf{cl}}$, i.e., for every preneighbourhood morphism $(T, \tau) \xrightarrow{h} (Y, \phi)$, the pullback f_h of f along h is a closed morphism. The symbol $\mathbb{A}_{\mathsf{pr}}$ denotes the (possibly large) set of proper morphisms in $\mathbb{A}$.

Theorem 6.2. *Let* $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$, $(T, \tau) \xrightarrow{h} (Y, \phi)$ *be preneighbourhood morphisms.*

(a) The preneighbourhood morphism f *is a proper morphism if and only if for any internal preneighbourhood space* (T, τ) *every corestriction of* $X \times T \xrightarrow{f \times \mathbf{1}_T} Y \times T$ *along a section of the second product projection* p_2 *is a closed morphism.*

(b) The set $\mathbb{A}_{\mathsf{pr}}$ *is a pullback stable set, is closed under compositions.*

(c) If all preneighbourhood morphisms are continuous then $\mathbb{A}_{\mathsf{clemb}} \subseteq \mathbb{A}_{\mathsf{pr}}$.

(d) If $g \circ f \in \mathbb{A}_{\mathsf{pr}}$ *and* f *is stably in* E *and is stably continuous, i.e., for any morphism* h *the pullback* f_h *of* f *along* h *is in* E *and is continuous, then* $g \in \mathbb{A}_{\mathsf{pr}}$.

(e) If $g \circ f \in \mathbb{A}_{\mathsf{pr}}$ *and* $g \in \mathtt{Mono}(\mathbb{A})$ *then* $f \in \mathbb{A}_{\mathsf{pr}}$.

(f) If f *is a proper morphism then* $f \times f$ *is a proper morphism.*

Proof. Firstly, sections of the second product projection $Y \times T \xrightarrow{p_2} T$ are precisely determined by $(T, \tau) \xrightarrow{h} (Y, \phi)$, namely $(h, \mathbf{1}_T)$. Towards the proof of (a), consider the commutative diagram:

$$\begin{tikzcd}
P \arrow[rr, "f_h"] \arrow[dr, "{(h_f,f_h)}"] \arrow[dd, "h_f"'] & & T \arrow[dr, "{(h,\mathbf{1}_T)}"] \arrow[dd, "h"] & \\
& X \times T \arrow[rr, "f\times\mathbf{1}_T"] \arrow[dl, "p_1"] & & Y \times T \arrow[dl, "p_1"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

in which p_1's are product projections, the horizontal square is the pullback of p_1 along f. From properties of pullbacks, f_h is the pullback of f along h if and only if f_h is the corestriction of $f \times \mathbf{1}_T$ along the section $(h, \mathbf{1}_T)$ of p_2. This completes the proof of (a). Since for any $m \in \mathfrak{C}_\phi$, $f^{-\mathbf{1}}m \in \mathfrak{C}_\mu$ when f is continuous (Proposition 3.15), closed embeddings are proper when every morphism is continuous, proving (c). For the first part of (b), $\mathbb{A}_{\mathsf{pr}}$ is the largest pullback stable subset of $\mathbb{A}_{\mathsf{cl}}$. If f and g are proper preneighbourhood morphisms then from the diagram

$$\begin{tikzcd}
R \arrow[r, "f_{w_g}"] \arrow[d, "w_{g\circ f}"'] & S \arrow[r, "g_w"] \arrow[d, "w_g"] & W \arrow[d, "w"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where $(W, \omega) \xrightarrow{w} (Z, \psi)$ is a preneighbourhood morphism, the right hand square is the pullback of w along g and the left hand square is the pullback of w_g along f, the outer square is the pullback of w along $g{\circ}f$. If g and f are proper morphisms, g_w and f_{w_g} are both closed morphisms and hence their composite $g_w{\circ}f_{w_g}$ is closed (Theorem 4.2(d)), proving $g{\circ}f$ is a proper morphism. This proves the second part of (b). On the other hand if the composite $g{\circ}f$ is a proper morphism then $g_w{\circ}f_{w_g}$ is a closed morphism. Further if f is a morphism stably continuous and stably in E, f_{w_g} is a continuous morphism stably in E; hence f_{w_g} is a formally surjective continuous morphism. Hence g_w is a closed morphism (Theorem 4.2 (e)) proving g to be a proper morphism. This proves (d). If $g{\circ}f$ is a proper morphism and g

is a monomorphism then from the commutative diagram:

$$\begin{tikzcd}
T' \arrow[r, "v"] \arrow[d, "{(u,v)}"'] & T \arrow[r, equal] \arrow[d, "{(h,\mathbf{1}_T)}"'] & T \arrow[d, "{(g\circ h,\mathbf{1}_T)}"] \\
X \times T \arrow[r, "f\times\mathbf{1}_T"'] & Y \times T \arrow[r, "g\times\mathbf{1}_T"'] & Z \times T
\end{tikzcd}$$

in which the left hand square is the pullback of $f \times \mathbf{1}_T$ along $(h, \mathbf{1}_T)$, since g is a monomorphism the right hand square is a pullback square, implying the outer square is the pullback of $(g\circ h, \mathbf{1}_T)$ along $(g\circ f) \times \mathbf{1}_T$. Since $g\circ f$ is proper, using (a) on the outer pullback square, v is a closed morphism. Hence using (a) again, f is a proper morphism. This proves (e).

Finally, towards the proof of (f), given $(T, \tau) \xrightarrow{(t,s)} (Y \times Y, \phi \times \phi)$ consider the diagrams:

$$\begin{tikzcd}
Q \arrow[r, "v"] \arrow[d, "u"'] & Q' \arrow[r, "s_f"] \arrow[d, "f_s"] & X \arrow[d, "f"] \\
P \arrow[r, "f_t"] \arrow[d, "t_f"'] & T \arrow[r, "s"'] \arrow[d, "t"] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Q \arrow[r, "f_t\circ u"] \arrow[d, "{(t_f\circ u, s_f\circ v)}"'] & T \arrow[d, "{(t,s)}"] \\
X \times X \arrow[r, "f\times f"'] & Y \times Y
\end{tikzcd}
\quad (\star\star)$$

where in the left hand diagram each square is a pullback square. Since f is proper then f_t, f_s, u and v are each closed morphisms. Clearly, the right hand square of $(\star\star)$ is a pullback square. Hence the composite $f_t\circ u$, which is the pullback of $f \times f$ along (t, s), is a closed morphism (Theorem 4.2(d)). Therefore $f \times f$ is a proper morphism, proving (f).

□

6.2 This section exhibit examples of proper maps.

Example 6.3. In the context (Set, Surjections, Injections) the proper maps of internal neighbourhood spaces are precisely the proper maps of topological spaces. In the context (Top, Epi, ExtMon) the proper maps are precisely the proper maps between the second topological spaces. In the context (Loc, Epi, RegMon) the proper maps between the internal T-neighbourhood spaces are precisely the usual localic proper maps [59, 65, 66].

Example 6.4. In the context $(\mathtt{Grp}, \mathsf{RegEpi}, \mathsf{Mon})$ (respectively, in the context $(\mathtt{CRng}, \mathsf{RegEpi}, \mathsf{Mon})$) if $(X, \Phi(\mathtt{ncl}_X)) \xrightarrow{f} (Y, \Phi(\mathtt{ncl}_Y))$ (respectively, $(X, \Phi(\mathtt{idl}_X)) \xrightarrow{f} (Y, \Phi(\mathtt{idl}_Y))$) with $f \in \mathsf{RegEpi}$ then f is proper.

6.3 This section introduce the *compact preneighbourhood spaces.*

Definition 6.5. An internal preneighbourhood space (X, μ) is *compact* if the unique morphism $(X, \mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1}, \nabla_{\mathtt{1}})$ is proper. The full subcategory of all compact objects is denoted by $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$.

Remark 6.6. Immediately from Theorem 6.2(a): an internal preneighbourhood space (X, μ) is compact if and only if for every internal preneighbourhood space (Y, ϕ), the projection $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ is a closed morphism, in fact a proper morphism.

Theorem 6.7. *(a) If (Y, ϕ) is compact and $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a proper morphism then (X, μ) is compact.*

(b) If (X, μ) is compact and $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism with f stably continuous and stably in E then (Y, ϕ) is compact.

(c) The category $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$ is finitely productive; if every preneighbourhood morphism is continuous then $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$ is closed hereditary, i.e., if (X, μ) is compact and $m \in \mathfrak{C}_\mu$ then $\left(M, (\mu|_M)\right)$ is compact.

Proof. Since $\mathtt{t}_X = \mathtt{t}_Y \circ f$, (a) & (b) follow from Definition and Theorem 6.2((b) & (d)). Since every isomorphism is a proper morphism, $(\mathtt{1}, \nabla_{\mathtt{1}})$ is compact. Further, binary products of compact objects is proper from (a) and Definition 6.5. Hence $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$ is finitely productive. Since every closed embedding is a proper morphism if every preneighbourhood morphism is continuous (Theorem 6.2(c)) the closed heredity of $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$ follows from (a). □

A detailed treatment of $\mathtt{K}[\mathbf{pNbd}[\mathbb{A}]]$ shall be done in a later paper.

7 Separated morphisms

Given the context $(\mathbb{A}, \mathsf{E}, \mathsf{M})$, let $(X, \mu) \xrightarrow{f} (Y, \phi)$ be a preneighbourhood morphism with $\mathtt{kerp}\,f \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$ its kernel pair. Since

$$\Big(X, ((\mu \times_\phi \mu)\big|_{d_f})\Big) \xrightarrow{d_f=(\mathbf{1}_X,\, \mathbf{1}_X)} (\mathtt{kerp}\,f, \mu \times_\phi \mu)\ ,$$

X is an admissible subobject of $\mathtt{kerp}\,f$. Evidently, any morphism $T \xrightarrow{t} \mathtt{kerp}\,f$ is determined by the pair $T \underset{t_2}{\overset{t_1}{\rightrightarrows}} X$ of morphisms such that $f_i{\circ}t = t_i$ $(i = 1, 2)$ and $t_i = t_i^{\mathsf{E}}{\circ}t_i^{\mathsf{M}}$ is the (E, M)-factorisation of t_i $(i = 1, 2)$. If $t = (t_1, t_2) \in \mathtt{Sub}_{\mathsf{M}}(\mathtt{kerp}\,f)$ and $[t_1{=}t_2] \xrightarrow{m_t} T \underset{t_2}{\overset{t_1}{\rightrightarrows}} X$ is the equaliser of the pair (t_1, t_2), then:

$$d_f{}^{-\mathbf{1}}t = t_1{\circ}m_t = t_2{\circ}m_t, \tag{7.1}$$

$$d_f \wedge t = (t_1{\circ}m_t, t_2{\circ}m_t), \tag{7.2}$$

and

$$\mu(d_f{}^{-\mathbf{1}}t) \supseteq \mu(t_1^{\mathsf{M}}) \vee \mu(t_2^{\mathsf{M}}). \tag{7.3}$$

The statements in (7.1)&(7.2) are trivial computations; for (7.3), $t_i{\circ}m_t = t_i^{\mathsf{M}}{\circ}t_i^{\mathsf{E}}{\circ}m_t$ implies $t_i{\circ}m_t \le t_i^{\mathsf{M}}$, $(i = 1, 2)$ yielding the result from (7.1). An use of (7.3) shows $((\mu \times_\phi \mu)\big|_{d_f}) \le \mu$. Since for every $v \in \mu(u)$, $v = d_f{}^{-\mathbf{1}}(f_1{}^{-\mathbf{1}}v \wedge f_2{}^{-\mathbf{1}}v) \Leftrightarrow d_f{\wedge}f_1{}^{-\mathbf{1}}v{\wedge}f_2{}^{-\mathbf{1}}v \le d_f{\circ}v = (v, v)$, $v \in ((\mu \times_\phi \mu)\big|_{d_f})(u)$; hence $\mu = ((\mu \times_\phi \mu)\big|_{d_f})$.

7.1 This section discuss the notion of *separated morphisms.*

Definition 7.1. A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is said to be a *separated morphism* if d_f is a proper morphism. The symbol $\mathbb{A}_{\mathsf{sep}}$ denotes the (possibly large) set of all separated morphisms of $\mathbb{A}$.

Remark 7.2. In case when every preneighbourhood morphism is continuous an use of Theorem 6.2 shows, a preneighbourhood morphism f is separated if and only of d_f is a closed embedding.

Example 7.3. In the context $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$ the separated morphisms between internal neighbourhood spaces are precisely those continuous maps in whose fibres distinct points are separated by disjoint neighbourhoods. In the context $(\mathtt{Top}, \mathsf{Epi}, \mathsf{ExtMon})$, the separated morphisms between the internal neighbourhood spaces are precisely the separated maps with respect to the second topologies.

Theorem 7.4. *Let* $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$ *be preneighbourhood morphisms.*

(a) The set $\mathbb{A}_{\mathsf{sep}}$ *contain all monomorphisms.*

(b) The set $\mathbb{A}_{\mathsf{sep}}$ *is pullback stable.*

(c) If $g, f \in \mathbb{A}_{\mathsf{sep}}$ *then* $g \circ f \in \mathbb{A}_{\mathsf{sep}}$*.*

(d) If $g \circ f \in \mathbb{A}_{\mathsf{sep}}$ *and* f *is a proper morphism stably continuous and stably in* E *then* $g \in \mathbb{A}_{\mathsf{sep}}$*.*

(e) If $g \circ f \in \mathbb{A}_{\mathsf{sep}}$ *then* $f \in \mathbb{A}_{\mathsf{sep}}$*.*

Proof. Since the kernel pair of a monomorphism f is trivial, d_f is an isomorphism. Hence every monomorphism is separated, proving (a). For the rest of the proof, let $\mathtt{kerp}\, f \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$, $\mathtt{kerp}\, g \overset{g_1}{\underset{g_2}{\rightrightarrows}} Y$, $\mathtt{kerp}\, h \overset{h_1}{\underset{h_2}{\rightrightarrows}} X$ the kernel pairs, where $h = g \circ f$. Evidently:

$$\begin{tikzcd}\mathtt{kerp}\, f \arrow[r, "(f_1, f_2)"] & \mathtt{kerp}\, h \arrow[r, "(f \circ h_1, f \circ h_2)"] & \mathtt{kerp}\, g\end{tikzcd} .$$

Consider the commutative the diagram

$$\begin{tikzcd}
X \arrow[r, tail, "d_f"] \arrow[d, equal] & \mathtt{kerp}\, f \arrow[r, "f\circ f_2"] \arrow[d, "{(f_1, f_2)}"'] & Y \arrow[d, tail, "d_g"] \\
X \arrow[r, tail, "d_h"'] & \mathtt{kerp}\, h \arrow[r, "{(f\circ h_1, f\circ h_2)}" description] \arrow[d, tail, "{(h_1, h_2)}"'] & \mathtt{kerp}\, g \arrow[d, tail, "{(g_1, g_2)}"] \\
& X\times X \arrow[r, "f\times f"'] & Y\times Y
\end{tikzcd} \quad . \ (\star)$$

The top right hand square is a pullback square — if $P \xrightarrow{p} Y$ and $P \overset{q}{\underset{r}{\rightrightarrows}} X$ be morphisms such that $h\circ q = h\circ r$ and $(p, p) = d_g\circ p = (f\circ h_1, f\circ h_2)\circ(q, r)$ then $f\circ q = p = f\circ r$. Hence, $P \xrightarrow{(q, r)} \mathtt{kerp}\, f$ is the unique morphism such that $(f_1, f_2)\circ(q, r) = (q, r)$ and $f\circ f_2\circ(q, r) = f\circ r = p$, proving the assertion. On the other hand the top outer square is trivially a pullback square. Hence using properties of pullback squares the pullback of (f_1, f_2) along d_h is $\mathbf{1}_X$. Finally the bottom right hand square is trivially a pullback square. If f is a proper morphism then so also is $f\times f$ (Theorem 6.2(f)); from the right hand bottom pullback square of $(\star)$, $(f\circ h_1, f\circ h_2)$ is a proper morphism. From the top outer square in $(\star)$, since $d_g\circ f = (f\circ h_1, f\circ h_2)\circ d_h$, if h is a separated morphism then $d_g\circ f$ is a proper morphism (Theorem 6.2(b)). Hence, if f is a proper morphism stably continuous and stably in E, from Theorem 6.2 (d), d_g is proper morphism, i.e., g is separated, proving (d). If h is separated then d_h is a proper morphism. Hence from the left hand pullback square, d_f being the pullback of d_h along (f_1, f_2) is a proper morphism. Hence f is a separated, proving (e). If g is separated, d_g is a proper morphism and hence (f_1, f_2) is proper. Further if f is separated, $d_h = (f_1, f_2)\circ d_f$ is a proper morphism. Hence h is a separated morphism proving (c). Towards (b), consider the diagram in (†) with f a proper morphism. In (†) the front vertical and the right hand vertical squares depict the pullback of f along $(Z, \psi) \xrightarrow{p} (Y, \phi)$, the base horizontal square is the kernel pair of f and the top horizontal square is the kernel pair of f_p the pullback of f along p. Since $f\circ p_f\circ(f_p)_2 = p\circ f_p\circ(f_p)_2 = p\circ f_p\circ(f_p)_1 = f\circ p_f\circ(f_p)_1$, there exists the unique morphism $\mathtt{kerp}\, f_p \xrightarrow{v} \mathtt{kerp}\, f$ such that $f_1\circ v = p_f\circ(f_p)_1$ and $f_2\circ v = p_f\circ(f_p)_2$.

Furthermore, using properties of pullbacks squares, all the faces of the cube are pullback squares.

$$\begin{tikzcd}
P \arrow[rr, tail, "d_{f_p}"] \arrow[dr, equal] \arrow[dd, dashed, "!\,w"'] & & \mathtt{kerp}\, f_p \arrow[dl, "(f_p)_1"'] \arrow[rr, "(f_p)_2"] \arrow[dd, dotted, "!\,v"] & & P \arrow[dl, "f_p"'] \arrow[dd, "p_f"] \\
& P \arrow[rr, "f_p"] \arrow[dd, "p_f"'] & & Z \arrow[dd, "p"] & \\
X \arrow[rr, tail, "d_f"] \arrow[dr, equal] & & \mathtt{kerp}\, f \arrow[dl, "f_1"] \arrow[rr, "f_2"] & & X \arrow[dl, "f"] \\
& X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

(†)

Since d_f is the equaliser of (f_1, f_2) and $f_1 \circ v \circ d_{f_p} = f_2 \circ v \circ d_{f_p}$, there exists a unique morphism $P \xrightarrow{w} X$ such that $d_f \circ w = v \circ d_{f_p}$. Hence $w = f_1 \circ d_f \circ w = f_1 \circ v \circ d_{f_p} = p_f \circ (f_p)_1 \circ d_{f_p} = p_f$. Furthermore, since the left most square is trivially a pullback square it follows from properties of pullbacks that p_f is the pullback of v along d_f. Consequently, if f is separated, then d_{f_p} being the pullback of d_f along v is also a proper morphism. This proves f_p a separated morphism whenever f is a separated morphism. Hence $\mathbb{A}_{\mathsf{sep}}$ is stable under pullbacks. This completes the proof. ☐

7.2 Given $(X, \mu) \xrightarrow{f} (Y, \phi)$, f is separated if and only if in the context $(\mathcal{A} \downarrow Y)$, the internal preneighbourhood space $((X, f), (\mu \downarrow Y))$ has the property: the unique morphism $f = \mathtt{t}_{(X,f)}$ is separated, i.e., the diagonal morphism from (X, f) to $(X, f) \times (X, f)$ is a proper morphism.

Definition 7.5. An internal preneighbourhood space (X, μ) is *Hausdorff* if the unique morphism $(X, \mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1}, \nabla_{\mathtt{1}})$ is separated.

Evidently, if (X, μ) is Hausdorff and $\mu' \geq \mu$, then (X, μ') is also Hausdorff (Theorem 7.4(a)).

Theorem 7.6. *The following are equivalent for any internal preneighbourhood space (X, μ) of $\mathbb{A}$:*

(a) *(X, μ) is an internal Hausdorff space.*

(b) *The diagonal morphism $(X, \mu) \overset{d_X}{\rightarrowtail} (X \times X, \mu \times \mu)$ is a proper morphism.*

(c) *Every preneighbourhood morphism with (X, μ) as domain is separated.*

(d) *There exists a separated preneighbourhood morphism from (X, μ) to an internal Hausdorff space.*

(e) *For every proper morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ with f stably continuous and stably in E, (Y, ϕ) is an internal Hausdorff space.*

(f) *The product projection $(X \times Y, \mu \times \phi) \xrightarrow{p_Y} (Y, \phi)$ is a separated morphism for every internal preneighbourhood space (Y, ϕ).*

(g) *For every internal Hausdorff space (Y, ϕ), $(X \times Y, \mu \times \phi)$ is an internal Hausdorff space.*

(h) *If $(E, (\psi\big|_E)) \xrightarrow{e} (Z, \psi) \overset{f}{\underset{g}{\rightrightarrows}} (X, \mu)$ be the equaliser diagram for f and g then e is a proper morphism.*

Proof. Evidently (a) and (b) are equivalent by definition. Given any preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ since $\mathtt{t}_X = \mathtt{t}_Y \circ f$, an use of Theorem 7.4(e), shows (a) implies (c). On the other hand, (c) evidently implies (a). Since $(\mathtt{1}, \nabla_{\mathtt{1}})$ is already an internal Hausdorff space, (a) automatically implies (d). On the other hand if (Y, ϕ) be an internal Hausdorff space and $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a separated preneighbourhood morphism then $\mathtt{t}_X = \mathtt{t}_Y \circ f$ implies from Theorem 7.4(c), X is an internal Hausdorff space. Hence (d) implies (a). Given any proper morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ with f stably continuous and stably in E an use of Theorem 7.4(b) prove from $\mathtt{t}_X = \mathtt{t}_Y \circ f$ the implication of (e) from (a). On the contrary, assuming (e) and considering $Y = X$, $f = \mathbf{1}_X$, (a) follows. Since the product projection $(X \times Y, \mu \times \phi) \xrightarrow{p_Y} (Y, \phi)$ is the pullback of $\mathtt{t}_X$ along $\mathtt{t}_Y$, (a) implies (f) from pullback stability of separated morphisms (Theorem 7.4(b)). If (Y, ϕ) is an internal Hausdorff space and $(X \times Y, \mu \times \phi) \xrightarrow{p_Y} (Y, \phi)$ is the product projection, then $\mathtt{t}_{X\times Y} = \mathtt{t}_Y \circ p_Y$ implies (g) from (f) & Theorem 7.4(c). Since any internal preneighbourhood space isomorphic to an internal Hausdorff space is also an internal Hausdorff space, (g) evidently implies (a). Since $\begin{tikzcd} E \arrow[r, "e"] & Z \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$ is an equaliser diagram if and only if $f \circ e$ is the pullback of $Z \xrightarrow{(f,g)} X \times X$ along d_X, (b) implies (h) from the pullback stability of separated morphisms proved in Theorem 7.4(b). On the other hand, since d_X is the equaliser of the product projections, (h) implies (b). This completes the proof of Theorem.

□

Corollary 7.7. *The category* $\mathtt{Haus}[\mathtt{pNbd}[\mathbb{A}]]$ *is a finitely complete subcategory of* $\mathtt{pNbd}[\mathbb{A}]$ *closed under subobjects and images of preneighbourhood morphisms stably continuous and stably in* E.

Proof. Since $(\mathtt{1}, \nabla_{\mathtt{1}})$ is an internal Hausdorff space, from (g) & (h) of Theorem the category $\mathtt{Haus}[\mathbb{A}]$ is closed under finite products and regular subobjects. Hence $\mathtt{Haus}[\mathbb{A}]$ is finitely complete. Let (X, μ) be an internal Hausdorff space and $Y \xrightarrow{f} X$ be a monomorphism. Let μ_f be the smallest preneighbourhood system on Y making f a preneighbourhood morphism. Then $(Y, \mu_f) \xrightarrow{f} (X, \mu)$ is separated morphism (Theorem 7.4(a)) and hence (Y, μ_f) is an internal Hausdorff space from (d) of Theorem. Finally from (e) of Theorem if (X, μ) is an internal Hausdorff space and (Y, ϕ) be an internal preneighbourhood space such that $Y = \exists_f X$ for some preneighbourhood morphism f stably continuous and stably in E then (Y, ϕ) is also an internal Hausdorff space. □

8 Perfect morphisms

This section discuss *perfect morphisms*, i.e., morphisms which are both proper and separated.

8.1 Firstly, some results for preneighbourhood morphisms between compact and Hausdorff preneighbourhood spaces.

Theorem 8.1. *(a) Every preneighbourhood morphism from a compact preneighbourhood space to a Hausdorff preneighbourhood space is proper.*

(b) A preneighbourhood morphism with a compact Hausdorff codomain is proper if and only if the domain is compact.

(c) Every compact admissible subobject of a Hausdorff preneighbourhood space is closed.

Proof. The statement in (b) follows from (a) and composition closed property of proper morphisms (Theorem 6.2(b)). Since an admissible subobject is proper if and only if it is closed, the statement in (c) follows from (a). Towards the proof of (a), if $(X,\mu)\xrightarrow{f}(Y,\phi)$ is a preneighbourhood morphism from a compact preneighbourhood space (X,μ) to a Hausdorff preneighbourhood space (Y,ϕ), and $X\times Y\xrightarrow{p_Y}Y$ is the product projection, then from the pullback square

$$\begin{tikzcd} X \arrow[r, tail, "(\mathbf{1}_X, f)"] \arrow[d, "f"'] & X\times Y \arrow[d, "f\times\mathbf{1}_Y"] \\ Y \arrow[r, tail, "d_Y"'] & Y\times Y \end{tikzcd}$$

, since (Y,ϕ) is Hausdorff the diagonal d_Y is proper (Theorem 7.6(b)) implying $(\mathbf{1}_X, f)$ is proper. Since (X,μ) is compact, p_Y is proper (Remark 6.6). Hence the composite $f=p_Y\circ(\mathbf{1}_X,f)$ is proper (Theorem 4.2(d)), proving (a). ☐

Definition 8.2. A preneighbourhood morphism $(X,\mu)\xrightarrow{f}(Y,\phi)$ is a *perfect morphism* if it is both proper and separated. The symbol $\mathbb{A}_{\mathsf{per}}=\mathbb{A}_{\mathsf{pr}}\cap\mathbb{A}_{\mathsf{sep}}$ is the (possibly large) set of all perfect morphisms of $\mathbb{A}$.

8.2 As an immediate consequence of Theorem 6.2 & 7.4:

Theorem 8.3. *The set $\mathbb{A}_{\mathsf{per}}$ of all perfect morphisms of $\mathbb{A}$ is a pullback stable set, is closed under composition and satisfies the properties:*

(a) If every preneighbourhood morphism is continuous then $\mathbb{A}_{\mathsf{clemb}}\subseteqq\mathbb{A}_{\mathsf{per}}$.

(b) If $g\circ f\in\mathbb{A}_{\mathsf{per}}$ and f is a proper morphism, stably continuous and stably in E then $g\in\mathbb{A}_{\mathsf{per}}$.

(c) If $g\circ f\in\mathbb{A}_{\mathsf{pr}}$ and $g\in\mathbb{A}_{\mathsf{sep}}$ then $f\in\mathbb{A}_{\mathsf{pr}}$.

(d) If $g{\circ}f \in \mathbb{A}_{\mathsf{per}}$ *and* $g \in \mathbb{A}_{\mathsf{sep}}$ *the* $f \in \mathbb{A}_{\mathsf{per}}$.

Proof. It is enough to prove (c). Let $(X,\mu) \xrightarrow{f} (Y,\phi) \xrightarrow{g} (Z,\psi)$ be preneighbourhood morphisms such that $g{\circ}f$ is proper and g is separated. Evidently, in $(\mathcal{A} \downarrow Z)$, $((X, g{\circ}f), (\mu \downarrow Z)) \xrightarrow{f} ((Y,g),(\phi \downarrow Z))$ is a preneighbourhood morphism from a compact preneighbourhood space to a Hausdorff preneighbourhood space. Hence from Theorem 8.1(a), f is proper in $(\mathcal{A} \downarrow Z)$, and hence proper in $\mathcal{A}$. The statement in (d) follows from (c) and Theorem 7.4(e). □

8.3 Three types of internal preneighbourhood spaces In conclusion, three important types of internal preneighbourhood spaces are defined. Detailed investigation of these spaces shall be done in later papers.

Definition 8.4. An internal preneighbourhood space (X,μ) is:

(a) *compact Hausdorff* if $(X,\mu) \xrightarrow{\mathtt{t}_X} (\mathtt{1}, \nabla_{\mathtt{1}})$ is a perfect morphism.

(b) *Tychonoff* if there exists a morphism $(X,\mu) \rightarrowtail^{m} (Y,\phi)$, where (Y,ϕ) is a compact Hausdorff internal preneighbourhood space and $m \in \mathsf{M}$.

(c) *absolutely closed* if for every morphism $(X,\mu) \rightarrowtail^{m} (Y,\phi)$ where (Y,ϕ) is a Hausdorff internal preneighbourhood space and $m \in \mathsf{M}$ the morphism $m \in \mathfrak{C}_{\phi}$.

The symbols `KHaus`[`pNbd`[$\mathbb{A}$]], `Tych`[`pNbd`[$\mathbb{A}$]], `AbCl`[`pNbd`[$\mathbb{A}$]] respectively denote the full subcategories of compact Hausdorff, Tychonoff, absolutely closed internal preneighbourhood spaces.

9 Concluding Remarks

Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context.

Given (possibly large) sets $\mathfrak{a}$, $\mathfrak{b}$ of morphisms of $\mathbb{A}$, the phrases $\mathfrak{b}$ *is composition closed* or $\mathfrak{b}$ *is (pullback) stable* is well known; the set $\mathfrak{b}$ shall be said to be *left* $\mathfrak{a}$ *cancellative* (respectively, *right* $\mathfrak{a}$ *cancellative*) if $g{\circ}f \in \mathfrak{b}$ and $g \in \mathfrak{a}$ (respectively, $f \in \mathfrak{a}$) implies $f \in \mathfrak{b}$ (respectively, $g \in \mathfrak{b}$). The set $\mathfrak{b}$ is *stably in* E if every pullback f_g of f along any morphism g is in E. If $\mathfrak{b}$ is a set of preneighbourhood morphisms then it is said to be *stably continuous* if for any μ-ϕ continuous preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ in $\mathfrak{b}$, and for any preneighbourhood morphism $(Z,\psi) \xrightarrow{g} (Y,\phi)$, the pullback $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z,\psi)$ of f along g is $(\mu \times_\phi \psi)$-ψ continuous and is also in $\mathfrak{b}$. Table 1 summarise the properties deduced in this paper. The cells highlighted in this colour are the properties where the *continuity* condition is required; the others do not require *continuity* of the involved preneighbourhood morphism, and hence are purely consequences of the preneighbourhood morphism property.

The following definition appears in §2 [54]:

Definition 9.1. A pullback stable (possibly large) set $\mathfrak{a}$ of morphisms of $\mathbb{A}$ is called a *topology* if it contains isomorphisms and is closed under compositions.

If $\mathfrak{a}$ is a topology and right $\mathfrak{a}$ cancellative, a topology $\mathfrak{b}$ is called a $\mathfrak{a}$-*topology* if it is right $\mathfrak{a}$ cancellative.

Drawing inspiration from [32], it is observed in (see §2 [54]) that in case when a finitely complete category $\mathbb{A}$ with a proper (E,M)-factorisation system has a set $\mathbb{A}_{\mathsf{cl}}$ of closed morphisms described by axioms (see Axioms (F3)-(F5) [32]), then the set of proper morphisms (i.e., morphisms stably in $\mathbb{A}_{\mathsf{cl}}$) is a $\mathfrak{s}$-topology, where $\mathfrak{s}$ is the set of morphisms stably in E.

In terms of Definition, Table 1 shows the set $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ is a right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative topology and each of the sets $\mathbb{A}_{\mathsf{pr}}$, $\mathbb{A}_{\mathsf{sep}}$, $\mathbb{A}_{\mathsf{per}}$ are $\mathbb{A}_{\mathtt{st(E,c,cl)}}$-topologies. The difference between the two approaches arises from the fact that in the present case $\mathbb{A}_{\mathsf{cl}}$ is right $\mathbb{A}_{\mathtt{fsc}}$ $(\subset \mathsf{E})$ cancellative, while the axioms of [32] assert $\mathbb{A}_{\mathsf{cl}}$ is right E cancellative. In case when $\mathcal{A}$ is RZC and E is pullback stable the present case reduces to the situation considered in [32].

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	contains	stability	closed under composition	cancellation properties
$\mathbb{A}_{\mathsf{cl}}$	$\mathtt{Iso}(\mathbb{A})$	$m \in \mathbb{A}_{\mathsf{clemb}}, f \in \mathbb{A}_{\mathtt{clc}} \Rightarrow f_m \in \mathbb{A}_{\mathsf{cl}}$ $m \in \mathbb{A}_{\mathsf{clemb}}, f \in \mathbb{A}_{\mathsf{c}} \Rightarrow f^{-1}m \in \mathbb{A}_{\mathsf{clemb}}$	composition closed	right $\mathbb{A}_{\mathtt{fsc}}$ cancellative
$\mathbb{A}_{\mathsf{d}}$	E		$g \in \mathbb{A}_{\mathsf{dc}}, f \in \mathbb{A}_{\mathsf{d}} \Rightarrow g \circ f \in \mathbb{A}_{\mathsf{d}}$	right $\mathbb{A}_1$ cancellative
$\mathbb{A}_{\mathsf{pr}}$	$\mathbb{A}_{\mathsf{clemb}}$, in $\mathtt{RZC}$	pullback stable	composition closed	right $\mathbb{A}_{\mathtt{st(E,c)}}$ cancellative left $\mathtt{Mono}(\mathbb{A})$ cancellative
$\mathbb{A}_{\mathsf{sep}}$	$\mathtt{Mono}(\mathbb{A})$	pullback stable	composition closed	right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative left $\mathbb{A}_1$ cancellative
$\mathbb{A}_{\mathsf{per}}$	$\mathbb{A}_{\mathsf{clemb}}$, in $\mathtt{RZC}$	pullback stable	composition closed	right $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ cancellative left $\mathbb{A}_{\mathsf{per}}$ cancellative

[1] $\mathbb{A}_1$ is the (possibly large) set of all morphisms

[2] $\mathbb{A}_{\mathsf{cl}}$ is the (possibly large) set of all closed morphisms

[3] $\mathbb{A}_{\mathrm{clemb}}$ is the (possibly large) set of all closed embeddings

[4] $\mathbb{A}_{\mathsf{d}}$ is the (possibly large) set of all dense preneighbourhood morphisms

[5] $\mathbb{A}_{\mathsf{pr}}$ is the (possibly large) set of all proper preneighbourhood morphisms

[6] $\mathbb{A}_{\mathsf{sep}}$ is the (possibly large) set of all separated preneighbourhood morphisms

[7] $\mathbb{A}_{\mathsf{per}}$ is the (possibly large) set of all perfect preneighbourhood morphisms

[8] $\mathbb{A}_{\mathsf{c}}$ is the (possibly large) set of all continuous preneighbourhood morphisms

[9] $\mathbb{A}_{\mathsf{dc}}$ is the (possibly large) set of all dense and continuous preneighbourhood morphisms

[10] $\mathbb{A}_{\mathtt{fsc}}$ is the (possibly large) set of all formally surjective and continuous preneighbourhood morphisms

[11] $\mathbb{A}_{\mathtt{clc}}$ is the (possibly large) set of all closed and continuous preneighbourhood morphisms

[11] $\mathbb{A}_{\mathtt{st(E,c)}}$ is the (possibly large) set of all preneighbourhood morphisms which are stably continuous and stably in E

[12] $\mathbb{A}_{\mathtt{st(E,c,cl)}}$ is the (possibly large) set of all preneighbourhood morphisms which are stably continuous, stably in E and stably closed

[13] `RZC` abbreviates *reflecting zero context*

[14] the cells in *this colour* indicate the presence of *continuity* in the assertion

[15] additionally, every `RZC` with continuous preneighbourhood morphisms has $(\mathbb{A}_{\mathsf{d}}, \mathbb{A}_{\mathrm{clemb}})$ factorisation structure

Table 1: Comparative list of properties

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