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Classification of 1-absorbing comultiplication modules over a pullback ring

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Abstract. One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. In this paper, we introduce the concept of 1-absorbing comultiplication modules and classify 1-absorbing comultiplication modules over local Dedekind domains and we study it in detail from the classification problem point of view. The main purpose of this article is to classify all those indecomposable 1-absorbing comultiplication modules with finite-dimensional top over pullback rings of two local Dedekind domains and establish a connection between the 1-absorbing comultiplication modules and the pure-injective modules over such rings. In fact, we extend the definition and results given in [17] to a more general 1-absorbing comultiplication modules case.

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1 Introduction and Preliminaries

The idea of investigating a mathematical structure via its representations in simpler structures is commonly used and often successful. The reader is referred to [2], [38, Chapter 1 and 14], [39] and [22] for a detailed discussion problems, their representation types (finite, tame, or wild), and useful computational reduction procedures.

Let $v_1 : R_1 \longrightarrow \bar{R}$ and $v_2 : R_2 \longrightarrow \bar{R}$ be a homomorphism of two local Dedekind domains R_i, $i = 1, 2$, onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $Ker(R \longrightarrow \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \longrightarrow P_i \longrightarrow R \longrightarrow R_j \longrightarrow 0$ is a exact sequence of R-modules (see [27]).

We know that every module is an elementary substructure of a pure-injective module. In fact, there is a minimal pure-injective elementary extension of each module M, denoted by $h(M)$, called the pure-injective hull of M and it is unique up to isomorphism fixing M. The class of pure-injective modules is closed under direct summands and finite direct sums, but an infinite direct sum of pure-injective modules need not be a pure-injective module. Observe that any finite module is pure-injective. In a sense, then, pure-injective modules are model theoretically typical: for example, classification of the complete theories of R-modules reduces to classify the (complete theories of) pure-injective modules. Also, for some rings, "small" (finite-dimensional, finitely generated, ...) modules are classified and in many cases this classification can be extended to give a classification of (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules. Therefore, pure-injective modules are very important (see [26], [35] and [41]). One point of this paper is to introduce a subclass of pure-injective modules.

Modules over pullback rings have been studied by several authors (see for example, [6], [8], [10], [12], [13], [15], [17], [18], [19], [32] and [42]). The important work of Levy [28] provides a classification of all finitely generated indecomposable modules over Dedekind-like rings. L. Klingler [24] extended this classification to lattices over certain non-commutative Dedekind-like rings, and L. Klingler and J. Haefner (see [20], [21]) classified lattices over certain non-commutative pullback rings, which they called special quasi triads. Common to all these classifications is the reduction to a "matrix proplem" over a division ring (see [34], [38] and [40] for background on matrix problems and their applications).

In the present article, we introduce a new class of R-modules, called 1-absorbing comultiplication modules (see Definition 2.5), and we study it in detail from the classification problem point of view. We are mainly interested in case either R is a Dedekind domain or R is a pullback of two local Dedekind domains. First, we give a complete description of the 1-absorbing comultiplication modules over a local Dedekind domain. Let R be a pullback of two local Dedekind domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable 1-absorbing comultiplication R-modules with finite-dimensional top over $R/Rad(R)$ (for any module M we define its top as $M/Rad(R)M$). In fact, we extend the definition and results given in [17] to a more general 1-absorbing comultiplication modules case.

The classification is divided into two stages: the description of all indecomposable separated 1-absorbing comultiplication R-modules and then, using this list of separated 1-absorbing comultiplication modules, we show that non-separated indecomposable 1-absorbing comultiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable 1-absorbing comultiplication R-modules. Then we use the classification of separated indecomposable 1-absorbing comultiplication modules from Section 3, together with results of Levy [29] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable 1-absorbing comultiplication modules M with finite-dimensional top (see Theorem 4.8).

For the sake of completeness, we state some definitions and notations used throughout. In this article all rings are commutative with identity and all modules unitary. Let R be the pullback ring as mentioned in the beginning of introduction. An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \longrightarrow S/PS \longleftarrow S/P_1S)$ [27, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [27, Theorem 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \longrightarrow M)$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \longrightarrow M$ with S' separated, then f is one-to-one. The module $K = Ker(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [27, Proposition 2.3]. An exact sequence $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [27, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [27, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [27, Theorem 2.6].

Definition 1.1. (a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M.

(b) A proper submodule N of a module M over a ring R is said to be primary submodule (resp., prime submodule) if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some n (resp., $m \in N$ or $r \in (N : M)$), so $Rad(N : M) = P$ (resp., $(N : M) = P'$) is a prime ideal of R, and N is said to be P-primary (resp., P'-prime) submodule. The set of all primary submodules (resp., prime submodules) in an R-module M is denoted $pSpec(M)$ (resp., $Spec(M)$).

(c) A proper submodule N of an R-module M is called to be 2-absorbing, if for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $ab \in (N : M)$ or $am \in N$ or $bm \in N$. So $(N : M)$ is a 2-absorbing ideal of R. The set of all 2-absorbing submodules in an R-module M is denoted by $2 - abSpec(M)$ (see [33]).

(d) A proper ideal I of a commutative ring R is said to be 1-absorbing prime, if for all non-unit elements $a, b, c \in R$, $abc \in I$, then $ab \in I$ or $c \in I$ (see [43]).

(e) An R-module M is defined to be a comultiplication module if for each submodule N of M, $N = (0 :_M I)$, for some ideal I of R. In this case, we can take $N = (0 :_M ann(N))$.

(f) An R-module M is defined to be a weak comultiplication module if $Spec(M) = \emptyset$ or for every prime submodule N of M, $N = (0 :_M I)$, for some ideal I of R (see [14]).

(g) An R-module M is defined to be a primary comultiplication module if $pSpec(M) = \emptyset$ or for every primary submodule N of M, $N = (0 :_M I)$, for some ideal I of R (see [15]).

(h) An R-module M is defined to be an absorbing comultiplication module if $2 - abSpec(M) = \emptyset$ or for every 2-absorbing submodule N of M, $N = (0 :_M I)$, for some ideal I of R (see [10]).

(i) A submodule N of an R-module M is called pure submodule, if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$ (see [35, 41]).

(j) A module M is pure-injective if it has the injective property relative to all pure exact sequences (see [35, 41]).

(k) Let R be a commutative ring and M be an R-module. A proper submodule N of M is said to be 1-absorbing prime, if for all non-unit elements $a, b \in R$ and $m \in M$, $abm \in N$, then $m \in N$ or $ab \in (N : M)$. The set of all 1-absorbing prime submodules in an R-module M is denoted by $abpSpec(M)$.

(l) An R-module M is called to be 1-absorbing prime, if its zero submodule is a 1-absorbing prime submodule of M.

Remark 1.2. (i) An R-module is pure-injective if and only if it is algebraically compact (see [42]).

(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M (see [41]).

(iii) Let N be an R-submodule of M. It is clear that N is an RD-submodule of M if and only if for all $m \in M$ and $r \in R$, $rm \in N$

implies that $rm = rn$ for some $n \in N$. Furthermore, if M is torsion-free, then N is an RD-sumodule if and only if for all $m \in M$ and for all non-zero $r \in R$, $rm \in N$ implies that $m \in N$. In this case, N is an RD-submodule if and only if N is a prime submodule.

2 1-absorbing comultiplication modules over a local Dedekind domain

The aim of this section is to classify 1-absorbing comultiplication modules over a local Dedekind domain. First, we collect basic properties concerning 1-absorbing comultiplication modules.

Note that every prime submodule is a 1-absorbing prime submodule and every 1-absorbing prime submodule is a 2-absorbing prime submodule. But the converse is not true in general. See the following examples.

Example 2.1. (**1-absorbing prime submodule that is not prime**) Consider $\mathbb{Z}_4$-module $\mathbb{Z}_4[X]$ and the submodule $N = \langle X \rangle$. Thus N is a 1-absorbing prime submodule, but N is not a prime submodule of $\mathbb{Z}_4[X]$. Because $\bar{4} = \bar{2}.\bar{2} \in N$, but $\bar{2} \notin N$ and $\bar{2} \notin (N :_{\mathbb{Z}_4} \mathbb{Z}_4[X])$.

Example 2.2. (**2-absorbing submodule that is not 1-absorbing prime**) Consider $\mathbb{Z}$-module $\mathbb{Z}_{30}$. Suppose that $N = \langle 6 \rangle$ is the cyclic submodule of $\mathbb{Z}$-module $\mathbb{Z}_{30}$. It is clear that $N = \langle 6 \rangle$ is a 2-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}_{30}$, but it is not a 1-absorbing prime submodule of $\mathbb{Z}_{30}$. Indeed, $2 \cdot 2 \cdot 3 \in \langle 6 \rangle$ but $4 \notin (N : \mathbb{Z}_{30})$ and $3 \notin \langle 6 \rangle$.

Proposition 2.3. *Let M be an R-module. Then*

(i) *If N is a 1-absorbing prime submodule of M, then $(N : M)$ is a 1-absorbing prime ideal of R.*

(ii) *If N is a 1-absorbing prime submodule of M and I, J are ideals of R and K is a submodule of M such that $IJK \subseteq N$, then $K \subseteq N$ or $IJ \subseteq (N : M)$.*

(iii) *Let $K \subset N$ be submodules of M. Then N is a 1-absorbing prime submodule of M if and only if N/K is a 1-absorbing prime submodule of M/K.*

(iv) *If N is a 1-absorbing prime submodule of M, then M/N is a 1-absorbing prime R-module.*

Proof. The proof is straightforward. □

Lemma 2.4. *Let M be an R-module, N a 1-absorbing prime submodule of M and I an ideal of R with $I \subset (0 : M)$. Then N is a 1-absorbing prime submodule of M as an R/I-module.*

Proof. By Proposition 2.3, M/N is a 1-absorbing prime R-module. Let $(a+I)(b+I)m \in N$ for some $m \in M$ and $a+I, b+I \in R/I$, so $ab(m+N) = 0$, hence $m \in N$ or $ab \in (0 : M/N) = (N : M)$, as needed. □

Definition 2.5. Let R be a commutative ring. An R-module M is said to be a 1-absorbing comultiplication module, if $abpSpec(M) = \emptyset$ or for every 1-absorbing prime submodule N of M, $N = (0 :_M I)$, for some ideal I of R.

Note that the class of 2-absorbing comultiplication modules contains the class of 1-absorbing comultiplication modules, and the class of 1-absorbing comultiplication modules contains the class of weak comultiplication modules.

Lemma 2.6. *Let M be a 1-absorbing comultiplication module over a commutative ring R. Then the following hold:*

(i) *If N is a pure submodule of M, then M/N is a 1-absorbing comultiplication R-module.*

(ii) *Every direct summand of M is a 1-absorbing comultiplication module.*

Proof. (i) Let K/N be a 1-absorbing prime submodule of M/N. Then by Proposition 2.3, K is a 1-absorbing prime submodule of M, then $K = (0 :_M I)$ for some ideal I of R. An inspection will show that $K/N = (0 :_{M/N} I)$.

(ii) Since every direct summand of M is a pure submodules of M, then the proof follows from (i). □

Lemma 2.7. *Let R and R' be any commutative rings, $f : R \to R'$ a surjective homomorphism and M an R'-module. Then the following hold:*

(i) *If M is a 1-absorbing prime as an R-module, then M is 1-absorbing prime as an R'-module.*

(ii) *If N is a 1-absorbing prime R-submodule of M, then N is a 1-absorbing prime R'-submodule of M.*

(iii) *If M is a 1-absorbing comultiplication R'-module, then M is a 1-absorbing comultiplication R-module.*

Proof. (i) It is obvious.

(ii) Clearly, M/N is a 1-absorbing prime R-module, so M/N is a 1-absorbing prime R'-module by (i), hence N is a 1-absorbing prime R'-submodule of M.

(iii) By part (ii), if N is a 1-absorbing prime R-submodule of M, then it is a 1-absorbing prime R'-submodule of M. Assume that M is a 1-absorbing comultiplication R'-module and let N be a 1-absorbing R-submodule of M. Then $N = (0 :_M J)$, where $J = (0 :_{R'} N)$; so $I = f^{-1}(J)$ is an ideal of R with $f(I) = J$. It is enough to show that $(0 :_M J) = (0 :_M I)$. Let $m \in (0 :_M J)$. If $r \in I$, then $f(r) \in J$, so $f(r)m = 0$. Thus $rm = 0$ for every $r \in I$; hence $m \in (0 :_M I)$. For the reverse inclusion, assume that $x \in (0 :_M I)$ and $s \in J$. Then $s = f(a)$ for some $a \in I$. It follows that $sx = g(a)x = ax = 0$ for every $s \in J$; hence $x \in (0 :_M J)$, and we have equality. □

Remark 2.8. Assume that R is a local Dedekind domain with maximal ideal $P = Rp$ and let $M = R$ (as an R-module). Since P is a 1-absorbing prime submodule of M with $(0 :_M ann(P)) = R$. Therefore, M is not a 1-absorbing comultiplication R-module.

Proposition 2.9. *Let R be a local Dedekind domain with unique maximal ideal $P = Rp$. Then $E = E(R/P)$, the injective hull of R/P, and $Q(R)$, the field of fractions of R, are 1-absorbing comultiplication R-modules.*

Proof. By [7, Lemma 2.6], every non-zero proper submodule L of E is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. However, no A_n is a 1-absorbing prime submodule of E, for if n is a positive integer, then $P^2 A_{n+2} = A_n$, but $A_{n+2} \nsubseteq A_n$ and $P^2 \nsubseteq (A_n : E) = 0$. Now we conclude that $abpSpec(E) = \emptyset$. Thus E is a 1-absorbing comultiplication module.

Clearly, 0 is a 1-absorbing prime (prime) submodule of $Q(R)$. To show that 0 is the only 1-absorbing prime submodule of $Q(R)$, we assume the contrary and let N be a non-zero 1-absorbing prime submodule of $Q(R)$.

Since N is a non-zero submodule, there exists $0/1 \neq a/b$, where $a, b \in R$, so that $a/b \in N$. Clearly, $1/ab \notin N$ (otherwise, $ab/ab = 1/1 \in N$, which is a contradiction). Now we have $a^2(1/ab) \in N$, but $1/ab =\notin N$ and $a^2 \notin (N :_R Q(R)) = 0$. Thus $abpSpec(Q(R)) = \{0\}$ and hence $Q(R)$ is a 1-absorbing comultiplication module. □

The following theorem represents a generalization of [17, Theorem 1.4] in the comultiplication module case.

Theorem 2.10. *Let R be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the following is a complete list, up to isomorphism, of the indecomposable 1-absorbing comultiplication modules:*

(i) *R/P^n $(n \geq 1)$ the indecomposable torsion module;*

(ii) *$E(R/P)$, the injective hull of R/P;*

(iii) *$Q(R)$, the field of fractions of R.*

Proof. First, we note that each of the preceding modules is indecomposable (by [6, Proposition 1.3]). $Q(R)$ and $E(R/P)$ are 1-absorbing comultiplication modules by Proposition 2.9. Moreover, R/P^n is a 1-absorbing comultiplication module since it is a comultiplication module (see [11]).
Now let M be an indecomposable 1-absorbing comultiplication module, and choose any non-zero element $a \in M$. consider $ann(a) = \{r \in R : \ ra = 0\}$ and the height $h(a) = sup\{n \mid a \in P^n M\}$ (so $h(a)$ is a non-negative integer or ∞). If $(0 : a) = P^{m+1}$, then $ann(ap^m) = P$. So, replacing a if necessary, it may be supposed that $ann(a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $ann(a)$.

Case 1. $abpSpec(M) = \emptyset$. Since $Spec(M) \subseteq abpSpec(M)$, it follows from [30, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [7, Proposition 2.7, Case 2(a)], $M \cong E(R/P)$. So we may assume that $abpSpac(M) \neq \emptyset$.

Case 2. $h(a) = n$, $ann(a) = P$. Since $h(a) = n$, there is an element $b \in M$ such that $a = p^n b$. So $p^n b \neq 0$ and the maximal power of p dividing $p^n b$ is just p^n. Moreover, $(0 : b) = p^{n+1}R$ gives $Rb \cong R/P^{n+1}$. Now we show that Rb is a pure submodule of M. Since R is a Dedekind domain, it suffices to show that for all integers h, k if $p^h | p^k b$ in M, then $p^h | p^k b$ in

Rb. First, we show that if $p^h | p^k b$ in M, then $h \leq k$. Clearly, $k \leq n$ and $p^{n-k+h} | p^{n-k+k} b = p^n b$ gives $n - k + h \leq n$ by maximality of n, that is, $h \leq k$. Then we can write $p^k b = p^h p^{k-h} b$. Thus, Rb is pure in M. By assumption, $Rb = P^s M$ for some s. Then there is an element $c \in M$ such that $b = p^s c$, so $a = p^{n+s} c$; hence $s = 0$ and $R/P^{n+1} \cong Rb = P^0 M = M$.

Case 3. $h(a) = \infty$, $ann(a) = P$. By argument in [7, Proposition 2.7], we get $M \cong E(R/P)$, hence $abpSpac(M) = \emptyset$ by Proposition 2.9, which is a contradiction.

Case 4. $h(a) = \infty$, $ann(a) = 0$. By [7, Proposition 2.7], we obtain $M \cong Q(R)$. □

Corollary 2.11. *Let M be a 1-absorbing comultiplication module over a local Dedekind domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$ and K is a direct sum of copies of $E(R/P)$ and $Q(R)$.*

3 The separated 1-absorbing comultiplication modules

Throughout this section, we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \qquad (1)$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated by p_1, p_2, respectively, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$). Let $a = (r, s) \in R$ with $r \neq 0$ and $s \neq 0$. Then we can write $a = (p_1^m, p_2^n)$ for some positive integers m, n, so $ann(a) = 0$; hence $Ra \cong R$. If $a = (0, p_2^m)$ for some positive integer m, then $ann(a) = P_1 \oplus 0$, and so $R(0 : p_2^m) \cong R/(P_1 \oplus 0) \cong R_2$. Similarly, $R(p_1^n, 0) \cong R/(P_1 \oplus 0) \cong R_2$. The other ideals I of R are of the form $I = P_1^m \oplus P_2^n = (\langle p_1^m, p_2^n \rangle)$ for some positive integers m, n since $I \subseteq P = P_1 \oplus P_2 = (< p_1 >, < p_2 >)$ and $p_1 p_2 = 0$ (see [6, page 4054]).

Remark 3.1. ([15, Remark 3.1]) Let R be the pullback ring as in (1), and let T be an R-submodule of a separated module

$$S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$$

, with projection maps $\pi_i : S \to S_i$. Set $T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T$ for some $t_2 \in S_2\}$ and $T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T$ for some $t_1 \in S_1\}$.

Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\acute{\pi_1} = \pi_1|T : T \to T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus Ker(f_2)) \cap T \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$).

We need the following proposition proved in [19, Proposition 3.2].

Proposition 3.2. *Let $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ be a proper submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ over the pullback ring as in* (1)*. Then the following hold:*

(i) *$Rad(T : S) = I \oplus J$ if and only if $Rad(T_1 : S_1) = I$ and $Rad(T_2 : S_2) = J$, where $I \neq 0$ and $J \neq 0$.*

(ii) *$Rad(T : S) = P_1 \oplus 0$ if and only if $Rad(T_1 : S_1) = P_1$ and $Rad(T_2 : S_2) = 0$.*

(iii) *$Rad(T : S) = 0 \oplus P_2$ if and only if $Rad(T_1 : S_1) = 0$ and $Rad(T_2 : S_2) = P_2$.*

Lemma 3.3. *Let I be a 1-absorbing prime ideal of R. Then $Rad(I) = P$ is a prime ideal of R such that $P^2 \subseteq I$.*

Proof. It follows from [43, Theorem 2.3 and Lemma 2.8]. □

Proposition 3.4. *Let R be the pullback ring as in* (1) *and let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be any separated R-module. Then the following hold:*

(i) *If S has a 1-absorbing prime submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ with $Rad(T : S) = P = P_1 \oplus P_2$ and $P^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = P_2$.*

(ii) *If S has a 1-absorbing prime submodule T with $Rad(T : S) = P_1 \oplus 0$ and $(P_1 \oplus 0)^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = 0$.*

(iii) *If S has a 1-absorbing prime submodule T with $Rad(T : S) = 0 \oplus P_2$ and $(0 \oplus P_2)^2 \subseteq (T : S)$, then T_1 is a 1-absorbing prime submodule of S_1 with $Rad(T_1 : S_1) = 0$ and T_2 is a 1-absorbing prime submodule of S_2 with $Rad(T_2 : S_2) = P_2$.*

Proof. (i) Let $a_1b_1s_1 \in T_1$ where $a_1, b_1 \in R_1$ and $s_1 \in S_1$. Then $v_1(a_1) = v_2(a_2)$, $v_1(b_1) = v_2(b_2)$ and $f_1(s_1) = f_2(s_2)$ for some $a_2, b_2 \in R_2$ and $s_2 \in S_2$. Hence $(a_1, a_2)(b_1, b_2)(s_1, s_2) \in P^2S \subseteq T$. Therefore, since T is a 1-absorbing prime submodule, we have $(s_1, s_2) \in T$ or $(a_1, a_2)(b_1, b_2) \in (T :_R S)$. So $s_1 \in T_1$ or $a_1b_1 \in (T_1 : S_1)$. Similarly, T_2 is a 1-absorbing prime submodule of S_2.

(ii) Let $a_1b_1s_1 \in T_1$ where $a_1, b_1 \in R_1$ and $s_1 \in S_1$. So $(a_1, 0), (b_1, 0) \in R$ and there exists $s_2 \in S_2$ such that $f_1(s_1) = f_2(s_2)$. Then

$$(a_1, 0)(b_1, 0)(s_1, s_2) \in (P_1 \oplus 0)^2S \subseteq T$$

by hypothesise. Hence $(a_1, 0)(b_1, 0)(s_1, s_2) \in P^2S \subseteq T$. Therefore $(s_1, s_2) \in T$ or $(a_1, 0)(b_1, 0) \in (T :_R S)$. So $s_1 \in T_1$ or $a_1b_1 \in (T_1 :_{R_1} S_1)$. Now we show that T_2 is a 1-absorbing prime submodule of S_2. Suppose that $a_2b_2s_2 \in T_2$ and $a_2b_2 \notin (T_2 : S_2) = 0$ where $a_2, b_2 \in R_2$. There exists $s_1 \in S_1$ such that $(s_1, s_2) \in S$. Therefore, $a_2b_2 \neq 0$, and so $a_2 \neq 0$ and $b_2 \neq 0$ since R_2 is a domain. Hence $(p_1, a_2)(p_1, b_2)(s_1, s_2) \in T$, because $p_1^2s_1 \in T_1 \cap P_1S_1$, $a_2b_2s_2 \in T_2 \cap P_2S_2$ and $f_1(p_1^2s_1) = 0 = f_2(a_2b_2s_2)$. Since T is a 1-absorbing prime submodule of S, then $(s_1, s_2) \in T$. Thus $s_2 \in T_2$, as required.

(iii) It is similar to that (ii). □

Proposition 3.5. *Let*

$$S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$$

be any separated module over the pullback ring as (1)*. $abpSpec(S) = \emptyset$ if only if $abpSpec(S_i) = \emptyset$ for $i = 1, 2$.*

Proof. For the necessarily, assume that $abpSpec(S) = \emptyset$ and let π be the projection map of R onto R_i. Suppose that $abpSpec(S_1) \neq \emptyset$ and let T_1 be a 1-absorbing prime submodule of S_1, so T_1 is a 1-absorbing prime R-submodule of $S/(0 \oplus P_2)S \cong S_1$, by Proposition 2.3; hence $abpSpec(S) \neq \emptyset$, which is a contradiction. Similarly, $abpSpec(S_2) = \emptyset$. The sufficiency by Proposition 3.4. □

Proposition 3.6. *Let R be the pullback ring as in* (1) *and let*

$$S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$$

be any separated R-module. then the following hold:

(i) *If L_1 is a non-zero 1-absorbing prime submodule of S_1, then there exists a separated submodule T of S such that $T + (0 \oplus P_2)S$ is a 1-absorbing prime submodule of S.*

(ii) *If L_2 is a non-zero 1-absorbing prime submodule of S_2, then there exists a separated submodule T' of S such that $T' + (P_1 \oplus 0)S$ is a 1-absorbing prime submodule of S.*

Proof. (i) If L_1 is a non-zero 1-absorbing prime submodule of S_1, then there exists a separated submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ of S, where $T_1 = L_1$. By Remark 3.1, $T_1 \cong (T + (0 \oplus P_2)S)/(0 \oplus P_2)S \subseteq S/(0 \oplus P_2)S$. Thus $(T + (0 \oplus P_2)S)/(0 \oplus P_2)S$ is a 1-absorbing prime R-submodule of $S/(0 \oplus P_2)S$. Hence $T + (0 \oplus P_2)S$ is a 1-absorbing prime R-submodule of S by Proposition 2.3. The proof of (ii) is similar that (i). $\square$

The following theorem represents a generalization of [17, Theorem 2.8] in the comultiplication module case.

Theorem 3.7. *Let*

$$S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$$

be any separated module over the pullback ring as (1). *Then S is a 1-absorbing comultiplication R-module if and only if S_i is a 1-absorbing comultiplication R_i-module, $i = 1, 2$.*

Proof. Note that by Proposition 3.5, $abpSpec(S) = \emptyset$ if only if for $i = 1, 2$, $abpSpec(S_i) = \emptyset$. So we may assume that $abpSpec(S) \neq \emptyset$. Let S be a separated 1-absorbing comultiplication R-module and let L be a non-zero 1-absorbing submodule of S_1. By Proposition 3.6, there exists a submodule $T = (T_1 \to \bar{T} \leftarrow T_2)$ of S such that $L = T_1$ and $T' = T + (0 \oplus P_2)S$ is a 1-absorbing submodule of S. Clearly, $ann(T') = ann(T) \cap ann((0 \oplus P_2)S) = 0$ or $P_1^n \oplus 0$ for some positive integer n. Since $S = (0 :_S 0)$, S is a 1-absorbing comultiplication module gives $T' = (0 :_S P_1^n \oplus 0)$. It suffices to show that $L = T_1 = (0 :_{S_1} p_1^n)$. Let $t \in T_1$. There exists $t_2 \in T_2$ such that $(t_1, t_2) \in T \subseteq T'$; so $(P_1^n \oplus 0)(t_1, t_2) = 0$. It then follows that $T_1 \subseteq (0 :_{S_1} p_1^n)$. For the reverse inclusion let $s_1 \in (0 :_{S_1} p_1^n)$. Then there is an element $s_2 \in S_2$ such that $(s_1, s_2) \in S$ and $(P_1^n \oplus 0)(s_1, s_2) = 0$. Hence $(s_1, s_2) \in T'$. Thus $s_1 \in T_1$ and we have equality. Therefore S_1 is a 1-absorbing comultiplication module. Similarly, S_2 is a 1-absorbing comultiplication module. Conversely, assume that S_1, S_2 are 1-absorbing comultiplication modules and let T be a 1-absorbing submodule of S. Hence $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ where T_i is an R_i-submodule of S_i for each $i = 1, 2$. By Proposition 3.4, T_1, T_2 are 1-absorbing submodules of S_1, S_2, respectively. By assumption, $T_1 = (0 :_{S_1} P_1^n)$ and $T_2 = (0 :_{S_2} P_2^m)$ for some integers n, m. An inspection will show that $T = (0 :_S P_1^n \oplus P_2^m)$. Thus S is a 1-absorbing comultiplication R-module. □

Lemma 3.8. *Let R be the pullback ring as in* (1)*. The following separated R-modules are indecomposable and 1-absorbing comultiplication:*

(1) $S = (E(R_1/P_1) \longrightarrow 0 \longleftarrow 0)$, $(0 \longrightarrow 0 \longleftarrow E(R_2/P_2)$, *where* $E(R_i/P_i)$ *is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*

(2) $S = (Q(R_1) \longrightarrow 0 \longleftarrow 0)$, $(0 \longrightarrow 0 \longleftarrow Q(R_2)$, *where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;*

(3) $R = (R_1/P_1^n \longrightarrow \bar{R} \longleftarrow R_2/P_2^m)$, *for all positive integers m, n.*

Proof. By [6, Lemma 2.8], these modules are indecomposable and 1-absorbing comultiplicativity follows from Theorem 2.10 and Theorem 3.7. □

Theorem 3.9. *Let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be an indecomposable separated 1-absorbing comultiplication module over the pullback ring as* (1)*. Then S is isomorphic to one of the modules listed in Lemma* 3.8.

Proof. If $abpSpec(S) = \emptyset$, then $abpSpec(S_i) = \emptyset$ by Proposition 3.5, so $S_i = P_i S_i$ for each $i = 1, 2$ (see Theorem 2.10, Case 1); hence $S = PS = P_1 S_1 \oplus P_2 S_2 = S_1 \oplus S_2$. Therefore $S = S_1$ or S_2 and so S is of type (1) in the list of Lemma 3.8. So we may assume that $abpSpec(S) \neq \emptyset$. If $S = PS$, then by [6, Lemma 2.7(i)], $S = S_1$ or S_2 and so S is an indecomposable 1-absorbing comultiplication R_i-module for some i, and since $PS = S$, it is of type (2) by Theorem 2.10. So we may assume that $S \neq PS$. By Theorem 3.7, S_i is a 1-absorbing comultiplication R_i-module, for each $i =$

$1, 2$. Hence by the structure of 1-absorbing comultiplication modules over a local Dedekind domain (see Theorem 2.11), $S_i = M_i \oplus N_i$ where N_i is a direct sum of copies of R_i/P_i^i $(n \geq 1)$ and M_i is a direct sum of copies $E(R_i/P_i)$ and $Q(R_i)$. Then we have

$$S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2) \oplus (M_1 \longrightarrow 0 \longleftarrow 0) \oplus (0 \longrightarrow 0 \longleftarrow M_2).$$

Since S is indecomposable and $S/PS \neq 0$ it follows that $S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2)$. We will see that each $S_i(= N_i)$ is indecomposable. There exist positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^k S_2 = 0$ and $P^n S = 0$. Now choose $t \in S_1 \cup S_2$ with $\bar{t} \neq 0$ and let $o(t)$ denote the least positive integer k such that $P^k t = 0$ if there is such k and if no such k $o(t) = \infty$ and $o(t)$ minimal among such t. Let $t \in S_2$ and so write $t = t_2$ and $m = k = o(t_2)$. Now pick $t_1 \in S_1$ with $\bar{t_1} = \bar{t_2} = \bar{t}$ and $o(t) = n$ minimal (so $o(t_2) \neq \infty$ and $o(t_1) \neq \infty$). There exists a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in S_i for $i = 1, 2$ (see [6, Theorem 2.9]). Therefore, $R_1 t_1 \cong R_1/P_1^m$ (resp. $R_2 t_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i t_i$ is a pure-injective module [6]. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let

$$M = (R_1 t_1 = M_1 \longrightarrow \bar{M} \longleftarrow M_2 = R_2 t_2).$$

Then M is an R-submodule of S which is 1-absorbing comultiplication by Lemma 3.8, and is a direct summand of S (see [6, Theorem 2.9]); this implies that $S = M$, and S is as in (3). $\square$

Corollary 3.10. *Let R be the pullback ring as in* (1). *Then*

(1) *Every separated 1-absorbing comultiplication R-module S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in* (3)*, and N is a direct sum of copies of the modules as in (1)-(2) of Lemma*3.8.

(2) *Every separated 1-absorbing comultiplication R-module is pure-injective.*

Proof. It follows from Theorem 3.9 and [6, Theorem 2.9]. $\square$

4 The non-separated 1-absorbing comultiplication modules

We continue to use the notion already established, so R is the pullback ring as in (1). In this section, we find the indecomposable non-separated 1-absorbing comultiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable 1-absorbing comultiplication modules.

Proposition 4.1. *Let R be the pullback ring as in* (1). *Then $E(R/P)$ is a non-separated* 1*-absorbing comultiplication R-module.*

Proof. It suffices to show that $abpSpac(E(R/P)) = \emptyset$. Assume that L is any submodule of $E(R/P)$ described in [17, Proposition 3.1]. However, no L, say $E_1 + A_n$ is a 1-absorbing prime submodule of $E(R/P)$, for if n is any positive integer, then $P^2(E_1 + A_{n+2}) = E_1 + A_n$, but $(E_1 + A_{n+2}) \nsubseteq E_1 + A_n$ and $P^2 \nsubseteq (E_1 + A_n : E(R/P)) = 0$. Therefore $E(R/P)$ is a non-separated 1-absorbing comultiplication R-module (see [6, page 4053]). □

Proposition 4.2. *Let R be the pullback ring as in* (1), *and let M be any R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then $abpSpec(S) = \emptyset$ if and only if $abpSpec(M) = \emptyset$.*

Proof. First suppose that $abpSpec(S) = \emptyset$ and let $abpSpec(M) \neq \emptyset$. So $M \cong S/K$ has a 1-absorbing prime submodule, say T/K, where T is a 1-absorbing prime submodule of S by Proposition 2.3, which is a contradiction. Next suppose that $abpSpec(M) = \emptyset$ and let $abpSpec(S) \neq \emptyset$. Let T be a non-zero 1-absorbing prime submodule of S. Then by [11, Proposition 4.3(ii)], $K \subseteq T$; hence T/K is a 1-absorbing prime submodule of M, which is a contradiction. □

Lemma 4.3. *Let R be the pullback ring as in* (1) *and let M be any R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of an R-module M.*

(i) *If $(0 :_R S) = P_1^m \oplus 0$ for some positive integer m, then M is separated.*

(ii) *If $(0 :_R S) = 0 \oplus P_1^m$ for some positive integer m, then M is separated.*

(iii) *If $(0 :_R S) = 0$, then M is separated.*

Proof. It follows from [10, Lemma 4.2]. □

Proposition 4.4. *Let R be the pullback ring as in* (1)*, and let M be a 1-absorbing comultiplication non-separated R-module. Let*

$$0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} M \longrightarrow 0$$

be a separated representation of M. If N is a non-zero R-submodule of M, then M/N is a 1-absorbing comultiplication R-module.

Proof. Let L/N be a 1-absorbing prime submodule of M/N. Then L is a 1-absorbing prime submodule of M by Proposition 2.3, so $L = (0 :_M ann(L))$. Since $ann(M) \subseteq ann(L) \neq 0$ and M is a non-separated R-module, Lemma 4.3 gives $ann(L) = P_1^n \oplus P_2^m$ for some positive integers m, n (note that if $ann(M) = 0$, then $ann(S) \subseteq (K :_R S) = ann(M) = 0$). We show that $L/N = (0 :_{M/N} (P_1^n \oplus P_2^m))$. Let $x + N \in L/N$. Then $(P_1^n \oplus P_2^m)x = 0$ gives $(P_1^n \oplus P_2^m)(x + N) = 0$; so $x + N \in (0 :_{M/N} (P_1^n \oplus P_2^m))$. For the reverse inclusion, assume that $y + N \in (0 :_{M/N} (P_1^n \oplus P_2^m))$. Then $(P_1^n \oplus P_2^m)y \subseteq N \subseteq L$. We claim that $(P_1^n \oplus P_2^m)y = 0$. Assume to the contrary, $0 \neq (P_1^n \oplus P_2^m)y \subseteq L$. Then $(P_1^{2n} \oplus P_2^{2m})y = 0$. Let t be the least positive integer such that $P^t y = 0$ (so $P^{t-1}y \neq 0$). So there exists $x \in S$ such that $y = \varphi(x)$ and $\varphi(P^t x) = 0$; so $\varphi(P_1^t x) = \varphi(P_2^t x) = 0$. By [28, Proposition 2.3], φ is one-to-one on P_iS for each i, we find that $P_2^t x = P_1^t x = 0$; hence $P^t x = 0$. Set $U = P^{t-1}y$. Then $0 \to K \to \varphi^{-1}(U) = P^{t-1}x \to U \to 0$ is a separated representation of U by [8, Lemma 3.1], such that $K \subseteq P(P^{t-1}x) = 0$ which is a contradiction. Thus $(P_1^n \oplus P_2^m)y = 0$, and so we have equality. □

The following theorem gets a generalization of [17, Proposition 3.7] in the comultiplication module case.

Theorem 4.5. *Let R be the pullback ring as in* (1)*, and let M be any non-separated R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then S is a* 1*-absorbing comultiplication module if and only if M is a* 1*-absorbing comultiplication module.*

Proof. By Proposition 4.2, we may assume that $abpSpec(S) \neq \emptyset$. Suppose that M is a 1-absorbing comultiplication R-module and let T be a non-zero 1-absorbing prime submodule of S. By [11, Proposition 4.3], $K \subseteq T$ and so T/K is a 1-absorbing prime submodule of S/K. Since $M \cong S/K$ is a 1-absorbing comultiplication module, we must have $T/K = (0 :_{S/K} P_1^n \oplus P_2^m)$

for some integers m, n. By an argument like that in [11, Theorem 4.4], we find that $T = (0 :_S P_1^n \oplus P_2^m)$ and so S is a 1-absorbing comultiplication module. Conversely, if S is a 1-absorbing comultiplication module, then $M \cong S/K$ is a 1-absorbing comultiplication module by Proposition 4.4, and this complete the proof. □

Proposition 4.6. *Let R be the pullback ring as in* (1)*, and let M be an indecomposable* 1*-absorbing comultiplication non-separated R-module with M/PM finite dimensional top over $\bar{R}$. If*

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

is a separated representation of M, then S has finite-dimensional top and is pure-injective.

Proof. By [6, Proposition 2.6(i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Theorem 4.5 and Corollary 3.10. □

Let R be the pullback ring as in (1), and let M be an indecomposable 1-absorbing comultiplication non-separated R-module with M/PM finite dimensional top over $\bar{R}$. Consider the separated representation

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0.$$

By Proposition 4.6, S is a pure-injective module. So in the proofs of [6, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [6, Proposition 2.6]), we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable 1-absorbing comultiplication non-separated R-module", because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following result:

Corollary 4.7. *Let R be the pullback ring as in* (1)*, and let M be an indecomposable 1-absorbing comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let*

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M. Then the following hold:

(i) *The quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summand of S;*

(ii) *S is a direct sum of finitely many indecomposable 1-absorbing comultiplication modules;*

(iii) *At most two copies of modules of finite length can occur among the indecomposable summands of S.*

Let R be a pullback ring as in (1). Let M be any R-module and let $0 \to K \to S \to M \to 0$ be a separated representation of M. We have already shown that if M is indecomposable 1-absorbing comultiplication with M finite-dimensional top then S is a direct sum of just finitely many indecomposable separated 1-absorbing comultiplication modules and these are known by Theorem 3.9. In any separated representation $0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} M \longrightarrow 0$ the kernel of the map φ to M is annihilated by P, hence is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socles of the various direct summands of S. This explains Corollary 4.7(i): the modules $Q(R_1)$ and $Q(R_2)$ have zero socle and so cannot be amalgamated with any other direct summands of S and hence cannot occur in a separated (hence "minimal") representation. So the questions are: does this provide any further condition on the possible direct summands of S? How can these summands be amalgamated in order to form M? For the case of finitely generated R-modules M these questions are answered by Levy's description [29], see also [28, Section 11]. Levy shows that the indecomposable finitely generated R-modules are of two nonoverlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type which are most relevant to us. Such a module is obtained from a direct sum, S, of indecomposable separated modules by amalgamating the direct summands of S in pairs to form a chain but leaving the two ends unamalgamated. Reflecting the fact that the dimension over $\bar{R}$ of the socle of any finitely generated indecomposable separated module is ≤ 2 each indecomposable summand of S may be amalgamated with at most two other indecomposable summands. Consider the indecomposable separated R-modules $S(n,m) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ with $n, m \geq 2$ (it is generated over R by $(1 + P_1^n, 1 + P_2^m)$). Actually, separated indecomposable R-modules also include R_1/P_1^n for $n \geq 2$, which can be regarded up to isomorphism as $S(n,1) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2)$. Similarly, for

$m \geq 2$, $S(1,m) = (R_1/P_1 \to \bar{R} \leftarrow R_2/P_2^m)$ is a separated indecomposable R-module. Moreover, R_1, R_2 and R themselves can be viewed as separated indecomposable R-modules, corresponding to the cases $n = \infty$ and $m = 1$, $n = 1$ and $m = \infty$, $n = m = \infty$. Deleted cycle indecomposable R-modules are introduced as follows: Let S be a direct sum of finitely many modules $S(i) = S(n_{i,1}, n_{i,2})$ (with $i < s$ a non-negative integer). Here $n_{i,j} \geq 2$ for every $j < s$ and $j = 1, 2$, with two possible exceptions $i = 0$, $j = 1$ and $i = s-1$ and $j = 2$, where the values $n_{i,j} = 1$ or ∞ are allowed. Then amalgamate the direct summands in S by identifying the P_2-part of the socle of $S(i)$ and the P_1-part of the socle $S(i+1)$ for every $i < s-1$. For instance, given the separated modules $S_1 = (R_1 \to \bar{R} \leftarrow R_2/P_2^3) = Ra$ with $P_2^3 a = 0$ and $S_1 = (R_1/P_1^7 \to \bar{R} \leftarrow R_2/P_2^2) = Ra$ with $P_1^7 a' = 0 = P_2^2 a'$. Then one can form the non-separated module $(S_1 \oplus S_2)/(R(p_2^2 a - p_1^6 a') = Rc + Rc'$ where $c = a + R(p_2^2 a - p_1^6 a')$, $c' = a' + R(p_2^2 a - p_1^6 a')$, $P_2^3 c = 0 = P_1^7 c' = P_2^2 c$ and $P_2^2 c = P_1^6 c'$ which is obtained by identifying the P_2-part of the socle of S_1 with the P_1-part of the socle of S_2. We will use that same description, but with 1- absorbing comultiplication separated modules in place of the finitely generated ones, gives us the non-zero indecomposable 1-absorbing comultiplication non-separated R-modules. As a consequence, any non-zero indecomposable 1-absorbing comultiplication separated module with 1-dimensional socle may occur only at one of the ends of the amalgamation chain (see [6, Proposition 3.4]). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable 1-absorbing comultiplication modules. We do that now and thus complete the classification of the indecomposable 1-absorbing comultiplication non-separated modules with finite-dimensional top.

Theorem 4.8. *Let $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ be the pullback ring of two local Dedekind domains R_1, R_2 with common factor field $\bar{R}$. Then the class of indecomposable non-separated 1-absorbing comultiplication modules with finite-dimensional top up to isomorphism, are the following:*

(i) *$M = E(R/P)$, the injective hull of R/P;*

(ii) *The indecomposable modules of finite length (apart from R/P which is separated), that is, $M = \sum_{i=1}^{s} Ra_i$ with $p_1^{n_s} a_s = 0 = p_2^{m_1} a_1$, $p_1^{n_{i-1}} a_i = p_2^{m_{i+1}-1} a_{i+1}$ $(1 \leq i \leq s-1)$, $m_i, n_i \geq 2$ except for $m_1 \geq 1$, $n_s \geq 1$.*

(iii) *$M = E_1 + \sum_{i=1}^{s} Ra_i + E_2$ with $a_0 = p_2^{m_1-1} a_1$, $b_0 = p_1^{n_s-1} a_s$, $p_1 a_0 =$*

$0 = p_2 b_0$ *and* $p_1^{n_i-1} a_i = p_2^{m_{i+1}-1} a_{i+1}$ *for all* $1 \leq i \leq s-1$, *where* $E_1 \cong E(Rb_0) \cong E(R_1/P_1)$, $E_2 \cong E(Rb_0) \cong E(R_2/P_2)$ *and* $m_i, n_i \geq 2$ *except for* $m_1 \geq 1$ *and* $n_s \geq 1$.

(iv) $M = E_1 + \sum_{i=1}^{s} Ra_i$ *with* $p_1^{n_s-1} a_s = 0$, $a_0 = p_2^{m_1-1} a_1$, $p_1 a_0 = 0$ *and* $p_1^{n_i-1} a_i = p_2^{m_{i+1}-1} a_{i+1}$ *for all* $1 \leq i \leq s-1$, *where* $E_1 \cong E(Rb_0) \cong E(R_1/P_1)$, *and* $m_i, n_i \geq 2$ *except for* $n_s \geq 1$.

(v) $M = \sum_{i=1}^{s} Ra_i + E_2$ *with* $p_2^{m_s} a_s = 0$, $b_0 = p_1^{n_1-1} a_1$, $p_2 b_0 = 0$ *and* $p_2^{m_i-1} a_i = p_2^{n_{i+1}-1} a_{i+1}$ *for all* $1 \leq i \leq s-1$, *where* $E_2 \cong E(Rb_0) \cong E(R_2/P_2)$, *and* $m_i, n_i \geq 2$ *except for* $m_s \geq 1$.

Proof. Let M be an indecomposable non-separated 1-absorbing comultiplication R-module with finite-dimensional top and let

$$0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} M \longrightarrow 0$$

be a separated representation of M. By Corollary 4.7(iii), S is a direct sum of finitely many indecomposable 1-absorbing comultiplication separated modules. We know already that every indecomposable 1-absorbing comultiplication non-separated module has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable 1-absorbing comultiplication modules. (i) follows from Proposition 4.1(i). Since a quotient of any 1-absorbing comultiplication R-module is 1-absorbing comultiplication by Proposition 4.4, they are 1-absorbing comultiplication modules. The indecomposability follows from [29, 1.9] and [6, Theorem 3.5]. □

Corollary 4.9. *Let* $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ *be the pullback ring of two local Dedekind domains* R_1, R_2 *with common factor field* $\bar{R}$. *Then*

(i) *Every indecomposable* 1*-absorbing comultiplication* R*-module with finite-dimensional top is pure-injective.*

(ii) *This paper includes the classification of indecomposable* 1*-absorbing comultiplication modules with finite-dimensional top over* k*-algebra* $k[x, y : xy = 0]_{(x,y)}$.

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