Quantum determinants in ribbon category

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Abstract. The aim of this paper is to introduce an abstract notion of determinant which we call quantum determinant, verifying the properties of the classical one. We introduce $\mathcal{R}$–basis and $\mathcal{R}$–solution on rigid objects of a monoidal $Ab$–category, for a compatibility relation $\mathcal{R}$, such that we require the notion of duality introduced by Joyal and Street, the notion given by Yetter and Freyd and the classical one, then we show that $\mathcal{R}$–solutions over a semisimple ribbon $Ab$–category form as well a semisimple ribbon $Ab$–category. This allows us to define a concept of so-called quantum determinant in ribbon category. Moreover, we establish relations between these and the classical determinants. Some properties of the quantum determinants are exhibited.

1 Introduction

The theory of monoidal categories was studied and developed by many authors [1, 7], see also [8, 9]. In particular duality in such categories introduced by Joyal and Street [8], (see also [2, 11]) as well as the concept of braiding -as a weaker version of commutativity- which came along firstly with Joyal and Street [8]. The notion of determinant dates a long time as an essential tool in linear algebra. Since then,
many versions and analogs were introduced and developed in the setting of square matrices of commutative entries as well as for non commutative entries (among widely used ones: q–determinants, Dieudonné determinant, quasideterminants...). It is well known that the notion of trace has been generalized to the context of categories (with tensor product and duality) [3–6, 13]. In particular, every ribbon category [10] or called tortile tensor category (in [15]), admits a canonical notion of (quantum) trace (well behaved: cyclicity and multiplicativity) and dimension [10], in the way that it generalizes the classical one of vector spaces in linear algebra. These traces are used to construct quantum invariants of links and 3–manifolds. Motivated by that, this paper introduces an abstract notion of determinant which we call “quantum determinant”, verifying the properties of the classical one. The name quantum here is justified by the fact that this element uses the quantum trace; in fact it is nothing but the quantum trace of the endomorphism $f^n \Lambda^n_A$ (Proposition 5.1).

We begin with the introduction of a concept of $\mathcal{R}$–basis on an object $V$ of a monoidal $Ab$–category $C$, for a (compatible) congruence relation $\mathcal{R}$, as a family of morphisms

$$d^i_V : V^* \otimes V \longrightarrow I , \quad b^i_V : I \longrightarrow V \otimes V^* \quad \text{and} \quad \pi^i_V : V \longrightarrow V ; \forall i \in J$$

for a finite index set $J$, such as they verify some axioms. We prove that it coincides with the usual basis when we consider the category of finite dimensional vector spaces over a certain field. The existence of such $\mathcal{R}$–basis in the context of semisimple ribbon $Ab$–categories, is ensured. In fact, we show that semisimplicity gives rise to an $\mathcal{R}$–basis on every object of the category. Moreover, we define a notion of $\mathcal{R}$–solution on an object $V$ as a quadruple $(V; d_V; b_V; \pi_V)$ obeing to some axioms.

Finally, we introduce the notion of quantum determinant in a semisimple ribbon $Ab$–category and show that its formula is independent of the choice of the $\mathcal{R}$–solution on the object. We prove that in fact, the quantum determinant of an endomorphism $f \in \text{End}_C/R(V)$ coincides with the classical determinant of an associated square matrix $M_f$ over the ground commutative ring $End_C(I)$ of $C$, denoted by $K_C$ and that under some conditions, there is a bijective correspondence between the $K_C$–algebras $\text{End}_C(V)$ and $M_n(K_C)$ of square matrices over $K_C$. In this case, $f \in \text{End}_C/R(V)$ is an automorphism, if and only if, its quantum determinant is invertible in $K_C$. 
2 Preliminaries

Throughout this paper, \( K \) states for a base field with unit and \( C \) for a strict monoidal category \( (C; \otimes; I) \) with unit object \( I \).

We recall some notions from the theory of monoidal categories. For more details, we refer to [12] and [17].

A monoidal category \( C = (C; \otimes; I; \alpha; l; r) \) consists of a category \( C \), a bifunctor \( \otimes : C \times C \rightarrow C \), a unit object \( I \) and natural isomorphisms \( \alpha : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C), l : I \otimes A \rightarrow A \) and \( r : A \otimes I \rightarrow A \) called associativity constraint, left and right unitality constraints respectively such that the pentagon and triangle axioms hold.

\( C \) is called strict if all components \( \alpha, l \) and \( r \) are identities.

A result due to Mac-Lane’s (see [12]) coherence Theorem, asserts that any monoidal category is necessarily equivalent to a strict one.

\( C \) is called an \( Ab \)–category provided that the hom sets \( Hom_C(U, V) \) are additive abelian groups and the composition map \( Hom_C(U, V) \times Hom_C(V, W) \rightarrow Hom_C(U, W), (f, g) \mapsto g \circ f \) is bilinear.

A braiding (firstly introduced in [8]) for a monoidal category \( C \) consists of a family of natural isomorphisms

\[
\rho_{V;W} : V \otimes W \rightarrow W \otimes V
\]

for all \( V \) and \( W \) in \( C \), such that for any three objects \( U, V \) and \( W \) we have

\[
\rho_{U;V \otimes W} = (id_V \otimes \rho_{U;W})(\rho_{U;V} \otimes id_W)
\]

\[
\rho_{U \otimes V;W} = (\rho_{U;W} \otimes id_V)(id_U \otimes \rho_{V;W}).
\]

For a monoidal category \( C \) with a braiding \( \rho \); a twist (see [7]) consists of a family of isomorphisms

\[
\theta_V : V \rightarrow V, \ V \in Ob(C)
\]

such that \( \theta_I = id_I \) and for any two objects \( V \) and \( W \) of \( C \) we have

\[
\theta_{X \otimes Y} = \rho_{Y;X}(\theta_Y \otimes \theta_X)\rho_{X;Y}
\]

The naturality of the twist \( \theta \) means that for any morphism \( f : U \rightarrow V \) of \( C \), we have \( \theta_V f = f \theta_U \).
Let \((C; \otimes; I)\) be a strict monoidal category with tensor product \(\otimes\) and unit \(I\). It is a monoidal category with \emph{left duality} if for each object \(V\) of \(C\) there exists an object \(V^*\) and morphisms \(b_V : I \to V \otimes V^*\) and \(d_V : V^* \otimes V \to I\) in \(C\) such that

\[(id_V \otimes d_V)(b_V \otimes id_{V^*}) = id_V \quad \text{and} \quad (d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*}.\]

For any morphism \(f : U \to V\), we define its \emph{dual} morphism \(f^* : V^* \to U^*\) by

\[f^* = (d_V \otimes id_{U^*})(id_{V^*} \otimes f \otimes id_{U^*})(id_{V^*} \otimes b_U)\]

and the morphism \(\lambda_{U;V} : V^* \otimes U^* \to (U \otimes V)^*\) (see [10, page 344], for more details) defined by

\[\lambda_{U;V} = (d_V \otimes id_{(U \otimes V)^*})(id_{V^*} \otimes d_U \otimes id_{V^* (U \otimes V)^*})(id_{V^* \otimes U^*} \otimes b_{U \otimes V}) \quad (2.1)\]

is an isomorphism for any two objects \(U\) and \(V\) of \(C\).

We say that duality is compatible with the braiding \(c\) and the twist \(\theta\) if for any object \(V\) of \(C\) we have

\[\theta_{V^*} = (\theta_V)^*.\]

In this case, the double dual \(V^{**} := (V^*)^*\) of an object \(V\) is canonically isomorphic to \(V\).

A \emph{ribbon category} is a monoidal category \(C\) equipped with a twist \(\theta\), a braiding \(c\) and a compatible duality \((*; b; d)\).

Let \(C\) be a ribbon category with unit \(I\). For any endomorphism \(f\) of an object \(V\) of \(C\), we define the \emph{quantum trace} \(Tr_q(f)\) of \(f\) as the element

\[Tr_q(f) = d_V c_{V;V^*}(\theta_V f \otimes id_{V^*})b_V \in \mathbb{K}C\]

When \(C\) is the category \(\text{vect}_K\) of finite dimensional vector spaces over a field \(K\), this concept of quantum trace coincides with the usual one.

We collect in following Theorem the principal properties of the quantum trace.

**Theorem 2.1.** ([10]) \textit{For any morphisms} \(f\) \textit{and} \(g\) \textit{in a ribbon category, we have}

(a) \(Tr_q(fg) = Tr_q(gf)\);

(b) \(Tr_q(f \otimes g) = Tr_q(f)Tr_q(g)\);

(c) \(Tr_q(f^*) = Tr_q(f)\) and

(d) \(Tr_q(k) = k\) \textit{for any} \(k \in \mathbb{K}C\).
Proof. See [10] for proof, where string diagrams are used to simplify the proof and make the passages more obvious.

For any object $V$ of $C$, the quantum dimension $\dim_q(V)$ of $V$ is the element defined by $\dim_q(V) = \text{Tr}_q(id_V)$ and we have

$$\dim_q(V \otimes W) = \dim_q(V) \dim_q(W) \quad \text{and} \quad \dim_q(V^*) = \dim_q(V).$$

Let $C$ be a ribbon category and $V$ an object of $C$ (for the following setup, we mainly follow the terminology adopted in [17]).

$V$ is called simple provided that the map

$$K : K_C \longrightarrow \text{End}_C(V); \quad k \longmapsto k \otimes id_V$$

is a bijection and $\dim_q(V)$ is invertible in $K_C$.

An object $V$ of a ribbon $Ab$–category $C$ is said to be dominated by $n$ simple objects $\{V_i\}_{i=1}^n$ of $C$, if there exists a finite family of morphisms $\{\epsilon^i_V : V \longrightarrow V_i ; \mu^i_V : V_i \longrightarrow V\}_{1 \leq i \leq n}$ such that the endomorphism

$$\sum_i \mu^i_V \epsilon^i_V - id_V$$

of $V$, is negligible as defined below.

The set of negligible morphisms between two objects $U$ and $V$ is denoted $\text{Negl}(U; V)$ and it is defined as

$$\text{Negl}(U; V) := \{f \in \text{Hom}_C(U, V) \mid \forall g \in \text{Hom}_C(V, U), \ \text{Tr}_q(fg) = 0\}.$$

Obviously $\text{Negl}(I; I) = \{0\}$.

We call a ribbon $Ab$–category semisimple provided that every object is dominated by a finite set of simple ones.

Recall from [12, page 52], that a relation $\mathcal{R}$, is a congruence on a category $C$ if for any objects $X$ and $Y$ of $C$, $\mathcal{R}_{X,Y}$ is an equivalence relation on the hom set $\text{Hom}_C(X,Y)$ and for all $f, g : X \longrightarrow Y$ such that $f \mathcal{R}_{X,Y} g$, we have $(vfu)\mathcal{R}_{A,B}(vgu)$, for any morphisms $u : A \rightarrow X$ and $v : Y \rightarrow B$ of $C$.

3 Notion of $\mathcal{R}$–basis

Definition 3.1. We call compatibility relation on a monoidal category $C$, any congruence relation $\mathcal{R}$ on $C$ verifying the following
(i) For any morphisms \( f \) and \( g \) between dualizable objects \( U \) and \( V \) of \( C \), such that \( f \mathcal{R}_{U,V} g \), we have \( f^* \mathcal{R}_{V^*,U^*} g^* \).

(ii) For any objects \( U, V, A \) and \( B \) of \( C \) and any morphisms \( f, g : U \rightarrow V \) and \( f', g' : A \rightarrow B \) such that \( f \mathcal{R}_{U,V} g \) and \( f' \mathcal{R}_{A,B} g' \), then \( (f \otimes f') \mathcal{R}_{U \otimes A, V \otimes B} (g \otimes g') \).

Lemma 3.2. Let \( C \) be a ribbon Ab–category. The relation \( \mathcal{R} \) defined on hom sets by

\[
\forall U, V \in \text{Ob}(C), \forall f, g : U \rightarrow V; f \mathcal{R}_{U,V} g \iff f - g \in \text{Negl}(U, V) \tag{3.1}
\]

is a compatibility relation on \( C \).

Proof. \( \mathcal{R}_{U,V} \) is clearly an equivalence relation on each hom set \( \text{Hom}_C(U,V) \). The axioms of Definition 3.1 hold by the fact that the dual of a negligible morphism is negligible and the tensor product of negligible morphisms is again negligible [17].

Remark 3.3. (a) Let \( C \) be a ribbon Ab–category and \( U \) and \( V \) two objects of \( C \). Then, \( \text{Negl}(U;V) \) is an ideal in \( C \) and the class modulo \( \mathcal{R}_{U,V} \) of the zero arrow is the ideal \( \text{Negl}(U;V) \). Recall [16] that a set \( R_X \) of arrows to an object \( X \) of \( C \), is called a right \( X \)–ideal if for all \( f, g : B \rightarrow X \) in \( R_X \), for all \( v : A \rightarrow B \), \( A, B \in \text{Ob}(C) \), one has \( (f + g)v \) is in \( R_X \). Left \( X \)–ideal is defined similarly. A set of arrows from an object \( X \) to an object \( Y \) of \( C \) is called an ideal if it is both a right \( X \)–ideal and a left \( Y \)–ideal.

(b) Let \( C \) be a ribbon Ab–category and \( f \in \text{Hom}_C(X,Y) \). Consider the sets

\[
R_X := \{g : A \rightarrow X, fg \in \text{Negl}(A;Y)\}_{A \in \text{Ob}(C)};
\]

\[
R_Y := \{g : Y \rightarrow B, gf \in \text{Negl}(X;B)\}_{B \in \text{Ob}(C)}.
\]

Then, \( R_X \) is a right \( X \)–ideal and \( R_Y \) is a left \( Y \)–ideal.

(c) Let \( C \) be a ribbon Ab–category. A set of arrows \( R \) is an ideal in \( C \), if and only if, the set

\[
R^* := \{f^*, f \in R\}
\]

is an ideal in the category \( \overline{C} \) defined by

\[
\text{Ob}(\overline{C}) := \{X^*, X \in \text{Ob}(C)\} \text{ and } \text{Mor}(\overline{C}) := \{f^*, f \in \text{Mor}(C)\}.
\]
From now on, \( \mathcal{R} \) will denote always the above compatibility relation (3.1), whenever \( C \) is considered as a ribbon Ab–category.

**Definition 3.4.** Let \( C \) be a monoidal Ab–category, equipped with a compatibility relation \( \mathcal{R} \) and \( V \) a dualizable object of \( C \) with duality structures \((V^*; d_V; b_V)\). An \( n - \mathcal{R} \)–basis \((V; d^i_V; b^i_V; \pi^i_V)_{1 \leq i \leq n}\) on \( V \), is a family of morphisms

\[
d^i_V : V^* \otimes V \rightarrow I, \ b^i_V : I \rightarrow V \otimes V^* \text{ and } \pi^i_V : V \rightarrow V
\]

such that for all \( 1 \leq i, j \leq n \), the following hold

\[
(id_V \otimes d^i_V)(b^j_V \otimes id_V) = \pi^j_V \pi^i_V;
\]

\[
(d^i_V \otimes id_{V^*})(id_{V^*} \otimes b^j_V) = (\pi^i_V \pi^j_V)^*\]

and

\[
\sum_{i=1}^{n} \pi^i_V = 1_V \ mod(\mathcal{R}_V, V).
\]

**Remark 3.5.** Let \( V \) be a dualizable object of \( C \) with dual \( V^* \) and consider an \( n - \mathcal{R} \)–basis \((V; d^i_V; b^i_V; \pi^i_V)_{1 \leq i \leq n}\) on \( V \).

(a) We know that the dual object in a monoidal category is unique up to a unique isomorphism [18, page 23]. Let \( V^\vee \) be another dual of \( V \) and \( f : V^* \rightarrow V^\vee \) be the unique isomorphism between the duals. Then, it is not difficult to verify that

\[
(V; d^i_V (f \otimes id_V); (f^{-1} \otimes id_V)b^i_V; \pi^i_V)_{1 \leq i \leq n}
\]

is another \( n - \mathcal{R} \)–basis on \( V \).

(b) Every sub family of an \( n - \mathcal{R} \)–basis is again an \( m - \mathcal{R} \)–basis with \( m \leq n \).

Remark 3.5. Let \( V \) be a dualizable object of \( C \) with dual \( V^* \) and consider an \( n - \mathcal{R} \)–basis \((V; d^i_V; b^i_V; \pi^i_V)_{1 \leq i \leq n}\) on \( V \).

(a) We know that the dual object in a monoidal category is unique up to a unique isomorphism [18, page 23]. Let \( V^\vee \) be another dual of \( V \) and \( f : V^* \rightarrow V^\vee \) be the unique isomorphism between the duals. Then, it is not difficult to verify that

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(V; d^i_V (f \otimes id_V); (f^{-1} \otimes id_V)b^i_V; \pi^i_V)_{1 \leq i \leq n}
\]

is another \( n - \mathcal{R} \)–basis on \( V \).

(b) Every sub family of an \( n - \mathcal{R} \)–basis is again an \( m - \mathcal{R} \)–basis with \( m \leq n \).

Note that this notion generalizes the standard notion of basis for vector spaces over a field \( K \).

**Example 3.6.** Every object \( V \) of the category \((\text{vec}_K, \otimes_K, K)\) of finite dimensional vector spaces over a field \( K \), admits an \( n - \mathcal{R} \)–basis where \( n \) is its dimension and \( \mathcal{R} \) is any compatibility relation.
Proof. Consider the family:
\[
d_l^V : V^* \otimes V \longrightarrow K, \quad e_j \otimes e_i \mapsto \delta_{lj}\delta_{li},
\]
\[
b_l^V : K \longrightarrow V \otimes V^*, \quad 1 \mapsto e_i \otimes e^i, \quad \pi^l_V : V \longrightarrow V
\]
for all 1 \leq l \leq n, where \{e_i\}_{1 \leq i \leq n} and \{e^i\}_{1 \leq i \leq n} are respectively a basis and its dual basis of \(V\) and its dual \(V^*\). Then, the family \((d_l^V; b_l^V; \pi^l_V)_{1 \leq l \leq n}\) defines an \(n - R\)–basis on \(V\).

Example 3.7. Let \(C\) be a monoidal \(Ab\)–category equipped with a compatibility relation \(R\) and \(V\) a dualizable object of \(C\) with duality structures denoted \((V^*; d_V; b_V)\). Then, \((V; d_V; b_V; 1_V)\) is a 1 – \(R\)–basis on \(V\).

Proposition 3.8. Any semisimple ribbon \(Ab\)–category \(C\) admits an \(R\)–basis on each of its objects.

Proof. An object \(V\) of a semisimple ribbon \(Ab\)–category is dominated by a finite family \((V_i)_{1 \leq i \leq n}\) of simple objects of \(C\), i.e, there exists a family of morphisms \(\{\varepsilon^i_V : V \longrightarrow V_i ; \mu^i_V : V_i \longrightarrow V\} _{i=1}^{i=n}\) such that \(\sum_i \mu^i_V \varepsilon^i_V - id_V\) is a negligible endomorphism of \(V\). Then
\[
(V; d_V, ((\mu^i_V)^* \otimes \varepsilon^i_V); (\mu^i_V \otimes (\varepsilon^i_V)^*)b_V; \mu^i_V \varepsilon^i_V)
\]
is an \(n - R\)–basis on \(V\). In fact, the following hold
\[
(1_V \otimes d_V, ((\mu^i_V)^* \otimes \varepsilon^i_V)((\mu^j_V \otimes (\varepsilon^j_V)^*)b_V) \otimes 1_V) = \mu^j_V \varepsilon^j_V \mu^j_V \varepsilon^j_V
\]
\[
(d_V((\mu^i_V)^* \otimes \varepsilon^i_V) \otimes 1_V)(1_{V^*} \otimes (\mu^j_V \otimes (\varepsilon^j_V)^*)b_V) = (\mu^j_V \varepsilon^j_V \mu^j_V \varepsilon^j_V)^*
\]
and \(\sum_i \mu^i_V \varepsilon^i_V = id_V \mod (R_{V,V})\).

Definition 3.9. Let \(C\) be a monoidal \(Ab\)–category, equipped with a compatibility relation \(R\) and let \(V\) be a dualizable object of \(C\). Denote by \(r_V\), the minimum cardinal, as explained below, of \(R\)–bases \((V; d^i_V; b^i_V; \pi^i_V)_{1 \leq i \leq n}\) on \(V\), \(n \in \mathbb{N}\); for which, \(\pi^i_V \neq 1_V\), for all \(i, 1 \leq i \leq n\).

\(r_V = n\) is a minimum cardinal in the sense that:
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(i) There exists an $R$–basis $(V; d_V^i; b_V^i; \pi_V^i)_{1 \leq i \leq n}$ on $V$ such that there exist no $\mathcal{A}$–morphisms $d_V^{n+1}, b_V^{n+1}$ and $\pi_V^{n+1}$ such that

$$(V; d_V^i; b_V^i; \pi_V^i)_{1 \leq i \leq n} \cup (V; d_V^{n+1}; b_V^{n+1}; \pi_V^{n+1})$$

is an $(n + 1) – R$–basis on $V$.

(ii) There is no $m – R$–basis on $V$ verifying (i) such that $m < n$.

Note that the condition on $\pi_V^i$ is just to avoid the trivial case when $C$ is rigid, where, for any object $V$ of $C$, $r_V = 1$ by Example 3.7.

The following lemmata will be useful in claiming forthcoming results on the integer $r_V$ introduced in the very definition.

**Lemma 3.10.** Let $C$ be a semisimple ribbon $Ab$–category and $A$ and $B$ be isomorphic objects in $C$. Then

(a) $A$ is dominated by $n$ simple objects, if and only if $B$ is;

(b) $(A; d_A^i; b_A^i; \pi_A^i)_{i=1}^n$ is an $R$–basis on $A$, if and only if

$$(B; d_A^i (f^* \otimes g); (f \otimes g^*) b_A^i; f \pi_A^i g)_{i=1}^n$$

is an $R$–basis on $B$.

**Proof.** (a) Let $f \in \text{Hom}_C(A; B)$ be an isomorphism with inverse $g$.

Assume that $A$ is dominated by $(V_i; \epsilon^i; \mu^i)_{i=1}^n$. Let $\epsilon^i = \epsilon_A^i g$ and $\mu^i = f \mu_A^i$.

Then, $B$ is dominated by $(V_i; \epsilon_B^i; \mu_B^i)_{i=1}^n$.

Inversely, if $B$ is dominated by $(V_i; \epsilon_B^i; \mu_B^i)_{i=1}^n$, one easily checks that $A$ is dominated by $(V_i; \epsilon_B^i f, g \mu_B^i)_{i=1}^n$.

(b) Let $(A; d_A^i; b_A^i; \pi_A^i)_{i=1}^n$ be an $R$–basis on $A$. Then

$$(B; d_A^i (f^* \otimes g); (f \otimes g^*) b_A^i; f \pi_A^i g)_{i=1}^n$$

is an $R$–basis on $B$. In fact, we have to prove the following three identities:

$$(id_B \otimes d_A^i (f^* \otimes g)) ((f \otimes g^*) b_A^i \otimes id_B) = (f \pi_A^i g) (f \pi_A^i g),$$

$$(d_A^i (f^* \otimes g) \otimes id_B^*) (id_B \otimes (f \otimes g^*) b_A^i) = (f \pi_A^i g)^* (f \pi_A^i g)^*$$
We have
\[
(1_B \otimes d^j_i (f^* \otimes g))((f \otimes g^*)(b^i_A \otimes 1_B))
\]
\[
= (1_B \otimes d^j_i)(1_B \otimes f^* \otimes g)(f \otimes g^* \otimes 1_B)(b^i_A \otimes 1_B)
\]
\[
= (1_B \otimes d^j_i)(f \otimes 1_A^* \otimes g)(b^i_A \otimes 1_B)
\]
\[
= f(1_A \otimes d^j_i)(b^i_A \otimes 1_A)g
\]
\[
= (f \pi^i_A g)(f \pi^i_A g)
\]
and
\[
(d^j_i(f^* \otimes g) \otimes id_{B^*})(id_{B^*} \otimes (f \otimes g^*)b^i_A)
\]
\[
= (d^j_i \otimes id_{B^*})(f^* \otimes g \otimes id_{B^*})(id_{B^*} \otimes f \otimes g^*)(id_{B^*} \otimes b^i_A)
\]
\[
= g^*(d^j_i \otimes id_{A^*})(id_{A^*} \otimes b^i_A)f^*
\]
\[
= (f \pi^i_A g)^*(f \pi^i_A g)^*.
\]

The third identity is obvious.
Inversely, if $B$ admits an $R$–basis on it, then by the same previous procedure interchanging the roles of $f$ and $g$, we get an $R$–basis on $A$.

\begin{lemma}
Let $C$ be a semisimple ribbon Ab–category and $A$ an object of $C$. Then
\begin{enumerate}
\item[(a)] $A$ is dominated by $n$ simple objects, if and only if $A^*$ is;
\item[(b)] $(A; d^i_A; b^i_A; \pi^i_A)_{i=1}^n$ is an $R$–basis on $A$, if and only if
\[
(A^*; (d^i_A)^*; (b^i_A)^*; (\pi^i_A)^*)_{i=1}^n
\]
is an $R$–basis on $A^*$.
\end{enumerate}
\end{lemma}

\begin{proof}
(a) Assume that $A$ is dominated by $(V^i_l; e^i_A; \mu^i_A)_{i=1}^n$. Then $A^*$ is dominated by $(V^*_l; (\mu^i_A)^*; (e^i_A)^*)_{i=1}^n$.
Inversely, this holds due to the fact that $A^{**} \simeq A$ is verified in light of the compatible duality of $C$.
\end{proof}
(b) Let \((A; d^i_A; b^j_A; \pi^i_A)_{i=1}^n\) be an \(R\)-basis on \(A\). Then

\[(d^i_A)_*: = d^i_1(b^i_A)^* \lambda_A; A^*, \quad (b^i_A)_*: = \lambda_A^{-1}(d^i_A)^* b_I \quad \text{and} \quad (\pi^i_A)_*: = (\pi^i_A)^*\]

for all \(1 \leq i \leq n\).

Along with the proof and the rest of the paper, by \(\lambda\) we mean \(\lambda_A; A^*\) and by \(\lambda^{-1}\) we mean \(\lambda_A^{-1}; A^*\) to reduce notations (where \(\lambda\) is defined as in (2.1)).

In fact, we prove the three identities:

\[
(id_A^* \otimes d^i_1(b^i_A)^* \lambda)(\lambda^{-1}(d^j_A)^* b_I \otimes id_A^*) = (\pi^j_A)^* (\pi^i_A)^* ,
\]

\[
(d^i_1(b^i_A)^* \lambda \otimes id_A^*) (id_A^* \otimes \lambda^{-1}(d^j_A)^* b_I) = ((\pi^j_A)^*) ((\pi^i_A)^*)
\]

and \(\sum_{i=1}^{n} (\pi^i_A)^* = 1_{A^* \mod (R_{A^*; A^*})}\).

The first equality is justified by the following commutative diagram:

\[
A^* = I \otimes A \xrightarrow{id_A^* \otimes id_A^*} I^* \otimes A^* \xrightarrow{(d^i_A)^* \otimes (id_A^*)} (A^* \otimes A)^* \xrightarrow{\lambda^{-1} \otimes id_A^*} A^* \otimes A^* \otimes A^*
\]

The two middle squares are commutative by the naturality of \(\lambda\) (where, \(\lambda_A; I = (d_1 \otimes id_A^*)(id_I^* \otimes d_A \otimes id_A^*) (id_A^* \otimes b_A) = (d_1 \otimes id_A^*)(d_A \otimes id_A^*)(id_A^* \otimes b_A) = d_1 \otimes id_A^*\) and the same thing for \(\lambda_A^{-1}; I\)).
With similar arguments one can prove the second identity which is justified by the following commutative diagram:

\[
\begin{align*}
A^{**} = A^{**} \otimes I & \xrightarrow{1 \otimes b_I} A^{**} \otimes I^* \\
& \xrightarrow{1 \otimes (d_I^I)^*} A^{**} \otimes (A^* \otimes A)^* \\
& \xrightarrow{1 \otimes A^{-1}} A^{**} \otimes A^* \otimes A^{**}
\end{align*}
\]

The third is obvious.

Inversely, an \(R\)-basis on \(A^*\) gives similarly an \(R\)-basis on \(A^{**}\) \((\simeq A)\). \(\square\)

**Proposition 3.12.** Let \(C\) be a semisimple ribbon \(Ab\)-category and \(A\) and \(B\) be isomorphic objects in \(C\). Assume that \(A\) (or \(B\)) is dominated by a finite set of simple objects. Then \(A\), \(A^*\), \(B\) and \(B^*\) admit \(R\)-bases on them and we have

(a) \(r_B = r_A\);

(b) \(r_{A^*} = r_A\).

**Proof.** \(A\) (resp. \(B\)) being dominated by simple objects ensures by using Lemma 3.10, the existence of \(R\)-bases on \(A\), \(A^*\), \(B\) and \(B^*\).

(a) Using Lemma 3.10, (ii), we obtain an \(r_A - R\)-basis on \(B\) which is minimal (among the cardinals of the other \(R\)-bases on \(B\)) and vice versa. Hence, \(r_B = r_A\).

(b) Identically to the above, using this time Lemma 3.11, (ii).

\(\square\)

**Definition 3.13.** Let \(C\) be a semisimple ribbon \(Ab\)-category and \(V\) an object of \(C\). We call quantum rank of \(V\) denoted by \(\text{ran}_q(V)\), the nonnegative integer defined as

\[
\text{ran}_q(V) = \min(n)
\]

where \(n\) runs over all finite cardinals of dominating families \((V_i; e_i^V, \mu_i^V)_{i=1}^n\) of simple objects of \(V\).

**Proposition 3.14.** Let \(C\) be a semisimple ribbon \(Ab\)-category and \(A\) and \(B\) be isomorphic objects in \(C\). Then
(a) \( \text{ran}_q(V) = 1 \) for every simple object \( V \) of \( C \);
(b) \( \text{ran}_q(B) = \text{ran}_q(A) \);
(c) \( \text{ran}_q(A^*) = \text{ran}_q(A) \).

**Proof.** Straightforward from Lemma 3.10 (i) and Lemma 3.11 (i).

### 4 Categorification of bilinear forms

**Definition 4.1.** Let \( C \) be a monoidal \( \mathbb{A}b \)–category equipped with a compatibility relation \( \mathcal{R} \) and \( V \) a dualizable object of \( C \). An \( \mathcal{R} \)–solution on \( V \), is a quadruple \((V; d_V; b_V; \pi_V)\), such that:

\[
(1_V \otimes d_V)(b_V \otimes 1_V) = \pi_V^2;
\]

\[
(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = (\pi_V^2)^*;
\]

\[
d_V(1_{V^*} \otimes \pi_V) = d_V(\pi_V^* \otimes 1_V);
\]

\[
(\pi_V \otimes 1_{V^*})b_V = (1_V \otimes \pi_V^*)b_V;
\]

\[
\pi_V = 1_V mod(\mathcal{R}_V, V).
\]

**Example 4.2.** Let \( V \) be an object of the category \((\text{vect}_K, \otimes, K)\) of finite dimensional vector spaces over a field \( K \) and \( \mathcal{R} \) any compatibility relation. Then, \( V \) admits an \( \mathcal{R} \)–solution on it.

**Proof.** \((V; d_V; b_V; 1_V)\) is an \( \mathcal{R} \)–solution on \( V \), where:

\[
d_V : V^* \otimes V \longrightarrow K \quad \text{and} \quad b_V : K \longrightarrow V \otimes V^*
\]

\[
e^j \otimes e_i \longmapsto \delta_{ij} \quad \text{and} \quad 1 \longmapsto \sum_i e_i \otimes e^i
\]

such that \{\(e_i\}_i\) and \{\(e^i\}_i\) are respectively a basis and its dual basis of \( V \) and its dual \( V^* \).

**Example 4.3.** The \( 1 - \mathcal{R} \)–basis in Example 3.7 of the previous section 3, is an \( \mathcal{R} \)–solution on \( V \).

**Proposition 4.4.** Let \( C \) be a monoidal \( \mathbb{A}b \)–category, \( \mathcal{R} \) a compatibility relation on \( C \) and \( V \) a dualizable object of \( C \). Then, for every morphism \( \pi_V : V \longrightarrow V \), such that \( \pi_V^2 = \pi_V \) and \( \pi_V = 1_V mod(\mathcal{R}_V, V) \), the quadruple

\[
(V; d_V(\pi_V^* \otimes \pi_V); (\pi_V \otimes \pi_V^*)b_V; \pi_V)
\]
is an $\mathcal{R}$–solution on $V$, where $d_V$ and $b_V$ are duality structures on $V$.

**Proof.** Straightforward. \hfill \qed

**Example 4.5.** Let $C$ be a semisimple ribbon $Ab$–category and $V$ an object of $C$ dominated by $(V_i; \varepsilon_i; \mu_i)_{i=1}^n$. Assume that $\varepsilon_i \mu_j = \delta_{i,j}$, for all $i, j; 1 \leq i, j \leq n$. Then

$$(V; d_V(T_V^* \otimes T_V); (T_V \otimes T_V^*)b_V; T_V)$$

is an $\mathcal{R}$–solution on $V$, where $T_V = \sum_i \mu_i \varepsilon_i$ and $d_V$ and $b_V$ are duality structures on $V$.

**Proof.** In fact, $\varepsilon_i \mu_j = \delta_{i,j}, 1 \leq i, j \leq n \Rightarrow T_V^2 = \sum_i \sum_j \mu_i \varepsilon_i \mu_j \varepsilon_j = T_V$. Hence, applying Proposition 4.4, the result holds. \hfill \qed

**Proposition 4.6.** Let $C$ be a monoidal $Ab$–category equipped with a compatibility relation $\mathcal{R}$ and $f : A \longrightarrow B$ be an isomorphism between dualizable objects in $C$. Then, the following are equivalent

(a) $(A; d_A; b_A; \pi_A)$ is an $\mathcal{R}$–solution on $A$;

(b) $(B; d_A(f^* \otimes f^{-1}); (f \otimes (f^{-1})^*)b_A; f \pi_A f^{-1})$ is an $\mathcal{R}$–solution on $B$.

**Proof.** (a)$\Rightarrow$(b): Let $(A; d_A; b_A; \pi_A)$ be an $\mathcal{R}$–solution on $A$. We have to prove

$$d_B(f^* \otimes \pi_A f^{-1}) = d_B((f \pi_A)^* \otimes f^{-1});$$

$$d_B(f^* \otimes f^{-1}) \otimes 1_B = (f \otimes (f^{-1})^*)b_A = (f \pi_A f^{-1})^*;$$

$$d_B(f^* \otimes \pi_A f^{-1}) = d_B((f \pi_A)^* \otimes f^{-1});$$

$$(\pi_A f^{-1} \otimes f^*)b_B = (f^{-1} \otimes (f \pi_A)^*)b_B$$

and $\pi_B = 1_B \mod (\mathcal{R}_{B,B}).$
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The proof of the first and second identities is similar to the one of Lemma 3.10 (ii).
For the third one, we have

\[ d_B(f^* \otimes \pi_A f^{-1}) = d_B(1_A^* \otimes \pi_A)(f^* \otimes f^{-1}) \]
\[ = d_B(\pi_A^* \otimes 1_A)(f^* \otimes f^{-1}) \]
\[ = d_B((f \pi_A)^* \otimes f^{-1}) \]

and similarly for the fourth one.
For the fifth, we have:

\[ \pi_A = 1_A \mod(\mathcal{R}_{A,A}) \Rightarrow \pi_B := f \pi_A f^{-1} = 1_B \mod(\mathcal{R}_{B,B}). \]

(b)\(\Rightarrow\)(a): Let \(d_A : A^* \otimes A \rightarrow I\), \(b_A : I \rightarrow A \otimes A^*\) and \(\pi_A : A \rightarrow A\) be morphisms such that

\[ (B; d_A(f^* \otimes f^{-1}); (f \otimes (f^{-1})^*)b_A; f \pi_A f^{-1}) \]

is an \(\mathcal{R}\)–solution on \(B\). Then

\[ (A; d_A(f^* \otimes f^{-1})((f^{-1})^* \otimes f); (f^{-1} \otimes f^*)(f \otimes (f^{-1})^*)b_A; f^{-1} f \pi_A f^{-1} f) \]

is an \(\mathcal{R}\)–solution on \(A\) (by the first sense), i.e: \((A; d_A; b_A; \pi_A)\) is an \(\mathcal{R}\)–solution on \(A\).

\[ \square \]

**Proposition 4.7.** Let \(C\) be a ribbon Ab–category equipped with a compatibility relation \(\mathcal{R}\) and \(A \in Ob(C)\) endowed with an \(\mathcal{R}\)–solution \((A; d_A; b_A; \pi_A)\) on it. Define the morphisms

\[ (d_A)_* := d_I b_A^* \lambda_{A;A^*}; \quad (b_A)_* := \lambda_{A^*;A}^{-1} d_A^* b_I \quad \text{and} \quad (\pi_A)_* := \pi_A^*. \]

Then, \((A^*; (d_A)_*; (b_A)_*; (\pi_A)_*)\) is an \(\mathcal{R}\)–solution on \(A^*\).

**Proof.** We have to prove the five identities

\[ (1_{A^*} \otimes d_I b_A^* \lambda)(\lambda^{-1} d_A^* b_I \otimes 1_{A^*}) = (\pi_A^*)^* \]
\[ (d_I b_A^* \lambda \otimes 1_{V^*})(1_{(A^*)^*} \otimes \lambda^{-1} d_A^* b_I) = (\pi_A^*)^* \]
\[ d_I b_A^* \lambda(1_{(A^*)^*} \otimes (\pi_A)_*) = d_I b_A^* \lambda(\pi_A^* \otimes 1_{A^*}) \]
\[ (\pi_A^* \otimes 1_{(A^*)^*}) \lambda^{-1} d_A^* b_I = (1_{A^*} \otimes \pi_A^*)(\lambda^{-1} d_A^* b_I) \]
\[ \pi_A^* = 1_{A^*} \mod(\mathcal{R}_{A^*,A^*}). \]
The proof of the first and second is exactly similar to the proof given in Lemma 3.12 (ii).
For the third one, we have
\[
d_{I}b_{A}^{*}A((\pi_{A^{*}})^{*} \otimes 1_{A^{*}}) = d_{I}b_{A}^{*}A\lambda^{-1}(1_{A} \otimes \pi_{A}^{*})^{*}\lambda
\]
\[
= d_{I}[(1_{A} \otimes \pi_{A})b_{A}]^{*}\lambda
\]
\[
= d_{I}[(\pi_{A} \otimes 1_{A^{*}})b_{A}]^{*}\lambda
\]
\[
= d_{I}b_{A}^{*}\lambda\lambda^{-1}(\pi_{A} \otimes 1_{A^{*}})^{*}\lambda
\]
\[
= d_{I}b_{A}^{*}A(1_{(A^{*})^{*}} \otimes \pi_{A}^{*}).
\]
The fourth: similar to the third.
The fifth identity is straightforward.

In order to study the properties of \(\mathcal{R}\)–solutions, we introduce the tensor product of bilinear forms in \(\mathcal{C}\).

**Proposition 4.8.** Let \(\mathcal{C}\) be a monoidal \(\mathcal{A}\)–category equipped with a compatibility relation \(\mathcal{R}\) and let \((A; d_{A}; b_{A}; \alpha)\) and \((B; d_{B}; b_{B}; \beta)\) be \(\mathcal{R}\)–solutions on two dualizable objects \(A\) and \(B\) of \(\mathcal{C}\). Then,

\[
(A \otimes B; d_{A} \otimes_{-} d_{B}; b_{A} \otimes_{+} b_{B}; \alpha \otimes \beta)
\]

is an \(\mathcal{R}\)–solution on \(A \otimes B\); where the tensor products \(\otimes_{-}\) of \(d_{A}\), \(d_{B}\) and \(\otimes_{+}\) of \(b_{A}, b_{B}\); are defined as

\[
d_{A} \otimes_{-} d_{B} := d_{B}(1_{B}^{*} \otimes d_{A} \otimes 1_{B})(\lambda_{A:B}^{-1} \otimes 1_{A} \otimes 1_{B});
\]

\[
b_{A} \otimes_{+} b_{B} := (1_{A} \otimes 1_{B} \otimes \lambda_{A:B})(1_{A} \otimes b_{B} \otimes 1_{A^{*}})b_{A}.
\]

**Proof.** The domains and codomains of the defined tensor products are as follows:

\[
d_{A} \otimes_{-} d_{B} : (A \otimes B)^{*} \otimes A \otimes B \rightarrow B^{*} \otimes A^{*} \otimes A \otimes B \rightarrow B^{*} \otimes B \rightarrow I
\]

and

\[
b_{A} \otimes_{+} b_{B} : I \rightarrow A \otimes A^{*} \rightarrow A \otimes B \otimes B^{*} \otimes A^{*} \rightarrow A \otimes B \otimes (A \otimes B)^{*}.
\]

Let’s prove the first identity:

\[
[1_{A \otimes B} \otimes d_{B}(1_{B}^{*} \otimes d_{A} \otimes 1_{B})(\lambda^{-1} \otimes 1_{A} \otimes 1_{B})][(1_{A} \otimes 1_{B} \otimes \lambda)(1_{A} \otimes b_{B} \otimes 1_{A^{*}})b_{A} \otimes
\]
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\[ 1_{A \otimes B} = (\alpha \otimes \beta)^2. \]

We have:
\[
[1_A \otimes 1_B \otimes d_B(1_B^* \otimes d_A \otimes 1_B)(\lambda^{-1} \otimes 1_A \otimes 1_B)][(1_A \otimes 1_B \otimes \lambda)(1_A \otimes b_B \otimes 1_{A^*})b_A \otimes 1_A \otimes 1_B]
\]
\[
= [1_A \otimes 1_B \otimes d_B(1_B^* \otimes d_A \otimes 1_B)][(1_A \otimes 1_B \otimes (\lambda^{-1} \otimes 1_A \otimes 1_B))][(1_A \otimes 1_B \otimes \lambda) \otimes 1_A \otimes 1_B] [(1_A \otimes b_B \otimes 1_{A^*})b_A \otimes 1_A \otimes 1_B]
\]
\[
= [1_A \otimes 1_B \otimes d_B][1_A \otimes b_B \otimes 1_B][1_A \otimes d_A \otimes 1_B][b_A \otimes 1_A \otimes 1_B]
\]
\[
= [1_A \otimes \beta^2][1_A \otimes d_A \otimes 1_B][b_A \otimes 1_A \otimes 1_B]
\]
\[
= \alpha^2 \otimes \beta^2
\]
\[
= (\alpha \otimes \beta)^2.
\]

The proof of the other identities is done similarly. \(\square\)

**Corollary 4.9.** Let \(C\) be a monoidal \(Ab\)-category equipped with a compatibility relation \(\mathcal{R}\) and let \((V_i; d_i^V; b_i^V; \pi_i^V)\) be \(1 - \mathcal{R}\)-bases on dualizable objects \(V_i\) of \(C\), for any \(i, 1 \leq i \leq n, n \geq 2.\) Then

\((V_1 \otimes \ldots \otimes V_n; d_1^V \otimes \ldots \otimes d_n^V; b_1^V \otimes \ldots \otimes b_n^V; \pi_1^V \otimes \ldots \otimes \pi_n^V)\)

is a \(1 - \mathcal{R}\)-basis on \(V_1 \otimes \ldots \otimes V_n.\)

**Proof.** By induction on \(n\), using Proposition 4.8 and remarking that in fact, an \(\mathcal{R}\)-solution on an object is in particular a \(1 - \mathcal{R}\)-basis on it. \(\square\)

The following definition serves to establish a forthcoming result.

**Definition 4.10.** Let \(C\) be a monoidal \(Ab\)-category; \(V\) an object of \(C\) and \((V; d_V; b_V; 1_V)\) a particular solution of the triangular system

\[(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V;\]

\[(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.\]

Let \((V; D_V; B_V; 1_V)\) be another solution of the same system. Then, for any automorphism \(f : V \rightarrow V\). Define the morphisms

\[f^*^1 = (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes b_V);\]

\[f^*^2 = (d_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes B_V);\]
and

\[
    f^{-1}.b_V := (f \otimes (f^{-1})^2)b_V : I \rightarrow V \otimes V^*;
\]

\[
    d_V.f := d_V(f^{-1} \otimes f^{-1}) : V^* \otimes V \rightarrow I.
\]

**Proposition 4.11.** Let \((V; d_V; b_V; 1_V)\) be a particular solution of the triangular system

\[
    (1_V \otimes d_V)(b_V \otimes 1_V) = 1_V;
\]

\[
    (d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.
\]

Then, all solutions are given by

\[
    (V; d_V.f; f^{-1}.b_V; 1_V), \ f \in \text{Aut}_C(V).
\]

**Proof.** Let \((V; d_V; b_V; 1_V)\) be a particular solution and \(f \in \text{Aut}_C(V)\). Then we get \((V; d_V.f; f^{-1}.b_V; 1_V)\) is a solution of the above system for any other solution \((V; D_V; B_V; 1_V)\) (including the fixed particular one). In fact, we have

\[
    (1_V \otimes d_V(f^{-1} \otimes f^{-1}))(f \otimes (f^{-1})^2)b_V \otimes 1_V
\]

\[
    = (1_V \otimes D_V)(1 \otimes 1_V \otimes 1_V \otimes 1_V)(1_V \otimes d_V \otimes 1_V \otimes 1_V)(f \otimes 1_V \otimes f^{-1} \otimes 1_V \otimes 1_V)(1_V \otimes 1_V \otimes B_V \otimes 1_V)(b_V \otimes 1_V)
\]

\[
    = (1_V \otimes D_V)(1 \otimes 1_V \otimes f)(1_V \otimes 1_V \otimes 1_V \otimes d_V)(1_V \otimes 1_V \otimes b_V \otimes 1_V)(1_V \otimes 1_V \otimes f^{-1})(f \otimes 1_V \otimes 1_V)(1_V \otimes d_V \otimes 1_V \otimes 1_V)(b_V \otimes 1_V \otimes 1_V \otimes 1_V)(f^{-1} \otimes 1_V \otimes 1_V)(B_V \otimes 1_V)
\]

\[
    = (1_V \otimes D_V)(1 \otimes 1_V \otimes f)(f^{-1} \otimes 1_V \otimes 1_V)(B_V \otimes 1_V)
\]

\[
    = 1_V.
\]

And

\[
    (d_V(f^{-1} \otimes f^{-1}) \otimes 1_{V^*})(1_{V^*} \otimes (f \otimes (f^{-1})^2)b_V)
\]

\[
    = (d_V \otimes 1_{V^*})(D_V \otimes 1_{V^*} \otimes 1_{V^*} \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_V)(1_{V^*} \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes B_V)(1_{V^*} \otimes b_V)
\]

\[
    = (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_V)(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V)
\]

\[
    = (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V)
\]

\[
    = 1_{V^*}.
\]
Now let \((V; D_V; B_V; 1_V)\) be a solution of the triangular system and let 
\[
f = (1_V \otimes d_V)(B_V \otimes 1_V) \quad (\text{resp}. \ f = (1_V \otimes D_V)(b_V \otimes 1_V)).
\]
Then, \(f\) is invertible and its inverse is 
\[
f^{-1} = (1_V \otimes D_V)(b_V \otimes 1_V) \quad (\text{resp}. \ f^{-1} = (1_V \otimes d_V)(B_V \otimes 1_V))
\]
and we have 
\[
d_V \cdot f = d_V(f^{-1} \otimes f^{-1})
\]
\[
= d_V((D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes b_V) \otimes f^{-1})
\]
\[
= D_V(1_{V^*} \otimes f)(1_{V^*} \otimes 1_V \otimes d_V)(1_{V^*} \otimes b_V \otimes 1_V)(1_{V^*} \otimes f^{-1})
\]
\[
= D_V
\]
and 
\[
f^{-1} \cdot b_V = (f \otimes (f^{-1})^*)b_V
\]
\[
= (f \otimes (d_V \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V))b_V
\]
\[
= (f \otimes 1_{V^*})(1_V \otimes d_V \otimes 1_{V^*})(b_V \otimes 1_V \otimes 1_{V^*})(f^{-1} \otimes 1_{V^*})B_V
\]
\[
= B_V.
\]

In general, we have the following.

**Proposition 4.12.** Let \((V; d_V; b_V; 1_V)\) be a particular solution of the triangular system 
\[
(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V;
\]
\[
(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.
\]

Denote by \(\text{Sol}_C(V)\) the set of all solutions of the above system on \(V\) and consider the map \(\varphi : \text{Aut}_C(V) \longrightarrow \text{Sol}_C(V), f \longmapsto (V; d_V \cdot f; f^{-1} \cdot b_V; 1_V).\) Then, \(\varphi\) is surjective but not injective.

**Proof.** Immediate from Proposition 4.11.
Definition 4.13. Let $C$ be a semisimple ribbon $Ab$–category. $R$–solutions over objects of $C$ form a category which is denoted by $Fin(C)$; the unit object is given by $\overline{I} = (I; d_I; b_I; 1_I)$, where $d_I$ and $b_I$ are duality structures on $I$.

A morphism

$$f : (A; d_A; b_A; \pi_A) \longrightarrow (B; d_B; b_B; \pi_B)$$

of $Fin(C)$, where $A$ and $B$ are two objects of $C$; consists of a morphism $f : A \longrightarrow B$ in $C$, such that

$$f.d_A = d_B.f \quad \text{and} \quad f.b_A = b_B.f$$

where, $f.d_A := d_A(f^* \otimes 1_A)$, $d_B.f := d_B(1_B \otimes f)$, $f.b_A := (f \otimes 1_{A^*})b_A$ and $b_B.f := (1_B \otimes f^*)b_B$ (notations here are independent from those of Definition 4.10).

Lemma 4.14. Let $C$ be a semisimple ribbon $Ab$–category and

$$f : (A; d_A; b_A; \pi_A) \longrightarrow (B; d_B; b_B; \pi_B)$$

a morphism in $Fin(C)$. Then, the dual morphism $f^*$ of $f$ defined a morphism

$$f^* : (A^*; (d_A)^*; (b_A)^*; (\pi_A)^*) \longrightarrow (B^*; (d_B)^*; (b_B)^*; (\pi_B)^*)$$

in $Fin(C)$.

Proof. We have to prove the following

$$(d_B)^*(f^{**} \otimes 1_{B^*}) = (d_A)^*(1_{A^{**}} \otimes f^*);$$

$$(1_{A^*} \otimes f^{**})(b_A)^* = (f^* \otimes 1_{B^{**}})(b_B)^*.$$
Proposition 4.15. Let $C$ be a semisimple ribbon $Ab$–category. Then, $Fin(C)$ is also a semisimple ribbon $Ab$–category.

Proof. The category $Fin(C)$ may be provided with canonical tensor product, duality and braiding (inherited from those of $C$), which makes it a braided monoidal category with duality.

The tensor product of a couple of $R$–solutions $(A; d_A; b_A; \pi_A)$ and $(B; d_B; b_B; \pi_B)$ is given by

$$(A; d_A; b_A; \pi_A) \otimes (B; d_B; b_B; \pi_B) = (A \otimes B; d_A \otimes d_B; b_A \otimes b_B; \pi_A \otimes \pi_B).$$

The category $Fin(C)$ is provided with canonical duality as follows: to each object $(A; d_A; b_A; \pi_A)$, there are associated an object

$$(A; d_A; b_A; \pi_A)^* := (A^*; (d_A)^*; (b_A)^*; (\pi_A)^*)$$

and morphisms

$$\overline{b_A} := b_{(A; d_A; b_A; \pi_A)} : \overline{I} \longrightarrow (A; d_A; b_A; \pi_A) \otimes (A^*; (d_A)^*; (b_A)^*; (\pi_A)^*)$$

and

$$\overline{d_A} := d_{(A; d_A; b_A; \pi_A)} : (A^*; (d_A)^*; (b_A)^*; (\pi_A)^*) \otimes (A; d_A; b_A; \pi_A) \longrightarrow \overline{I}$$

given by $b_A$ and $d_A$ respectively, such that the identities hold

$$(1 \otimes \overline{d_A})(\overline{b_A} \otimes 1) = 1$$

$$(\overline{d_A} \otimes 1)(1 \otimes \overline{b_A}) = 1$$

The dual $f^*$ of an arbitrary morphism

$$f : (A; X_A; Y_A; \alpha) \longrightarrow (B; Z_B; T_B; \beta)$$

is well defined by Lemma 4.14, and it is given by the formula

$$f^* = (Z_B \otimes 1_A^*)(1_{B^*} \otimes f \otimes 1_{A^*})(1_{B^*} \otimes Y_A).$$
It is easy to deduce that for any objects \((A; X_A; Y_A; \alpha)\) and \((B; Z_B; T_B; \beta)\) of \(\text{Fin}(C)\), there is a natural family of isomorphisms between

\[(B; Z_B; T_B; \beta)^* \otimes (A; X_A; Y_A; \alpha)^*\]

and

\[((A; X_A; Y_A; \alpha) \otimes (B; Z_B; T_B; \beta))^*\]

defined as

\[(\alpha \otimes \beta)^* (Z_B \otimes 1_{(A \otimes B)^*}) (1_{B^*} \otimes X_A \otimes 1_B \otimes 1_{(A \otimes B)^*}) (1_{B^*} \otimes 1_{A^*} \otimes Y_A \otimes 1_T) (\beta^* \otimes \alpha^*).\]

We provide \(\text{Fin}(C)\) with the braiding induced from \(C\). \(\text{Fin}(C)\) is twisted as follows: the twist \(\theta_{(A; d_A; b_A; \pi_A)}\) on an object \((A; d_A; b_A; \pi_A)\), consists of the twist \(\theta_A\). In fact, \(\theta_A.d_A = d_A.\theta_A\) and \(\theta_A.b_A = b_A.\theta_A\) by the naturality of \(\theta\).

Consequently, \((\ast; \overline{b_A}; \overline{d_A})\) is a compatible duality in \(\text{Fin}(C)\). Hence, the later is a ribbon category.

For semisimplicity, it is easy to verify that every object \((A; d_A; b_A; \pi_A)\) of \(\text{Fin}(C)\) is dominated by \(\{\{V_i; d_{V_i}; b_{V_i}; 1_{V_i}; \varepsilon_i; \mu_i\}_{i=1}^{i=n}\) where \(A\) is dominated by \((V_i; \varepsilon_i; \mu_i)_{i=1}^{i=n}\).

\[
\square
\]

5 The concept of a determinant

In all the sequel, we write \(\text{Tr}(f)\) instead of \(\text{Tr}_q(f)\) to reduce indices as well as for dimension and we identify \(V^n\) with \(V^\otimes n\) and \(f^\otimes n\) with \(f^n\), for all \(V \in \text{Ob}(C)\); \(f \in \text{End}_C(V)\).

Let \(C\) be a semisimple ribbon \(Ab\)–category and \(A\) an object of \(C\) of rank \(n\) dominated by simple objects \((V_i)_{1 \leq i \leq n}\) with domination morphisms denoted \(\{\varepsilon_i : V \rightarrow V_i; \mu_i : V_i \rightarrow V\}_{i}.\) Let \([1; n] \cap \mathbb{N} = I_1 \cup I_2 \cup \ldots \cup I_m\) be a partition of \([1; n] \cap \mathbb{N}\) into isomorphic classes. Denote \(\text{card}(I_j) = n_j\) for all \(1 \leq j \leq m\); \(W_j\) a representative of the isomorphic objects indexed by indices in \(I_j\) and \(C^{w_j}\) the identity endomorphism of \(W_j^{n_j}\).

We define the endomorphism \(\Lambda^n_A\) of \(A^n\) as:

\[
\Lambda^n_A = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \text{Tr}(C^{w_{n_1}})^{-1} D^1_{\sigma} \otimes \ldots \otimes \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \text{Tr}(C^{w_{n_m}})^{-1} D^m_{\sigma}
\]
where $D^j_\sigma$ is the endomorphism of $A^{n_j}$ defined by

$$D^j_\sigma = \mu_j e_{\sigma(j_1)} \otimes \ldots \otimes \mu_{j_{n_j}} e_{\sigma(j_{n_j})}$$  \hspace{1cm} (5.1)$$

with $I_j = [j_1, j_{n_j}] \cap \mathbb{N}$ and $\sigma$ a permutation of $\mathfrak{S}_{n_j}$.

If $n = 1$, we consider $\Lambda_1^A = \text{Tr}(id_A)^{-1}id_A$.

**Proposition 5.1.** Let $(A; X_A; Y_A; 1_A)$ be a particular solution on $A$ and $f \in \text{End}_C(A)$. Then, the quantum determinant, $\det^C_n(f)$, of $f$ defined by

$$\det^C_n(f) = X_A^{\otimes n}(f^{n^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}Y_A^{\otimes n}$$

is independent of the choice of the solution on $A$.

**Proof.** $(A; X_A; Y_A; 1_A)$ is a particular solution of the triangular system as in Proposition 4.11. If $(A; X_A; Y_A; 1_A)$ is another particular solution, then $(A^n; X_A^{\otimes n}; Y_A^{\otimes n}; 1)$ and $(A^n; \overline{X}_A^{\otimes n}; \overline{Y}_A^{\otimes n}; 1)$ are solutions on $A^n$ by Proposition 4.8. Using now Proposition 4.11, we obtain

$$\overline{X}_A^{\otimes n} = X_A^{\otimes n}.h := X_A^{\otimes n}(h^1 \otimes h^{-1}) \ ; \ \overline{Y}_A^{\otimes n} = Y_A^{\otimes n}.h^{-1} := (h \otimes (h^{-1})^{*2})Y_A^{\otimes n}$$

where $h$ is the automorphism $(1 \otimes X_A^{\otimes n})(\overline{Y}_A^{\otimes n} \otimes 1)$ of $A^n$. Hence, we have

$$\overline{X}_A^{\otimes n}(f^{n^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}Y_A^{\otimes n} = X_A^{\otimes n}(h^* \otimes h^{-1})(f^{n^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(h \otimes (h^{-1})^{*2})Y_A^{\otimes n}$$

$$= X_A^{\otimes n}(1 \otimes h)(1 \otimes 1 \otimes h)(1 \otimes Y_A^{\otimes n} \otimes 1)(1 \otimes h^{-1})(1 \otimes f^{n^*} \Lambda_A^n \theta_{A^n})$$

$$c_{A^n,(A^n)^*}(h \otimes 1)(1 \otimes X_A^{\otimes n} \otimes 1)(Y_A^{\otimes n} \otimes 1 \otimes 1)(h^{-1} \otimes 1)\overline{Y}_A^{\otimes n}$$

$$= X_A^{\otimes n}(1 \otimes h)(1 \otimes h^{-1})(1 \otimes f^{n^*} \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(h \otimes 1)(h^{-1} \otimes 1)\overline{Y}_A^{\otimes n}$$

$$= X_A^{\otimes n}(f^{n^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}Y_A^{\otimes n}$$

\hfill \Box

**Theorem 5.2.** Let $C$ be a semisimple ribbon Ab–category, $A$ an object of $C$ of rank $n$ dominated by a family $(V_i; e_i; \mu_i)_{1 \leq i \leq n}$ of simple objects and $f \in \text{End}_C(A)$. Then, the quantum determinant $\det^C_n(f)$ of $f$, verifies the following

(a) $\det^C_n(f) \in K_C$
(b) \( \det^C_1(V) = 1_I \) where \( V \) is a simple object;

(c) Assume that \( \epsilon_{i\mu_j} = \delta_{i\mu} \) for all \( 1 \leq i; j \leq n \). Then, \( \det^C_n(1_A) = 1_I \);

(d) \( \det^C_n(q \otimes f) = q^n \det^C_n(f) \) for all \( q \in U(K_C) \);

(e) \( \det^C_n(f^*) = \det^C_n(f) \).

Proof. (a) By definition.

(b) Straightforward.

(c) \( \det^C_n(1_A) \)

\[
= X_A^{\otimes n} (1_{(A^*)^n} \otimes \Lambda_A^n \theta_{A^n}) c_{A^n,(A^*)^n} (Y_A^{\otimes n}) \\
= Tr(\Lambda_A^n) \\
= \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n})^{-1} D^1_\sigma \otimes \ldots \otimes \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n m})^{-1} D^m_\sigma \\
= \epsilon(1_{\mathfrak{S}_n}) Tr(C_{\mathfrak{S}_n}^{w_1 n})^{-1} Tr(D^1_{1_{\mathfrak{S}_n}}) \ldots \epsilon(1_{\mathfrak{S}_n}) Tr(C_{\mathfrak{S}_n}^{w_1 n m})^{-1} Tr(D^m_{1_{\mathfrak{S}_n}}) \\
+ \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n m})^{-1} Tr(D^1_{\sigma}) \ldots \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n m})^{-1} Tr(D^m_{\sigma})
\]

(where in the second term of the summand, at least one of \( \sigma \in \mathfrak{S}_n \) is non identity for some \( 1 \leq i \leq n \))

\( = 1_I + 0 \)

\( = 1_I. \)

(d)

\[
\det^C_n(q \otimes f) = X_A^{\otimes n} ((q \otimes f)^n) \otimes \Lambda_A^n \theta_{A^n}) c_{A^n,(A^*)^n} (Y_A^{\otimes n}) \\
= X_A^{\otimes n} (1_{(A^*)^n} \otimes q^n f^n \Lambda_A^n \theta_{A^n}) c_{A^n,(A^*)^n} (Y_A^{\otimes n}) \\
= q^n X_A^{\otimes n} (1_{(A^*)^n} \otimes f^n \Lambda_A^n \theta_{A^n}) c_{A^n,(A^*)^n} (Y_A^{\otimes n}) \\
= q^n \det^C_n(f).
\]

(e) \( V^* \) is dominated by \( (V_i^*)_{1 \leq i \leq n} \) with \( \bar{\mu}_i = \epsilon_i^* \) and \( \bar{\epsilon}_i = \mu_i^* \). Then:

\[
\Lambda_A^n = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n})^{-1} D^1_\sigma \otimes \ldots \otimes \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) Tr(C_{\sigma}^{w_1 n m})^{-1} D^m_\sigma
\]

where

\[
D^i_\sigma = \bar{\mu}_{j_1} \bar{\epsilon}_{\sigma(j_1)} \otimes \ldots \otimes \bar{\mu}_{j_n} \bar{\epsilon}_{\sigma(j_n)}
\]

as in (5.1) and we have
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\[ det_n^C(f^*) = Tr((f^*)^n A_n^*) = Tr(c_{\sigma} Tr(C^{-1}_{w_{n1}})^{(f^*)^n D_1^\sigma} \otimes \ldots \otimes c_{\sigma} Tr(C^{-1}_{w_{nm}})^{(f^*)^n D_m^\sigma}) = Tr(c_{\sigma} Tr((C^{-1}_{w_{n1}})^*(f^*)^n (\mu_{11} E_{\sigma(11)} \otimes \ldots \otimes \mu_{1n_1} E_{\sigma(n_1)}))) = \sum_{c_{\sigma} \in \mathfrak{S}_{n_1}} c_{\sigma} Tr((C^{-1}_{w_{nm}})^* (f^*)^n (\mu_{11} E_{\sigma(11)} \otimes \ldots \otimes \mu_{m_{n_{nm}}}, E_{\sigma(m_{nm})})) \]

... ... \[ = \sum_{c_{\sigma} \in \mathfrak{S}_{n}} c_{\sigma} Tr((C^{-1}_{w_{nm}})^* (f^*)(\mu_{11} E_{\sigma(11)} \otimes \ldots \otimes \mu_{m_{n_{nm}}}, E_{\sigma(m_{nm})}))) = \sum_{c_{\sigma} \in \mathfrak{S}_{n}} c_{\sigma} Tr((C^{-1}_{w_{nm}})^* (f^*)^n (\mu_{11} E_{\sigma(11)} \otimes \ldots \otimes \mu_{m_{n_{nm}}}, E_{\sigma(m_{nm})}))) = \sum_{c_{\sigma} \in \mathfrak{S}_{n}} c_{\sigma} Tr((C^{-1}_{w_{nm}})^* (f^*)^n (\mu_{11} E_{\sigma(11)} \otimes \ldots \otimes \mu_{m_{n_{nm}}}, E_{\sigma(m_{nm})}))) = det_n^C(f). \]

\[ \Box \]

Theorem 5.3. Let \( C \) be a semisimple ribbon Ab–category, \( A \) an object of \( C \) of rank \( n \) dominated by a family \( (V_i; \epsilon_i; \mu_i)_{1 \leq i \leq n} \) of simple objects and \( f \in \text{End}_{C/\mathcal{R}}(A) \). To \( f \), we associate the matrix \( M_f^C = (a_{i,j}^f)_{1 \leq i,j \leq n} \), where

\[ a_{i,j}^f = \begin{cases} \frac{1}{Tr(\epsilon_i f \mu_j) \text{dim}(V_i)^{-1}} & \text{if } V_i \simeq V_j; \\ 0 & \text{else.} \end{cases} \]

Then

1. \( det_n^C(f) = det(M_f^C) \);
2. The map \( \text{End}_{C/\mathcal{R}}(A) \to \mathcal{K}_C, f \mapsto det_n^C(f) \) is multiplicative, i.e. \( det_n^C(fg) = det_n^C(f)det_n^C(g) \); \( \forall g \in \text{End}_{C/\mathcal{R}}(A) \);
3. The map \( \psi : \text{End}_{C/\mathcal{R}}(A) \to M_n(\mathcal{K}_C), f \mapsto M_f^C \) is a monomorphism of \( \mathcal{K}_C \)-algebras;
4. Assume that \( V_i \simeq V, \) for all \( 1 \leq i \leq n \), then \( \text{Tr}(f) = \text{dim}(V)\text{Tr}(M_f^C) \).

Proof. (1) \( M_f^C \) is a block diagonal matrix: \( M_f^C = \text{diag}(M_1, \ldots, M_m) \) where \( M_j = (Tr(\epsilon_l f \mu_k) \text{dim}(V_i)^{-1})_{j_1 \leq l, k \leq j_m} \) \( \forall \ 1 \leq j \leq m \). Then, we have
\[ \text{det}(M_f^C) = \text{det}(M_1) \ldots \text{det}(M_m) \]
\[ = \sum_{\sigma \in \Sigma_{n_1}} \epsilon(\sigma)(\text{dim}(V_1)^{-1})^{n_1} \text{Tr}(\epsilon_{1_1} f \mu \sigma(1_1)) \ldots \text{Tr}(\epsilon_{1_{n_1}} f \mu \sigma(1_{n_1})) \]
\[ = \sum_{\sigma \in \Sigma_{n_m}} \epsilon(\sigma)(\text{dim}(V_m)^{-1})^{n_m} \text{Tr}(\epsilon_{m_1} f \mu \sigma(m_1)) \ldots \text{Tr}(\epsilon_{m_{n_m}} f \mu \sigma(m_{n_m})) \]
\[ = \sum_{\sigma \in \Sigma_{n_1}} \epsilon(\sigma) \text{Tr}(C^{w_{n_1}})^{-1} \text{Tr}(f^{n_1} D_\sigma^1) \ldots \sum_{\sigma \in \Sigma_{n_m}} \epsilon(\sigma) \text{Tr}(C^{w_{n_m}})^{-1} \]
\[ \text{Tr}(f^{n_m} D_{\sigma}^m) \]
\[ = \text{Tr}(f^{n_1} (\sum_{\sigma \in \Sigma_{n_1}} \epsilon(\sigma) \text{Tr}(C^{w_{n_1}})^{-1} D_{\sigma}^1) \ldots \text{Tr}(f^{n_m} (\sum_{\sigma \in \Sigma_{n_m}} \epsilon(\sigma) \text{Tr}(C^{w_{n_m}})^{-1} D_{\sigma}^m))) \]
\[ = \text{det}_n^C(f). \]

(2) We have

\[ (a_{i,j}^{fg})_{i,j} = (\text{Tr}(\epsilon_i f 1_{Ag} \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\text{Tr}(\epsilon_i f \sum_{l=1}^{l=n} \mu_l \epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \ (\text{Negl}(I, I) = \{0\}) \]
\[ = (\sum_{l=1}^{l=n} \text{Tr}(\epsilon_i f \mu_l \epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\sum_{l=1}^{l=n} \text{Tr}(k_{i,l} \otimes 1_V \epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\sum_{l=1}^{l=n} \text{Tr}(k_{i,l} \otimes \epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\sum_{l=1}^{l=n} \text{Tr}(k_{i,l}) \text{Tr}(\epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\sum_{l=1}^{l=n} \text{Tr}(k_{i,l}) \text{dim}(V)^{-1} \text{Tr}(\epsilon_l g \mu_j) \text{dim}(V)^{-1})_{i,j} \]
\[ = (\sum_{l=1}^{l=n} a_{i,l}^f a_{l,j}^g)_{i,j}. \]
where \( k_{i,l} \) is unique in \( K_C \) because \( \varepsilon_i f \mu_l \) is an endomorphism of a simple object \( V \). Then

\[
det_n^C (f) \det_n^C (g) = \det(M_f^C) \det(M_g^C) = \det(M_f^C M_g^C) = \det(M_{fg}^C) = \det_n^C (fg).
\]

(3)

(i) \( \psi \) is a morphism of \( K_C \)-algebras. In fact, linearity is obtained by the fact that for any objects \( V \) and \( W \) of \( C \), the group \( \text{Hom}_C(V; W) \) acquires the structure of a \( K_C \)-module with bilinear composition of morphisms. Furthermore, we have \( \psi(fg) = \psi(f)\psi(g) \) by Theorem 5.3 (2).

(ii) Let \( f \in \text{End}_{C/R}(A) \) such that \( \psi(f) = 0 \). Then, \( Tr(\varepsilon_i f \mu_j) = 0 \) for all \( 1 \leq i, j \leq n \); but \( \varepsilon_i f \mu_j \) is a morphism of a simple object, then, \( \varepsilon_i f \mu_j = k_{i,j} \otimes 1_V \) for some unique \( k_{i,j} \in K_C \). Hence, for all \( 1 \leq i, j \leq n \) we have \( k_{i,j} = 0 \), because \( V \) is simple. Thus \( \mu_i \varepsilon_i f \mu_j \varepsilon_j = 0 \), and so \( \sum_{i,j} \mu_i \varepsilon_i f \mu_j \varepsilon_j = 0 \) (composition with \( \mu_i \) in left and \( \varepsilon_j \) in right, then entering summand). Therefore, \( f = 0 \mod (\mathcal{R}_A, A) \). Thus, \( \psi \) is injective.

(4)

\[
a_{i,i}^f = Tr(\varepsilon_i f \mu_i) \text{dim}(V)^{-1}; \quad 1 \leq i \leq n
\]

\[
\Leftrightarrow Tr(M_f^C) = \sum_{i=1}^{n} Tr(\varepsilon_i f \mu_i) \text{dim}(V)^{-1}
\]

\[
\Leftrightarrow \text{dim}(V) Tr(M_f^C) = \sum_{i=1}^{n} Tr(\mu_i \varepsilon_i f)
\]

\[
\Leftrightarrow \text{dim}(V) Tr(M_f^C) = Tr((\sum_{i=1}^{n} \mu_i \varepsilon_i) f)
\]

\[
\Leftrightarrow \text{dim}(V) Tr(M_f^C) = Tr(f).
\]

\[\square\]

**Remark 5.4.** From the above Theorem 5.3 (2) and under the same hypotheses; naturality of the quantum determinant is then trivial, i.e:

\[
\forall f \in \text{End}_{C/R}(A), \quad \det_n^C (g^{-1} fg) = \det_n^C (f), \quad \forall g \in \text{Aut}_{C/R}(A).
\]
Remark 5.5. We can construct in some cases dominating families of simple objects verifying \( \varepsilon_i \mu_j = \delta_{i,j} \), for all \( i, j, 1 \leq i, j \leq n \). In fact, let \( C \) be a semisimple ribbon Ab–category enriched over finite dimensional vector spaces over a field \( K \) (i.e., for any objects \( V \) and \( W \) of \( C \), \( \text{Hom}_C(V, W) \) is a finite dimensional \( K \– \)vector space) and let \( A \) be an object of \( C \) and \( V \) a simple one. The \( K \– \)vector space \( \text{Hom}_C(V; A) \) is dualizable and its dual is \( \text{Hom}_C(A; V) \); consider a basis \( (\mu_i)_{i=1}^{\dim(V)} \) of \( \text{Hom}_C(V; A) \) (where \( n \) is its dimension over \( K \)) and its dual basis \( (\varepsilon_i)_{i=1}^{\dim(V)} \) of \( \text{Hom}_C(A; V) \). Then, \( A \) is dominated by \( (V; \varepsilon_i; \mu_i)_{1 \leq i, j \leq n} \), \( \text{ran}(A) = n \) and moreover, we have \( \varepsilon_i \mu_j = \delta_{i,j} \), for all \( i, j, 1 \leq i, j \leq n \).

Corollary 5.6. Under the same hypotheses of Theorem 5.3. Assume moreover that \( \varepsilon_i \mu_j = \delta_{i,j} \), for all \( i, j, 1 \leq i, j \leq n \). Then

(a) The map \( \psi : \text{End}_{C/R}(A) \rightarrow M_n(K_C) \); \( f \mapsto M_f \) is an isomorphism of \( K_C \– \)algebras.

(b) \( f \) is invertible in \( C/R \), if and only if \( \text{det}_n^C(f) \) is invertible in \( K_C \).

Proof. (a) By Theorem 5.3 (3); we are just still have to show that \( \psi \) is surjective. Let \( M = (a_{i,j})_{1 \leq i, j \leq n} \) and \( f = \sum_{i,j} a_{i,j} \varepsilon_i \mu_j \). Then \( \text{Tr}(\varepsilon_{i_0} f \mu_{j_0}) \dim(V)^{-1} = a_{i_0, j_0} \), for all \( i_0, j_0, 1 \leq i_0, j_0 \leq n \), and so \( \psi(f) = M \).

(b) Assume that \( f \) is invertible in \( C/R \). Then:

\[
1_f = \text{det}_n^C(1_A)
= \text{det}_n^C(f f^{-1})
= \text{det}_n^C(f) \text{det}_n^C(f^{-1})
= \text{det}_n^C(f^{-1}) \text{det}_n^C(f).
\]

Hence, \( (\text{det}_n^C(f))^{-1} = \text{det}_n^C(f^{-1}) \).

Inversely, if \( \text{det}_n^C(f) \) is invertible, then \( M_f^C \) is invertible, so there exists \( N \in M_n(K_C) \) such that \( M_f^C N = NM_f^C = I_n \), but \( N = \psi(g) \) for some unique \( g \in \text{End}_{C/R}(A) \). Hence

\[
\psi(1_A) = I_n = M_f^C N = \psi(f) \psi(g) = \psi(g f)
\]

and similarly \( \psi(1_A) = \psi(g f) \), then \( f g = g f = 1_A \ mod(\mathcal{R}_{A,A}) \).
Example 5.7. Let $C = (\text{Proj}(R); \otimes_R; R)$ be the category of finitely generated and projective modules over a commutative ring $R$. This is a modular category with simple objects isomorphic to $R$. Let $V$ be a free finitely generated and projective $R$–module with basis $(x_i)_{i=1}^n$. By Corollary 5.6, $\text{ran}_q(V) = n$ and $\text{det}_C^n(f)$, $f \in \text{End}_C(V)$, coincides with its classical determinant, i.e of a representative matrix of $f$.

Example 5.8. This is due to Reshetikhin and Turaev [14]. It deals with the semisimple ribbon $Ab$–category (in fact modular [14]) associated to the Hopf algebra $\overline{U}_q$, i.e, the finite dimensional quotient of the Hopf algebra $U_q(Sl_2(\mathbb{C}))$ for $q$ a root of unity. Moreover, a general principe is given in [14] to construct modular categories upon categories of modules over quantum groups at roots of unity. The objects of $C$ are finite dimensional $\overline{U}_q$–modules and the simple objects are highest weight modules $\{V_\lambda\}_\lambda$ (see [10, 14], for more details). Hence, the quantum determinant of an endomorphism $f$ of an $\overline{U}_q$–module is computed via the associated square matrix of $f$, by Theorem 5.3 (1).

Acknowledgement

The authors gratefully would like to thank the referees for the valuable suggestions and the usefull comments, the thing which enabled us to improve the paper.

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