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On epimorphisms and structurally regular semigroups

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Abstract. In this paper we study epimorphisms, dominions and related properties for some classes of structurally (n, m)-regular semigroups for any pair (n, m) of positive integers. In Section 2, after a brief introduction of these semigroups, we prove that the class of structurallly (n, m)-generalized inverse semigroups is closed under morphic images. We then prove the main result of this section that the class of structurally (n, m)-generalized inverse semigroups is saturated and, thus, in the category of all semigroups, epimorphisms in this class are precisely surjective morphisms. Finally, in the last section, we prove that the variety of structurally (o, n)-left regular bands is saturated in the variety of structurally (o, k)-left regular bands for all positive integers k and n with $1 \leqslant k \leqslant n$.

C Introduction and preliminaries

A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ with $\alpha\beta = \alpha\gamma$ implies

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$\beta = \gamma$. In any category $\mathcal{C}$, one can easily verify that every surjective morphism is epi but the converse is not true in general. In the category of semigroups, there do exist non-surjective epimorphisms; for example, the inclusion $i : (0, 1] \to (0, \infty)$, where both intervals are multiplicative semigroups, is an epimorphism. In semigroups, epimorphisms are studied via dominions whose notion was first introduced by Isbell in [6].

Let U be a subsemigroup of a semigroup S. We say that U dominates an element $d \in S$ if for every semigroup T and all morphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all such elements is called the *dominion* of U in S and is denoted by $\mathrm{Dom}(U, S)$. It is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is said to be *closed* and *epimorphically embedded* in S if $\mathrm{Dom}(U, S) = U$ and $\mathrm{Dom}(U, S) = S$, respectively. A semigroup U is said to be *saturated* if $\mathrm{Dom}(U, S) \neq S$ for every properly containing semigroup S . It can be easily seen that $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi if and only if $Dom(S\alpha, T) = T$.

If $\mathcal{C}$ is a class of semigroups, then every epi from a member of $\mathcal{C}$ is onto if $\mathcal{C}$ is closed under morphic images and each member of $\mathcal{C}$ is saturated. A semigroup U is said to be $\mathcal{C} - saturated$ if for all $S \in \mathcal{C}$ with U as a proper subsemigroup of S, $\mathrm{Dom}(U, S) \neq S$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of semigroups with $\mathcal{C}_1 \subseteq \mathcal{C}_2$. We say that $\mathcal{C}_1$ is $\mathcal{C}_2 - saturated$ if every member of $\mathcal{C}_1$ is $\mathcal{C}_2$-saturated. A class $\mathcal{C}$ of semigroups is said to be saturated if every member of $\mathcal{C}$ is saturated.

Isbell provided the most useful characterization of semigroup dominions which is called as the Isbell's Zigzag Theorem and is as follows:

Theorem C.1. ([5], Theorem 8.3.5) Let U be subsemigroup of a semigroup S and $d \in S$. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there exists a system of equalities for d as follows:

$$\begin{aligned} d &= a_0 y_1 & a_0 &= x_1 a_1 \\ a_1 y_1 &= a_2 y_2 & x_1 a_2 &= x_2 a_3 \\ &\vdots & &\vdots \\ a_{2i-1} y_i &= a_{2i} y_{i+1} & x_i a_{2i} &= x_{i+1} a_{2i+1} \\ a_{2m-1} y_m &= a_{2m} & x_m a_{2m} &= d \end{aligned} \tag{C.1}$$

where $a_i \in U (0 \leqslant i \leqslant 2m)$ and $x_i, y_i \in S (1 \leqslant i \leqslant m)$.

The above system (C.1) of equalities is called as the *zigzag of length* m in S over U with value d. In whatever follows, by zigzag equations, we shall mean a system of equations of type (C.1). We further mention that the bracketed statements shall mean the statements dual to other.

Following results are also useful for our investigation:

Theorem C.2. ([7], Result 3) *Let U be a subsemigroup of a semigroup S. Take any $d \in S \setminus U$ such that $d \in Dom(U, S)$ and let* (1) be a zigzag of minimal length m over U with value d. Then $x_i, y_i \in S \setminus U (1 \leqslant i \leqslant m)$.

Theorem C.3. ([7], Result 4) *Let U be a subsemigroup of a semigroup S and $Dom(U, S) = S$. Then for any $d \in S \setminus U$ and any positive integer k, there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$ $[d = d_k b_k b_{k-1} \cdots b_1]$. In particular, $d \in S^k$ for every positive integer k.*

An element x of a semigroup S is said to be (von Neumann) regular if there exists an (inverse) element y in S such that $xyx = x$ and $yxy = y$ and semigroups consisting entirely of such elements are called *regular*. The set of all inverses of a regular element x is denoted by $V(x)$. An element x of S is said to be an *idempotent* if $x^2 = x$. The set of all idempotent elements of a semigroup S will be denoted by $E(S)$. Semigroups consisting entirely of idempotent elements are called *bands*. A band is called as a *normal band* if it satisfies the identity $axyb = ayxb$. A band is said to be *left [right] regular* if it satisfies the identity $xax = xa$ $[xax = ax]$. *Inverse semigroups* are the regular semigroups with unique inverses, or equivalently, they are regular semigroups with commuting idempotents ([5], Theorem 5.1.1). A *generalised inverse semigroup* is a regular semigroup S whose idempotents form a normal band; that is, $E(S)$ is a subsemigroup of S satisfying the identity $efgh = egfh$ [2].

The following countable family of congruences on a semigroup S was introduced by Samuel J. L. Kopamu in [8]. For each ordered pair (n, m) of non-negative integers, the congruence $\theta(n, m)$ is defined as

$$\theta(n, m) = \{(a, b) : zaw = zbw, \text{ for all } z \in S^n \text{ and } w \in S^m\},$$

where $S^1 = S$ and S^0 denotes the set containing the empty word. In particular,

$$\theta(0,m) = \{(a,b) : av = bv, \text{for all } v \in S^m\},$$

while $\theta(0,0)$ is the identity relation on S.

The concept of structurally regular semigroups was first given by Kopamu in [9]. He gave its characterization, presented some examples and explored its relationship with other known generalizations of the class of regular semigroups. A semigroup S is said to be *structurally regular* if there exists some ordered pair (n,m) of non-negative integers such that $S/\theta(n,m)$ is regular. The class of structurally regular semigroups is much larger than the class of regular semigroups. In fact it is different from each of the following well known extensions of the class of regular semigroups: eventually regular semigroups, locally regular semigroups, nilpotent extensions of regular semigroups and weakly regular semigroups (see [9], for details). Clearly every regular semigroup is structurally (structurally $(0,0)$) regular.

As in [8], for any class $\mathcal{C}$ of semigroups, $H(\mathcal{C})$, $S(\mathcal{C})$, $S_r(\mathcal{C})$, $P_f(\mathcal{C})$ and $P(\mathcal{C})$ denote, respectively, the classes of homomorphic images, subsemigroups, regular subsemigroups, finite direct products and direct products of members of $\mathcal{C}$. A class $\mathcal{C}$ is said to be an *existence variety* or an *e-variety* if it is closed under $H(\mathcal{C})$, $S_r(\mathcal{C})$, and $P_f(\mathcal{C})$. Let $\mathcal{RS}$ denote the e-variety of all regular semigroups. A class $\mathcal{C}$ is said to be a *pseudovariety* if and only if it is closed under $H(\mathcal{C})$, $S(\mathcal{C})$, and $P_f(\mathcal{C})$. Let $\mathcal{F}in$ denote the pseudovariety of all finite semigroups.

For any class $\mathcal{V}$ of regular semigroups, we say that a semigroup S is a *structurally* (n,m)-$\mathcal{V}$ semigroup if $S/\theta(n,m)$ belongs to $\mathcal{V}$. In particular, a semigroup S is said to be *structurally* (n,m)*-generalised inverse [inverse, band]* if $S/\theta(n,m)$ is generalised inverse [inverse, band]. More precisely, for any class $\mathcal{V}$ of semigroups and any $(n,m) \in \mathbb{N}^{\{0\}} \times \mathbb{N}^{\{0\}}$, we define class of semigroups $\mathcal{V}^{(n,m)} = \{S : S/\theta(n,m) \in \mathcal{V}\}$. Now, we have the following result.

Theorem C.4. ([8], Theorem 4.2) *We have the following statements about certain types of varieties of semigroups:*

(a) *If $\mathcal{V}$ is a variety of semigroups, then so is $\mathcal{V}^{(n,m)}$.*

(b) *If $\mathcal{V}$ ia an existence variety of regular semigroups, then the class $\mathcal{RS} \cap \mathcal{V}^{(n,m)}$ is also an existence variety.*

(c) *If $\mathcal{V}$ is a generalized variety, then so is $\mathcal{V}^{(n,m)}$.*

(d) *If $\mathcal{V}$ is pseudovariety, then so is $\mathcal{F}in \cap \mathcal{V}^{(n,m)}$.*

An element a of a semigroup S is said to be an $(n,m)-$*idempotent* if it is $\theta(n,m)$ related to a^2; that is, if $za^2w = zaw$ for all $z \in S^n$ and $w \in S^m$. We denote the set of all (n,m)-idempotents of S by

$$\begin{aligned} E_{(n,m)}(S) &= \{a \in S : (a, a^2) \in \theta(n,m)\} \\ &= \{a \in S : zaw = za^2w \ \forall z \in S^n, w \in S^m\}. \end{aligned}$$

Note that, to say that a is an (n,m)-idempotent in S is equivalent to say that, $a\theta(n,m)$ is idempotent in $S/\theta(n,m)$, so $E_{(n,m)}(S) = E(S/\theta(n,m))$. Also, since every idempotent of S is clearly an (n,m)-idempotent of S, so $E(S) \subseteq E_{(n,m)}(S)$. A semigroup S is *strucuturally* (n,m)-*band* if every element of $S/\theta(n,m)$ is idempotent, that is, every element of S is (n,m)-idempotent.

The next result gives the more useful characterization of structurally regular semigroups.

Theorem C.5. ([9], Theorem 2.1) *Let (n,m) be an ordered pair of non-negative integers. For any semigroup S, $S/\theta(n,m)$ is regular (and hence, S is structurally regular) if and only if for each element a of S, there exists a' in S such that*

$$zaa'aw = zaw \text{ and } za'aa'w = za'w, \text{ for all } z \in S^n \text{ and } w \in S^m.$$

The condition that for each element a there exists b such that $zaw = zabaw$ for all z in S^n and w in S^m implies that there exists an element $a^* = bab$ such that $zaw = zaa^*aw$ and $za^*w = za^*aa^*w$. Therefore the set

$$V_S(a;n,m) = \{a^* \in S : (aa^*a, a), (a^*aa^*, a^*) \in \theta(n,m)\}$$

is non-empty. We refer to each element of the set $V_S(a;n,m)$ as an $(n,m)-$*inverse* of a. Clearly $V(a) \subseteq V_S(a;n,m)$ and S is structurally (n,m)-regular if every element of S has an (n,m)-inverse in S. Note that, if a^* is an (n,m) inverse of a in a semigroup S, then aa^* and a^*a are in $E_{(n,m)}(S)$.

C Epis and structurally (n, m)-generalized inverse semigroups

In [2], Higgins proved that epis are onto for generalized inverse semigroups generalizing the well known result of Howie and Isbell [4] that inverse semigroups were absolutely closed. A corollary of Higgins result showed that normal bands were saturated. This result generalizes the result of Schebilch [11] that normal bands were closed. In this section, we prove that epis are onto for structurally (n, m)-generalized inverse semigroups by showing that such semigroups are saturated, thus extending the above mentioned results of Higgins [2] and Schebilch [11].

Let U and S be any semigroups. Then

$$\theta^S(n,m) = \{(a,b) \in S \times S : zaw = zbw \ \forall z \in S^n, w \in S^m\},$$
$$\theta^U(n,m) = \{(a,b) \in U \times U : zaw = zbw \ \forall z \in U^n, w \in U^m\}.$$

Before proving the main result of this section, we first prove that the class of structurally (n, m)-generalized inverse semigroups is closed under morphic images.

Proposition C.1. *If $\alpha : U \to S$ be an onto homomorphism. Then $\beta : U/\theta^U(n,m) \to S/\theta^S(n,m)$ given by $(u\theta^U(n,m))\beta = (u\alpha)\theta^S(n,m)$ is also an onto homomorphism.*

Proof. Clearly the natural homomorphism $\nu : S \to S/\theta^S(n,m)$ is onto. So, $\alpha\nu : U \to S/\theta^S(n,m)$ is onto. Now, for any $(u, u') \in \theta^U(n,m)$, we have

$$\begin{aligned} & u\theta^U(n,m) = u'\theta^U(n,m) \\ \Rightarrow \ & (u\theta^U(n,m))\beta = (u'\theta^U(n,m))\beta \\ \Rightarrow \ & (u\alpha)\theta^S(n,m) = (u'\alpha)\theta^S(n,m) \\ \Rightarrow \ & u\alpha\nu = u'\alpha\nu. \end{aligned}$$

This implies that $(u, u') \in (\alpha\nu) \circ (\alpha\nu)^{-1}$. Therefore $\theta^U(n,m) \subseteq ker(\alpha\nu)$. Thus, by ([5], Theorem 1.5.3), we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha\nu"] \arrow[d, "\nu|_U"'] & S/\theta^S(n,m) \\ U/\theta^U(n,m) \arrow[ur, "\beta"'] & \end{tikzcd}$$

such that $im\beta = im(\alpha\nu) = S/\theta^S(n,m)$. Hence β is onto, as required. $\square$

From Proposition C.1, we have the following immediate corollaries.

Corollary C.2. *Any morphic image of structurally (n,m)-regular semigroup is structurally (n,m)-regular.*

Proof. The proof follows from Proposition C.1, as homomorphic images of regular semigroups are regular. $\square$

Corollary C.3. *Any morphic image of structurally (n,m)-generalized inverse semigroup is structurally (n,m)-generalized inverse.*

Proof. The proof follows from Propsition C.1, as homomorphic image of generalized inverse semigroup is generalized inverse semigroup ([2], Result 2). $\square$

Following result gives a more useful characterisation of structurally (n,m)-generalized inverse semigroups.

Theorem C.4. *A semigroup S is structurally (n,m)-generalized inverse semigroup if and only if S is structurally (n,m)-regular and for any e,f,g and $h \in E_{(n,m)}(S)$,*

$$zefghw = zegfhw \text{ for all } z \in S^n,\ w \in S^m. \tag{C.1}$$

Proof. Suppose S is structurally (n,m)-generalised inverse semigroup. Then $S/\theta(n,m)$ is generalized inverse and, thus, regular. So, for any $e,f,g,h \in E_{(n,m)}(S)$, we have $e\theta(n,m)$, $f\theta(n,m)$, $g\theta(n,m)$ and $h\theta(n,m)$ are in $E(S/\theta(n,m))$ such that

$$e\theta(n,m)f\theta(n,m)g\theta(n,m)h\theta(n,m) = e\theta(n,m)g\theta(n,m)f\theta(n,m)h\theta(n,m).$$

This implies that $(efgh, egfh) \in \theta(n,m)$ and so

$$zefghw = zegfhw \text{ for all } z \in S^n,\ w \in S^m,$$

as required.

Conversely, if S is structurally (n,m)-regular and satisfies (C.1), then $S/\theta(n,m)$ is clearly generalized inverse semigroup and, so, S is structurally (n,m)-generalized inverse semigroup. $\square$

Also, by ([10], Corollary 2.2), it follows that a semigroup S is structurally (n, m) inverse if and only if S is structurally (n, m)-regular and for any $e, f \in E_{(n,m)}(S)$,

$$zefw = zfew \text{ for all } z \in S^n, w \in S^m.$$

In Lemma C.5, U is a structurally (n, m)-regular proper subsemigroup of a semigroup S such that $\mathrm{Dom}(U, S) = S$. For any semigroup S, $S^{(1)}$ denotes the semigroup S with identity adjoined.

Lemma C.5. *For any $x, y \in S \setminus U$ and $u, v \in U^{(1)}$*

$$xuavy = xuaa^*avy, \text{ and } xua^*vy = xua^*aa^*vy \text{ for all } a \in U.$$

Proof. We outline the proof of only the first equality as the proof of the second equality follows on the similar lines. Since $x, y \in S \setminus U$, by Theorem C.3, we can write

$$x = x^{'}z, \; y = wy^{'} \tag{C.2}$$

for some $x^{'}, y^{'} \in S \setminus U$ and $z \in U^n, w \in U^m$. Now

$$\begin{aligned} xuavy &= x^{'}zuavwy^{'} && \text{(by equalities (C.2))} \\ &= x^{'}zuaa^*avwy^{'} && \text{(since } U \text{ is structurally } (n,m)\text{-regular)} \\ &= xuaa^*avy && \text{(by equalities (C.2))}, \end{aligned}$$

as required. □

In order to prove the main result of this section, we first prove the following lemmas in which U is a proper structurally (n, m)-regular subsemigroup of a semigroup S satisfying

$$zseftw = zsfetw \tag{C.3}$$

for all $z \in U^n, w \in U^m$, $s, t \in S$ and $e, f \in E_{(n,m)}(U)$.

Lemma C.6. *If $a^* \in V_S(a; n, m)$ and $e \in E_{(n,m)}(U)$, then aea^*, $a^*ea \in E_{(n,m)}(U)$.*

Proof. For any $z \in U^n, w \in U^m$, we have

$$\begin{aligned} z(aea^*)^2w &= za(ea^*a)ea^*w \\ &= zaa^*aeea^*w \quad \text{(by Equation (C.3))} \\ &= zaea^*w \qquad \text{(since } U \text{ is structurally } (n,m)\text{-regular).} \end{aligned}$$

Thus $aea^* \in E_{(n,m)}(U)$ and similarly $a^*ea \in E_{(n,m)}(U)$. □

Lemma C.7. *If any $d \in S \setminus U$ has zigzag equations of type* (C.1) *in S over U of length m, then for each $r = 1, 2, \ldots, m-1$,*

$$u_{2r} = a^*_{2r}a_{2r-1}a^*_{2r-2}a_{2r-3}\cdots a^*_2a_1a^*_1a_2a^*_3a_4\cdots a^*_{2r-1}a_{2r} \in E_{(n,m)}(U). \quad \text{(C.4)}$$

Proof. We will prove this by induction on r. For $r = 1$ and for all $z \in U^n, w \in U^m$, we have

$$\begin{aligned} z(u_2)^2w &= z(a^*_2a_1a^*_1a_2)^2w \\ &= za^*_2a_1a^*_1a_2w \text{ (as } a^*_2a_1a^*_1a_2 \in E_{(n,m)}(U) \text{ by Lemma C.6)} \\ &= zu_2w. \end{aligned}$$

Thus the result holds for $r = 1$. Assume inductively that the result holds for $r = k < m-1$; that is, $u_{2k} \in E_{(n,m)}(U)$. We shall show that the result holds for $r = k+1$. Now

$$\begin{aligned} &z(u_{2(k+1)})^2w \\ &= z(a^*_{2k+2}a_{2k+1}a^*_{2k}a_{2k-1}\cdots a^*_2a_1a^*_1a_2a^*_3a_4\cdots a^*_{2k+1}a_{2k+2})^2w \\ &= z(a^*_{2k+2}a_{2k+1}(a^*_{2k}a_{2k-1}\cdots a^*_2a_1a^*_1a_2\cdots a^*_{2k-1}a_{2k})a^*_{2k+1}a_{2k+2})^2w \\ &= z(a^*_{2k+2}a_{2k+1}u_{2k}a^*_{2k+1}a_{2k+2})^2w \\ &= za^*_{2k+2}(a_{2k+1}u_{2k}a^*_{2k+1})(a_{2k+2}a^*_{2k+2})(a_{2k+1}u_{2k}a^*_{2k+1})a_{2k+2}w \\ &= za^*_{2k+2}(a_{2k+2}a^*_{2k+2})(a_{2k+1}u_{2k}a^*_{2k+1})(a_{2k+1}u_{2k}a^*_{2k+1})a_{2k+2}w \\ &\qquad \text{(by (C.3) as } a_{2k+1}u_{2k}a^*_{2k+1} \in E_{(n,m)}(U) \text{ by Lemma C.6} \\ &\qquad\qquad \text{together with inductive hypothesis)} \\ &= z(a^*_{2k+2}a_{2k+2}a^*_{2k+2})(a_{2k+1}u_{2k}a^*_{2k+1})(a_{2k+1}u_{2k}a^*_{2k+1})a_{2k+2}w \\ &= za^*_{2k+2}(a_{2k+1}u_{2k}a^*_{2k+1})^2a_{2k+2}w \\ &\qquad \text{(as } U \text{ is structurally } (n,m)\text{-regular)} \end{aligned}$$

$$\begin{aligned}
&= za_{2k+2}^*(a_{2k+1}u_{2k}a_{2k+1}^*)a_{2k+2}w \quad (\text{as } a_{2k+1}u_{2k}a_{2k+1}^* \in E_{(n,m)}(U) \\
&\qquad\qquad\qquad\qquad\qquad \text{by Lemma C.6 together with inductive hypothesis}) \\
&= z(a_{2k+2}^*a_{2k+1}a_{2k}^*a_{2k-1}\cdots a_2^*a_1a_1^*a_2a_3^*a_4\cdots a_{2k-1}^*a_{2k}a_{2k+1}^*a_{2k+2})w \\
&= zu_{2(k+1)}w.
\end{aligned}$$

Thus the result also holds for $r = k + 1$ and, so by induction, the result follows. □

Lemma C.8. *If any $d \in S \setminus U$ has zigzag equations of type* (1) *in S over U of length m and $u_{2r} = a_{2r}^*a_{2r-1}a_{2r-2}^*a_{2r-3}\cdots a_2^*a_1a_1^*a_2a_3^*a_4\cdots a_{2r-1}^*a_{2r}$ $\in E_{(n,m)}(U)$ $(1 \leqslant r \leqslant m-1)$, then the following are in $E_{(n,m)}(U)$:*

(i) $v_{2j} = a_{2j}^*a_{2j-1}u_{2j-2}a_{2j-1}^*a_{2j}$ $(2 \leqslant j \leqslant m)$*;*

(ii) $w_{2j} = a_{2j+1}u_{2j}a_{2j+1}^*$ $(1 \leqslant j \leqslant m-1)$.

Proof. We will prove the lemma for case (i); the other case follows by applying Lemma C.7 and Lemma C.6, respectively. We prove it by induction on j. For $j = 2$ and for all $z \in U^n, w \in U^m$, we have

$$\begin{aligned}
z(v_4)^2w &= z(a_4^*a_3u_2a_3^*a_4)^2w \\
&= z(a_4^*a_3u_2a_3^*a_4)(a_4^*a_3u_2a_3^*a_4)w \\
&= za_4^*(a_3u_2a_3^*)(a_4a_4^*)(a_3u_2a_3^*)a_4w \\
&= za_4^*(a_4a_4^*)(a_3u_2a_3^*)(a_3u_2a_3^*)a_4w \quad (\text{by (C.3) as } a_4a_4^*, a_3u_2a_3^* \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in E_{(n,m)}(U) \text{ by Lemma C.6}) \\
&= z(a_4^*a_4a_4^*)(a_3u_2a_3^*)(a_3u_2a_3^*)a_4w \\
&= za_4^*(a_3u_2a_3^*)a(a_3u_2a_3^*)a_4w \ (\text{as } U \text{ is structurally } (n,m)\text{-regular}) \\
&= za_4^*(a_3u_2a_3^*)^2a_4w \\
&= za_4^*(a_3u_2a_3^*)a_4w \quad (\text{as } a_3u_2a_3^* \in E_{(n,m)}(U) \text{ by Lemma C.6}) \\
&= z(a_4^*a_3u_2a_3^*a_4)w \\
&= zv_4w.
\end{aligned}$$

Thus the result is true for $j = 2$. Assume that the result is true for $j = k$ $(2 \leqslant k < m)$. Then, we have

$$v_{2k} = a_{2k}^*a_{2k-1}u_{2k-2}a_{2k-1}^*a_{2k} \in E_{(n,m)}(U).$$

From this, we show that the result also holds for $j = k+1$. Now

$$
\begin{aligned}
&z(v_{2(k+1)})^2w\\
&= z(a^*_{2k+2}a_{2k+1}u_{2k}a^*_{2k+1}a_{2k+2})^2w\\
&= z(a^*_{2k+2}a_{2k+1}(a^*_{2k}a_{2k-1}\cdots a^*_2a_1a^*_1a_2\cdots a^*_{2k-1}a_{2k})a^*_{2k+1}a_{2k+2})^2w\\
&\qquad\qquad\qquad\qquad(\text{as } u_{2k} = a^*_{2k}a_{2k-1}\cdots a^*_2a_1a^*_1a_2\cdots a^*_{2k-1}a_{2k})\\
&= z(a^*_{2k+2}a_{2k+1}(a^*_{2k}a_{2k-1}u_{2k-2}a^*_{2k-1}a_{2k})a^*_{2k+1}a_{2k+2})^2w\\
&\qquad\qquad\qquad(\text{as } u_{2k-2} = a^*_{2k-2}a_{2k-3}\cdots a^*_2a_1a^*_1a_2\cdots a^*_{2k-3}a_{2k-2})\\
&= z(a^*_{2k+2}a_{2k+1}v_{2k}a^*_{2k+1}a_{2k+2})^2w\\
&= za^*_{2k+2}(a_{2k+1}v_{2k}a^*_{2k+1})(a_{2k+2}a^*_{2k+2})(a_{2k+1}v_{2k}a^*_{2k+1})a_{2k+2}w\\
&= za^*_{2k+2}(a_{2k+2}a^*_{2k+2})(a_{2k+1}v_{2k}a^*_{2k+1})(a_{2k+1}v_{2k}a^*_{2k+1})a_{2k+2}w\\
&\qquad\qquad(\text{by (C.3) as } a_{2k+1}v_{2k}a^*_{2k+1} \in E_{(n,m)}(U) \text{ by Lemma C.6}\\
&\qquad\qquad\qquad\qquad\text{together with inductive hypothsesis})\\
&= z(a^*_{2k+2}a_{2k+2}a^*_{2k+2})(a_{2k+1}v_{2k}a^*_{2k+1})(a_{2k+1}v_{2k}a^*_{2k+1})a_{2k+2}w\\
&= za^*_{2k+2}(a_{2k+1}v_{2k}a^*_{2k+1})^2a_{2k+2}w \quad (\text{as } U \text{ is structurally } (n,m)\text{-regular})\\
&= za^*_{2k+2}(a_{2k+1}v_{2k}a^*_{2k+1})a_{2k+2}w \quad (\text{as } a_{2k+1}v_{2k}a^*_{2k+1} \in E_{(n,m)}(U) \text{ by}\\
&\qquad\qquad\qquad\qquad\text{Lemma C.6 together with inductive hypothsesis})\\
&= za^*_{2k+2}a_{2k+1}(a^*_{2k}a_{2k-1}u_{2k-2}a^*_{2k-1}a_{2k})a^*_{2k+1}a_{2k+2}w\\
&= za^*_{2k+2}a_{2k+1}(a^*_{2k}a_{2k-1}a^*_{2k-2}a_{2k-3}\cdots a^*_2a_1a^*_1a_2\cdots a^*_{2k-3}a_{2k-2}a^*_{2k-1}a_{2k})\\
&\qquad a^*_{2k+1}a_{2k+2}w\\
&= z(a^*_{2k+2}a_{2k+1}u_{2k}a^*_{2k+1}a_{2k+2})w\\
&= zv_{2(k+1)}w.
\end{aligned}
$$

Therefore, the result also holds for $j = k+1$ and, so by induction, the result follows. □

Lemma C.9. *If $Dom(U,S) = S$, then for any $x, y \in S \setminus U$ and for any e, f in $E_{(n,m)}(U)$,*

$$xuefy = xufey\ [xefvy = xfevy] \textit{ for all } u, v \in U.$$

Proof. Suppose that $\mathrm{Dom}(U,S) = S$. Since $x, y \in S \setminus U$, by Theorem C.3, we can write x, y as

$$x = \bar{x}zb, \quad y = cw\bar{y}, \tag{C.5}$$

for some $z \in U^n, w \in U^m, b, c \in U$ and $\bar{x}, \bar{y} \in S \setminus U$. Now

$$\begin{aligned} xuefy &= \bar{x}zbuefcw\bar{y} \quad \text{(by equalities (C.5))} \\ &= \bar{x}zbufecw\bar{y} \quad \text{(by Equation (C.3))} \\ &= xufey. \quad \text{(by equalities (C.5))} \end{aligned}$$

□

Theorem C.10. *Let U be a structurally (n, m)-regular proper subsemigroup of a semigroup S such that $zseftw = zsfetw$ for all $z \in U^n, w \in U^m$, $s, t \in S$ and $e, f \in E_{(n,m)}(U)$. Then $Dom(U, S) \neq S$.*

Proof. Suppose on the contrary that $\mathrm{Dom}(U, S) = S$. Take any $d \in S \setminus U$. Then by Theorem C.1, there exists a zigzag of type (C.1) in S over U with value d of minimum length m. In order to prove the theorem, we first prove the following lemma.

Lemma C.11. *For each $k = 1, 2, \ldots, m$, $d = (\prod_{i=0}^{k-1} a_{2i}a_{2i+1}^*)a_{2k-1}y_k$.*

Proof. We will prove this by induction on k. For $k = 1$, we have

$$\begin{aligned} d &= a_0y_1 \\ &= x_1a_1y_1 \quad \text{(by zigzag equations)} \\ &= x_1a_1a_1^*a_1y_1 \quad \text{(by Lemma C.5 as } x_1, y_1 \in S \setminus U) \\ &= a_0a_1^*a_1y_1 \quad \text{(by zigzag equations).} \end{aligned}$$

Thus the result is true for $k = 1$. Assume, inductively that the result is true for $k = j$ for $1 < j < m$. We now show that the result also holds for $k = j + 1$. For this, we have

$$
\begin{aligned}
d &= \Big(\prod_{i=0}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \quad \text{(by inductive hypothesis)}\\
&= a_0a_1^*a_2a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j\\
&= x_1a_1a_1^*a_2a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \quad \text{(by zigzag equations)}\\
&= x_1a_1a_1^*a_2a_2^*a_2a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \text{(by Lemma C.5 as } x_1, y_j \in S\setminus U)\\
&= x_1a_2a_2^*a_1a_1^*a_2a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \quad \text{(by Lemma C.9 as } x_1, y_j \in\\
&\qquad\qquad\qquad\qquad\qquad\qquad S\setminus U \text{ and } a_1a_1^*, a_2a_2^* \in E_{(n,m)}(U))\\
&= x_2a_3(a_2^*a_1a_1^*a_2)a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \text{ (by zigzag equations)}\\
&= x_2a_3u_2a_3^*\Big(\prod_{i=2}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \quad (\text{as } u_2 = a_2^*a_1a_1^*a_2)\\
&= x_2(a_3u_2a_3^*)a_4a_5^*\Big(\prod_{i=3}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j\\
&= x_2w_2a_4a_4^*a_4a_5^*\Big(\prod_{i=3}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \text{ (by Lemma C.5 as } x_2, y_j \in S\setminus U)\\
&= x_2a_4a_4^*w_2a_4a_5^*\Big(\prod_{i=3}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \quad \text{(by Lemma C.9 as } x_2, y_j \in\\
&\qquad\qquad\qquad\qquad\qquad\qquad S\setminus U \text{ and } a_4a_4^*, w_2 \in E_{(n,m)}(U))\\
&= x_3a_5(a_4^*a_3u_2a_3^*a_4)a_5^*\Big(\prod_{i=3}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \text{ (by zigzag equations)}\\
&= x_3a_5v_4a_5^*\Big(\prod_{i=3}^{j-1} a_{2i}a_{2i+1}^*\Big)a_{2j-1}y_j \text{ (by Lemma C.8 (i))}\\
&\;\;\vdots\\
&= x_ja_{2j-1}v_{2j-2}a_{2j-1}^*a_{2j-1}y_j\\
&= x_ja_{2j-1}v_{2j-2}a_{2j-1}^*a_{2j}y_{j+1} \quad \text{(by zigzag equations)}\\
&= x_ja_{2j-1}(a_{2j-2}^*a_{2j-3}u_{2j-4}a_{2j-3}^*a_{2j-2})a_{2j-1}^*a_{2j}y_{j+1} \quad \text{(by Lemma C.8 (i))}
\end{aligned}
$$

$$
\begin{aligned}
&= x_j a_{2j-1}(a^*_{2j-2}a_{2j-3}a^*_{2j-4}a_{2j-5}\cdots a_1a^*_1\cdots a^*_{2j-5}a_{2j-4}a^*_{2j-3}a_{2j-2})\\
&a^*_{2j-1}a_{2j}y_{j+1}\\
&= x_j(a_{2j-1}u_{2j-2}a^*_{2j-1})a_{2j}y_{j+1} \quad \text{(by Lemma C.7)}\\
&= x_jw_{2j-2}a_{2j}y_{j+1} \quad \text{(by Lemma C.8 (ii))}\\
&= x_jw_{2j-2}a_{2j}a^*_{2j}a_{2j}y_{j+1} \quad \text{(by Lemma C.5 as } x_j, y_{j+1} \in S\setminus U)\\
&= x_ja_{2j}a^*_{2j}w_{2j-2}a_{2j}y_{j+1} \quad \text{(by Lemma C.9 as } x_j, y_{j+1} \in S\setminus U \text{ and}\\
&\qquad\qquad w_{2j-2},\ a_{2j}a^*_{2j} \in E_{(n,m)}(U))\\
&= x_{j+1}a_{2j+1}a^*_{2j}w_{2j-2}a_{2j}y_{j+1} \quad \text{(by zigzag equations)}\\
&= x_{j+1}a_{2j+1}(a^*_{2j}a_{2j-1}u_{2j-2}a^*_{2j-1}a_{2j})y_{j+1} \quad \text{(by Lemma C.8 (ii))}\\
&= x_{j+1}a_{2j+1}v_{2j}y_{j+1} \quad \text{(by Lemma C.8 (i))}\\
&= x_{j+1}a_{2j+1}a^*_{2j+1}a_{2j+1}v_{2j}y_{j+1} \quad \text{(by Lemma C.5 as } x_{j+1}, y_{j+1} \in S\setminus U)\\
&= x_{j+1}a_{2j+1}v_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.9 as } x_{j+1}, y_{j+1} \in S\setminus U\\
&\qquad\qquad \text{and } v_{2j},\ a^*_{2j+1}a_{2j+1} \in E_{(n,m)}(U))\\
&= x_ja_{2j}v_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by zigzag equations)}\\
&= x_ja_{2j}a^*_{2j}(a_{2j-1}u_{2j-2}a^*_{2j-1})a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.8 (i))}\\
&= x_ja_{2j}a^*_{2j}w_{2j-2}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.8 (ii))}\\
&= x_jw_{2j-2}a_{2j}a^*_{2j}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.9 as } x_j, y_{j+1} \in S\setminus U\\
&\qquad\qquad \text{and } w_{2j-2}, a_{2j}a^*_{2j} \in E_{(n,m)}(U))\\
&= x_jw_{2j-2}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.5 as } x_j, y_{j+1} \in S\setminus U)\\
&= x_ja_{2j-1}u_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.8 (ii))}\\
&= x_{j-1}a_{2j-2}u_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by zigzag equations)}\\
&= x_{j-1}a_{2j-2}a^*_{2j-2}a_{2j-3}\cdots a_1a^*_1\cdots a^*_{2j-3}a_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1}\\
&= x_{j-1}a_{2j-2}a^*_{2j-2}(a_{2j-3}u_{2j-4}a^*_{2j-3})a_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1}\\
&= x_{j-1}a_{2j-2}a^*_{2j-2}w_{2j-4}a_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1}\\
&= x_{j-1}w_{2j-4}(a_{2j-2}a^*_{2j-2}a_{2j-2})a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \quad \text{(by Lemma C.9}\\
&\qquad\qquad \text{as } x_{j-1}, y_{j+1} \in S\setminus U \text{ and } a_{2j-2}a^*_{2j-2}, w_{2j-4} \in E_{(n,m)}(U))\\
&= x_{j-1}w_{2j-4}a_{2j-2}a^*_{2j-1}a_{2j}a^*_{2j+1}a_{2j+1}y_{j+1} \text{ (by Lemma C.5 as } x_{j-1}, y_{j+1} \in S\setminus U)\\
&= x_{j-1}a_{2j-3}u_{2j-4}a^*_{2j-3}(\prod_{i=j-1}^{j} a_{2i}a^*_{2i+1})a_{2j+1}y_{j+1}\\
&\vdots\\
&= x_2a_3u_2a^*_3(\prod_{i=2}^{j} a_{2i}a^*_{2i+1})a_{2j+1}y_{j+1}\\
&= x_1a_2u_2a^*_3(\prod_{i=2}^{j} a_{2i}a^*_{2i+1})a_{2j+1}y_{j+1} \quad \text{(by zigzag equations)}
\end{aligned}
$$

$$= x_1a_2a_2^*a_1a_1^*a_2a_3^*(\prod_{i=2}^{j} a_{2i}a_{2i+1}^*)a_{2j+1}y_{j+1} \quad (\text{as } u_2 = a_2^*a_1a_1^*a_2)$$

$$= x_1a_1a_1^*(a_2a_2^*a_2)a_3^*(\prod_{i=2}^{j} a_{2i}a_{2i+1}^*)a_{2j+1}y_{j+1} \quad (\text{by Lemma C.9 as } x_1,\ y_{j+1} \in S\setminus U \text{ and } a_1a_1^*, a_2a_2^* \in E_{(n,m)}(U))$$

$$= x_1a_1a_1^*(\prod_{i=1}^{j} a_{2i}a_{2i+1}^*)a_{2j+1}y_{j+1} \quad (\text{by Lemma C.5 as } x_1, y_{j+1} \in S\setminus U)$$

$$= a_0a_1^*(\prod_{i=1}^{j} a_{2i}a_{2i+1}^*)a_{2j+1}y_{j+1} \quad (\text{by zigzag equations})$$

$$= (\prod_{i=0}^{j} a_{2i}a_{2i+1}^*)a_{2j+1}y_{j+1}.$$

This shows that the result also holds for $k = j+1$. Hence by induction the lemma follows. □

Now to complete the proof of the theorem, letting $k = m$ in Lemma C.11, we have

$$d = (\prod_{i=0}^{m-1} a_{2i}a_{2i+1}^*)a_{2m-1}y_m$$
$$= (\prod_{i=0}^{m-1} a_{2i}a_{2i+1}^*)a_{2m} \quad (\text{by zigzag equations}).$$

Thus $d \in U$, a contradiction. Hence $\mathrm{Dom}(U,S) \neq S$, as required. □

Theorem C.12. *For any pair (n,m) of positive integers, the class $\mathcal{V}^{(n,m)}$ of structurally (n,m)-generalized inverse semigroups is saturated.*

Proof. Suppose that $U \in \mathcal{V}^{(n,m)}$ and S be any semigroup containing U properly such that $\mathrm{Dom}(U,S) = S$. Let $s,t \in S$ and $e,f \in E_(n,m)(U)$. If

$s, t \in U$, then for all $z \in U^n, w \in U^m$, we have

$$\begin{aligned} zseftw &= zss^*seftw \\ &\qquad \text{(as } eftw \in U^m \text{ and } U \text{ is structurally } (n,m)\text{-regular)} \\ &= zss^*seftt^*tw \\ &\qquad \text{(as } ss^*sef \in U^n \text{ and } U \text{ is structurally } (n,m)\text{-regular)} \\ &= zss^*sfett^*tw \\ &\qquad \text{(as } U \text{ is structurally } (n,m)\text{-generalised inverse)} \\ &= zsfetw \quad \text{(as } U \text{ is structurally } (n,m)\text{-regular).} \end{aligned}$$

Now, if $s \in S \setminus U$, then by the Theorem C.1, there exist $a \in U$, $x \in S$ such that $s = xa$. Similarly if $t \in T \setminus U$, then, again, by the Theorem C.1, there exist $b \in U$, $y \in S$ such that $t = by$. By Theorems C.2 and C.3, we can have $x = \bar{x}u$ and $y = v\bar{y}$ for some $u \in U^n$ and $v \in U^m$. Then, we have

$$\begin{aligned} zseftw &= z(xa)ef(byw) \\ &= z(xa)ef(by)w \\ &= z\bar{x}u(aefb)v\bar{y}w \\ &= z\bar{x}u(aa^*aefbb^*b)v\bar{y}w \quad \text{(as } U \text{ is structurally } (n,m)\text{-regular)} \\ &= z\bar{x}ua(a^*aefbb^*)bv\bar{y}w \\ &= z\bar{x}ua(a^*afebb^*)bv\bar{y}w \\ &\qquad \text{(as } U \text{ is structurally } (n,m)\text{-generalised inverse)} \\ &= z\bar{x}uafebv\bar{y}w \quad \text{(as } U \text{ is structurally } (n,m)\text{-regular)} \\ &= (xa)fe(by) \\ &= zsfetw. \end{aligned}$$

Thus $zseftw = zsfetw$ for all $z \in U^n, w \in U^m$ and $s, t \in S$. Therefore, by Theorem C.10, we deduce that $\mathrm{Dom}(U, S) \neq S$ for all S containing U as a proper subsemigroup. Hence U is saturated, as required. □

One can easily see that for any pair (n, m) of positive integers, structurally (n, m)-inverse semigroup is also structurally (n, m)-generalised inverse semigroup. So, we have the following corollaries.

Corollary C.13. *The class $\mathcal{V}^{(n,m)}$ of structurally (n, m)-inverse semigroups is saturated for each pair (n, m) of positive integers.*

Corollary C.14. *The class $\mathcal{V}^{(n,m)}$ of structurally (n,m)-normal bands is saturated for each pair (n,m) of positive integers.*

Corollary C.15. *In the category of all semigroups, for each pair (n,m) of positive integers, any epi from a structurally (n,m)-generalised inverse semigroup is surjective.*

C Epis and structurally $(0,k)$-bands

In [1], Alam and Khan proved that the variety of left [right] regular bands is closed. In this section, we show that the varieties of structurally (o,n)-left regular bands are saturated in the varieties of structurally (o,k)-left regular bands for any k and n with $1 \leqslant k \leqslant n$. This partially generalizes the result of Alam and Khan and Corollary C.14 of previous section.

One can easily see that for each k and n with $1 \leqslant k \leqslant n$, the class of structurally $(0,n)$ semigroups is contained in the class of structurally $(0,k)$ semigroups.

A semigroup S is said to be structurally $(0,k)$-band, if $S/\theta(0,k)$ is a band; that is, for any a in S, we have

$$aw = a^2w \text{ for all } w \in S^k.$$

A structurally $(0,k)$-band B is said to be structurally $(0,k)$-left regular band, if $B/\theta(0,k)$ is left regular band; that is, for any $a,x \in S$, we have

$$xaw = xaxw \text{ for all } w \in B^k.$$

Dually, a structurally $(k,0)$-right regular band may be defined.

Note that, by Theorem C.4, the class $\mathcal{V}^{(0,n)}$ of structurally $(0,n)$-left regular bands is a variety for each positive integer n. Also $\mathcal{V}^{(0,n)} \subseteq \mathcal{V}^{(0,k)}$ for each positive integers k and n.

In the following lemma, U and S are any $(0,n)$-bands with U as a proper subband of S such that $\mathrm{Dom}(U,S) = S$.

Lemma C.1. *For any $x,y \in S \setminus U$ and $a \in U$, $xay = xa^2y$.*

Proof. Since $x, y \in S \setminus U$, by Theorem C.3, we may write y as

$$y = w\bar{y} \tag{C.1}$$

for some $w \in U^n$ and $\bar{y} \in S \setminus U$. Now

$$\begin{aligned} xay &= xaw\bar{y} \quad \text{(by Equation (C.1))} \\ &= xaaw\bar{y} \quad \text{(as } U \text{ is structurally } (0,n)\text{-band)} \\ &= xa^2y \qquad \text{(by Equation (C.1))}, \end{aligned}$$

as required. □

In the next lemma, U is a structurally $(0, n)$-left regular proper subband of a structurally $(0, k)$-left regular band S such that $\mathrm{Dom}(U, S) = S$.

Lemma C.2. *Let $x, y \in S \setminus U$ and let $u \in U$. Then*

$$xauy = xaxuy \text{ for all } a \in U.$$

Proof. Since $y \in S \setminus U$, by Theorem C.3, we have

$$y = w\bar{y} \tag{C.2}$$

for some $w \in U^k$ and $\bar{y} \in S \setminus U$. Now

$$\begin{aligned} xauy &= xauw\bar{y} \quad \text{(by Equation (C.2))} \\ &= xaxuw\bar{y} \quad \text{(as } S \text{ is structurally } (0,k)\text{-left regular band)} \\ &= xaxuy, \end{aligned}$$

as required. □

Theorem C.3. *For each positive integers k and n with $1 \leqslant k \leqslant n$ the variety $\mathcal{V}^{(0,n)}$ of structurally (o, n)-left regular bands is saturated in the variety $\mathcal{V}^{(0,k)}$ of structurally (o, k)-left regular bands.*

Proof. Suppose on the contrary that the variety of structurally (o, n)-left regular bands is not saturated in the variety of structurally (o, k)-left regular bands, $(k \leqslant n)$. So, there exists a structurally (o, n)-left regular band U and a structurally (o, k)-left regular band S containing U as a subband and such that $\mathrm{Dom}(U, S) = S$. Take any $d \in S \setminus U$, then by Theorem C.1, d has a zigzag of type (C.1) in S over U of minimum length m. In order to prove the theorem we first prove the following lemma.

Lemma C.4. *For each $k = 1, 2, \dots, m$, $d = (\prod_{i=0}^{k-1} a_{2i})a_{2k-1}y_k$.*

Proof. We will prove this by induction on k. For $k = 1$, we have

$$\begin{aligned} d &= a_0y_1 \\ &= x_1a_1y_1 \quad \text{(by zigzag equations)} \\ &= x_1a_1a_1y_1 \quad \text{(by Lemma C.1 as } x_1, y_1 \in S \setminus U) \\ &= a_0a_1y_1 \quad \text{(by zigzag equations).} \end{aligned}$$

Thus the result holds for $k = 1$. Assume, inductively that the result holds for $k = j$ with $1 < j < m$. We now show that the result also holds for $k = j + 1$. For this, we have

$$\begin{aligned} d &= (\prod_{i=0}^{j-1} a_{2i})a_{2j-1}y_j \quad \text{(by inductive hypothesis)} \\ &= x_1a_1a_2(\prod_{i=2}^{j-1} a_{2i})a_{2j}y_{j+1} \quad \text{(by zigzag equations)} \\ &= x_1a_1x_1a_2(\prod_{i=2}^{j-1} a_{2i})a_{2j}y_{j+1} \quad \text{(by Lemma C.2 as } x_1, y_{j+1} \in S \setminus U) \\ &= x_1a_1x_2a_3(\prod_{i=2}^{j-1} a_{2i})a_{2j}y_{j+1} \quad \text{(by zigzag equations)} \\ &= (\prod_{i=1}^{2} x_ia_{2i-1})a_4(\prod_{i=3}^{j-1} a_{2i})a_{2j}y_{j+1} \\ &= (\prod_{i=1}^{2} x_ia_{2i-1})x_2a_4(\prod_{i=3}^{j-1} a_{2i})a_{2j}y_{j+1} \\ &\qquad\qquad \text{(by Lemma C.2 as } x_2, y_{j+1} \in S \setminus U) \\ &= (\prod_{i=1}^{2} x_ia_{2i-1})x_3a_5(\prod_{i=3}^{j-1} a_{2i})a_{2j}y_{j+1} \quad \text{(by zigzag equations)} \end{aligned}$$

$$= (\prod_{i=1}^{3} x_i a_{2i-1})(\prod_{i=3}^{j-1} a_{2i}) a_{2j} y_{j+1}$$

$$\vdots$$

$$= (\prod_{i=1}^{j} x_i a_{2i-1}) a_{2j} y_{j+1}$$

$$= (\prod_{i=1}^{j} x_i a_{2i-1}) x_j a_{2j} y_{j+1} \quad (\text{by Lemma C.2 as } x_j, y_{j+1} \in S \setminus U)$$

$$= (\prod_{i=1}^{j} x_i a_{2i-1}) x_{j+1} a_{2j+1} y_{j+1} \quad (\text{by zigzag equations})$$

$$= (\prod_{i=1}^{j} x_i a_{2i-1}) x_{j+1} a_{2j+1} a_{2j+1} y_{j+1}$$

$$(\text{by Lemma C.1 as } x_{j+1}, y_{j+1} \in S \setminus U)$$

$$= (\prod_{i=1}^{j} x_i a_{2i-1}) x_j a_{2j} a_{2j+1} y_{j+1} \quad (\text{by zigzag equations})$$

$$= (\prod_{i=1}^{j-1} x_i a_{2i-1}) x_j a_{2j-1} a_{2j} a_{2j+1} y_{j+1}$$

$$(\text{by Lemma C.2 as } x_j, y_{j+1} \in S \setminus U)$$

$$= (\prod_{i=1}^{j-1} x_i a_{2i-1}) x_{j-1} a_{2j-2} a_{2j} a_{2j+1} y_{j+1} \quad (\text{by zigzag equations})$$

$$= (\prod_{i=1}^{j-1} x_i a_{2i-1})(\prod_{i=j-1}^{j} a_{2i}) a_{2j+1} y_{j+1}$$

$$(\text{by Lemma C.2 as } x_j, y_{j+1} \in S \setminus U)$$

$$= (\prod_{i=1}^{j-2} x_i a_{2i-1}) x_{j-1} a_{2j-3} (\prod_{i=j-1}^{j} a_{2i}) a_{2j+1} y_{j+1}$$

$$= (\prod_{i=1}^{j-2} x_i a_{2i-1}) x_{j-2} a_{2j-4} (\prod_{i=j-1}^{j} a_{2i}) a_{2j+1} y_{j+1} \quad (\text{by zigzag equations})$$

$$= (\prod_{i=1}^{j-2} x_i a_{2i-1})(\prod_{i=j-2}^{j} a_{2i}) a_{2j+1} y_{j+1}$$

(by Lemma C.2 as $x_{j-1}, y_{j+1} \in S \setminus U$)

$$\vdots$$

$$= (\prod_{i=1}^{2} x_i a_{2i-1})(\prod_{i=2}^{j} a_{2i}) a_{2j+1} y_{j+1}$$

$$= x_1 a_1 x_2 a_3 (\prod_{i=2}^{j} a_{2i}) a_{2j+1} y_{j+1}$$

$$= x_1 a_1 x_1 a_2 (\prod_{i=2}^{j} a_{2i}) a_{2j+1} y_{j+1} \quad \text{(by zigzag equations)}$$

$$= x_1 a_1 a_2 (\prod_{i=2}^{j} a_{2i}) a_{2j+1} y_{j+1} \quad \text{(by Lemma C.2 as } x_1, y_{j+1} \in S \setminus U)$$

$$= a_0 a_2 (\prod_{i=2}^{j} a_{2i}) a_{2j+1} y_{j+1} \quad \text{(by zigzag equations)}$$

$$= (\prod_{i=0}^{j} a_{2i}) a_{2j+1} y_{j+1}.$$

This shows that the result also holds for $k = j + 1$. Hence by induction the lemma follows. □

Now to complete the proof of the theorem, letting $k = m$ in Lemma C.4, we have

$$\begin{aligned} d &= (\prod_{i=0}^{m-1} a_{2i}) a_{2m-1} y_m \\ &= (\prod_{i=0}^{m-1} a_{2i}) a_{2m} \quad \text{(by zigzag equations)} \\ &= \prod_{i=0}^{m} a_{2i}. \end{aligned}$$

Thus $d \in U$, a contradiction. This completes the proof of the theorem. $\square$

Dually, we may prove the following:

Theorem C.5. *For each positive integers k and n with $1 \leqslant k \leqslant n$. The variety $\mathcal{V}^{(n,0)}$ of structurally $(n,0)$-right regular bands is saturated in the variety $\mathcal{V}^{(k,0)}$ of structurally $(k,0)$-right regular bands.*

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