Pre-image of functions in $C(L)$

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Abstract. Let $C(L)$ be the ring of all continuous real functions on a frame $L$ and $S \subseteq \mathbb{R}$. An $\alpha \in C(L)$ is said to be an overlap of $S$, denoted by $\alpha \triangleright S$, whenever $u \cap S \subseteq v \cap S$ implies $\alpha(u) \leq \alpha(v)$ for every open sets $u$ and $v$ in $\mathbb{R}$. This concept was first introduced by A. Karimi-Feizabadi, A.A. Estaji, M. Robat-Sarpoushi in *Pointfree version of image of real-valued continuous functions* (2018). Although this concept is a suitable model for their purpose, it ultimately does not provide a clear definition of the range of continuous functions in the context of pointfree topology. In this paper, we will introduce a concept which is called pre-image, denoted by $\text{pim}$, as a pointfree version of the image of real-valued continuous functions on a topological space $X$. We investigate this concept and in addition to showing $\text{pim}(\alpha) = \bigcap\{S \subseteq \mathbb{R} : \alpha \triangleright S\}$, we will see that this concept is a good surrogate for the image of continuous real functions. For instance, we prove, under some achievable conditions, we have $\text{pim}(\alpha \lor \beta) \subseteq \text{pim}(\alpha) \lor \text{pim}(\beta)$, $\text{pim}(\alpha \land \beta) \subseteq \text{pim}(\alpha) \land \text{pim}(\beta)$, $\text{pim}(\alpha\beta) \subseteq \text{pim}(\alpha)\text{pim}(\beta)$ and $\text{pim}(\alpha + \beta) \subseteq \text{pim}(\alpha) + \text{pim}(\beta)$.

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1 Introduction and preliminaries

A complete lattice $L$ is said to be a frame if for any $a \in L$ and $B \subseteq L$, we have $a \land \bigvee B = \bigvee_{b \in B} (a \land b)$. We denote the top element and the bottom element of a frame $L$ by $\textbf{Top}$ and $\bot$, respectively. For every element $a$ of a frame $L$ the pseudocomplement of $a$ is $a^c = \bigvee \{x \in L : x \land a = \bot\}$. Let $L$ be a frame. The set of all prime ideals (respectively, maximal ideals) of $L$ is denoted by $\text{Spec}(L)$ (respectively, $(\text{Max}(L))$. An element $p \in L$ is called prime if $p < \textbf{Top}$, and $a \land b \leq p$ implies $a \leq p$ or $b \leq p$. Clearly, $a \in L$ is a prime element if and only if $\bot a = \{x \in L : x \leq a\}$ is a prime ideal of $L$. We denote by $\text{Sp}(L)$ the set of all prime element of $L$. For every $a \in L$, define $\mathcal{h}^c(a) = \{p \in \text{Sp}(L) : a \not\leq p\}$. It is easily seen that $\{\mathcal{h}^c(a) : a \in L\}$ is a topology on $\text{Sp}(L)$. Here after we use $\text{Sp}(L)$ equipped with this topology.

Let $X$ and $Y$ be two partial ordered sets and $f : X \to Y$ and $g : Y \to X$ be two increasing maps. We say $f$ is left adjoint of $g$ (respectively, $g$ is right adjoint of $f$) if $fg \leq I_Y$ and $gf \geq I_X$. It is easy to see that $g$ is uniquely determined by $f$ and vice versa. The right adjoint of a map $f : X \to Y$ (respectively, left adjoint of a map $g : Y \to X$), if there exists, is denoted by $f_*$ (resp., $g^*$). Supposing $X$ and $Y$ are complete lattices, one can easily see that $f : X \to Y$ is a left adjoint map if and only if $f$ preserves arbitrary joins and in this case $f_*(y) = \bigvee \{x \in X : f(x) \leq y\}$ for every $y \in Y$. A frame homomorphism is a map $f$ from a frame $L$ to a frame $L'$ such that it preserves finite meets and arbitrary joins; clearly in this case we have $f(\bot) = \bot$ and $f(\textbf{Top}) = \textbf{Top}$. Obviously, every frame homomorphism is a left adjoint map. We denote by $\mathcal{O}X$ and $\mathcal{O}_x$ the frames of all open subsets of a topological space $X$ and the set of all open neighborhoods of $x \in X$, respectively. If $X$ and $Y$ are two topological spaces, then for every continuous function $f : X \to Y$ we define $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ with $(\mathcal{O}f)(w) = f^{-1}(w)$ for every $w \in \mathcal{O}Y$. It is obvious that $\mathcal{O}$ is a contravariant functor from the category $\textbf{Top}$ to the category $\textbf{Frm}$. Let $L$ and $L'$ be two frames. For every frame homomorphism $f : L \to L'$ we can define $\text{Sp}(f) : \text{Sp}(L') \to \text{Sp}(L)$ with $(\text{Sp}(f))(q) = f_*(q)$. For any $a \in L$, we can write

$$(\text{Sp}(f))^{-1}(\mathcal{h}^c(a)) = \{q \in \text{Sp}(L') : f_*(q) \in \mathcal{h}^c(a)\}$$

$= \{q \in \text{Sp}(L') : a \not\leq f_*(q)\}$$

$= \{q \in \text{Sp}(L') : f(a) \not\leq q\} = \mathcal{h}^c(f(a)).$$
Therefore, $Spf$ is a continuous map. It is easy to see that $SpI_L = I_{SpL}$ and $Spfg = SpgSpf$ whenever $fg$ means the composition of $f$ and $g$. Thus, $Sp : \text{ Frm} \rightarrow \text{ Top}$ is a contravariant functor. In fact the functor $Sp$ is a right adjoint of the functor $O$.

Recall that an ordered ring is a ring $A$ with a partial order $\leq$ such that for every $a, b, c \in A$, from $a \leq b$ it follows that $a + c \leq b + c$ and if $a, b \geq 0$, then $ab \geq 0$. An ordered ring is called a lattice-ordered ring if $A$ is a lattice under the partial order on $A$. By an $f$-ring we mean a lattice-ordered ring $R$ with this property that $a(b \land c) = ab \land ac$ and $(b \land c)a = ba \land ca$ for every $a \in R^+$ and every $b, c \in R$. An algebra (over a field $F$) is a structure consisting of a set $A$ with two operations “+” and “.”, and also a scaler multiplication such that $(A, +, \cdot)$ is a ring and $A$ with addition and scaler multiplication is a vector space (over $F$), and in addition, for every $x, y \in A$ and every $c \in F$, we have

$$1_Fx = x \quad c(xy) = (cx)y = x(cy).$$

Finally, an $f$-algebra (over an ordered field) is an algebra with a partial order $\leq$ such that $(A, +, \cdot, \leq)$ is an $f$-ring, and $A$ with “+” and the scaler multiplication is a vector space (over $F$) in which $cx \geq 0$ for every $c \in F^+$ and every $x \in A^+$.

Suppose that $A$ is a lattice-ordered ring and $a \in A$. The positive part of $a$, negative part of $a$, and $|a|$ are defined as $a^+ = a \lor 0$, $a^- = -a \lor 0$ and $|a| = a \lor -a$, respectively. Clearly, if $A$ is an $f$-ring, then $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+a^- = 0$ and $|a|^2 = a^2$ for any $a \in A$.

In the present part of this paper, for convenience of readers, we give a short review of $C(L)$, at a slightly different perspective from what is stated in the main texts.

A frame homomorphism $\alpha : O\mathbb{R} \rightarrow L$ is called continuous real function on a frame $L$ and the set of all continuous real function on a frame $L$ is denoted by $C(L)$. Although, this concept was first introduced by R.N. Ball and A.W. Hager in [1], B. Banaschewski studied this concept deeply in [2]; he also showed in [3] that $C(L)$ is a class which strictly contains $C(X)$. Note that we work under the axiomatic system of $ZFC$ and in this system, we have $L(\mathbb{R}) \simeq O\mathbb{R}$. In this axiomatic system $C(L)$ has a simpler representation.

Supposing that $A, S \subseteq L$, we denote by $\downarrow A$ the set $\{x \in L : \exists a \in L\}$. Finally, an $f$-algebra (over an ordered field) is an algebra with a partial order $\leq$ such that $(A, +, \cdot, \leq)$ is an $f$-ring, and $A$ with “+” and the scaler multiplication is a vector space (over $F$) in which $cx \geq 0$ for every $c \in F^+$ and every $x \in A^+$.
A, \( x \leq a \}; we use \( \downarrow x \) instead of \( \downarrow \{x\} \) and \( \downarrow_S A \) instead of \( S \cap \downarrow A \). Clearly, for any \( S \subseteq L \), the map \( \downarrow_S : L \to P(S) \) is a meet-homomorphism but not a join-homomorphism, see [15]. A subset \( B \) of \( L \) is said to be a base for \( L \) if \( x = \bigvee \downarrow_B x \) for every \( x \in L \). Let \( L \) and \( L' \) be two frames and \( B \) be a base for \( L \). A map \( f : B \to L' \) is said to be conditional homomorphism if for every \( A \subseteq B \) and every finite \( F \subseteq B \) we have \( f(\bigvee A) = \bigvee f(A) \) and \( f(\bigwedge F) = \bigwedge f(F) \), provided that \( \bigvee A \in B \) and \( \bigwedge F \in B \). Supposing that \( B \) is a base for a frame \( L \), we call \( B \) a homomorphism maker if every conditional homomorphism from \( B \) to a frame \( L' \) has an extension homomorphism from \( L \) to \( L' \).

**Proposition 1.1.** Let \( B \) be a base for \( L \) closed under finite meets. Then \( B \) is homomorphism maker.

**Proof.** Let \( f : B \to L' \) be a conditional homomorphism. We define \( \tilde{f} : L \to L' \) with \( \tilde{f}(x) = \bigvee f(\downarrow_B (x)) \) and prove that \( \tilde{f} \) is a homomorphism extension of \( f \). Clearly, \( \tilde{f} \) is order preserving, \( \tilde{f}|_B = f \), \( f(\bot) = \bot \) and \( f(\text{Top}) = \text{Top} \). Assuming that \( x_\lambda \in L \) for every \( \lambda \in \Lambda \), since \( \tilde{f} \) is order preserving, we have \( \bigvee_{\lambda \in \Lambda} \tilde{f}(x_\lambda) \leq \tilde{f}(\bigvee_{\lambda \in \Lambda} x_\lambda) \). Conversely, for every \( b \in \downarrow_B (\bigvee_{\lambda \in \Lambda} x_\lambda) \),

\[
b = \bigvee_{\lambda \in \Lambda} b \land x_\lambda = \bigvee_{\lambda \in \Lambda} \bigvee \{c \in B : c \leq b \land x_\lambda\},
\]

which implies that

\[
f(b) = \bigvee_{\lambda \in \Lambda} \bigvee \{f(c) : c \in B, c \leq b \land x_\lambda\}
\]

\[
\leq \bigvee_{\lambda \in \Lambda} \bigvee \{f(c) : c \in B, c \leq x_\lambda\}
\]

\[
= \bigvee_{\lambda \in \Lambda} \tilde{f}(x_\lambda),
\]

and this shows that

\[
\tilde{f}(\bigvee_{\lambda \in \Lambda} x_\lambda) = \bigvee \{f(b) : b \in \downarrow \bigvee_{\lambda \in \Lambda} x_\lambda\} \leq \bigvee_{\lambda \in \Lambda} \tilde{f}(x_\lambda).
\]
Therefore, \( \bar{f}(\bigvee_{\lambda \in \Lambda} x_\lambda) = \bigvee_{\lambda \in \Lambda} \bar{f}(x_\lambda) \). Now, supposing that \( x, y \in L \), clearly
\[
\bar{f}(x \land y) = \bigvee \{f(c) : c \in B, c \leq x \land y\}
\]
\[
= \bigvee \{f(c_1 \land c_2) : c_1, c_2 \in B, c_1 \leq x, c_2 \leq y\}
\]
\[
= \bigvee \{f(c_1) \land f(c_2) : c_1, c_2 \in B, c_1 \leq x, c_2 \leq y\}
\]
\[
= \bigvee \{f(c_1) : c_1 \in B, c_1 \leq x\} \land \bigvee \{f(c_2) : c_2 \in B, c_2 \leq y\}
\]
\[
= \bar{f}(x) \land \bar{f}(y).
\]

In the above proposition, the condition “closedness under finite meets” cannot be omitted. For example, suppose that \( B = \{(a,b) : a,b \in \mathbb{Q}, a < b\} \) and \( f : B \rightarrow L \) with \( f(a,b) = \text{Top} \) for every \( (a,b) \in B \). Obviously, \( B \) is a base for \( \mathcal{OR} \), \( f \) is conditional homomorphism and \( B \) is not homomorphism maker.

**Corollary 1.2.** Let \( B = \{(r,s) : r,s \in \mathbb{Q}\} \cup \{\mathbb{R}\} \). Clearly, \( B \) is a base for \( \mathcal{OR} \) and closed under finite meets. Hence, \( B \) is a homomorphism maker. In other words, a map \( f : B \rightarrow L \) has an extension homomorphism \( \alpha \in C(L) \) if and only if \( f \) has the following properties.

(R1) \( f((p,q) \land (r,s)) = f(p,q) \land f(r,s) \), whenever \( p,q,r,s \in \mathbb{Q} \) and \( (p,q) \land (r,s) \neq \emptyset \).

(R2) \( f((p,q) \lor (r,s)) = f(p,q) \lor f(r,s) \), whenever \( p,q,r,s \in \mathbb{Q} \) and \( p \leq r < q \leq s \).

(R3) \( f(p,q) = \bigvee \{f(r,s) : r,s \in \mathbb{Q}, p < r < s < q\} \) for every \( p,q \in \mathbb{Q} \).

(R4) \( \text{Top} = f(\mathbb{Q}) = \bigvee \{f(p,q) : p,q \in \mathbb{Q}\} \).

Suppose that \( \diamond \) is an operation such as “+”, “-”, “\lor” and “\land”. For every \( \alpha, \beta \in C(L) \) and every \( p,q \in \mathbb{Q} \), we define
\[
(\alpha \diamond \beta)(p,q) = \bigvee \{\alpha(r,s) \land \beta(t,u) : r,s,t,u \in \mathbb{Q}, (r,s) \diamond (t,u) \subseteq (p,q)\},
\]
where \( (r,s) \diamond (t,u) = \{a \diamond b : a \in (r,s), b \in (t,u)\} \). It can be proved that \( \alpha \diamond \beta \) is a conditional homomorphism on \( B = \{(r,s) : r,s \in \mathbb{Q}\} \cup \{\mathbb{R}\} \) and hence \( \alpha \diamond \beta \in C(L) \), see [2] and [14]. Also, for every \( r \in \mathbb{R} \) it is defined that
\( r(w) = \text{Top} \) if \( r \in w \) and \( r(w) = \bot \) if \( r \notin w \). It is clear to see that \( r \in C(L) \).

Now, \( r.\alpha \) is defined by \( \alpha r \). Consequently \( C(L) \) is an \( f \)-algebra with these operations.

**Proposition 1.3.** For every \( \alpha, \beta \in C(L) \) and every \( w \in \mathcal{O} \mathbb{R} \), we have 

\[
(\alpha \circ \beta)(w) = \bigvee \{ \alpha(r, s) \land \beta(t, u) : r, s, t, u \in \mathbb{Q}, (r, s) \circ (t, u) \subseteq w \}
\]

\[
= \bigvee \{ \alpha(w_1) \land \beta(w_2) : w_1, w_2 \in \mathcal{O} \mathbb{R}, w_1 \circ w_2 \subseteq w \},
\]

where \( w_1 \circ w_2 = \{ a \circ b : a \in w_1, b \in w_2 \} \).

**Proof.** Assume that 

\[
A_{a,b} = \{ \alpha(r, s) \land \beta(t, u) : r, s, t, u \in \mathbb{Q}, (r, s) \circ (t, u) \subseteq (a, b) \},
\]

\[
A_w = \{ \alpha(r, s) \land \beta(t, u) : r, s, t, u \in \mathbb{Q}, (r, s) \circ (t, u) \subseteq w \}
\]

and

\[
B_w = \{ \alpha(w_1) \land \beta(w_2) : w_1, w_2 \in \mathcal{O} \mathbb{R}, w_1 \circ w_2 \subseteq w \}.
\]

Since \( (\alpha \circ \beta) \in C(L) \), it follows that

\[
(\alpha \circ \beta)(w) = (\alpha \circ \beta) \left( \bigcup \{ (a, b) : a, b \in \mathbb{Q}, (a, b) \subseteq w \} \right)
\]

\[
= \bigvee \{ (\alpha \circ \beta)(a, b) : a, b \in \mathbb{Q}, (a, b) \subseteq w \}
\]

\[
= \bigvee \left\{ \bigvee A_{a,b} : (a, b) \subseteq w \right\}.
\]

Therefore, clearly, \( (\alpha \circ \beta)(w) \leq \bigvee A_w \leq \bigvee B_w \). Now, suppose that \( \alpha(r, s) \land \beta(t, u) \in A_w \). Obviously, there exist \( a, b \in \mathbb{Q} \) such that \( (r, s) \circ (t, u) \subseteq (a, b) \subseteq w \). Hence, \( \alpha(r, s) \land \beta(t, u) \in A_{a,b} \) and consequently \( \bigvee A_w \leq \bigvee A_{a,b} \leq (\alpha \circ \beta)(w) \) and so \( \bigvee A_w = (\alpha \circ \beta)(w) \).

Finally, assume that \( \alpha(w_1) \land \beta(w_2) \in B_w \), where \( w_1 \circ w_2 \subseteq w \). Clearly, \( w_1 = \bigcup_{i \in I} (r_i, s_i) \) and \( w_2 = \bigcup_{j \in J} (t_j, u_j) \), where \( r_i, s_i, t_j, u_j \in \mathbb{Q} \) for every \( i \in I \) and every \( j \in J \). Thus,

\[
\bigcup_{i \in I} (r_i, s_i) \circ (t_j, u_j) = \bigcup_{i \in I} (r_i, s_i) \circ \bigcup_{j \in J} (t_j, u_j) = w_1 \circ w_2 \subseteq w
\]

and so \( (r_i, s_i) \circ (t_j, u_j) \subseteq w \) for every \( i \in I \) and every \( j \in J \). Therefore, it is easy to see that \( \alpha(w_1) \land \beta(w_2) = \bigvee_{i \in I} \bigvee_{j \in J} \alpha(r_i, s_i) \land \beta(t_j, u_j) \leq \bigvee A_w \).

Hence, \( \bigvee B_w \leq \bigvee A_w \) and so \( \bigvee B_w = \bigvee A_w \). 

\[ \square \]
Throughout the paper, the notations \( L \) and \( C(L) \) stand for a frame and the \( f \)-algebra of all continuous real functions on the frame \( L \), respectively. The reader is referred to [2], [14], and [12], for more information about frames and \( C(L) \). Also, see [4], [5], [11], [15], and [10] for more information about general lattice theory and rings of continuous functions, respectively.

We need the following proposition which can be found in the literature.

**Proposition 1.4.** Let \( \alpha, \beta \in C(L) \) and \( a \in \mathbb{R} \). The following statements hold.

(a) If \( \alpha \geq 0 \), then \( \alpha(-\infty, x) = \bot \) for every \( x \leq 0 \).

(b) If \( \alpha \geq 0 \), then \( \alpha(x, +\infty) = \text{Top} \) for every \( x < 0 \).

(c) \( (\alpha \lor \beta)(x, +\infty) = \alpha(x, +\infty) \lor \beta(x, +\infty) \) for every \( x \in \mathbb{R} \).

(d) \( (\alpha \lor \beta)(-\infty, x) = \alpha(-\infty, x) \land \beta(-\infty, x) \) for every \( x \in \mathbb{R} \).

(e) \( (\alpha \land \beta)(x, +\infty) = \alpha(x, +\infty) \land \beta(x, +\infty) \) for every \( x \in \mathbb{R} \).

(f) \( (\alpha \land \beta)(-\infty, x) = \alpha(-\infty, x) \lor \beta(-\infty, x) \) for every \( x \in \mathbb{R} \).

(g) \( (c\alpha)(w) = \alpha(\frac{1}{c}w) \) for every \( w \in O\mathbb{R} \) and each \( c \neq 0 \), where \( bw = \{ bx : x \in w \} \).

(h) \( (e + \alpha)(w) = \alpha(w - c) \) for each \( w \in O\mathbb{R} \) and each \( c \in \mathbb{R} \), where \( w + b = \{ x + b : x \in w \} \).

2 Pre-image of a continuous real function on \( L \)

In [13], although it does not introduce a determined definition for pointfree version of the “image” of continuous real functions, using a concept, called “overlap”, an attempt has been made to fill the vacuum of the concept of image of continuous real functions in pointfree topology. In this main section, we give a determined version of the image of continuous real functions on a topological space \( X \) in the pointfree topology and we show that this is independent of what we see in [13].

**Definition 2.1.** For every \( \alpha \in C(L) \), we define \( \text{pim}(\alpha) \), called pre-image of \( \alpha \), as

\[ \text{pim}(\alpha) = \bigcap \{ w \in O\mathbb{R} : \alpha(w) = \text{Top} \}. \]

At below we provide an example in which we demonstrate that \( \text{pim}(\alpha) \) is an appropriate model of image of the real-valued functions in pointfree topology.
Example 2.2. Let $C(X)$ be the ring of real-valued continuous functions on a topological space $X$. We know that for all $f \in C(X)$ we have $\mathcal{O}f \in C(\mathcal{O}X)$ and clearly, we can write

$$\text{Im}(f) = f(X) = \bigcap_{f(X) \subseteq w} w = \bigcap_{f^{-1}(w) = X} \mathcal{O}w : \mathcal{O}f(w) = \text{Top}.$$ 

Therefore, $\text{Im}(f) = \text{pim}(\mathcal{O}f)$.

Hereinafter, by $\mathbb{R}_x$, we mean $\mathbb{R} \setminus \{x\}$.

Proposition 2.3. For every $\alpha \in C(L)$, the following statements hold:

(a) $\text{pim}(\alpha) = \bigcap \{\mathbb{R}_x : \alpha(\mathbb{R}_x) = \text{Top}\}$.

(b) $x \notin \text{pim}(\alpha)$ if and only if $\alpha(\mathbb{R}_x) = \text{Top}$.

Proof. (a): Suppose that $\mathcal{B} = \{\mathbb{R}_x : \alpha(\mathbb{R}_x) = \text{Top}\}$. Obviously $\text{pim}(\alpha) \subseteq \bigcap \mathcal{B}$. Now, assuming $x \notin \text{pim}(\alpha)$, there exists $w \in \mathcal{O}\mathbb{R}$ such that $x \notin w$ and $\alpha(w) = \text{Top}$. Hence, $w \subset \mathbb{R}_x$, consequently $\alpha(\mathbb{R}_x) = \text{Top}$ and so $x \notin \mathbb{R}_x \in \mathcal{B}$. Therefore, $\bigcap \mathcal{B} \subseteq \text{pim}(\alpha)$ and subsequently $\text{pim}(\alpha) = \bigcap \mathcal{B}$.

(b): According to (a), it is obvious that we can write

$$x \notin \text{pim}(\alpha) \Rightarrow \exists \mathbb{R}_y, \alpha(\mathbb{R}_y) = \text{Top}, \ x \notin \mathbb{R}_y.$$ 

Since $x \notin \mathbb{R}_y$, $x = y$ and consequently $\alpha(\mathbb{R}_x) = \text{Top}$. Conversely, assume that $\alpha(\mathbb{R}_x) = \text{Top}$. Thus, $\text{pim}(\alpha) \subseteq \mathbb{R}_x$ and so $x \notin \text{pim}(\alpha)$. \qed

Estaji and at al. in [8], put

$R_\alpha = \{r \in \mathbb{R} : \text{coz}(\alpha - r) \neq \text{Top}\}$

for every $\alpha \in C(L)$, and they studied some of its properties. By Proposition 2.3, it is evident that $R_\alpha = \text{pim}(\alpha)$.

Recall that $w^* = \mathbb{R} \setminus \overline{w}$ and $\overline{w} = \bigcap_{x \in w^*} \mathbb{R}_x$ for every $w \in \mathcal{O}\mathbb{R}$.

Proposition 2.4. For every $w \in \mathcal{O}\mathbb{R}$ and every $\alpha \in C(L)$, the following statements hold:

(a) If $\alpha(w^*) = \perp$, then $\alpha(\mathbb{R}_x) = \text{Top}$ for all $x \in w^*$.

(b) If $\alpha(w^*) = \perp$, then $\text{pim}(\alpha) \subseteq \overline{w}$.

(c) If $r \in \text{pim}(\alpha)$ and $w \in \mathcal{O}_r$, then $\alpha(w) \neq \perp$. 

Proof. (a): Suppose that $w \in \mathcal{O}R$ and $\alpha \in C(L)$. Then for every $x \in w^*$, we can write
\[
\mathbb{R}_x \cup w^* = \mathbb{R} \quad \Rightarrow \quad \alpha(\mathbb{R}_x) = \alpha(\mathbb{R}) = \alpha(\mathbb{R}_x \cup w^*) = \alpha(\mathbb{R}) = \text{Top}.
\]

(b): Since $\alpha(w^*) = \bot$, by part (a), for all $x \in w^*$, we have $\alpha(\mathbb{R}_x) = \text{Top}$ and so
\[
pim(\alpha) = \bigcap \{ \mathbb{R}_x : \alpha(\mathbb{R}_x) = \text{Top} \} \subseteq \bigcap_{x \in w^*} \mathbb{R}_x = w.
\]

(c): Suppose that $r \in \overline{\text{pim}(\alpha)}$ and $w \in \mathcal{O}_r$. Thus, there exists $y \in w \cap \text{pim}(\alpha)$ and therefore
\[
\text{Top} = \alpha(\mathbb{R}) = \alpha(\mathbb{R}_y \cup w) = \alpha(\mathbb{R}_y) \vee \alpha(w).
\]

On the other hand, since $y \in \text{pim}(\alpha)$, $\alpha(\mathbb{R}_y) \neq \text{Top}$ and so $\alpha(w) \neq \bot$. \qed

By Example 2.2, it is easy to see that if $\text{pim}(\mathcal{O}f) \subseteq w \in \mathcal{O}R$, then $\mathcal{O}f(w) = \text{Top}$. Also, if $\mathcal{O}f(w) \neq \bot$, for every $w \in \mathcal{O}_r$, $r \in \text{pim}(\alpha)$. So here are two natural questions.

**Question 1**: Suppose that $\alpha \in C(L)$ and $w \in \mathcal{O}R$. Can we imply $\alpha(w) = \text{Top}$ from $\text{pim}(\alpha) \subseteq w$?

**Question 2**: Suppose that $\alpha(w) \neq \bot$, for every $w \in \mathcal{O}_r$. Can we conclude that $r \in \overline{\text{pim}(\alpha)}$?

Example 2.8 shows that the answer to these two questions is generally negative (in the first question, even if $w$ is an unbounded interval in $\mathbb{R}$). But, in the following proposition, we will find that the answer to the first question is positive under some conditions.

**Proposition 2.5.** Let $\alpha \in C(L)$, $w \in \mathcal{O}R$ and $\text{pim}(\alpha) \subseteq w$, then the following statements hold:

(a) If $w$ is dense in $\mathbb{R}$ and the boundary of $w$ is finite, then $\alpha(w) = \text{Top}$.

(b) Let $\mathcal{U} \subseteq \mathcal{O}R$ be such that one of these families is bounded, $\text{pim}(\alpha) \subseteq \bigcap \mathcal{U}$ and $\alpha(u) = \text{Top}$ for every $u \in \mathcal{U}$. If $\bigcap_{u \in \mathcal{U}} \overline{u} \subseteq w$, then it follows that $\alpha(w) = \text{Top}$. 
Proof. (a): It is clear.
(b): Without loss of generality, we can suppose that \( u \) is compact for all \( u \in \mathcal{U} \). Now, it is easy to see that there exist \( u_1, \ldots, u_n \in \mathcal{U} \) such that \( \bigcap_{i=1}^n u_i \subseteq w \). Therefore,
\[
\text{Top} = \bigwedge_{i=1}^n \alpha(u_i) = \alpha\left(\bigcap_{i=1}^n u_i\right) \leq \alpha(w) \Rightarrow \alpha(w) = \text{Top}.
\]

Suppose that \( \alpha \in C(L) \) and \( S \subseteq \mathbb{R} \). We recall from [13] that \( \alpha \) is an overlap of \( S \), denoted by \( \alpha \rhd S \), whenever \( i(u) \subseteq i(v) \) implies \( \alpha(u) \leq \alpha(v) \); that is, \( u \cap S \subseteq v \cap S \) implies \( \alpha(u) \leq \alpha(v) \). In the following propositions and example, we will see that although this concept and \( \text{pim}(\alpha) \) are closely related, but they are different from each other.

**Proposition 2.6.** Suppose that \( \alpha \in C(L) \) and \( OV(\alpha) = \{S \subseteq \mathbb{R} : \alpha \rhd S\} \). Then \( \text{pim}(\alpha) = \bigcap_{S \in OV(\alpha)} S \).

**Proof.** Let \( S \in OV(\alpha) \) and \( x \notin S \). Thus, \( \mathbb{R}_x \cap S = S \cap S \) and so \( \text{Top} = \alpha(\mathbb{R}) = \alpha(\mathbb{R}_x) \); that, \( x \notin \text{pim}(\alpha) \). Therefore, \( \text{pim}(\alpha) \subseteq \bigcap_{S \in OV(\alpha)} S \). Conversely, suppose \( x \notin \text{pim}(\alpha) \); it suffices to show that \( \mathbb{R}_x \in OV(\alpha) \). To see this, for every \( u, v \in \mathcal{O}\mathbb{R} \), we can write
\[
u \cap \mathbb{R}_x \subseteq v \cap \mathbb{R}_x \Rightarrow \alpha(u) = \alpha(u) \wedge \text{Top} = \alpha(u) \wedge \alpha(\mathbb{R}_x)
= \alpha(u \cap \mathbb{R}_x) \leq \alpha(v \cap \mathbb{R}) = \alpha(v).
\]

**Proposition 2.7.** Suppose that \( \alpha \in C(L) \), \( w \in \mathcal{O}\mathbb{R} \) and \( \alpha(w) = \text{Top} \), then \( \alpha \rhd w \).

**Proof.** Let \( u, v \in \mathcal{O}\mathbb{R} \) and \( u \cap w \subseteq v \cap w \). Hence
\[
\alpha(u) = \alpha(u) \wedge \text{Top} = \alpha(u) \wedge \alpha(w)
= \alpha(u \cap w) \leq \alpha(v \cap w) = \alpha(v) \wedge \alpha(w) = \alpha(v) \wedge \text{Top} = \alpha(v).
\]
In this way, it turns out that the following equality is in place, too.

\[ \text{pim}(\alpha) = \bigcap \{ w \in \mathcal{O}_R : \alpha \upharpoonright w \}. \]

**Example 2.8.** There is a frame \( L \) and \( \beta \in C(L) \) such that \( \beta \upharpoonright \text{pim}(\beta) \). To see this, let \( L, \beta \) and the family \( \{S_c\}_{c \in \mathcal{I}} \) be same as in [13, Example 3.18]. Then, \( \text{pim}(\beta) \subseteq \bigcap_{c \in \mathcal{I}} S_c = \emptyset \). Thus, \( \beta \upharpoonright \text{pim}(\beta) \) does not hold. Furthermore, since \( \beta(\emptyset) = \bot \), there exists \( w \in \mathcal{O}_R \) such that \( \beta(w) \neq \top \). Thus, the answer to Question 1 is negative. Also, since \( \beta(\top) = \top \), there exists an element \( r \in \mathbb{R} \) such that for every \( w \in \mathcal{O}_R \), we have \( \beta(w) \neq \bot \), whereas \( r \notin \emptyset = \text{pim}(\beta) \). Therefore, the answer to Question 2 is also negative.

Now, we want to find the relationship between \( \text{pim}(|\alpha|) \) and \( \text{pim}(\alpha) \).

**Lemma 2.9.** For every \( \alpha \in C(L) \) and every \( x \in \mathbb{R} \), we have

\[ |\alpha|(\mathbb{R}_x) = \left( \alpha(x, +\infty) \lor \alpha(-\infty, |x|) \right) \land \left( \alpha(-|x|, +\infty) \lor \alpha(-\infty, -x) \right). \]

**Proof.** By Proposition 1.4, the proof is straightforward. \( \square \)

The following corollary is followed from the above lemma immediately.

**Corollary 2.10.** Assume that \( \alpha \in C(L) \) and \( x \in \mathbb{R} \). Then the following statements hold:

(a) If \( x < 0 \), then \( |\alpha|(\mathbb{R}_x) = \top \).

(b) If \( x \geq 0 \), then \( |\alpha|(\mathbb{R}_x) = \alpha(\mathbb{R}_x) \land \alpha(\mathbb{R} \setminus x) \).

(c) \( \text{pim}(|\alpha|) \subseteq \mathbb{R}^+ \).

**Proposition 2.11.** \( \text{pim}(|\alpha|) = \{ |x| : x \in \text{pim}(\alpha) \} \) for every \( \alpha \in C(L) \).

**Proof.** Supposing \( A = \{ |x| : x \in \text{pim}(\alpha) \} \), clearly, \( A = \{ x \in \mathbb{R}^+ : x \in \text{pim}(\alpha) \text{ or } -x \in \text{pim}(\alpha) \} \). Accordingly to Lemma 2.9, for every \( x \geq 0 \), we can write

\[
x \notin A \iff x, -x \notin \text{pim}(\alpha) \iff \alpha(\mathbb{R}_x) = \alpha(\mathbb{R} \setminus x) = \top \iff |\alpha|(\mathbb{R}_x) = \top \iff x \notin \text{pim}(|\alpha|).
\]

\( \square \)
Proposition 2.12. The following relations are true for each \( \alpha \in C(L) \) and each \( r \in \mathbb{R} \):

(a) \( \text{pim}(r) = \{r\} \).
(b) \( \text{pim}(r\alpha) = r \text{pim}(\alpha) \).
(c) \( \text{pim}(r + \alpha) = r + \text{pim}(\alpha) \).

Proof. (a): Clearly, for every \( r \in \mathbb{R} \), we can write
\[
 r(\mathbb{R}_x) = \text{Top} \iff x \neq r.
\therefore \text{pim}(r) = \bigcap_{x \neq r} \mathbb{R}_x = \{r\}. 
\]

(b): For every \( r \in \mathbb{R} \), we can write (without loss of generality, assume that \( r \neq 0 \))
\[
 \text{pim}(r\alpha) \subseteq \mathbb{R}_x \iff (r\alpha)(\mathbb{R}_x) = \text{Top} \iff \alpha(\frac{1}{r}\mathbb{R}_x) = \alpha(\mathbb{R}_{\frac{1}{r}}) = \text{Top} 
\iff \text{pim}(\alpha) \subseteq \mathbb{R}_{\frac{1}{r}} \iff r \text{pim}(\alpha) \subseteq \mathbb{R}_x 
\iff \text{pim}(r) \cdot \text{pim}(\alpha) \subseteq \mathbb{R}_x.
\]

(c): For every \( r \in \mathbb{R} \), we can write
\[
 \text{pim}(r + \alpha) \subseteq \mathbb{R}_x \iff (r + \alpha)(\mathbb{R}_x) = \text{Top} \iff \alpha(-r + \mathbb{R}_x) = \alpha(\mathbb{R}_{x-r}) = \text{Top} 
\iff \text{pim}(\alpha) \subseteq \mathbb{R}_{x-r} = -r + \mathbb{R}_x \iff r + \text{pim}(\alpha) \subseteq \mathbb{R}_x 
\iff \text{pim}(r) + \text{pim}(\alpha) \subseteq \mathbb{R}_x.
\]

Now, we state the relation between \( \text{pim}(\alpha) \), \( \text{pim}(\alpha^+) \), and \( \text{pim}(\alpha^-) \) in the following.

Proposition 2.13. For every \( \alpha \in C(L) \), the following relations hold:

(a) \( \text{pim}(\alpha) \cap (0, +\infty) = \text{pim}(\alpha^+) \setminus \{0\} \).
(b) \( \text{pim}(\alpha) \cap (-\infty, 0) = \text{pim}(\alpha^-) \setminus \{0\} \).
(c) \( \text{pim}(\alpha) \setminus \{0\} = ((\text{pim}(\alpha^+) \cup \text{pim}(\alpha^-)) \setminus \{0\} \).

Proof. (a): For every \( x > 0 \), by Proposition 1.4, we have
\[
 \alpha^+(-\infty, x) = (\alpha \lor \mathbb{0})(-\infty, x) = \alpha(-\infty, x) \wedge \mathbb{0}(-\infty, x) = \alpha(-\infty, x)
\]
and similarly,
\[ \alpha^+(x, +\infty) = (\alpha \lor 0)(x, +\infty) = \alpha(x, +\infty) \lor 0(x, +\infty) = \alpha(x, +\infty). \]

Therefore, for every \( x > 0 \), we can deduce that
\[ \alpha(\mathbb{R}_x) = \alpha(-\infty, x) \lor \alpha(x, +\infty) = \alpha^+(-\infty, x) \lor \alpha^+(x, +\infty) = \alpha^+(\mathbb{R}_x). \]

Hence, \((0, +\infty) \cap \text{pim}(\alpha) = \text{pim}(\alpha^+) \setminus \{0\}\). 

(b): For every \( x < 0 \), by part (a), we can write
\[
-\alpha^-(\mathbb{R}_x) = -\alpha^-([-\infty, x) \lor (x, +\infty)] \\
= -\alpha^-(-\infty, x) \lor -\alpha^-(x, +\infty) \\
= \alpha^-(-x, +\infty) \lor \alpha^-(-\infty, -x) \\
= (-\alpha)^+(-x, +\infty) \lor (-\alpha)^+(-\infty, -x) \\
= -\alpha(-x, +\infty) \lor -\alpha(-\infty, -x) \\
= \alpha(-\infty, x) \lor \alpha(x, +\infty) = \alpha(\mathbb{R}_x).
\]

Therefore, \((-\infty, 0) \cap \text{pim}(\alpha) = \text{pim}(-\alpha^-) \setminus \{0\}\).

(c): Straightforward from (a) and (b), it is concluded that
\[ \text{pim}(\alpha) \setminus \{0\} = ((\text{pim}(\alpha^+) \cup \text{pim}(-\alpha^-)) \setminus \{0\}. \]

\[ \square \]

**Question 3:** Now, this question arises whether the following relations, similar to what we have for real functions on topological spaces, hold.

\[ \text{pim}(\alpha \lor \beta) \subseteq \text{pim}(\alpha) \cup \text{pim}(\beta) \quad \text{pim}(\alpha \land \beta) \subseteq \text{pim}(\alpha) \cap \text{pim}(\beta) \]
\[ \text{pim}(\alpha + \beta) \subseteq \text{pim}(\alpha) + \text{pim}(\beta) \quad \text{pim}(\alpha \beta) \subseteq \text{pim}(\alpha)\text{pim}(\beta). \]

We show that under some achievable conditions, the answer is positive. But first we need some preparations.

**Definition 2.14.** An ideal \( I \) in a frame \( L \) is called \( \lor \)-complete (countably \( \lor \)-complete) if from \( D \subseteq I \) (countable set \( D \subseteq I \)), it follows that \( \bigvee D \in I \).
Example 2.15. (a) Every principal ideal is $\vee$-complete.
(b) Suppose that $\omega_1$ is the first uncountable ordinal and $L = \downarrow \omega_1$. Clearly $L$ is a frame and if we put $P = L \setminus \{\text{Top}\}$, then $P$ is a countably $\vee$-complete ideal whereas it is not a $\vee$-complete ideal.

Definition 2.16. For every $P \in \text{Spec}(L)$, we define $A_P(\alpha) = \{x \in \mathbb{R} : \alpha(x, +\infty) \in P\}$ and $B_P(\alpha) = \{x \in \mathbb{R} : \alpha(-\infty, x) \in P\}$.

Because these two sets $A_P(\alpha)$ and $B_P(\alpha)$ are important in our work, we discuss them briefly.

Lemma 2.17. Let $P \in \text{Spec}(L)$ and $\alpha \in C(L)$. Then
(a) $A_P(\alpha) \cup B_P(\alpha) = \mathbb{R}$.
(b) Any element of $A_P(\alpha)$ is an upper bound of $B_P(\alpha)$ and any element of $B_P(\alpha)$ is a lower bound of $A_P(\alpha)$.
(c) $\uparrow A_P(\alpha) = A_P(\alpha)$ and $\downarrow B_P(\alpha) = B_P(\alpha)$.

Proof. (a): Assuming $x \notin A_P(\alpha)$, it follows that $\alpha(x, +\infty) \notin P$. Since $P$ is prime and $\alpha(x, +\infty) \land \alpha(-\infty, x) = \perp \in P$, we deduce that $\alpha(-\infty, x) \in P$. Hence $x \in B_P(\alpha)$.

(b): Assume that $x \in A_P(\alpha)$ and, on the contrary, there exists an element $c \in B_P(\alpha)$ such that $x < c$. Therefore, $\text{Top} = \alpha(\mathbb{R}) = \alpha(-\infty, c) \lor \alpha(x, +\infty) \in P$ and this is a contradiction. Similarly, any element of $B_P(\alpha)$ is a lower bound of $A_P(\alpha)$.

(c): Supposing $x \in \uparrow A_P(\alpha)$, there exists an element $a \in A_P(\alpha)$ such that $a \leq x$. Thus, $\alpha(x, +\infty) \leq \alpha(a, +\infty) \in P$ and consequently $x \in A_P(\alpha)$. \qed

Corollary 2.18. Let $P \in \text{Spec}(L)$ and $\alpha \in C(L)$. Then the following statements are equivalent:
(a) $\inf A_P(\alpha) \in \mathbb{R}$
(b) $A_P(\alpha) \neq \emptyset \neq B_P(\alpha)$.
(c) $\sup B_P(\alpha) \in \mathbb{R}$
(d) There exists an element $x \in \mathbb{R}$ such that

$$(x, +\infty) \subseteq (\inf A_P(\alpha), +\infty) \subseteq [x, +\infty) \text{ and }$$

$$(-\infty, x) \subseteq (-\infty, \sup B_P(\alpha)) \subseteq (-\infty, x].$$

(e) $\inf A_P(\alpha) = \sup B_P(\alpha) \in \mathbb{R}$. 

Proof. (a) ⇒ (b): By hypothesis, clearly, \( A_P(\alpha) \neq \emptyset \) and there exists an element \( x \in \mathbb{R} \) such that \( x \notin A_P(\alpha) \). By Lemma 2.17, \( x \in B_P(\alpha) \). Thus, \( B_P(\alpha) \) is also non-empty.

(b) ⇒ (c): By Lemma 2.17, it is clear.

(c) ⇒ (d): Similar to (a) ⇒ (b), it follows that \( A_P(\alpha) \neq \emptyset \neq B_P(\alpha) \).

Hence, by part (b) of Lemma 2.17, \( A_P(\alpha) \) and \( B_P(\alpha) \) are non-empty closed subsets in \( \mathbb{R} \). If \( P \) is real with respect to every \( \alpha \in C(L) \), then we say \( P \) is real.

Definition 2.19. \( P \in \text{Spec}(L) \) is said to be real with respect to \( \alpha \in C(L) \) if \( A_P(\alpha) \) and \( B_P(\alpha) \) are non-empty closed subsets in \( \mathbb{R} \). If \( P \) is real with respect to every \( \alpha \in C(L) \), then we say \( P \) is real.

Lemma 2.20. Assume that \( P \in \text{Spec}(L) \) and \( \alpha \in C(L) \). Then, the following statements are equivalent:

(a) \( P \) is real with respect to \( \alpha \).
(b) \( \inf A_P(\alpha) \in A_P(\alpha) \) and \( \sup B_P(\alpha) \in B_P(\alpha) \).
(c) There is an element \( x \in \mathbb{R} \) such that \( A_P(\alpha) \cap B_P(\alpha) = \{x\} \).
(d) There exists an element \( x \in \mathbb{R} \) such that \( \alpha(\mathbb{R}_x) \in P \).

Proof. By Corollary 2.18, it is clear.

Lemma 2.21. Let \( P \in \text{Spec}(L) \) be countably \( \lor \)-complete. Then \( P \) is real.

Proof. Suppose that \( \alpha \in C(L) \). Since \( P \) is countably \( \lor \)-complete, it follows that \( \inf A_P(\alpha) \in \mathbb{R} \) and so, by Corollary 2.18, there exists an element \( x \in \mathbb{R} \) such that
\[
(x, +\infty) \subseteq (\inf A_P(\alpha), +\infty) \subseteq [x, +\infty)
\]
and
\[
(-\infty, x) \subseteq (-\infty, \sup B_P(\alpha)) \subseteq (-\infty, x].
\]
By Lemma 2.20, it is enough to show that \( x \in A_P(\alpha) \cap B_P(\alpha) \). This is obvious, since \( P \) is countably \( \lor \)-complete and \( \mathbb{Q} \) is dense in \( \mathbb{R} \).
By the above lemma, $\downarrow p$ is real for each $p \in \text{Sp}L$.

We need the following lemma for the next theorem.

**Lemma 2.22.** Let $P$ be prime ideal in a frame $L$ and $\alpha \in C(L)$. The following statements hold:

(a) $A_P(-\alpha) = -B_P(\alpha)$ and $B_P(-\alpha) = -A_P(\alpha)$.

(b) $B_P(\alpha^+) = (-\infty, 0) \cup B_P(\alpha)$.

(c) $A_P(\alpha^+) = (0, +\infty) \cap A_P(\alpha)$.

(d) $B_P(\alpha^-) = (-\infty, 0) \cup -A_P(\alpha)$.

(e) $A_P(\alpha^-) = (0, +\infty) \cap -B_P(\alpha)$.

If, in addition, $\hat{P}(\alpha) = \inf A_P(\alpha) \in \mathbb{R}$, then

(f) $\hat{P}(\alpha^+) = (\hat{P}(\alpha))^+$;

(g) $\hat{P}(\alpha^-) = (\hat{P}(\alpha))^-$.

**Proof.** (a): It is clear that

$$A_P(-\alpha) = \{x \in \mathbb{R} : -\alpha(x, +\infty) \in P\} = \{x \in \mathbb{R} : \alpha(-\infty, -x) \in P\} = -\{y \in \mathbb{R} : \alpha(-\infty, y) \in P\} = -B_P(\alpha).$$

Similarly, we conclude that $B_P(-\alpha) = -A_P(\alpha)$.

(b): We can write

$$B_P(\alpha^+) = \{x \in \mathbb{R} : \alpha^+(-\infty, x) \in P\} = \{x \in \mathbb{R} : 0(-\infty, x) \land \alpha(-\infty, x) \in P\} = \{x \in \mathbb{R} : 0(-\infty, x) \in P\} \cup \{x \in \mathbb{R} : \alpha(-\infty, x) \in P\} = (-\infty, 0) \cup B_P(\alpha).$$

(c): We can write

$$A_P(\alpha^+) = \{x \in \mathbb{R} : \alpha^+(x, +\infty) \in P\} = \{x \in \mathbb{R} : 0(x, +\infty) \lor \alpha(x, +\infty) \in P\} = \{x \in \mathbb{R} : 0(x, +\infty) \in P\} \cap \{x \in \mathbb{R} : \alpha(x, +\infty) \in P\} = (0, +\infty) \cap A_P(\alpha).$$

(d): By parts (a) and (b), it follows that

$$B_P(\alpha^-) = B_P((-\alpha)^+) = (-\infty, 0) \cup B_P(-\alpha) = (-\infty, 0) \cup -A_P(\alpha).$$

(e): Using (a) and (c), we do similar to (d).

(f): By part (b) and Corollary 2.18, we can write

$$(\hat{P}(\alpha))^+ = 0 \lor \hat{P}(\alpha) = \sup(-\infty, 0) \lor \sup B_P(\alpha) = \sup B_P(\alpha^+) = \hat{P}(\alpha^+).$$
(g): By part (d) and Corollary 2.18, we can write
\[(\hat{P}(\alpha))^- = 0 \lor -\hat{P}(\alpha) = \sup((-\infty, 0) \cup -A_P(\alpha)) = \sup B_P(\alpha^-) = \hat{P}(\alpha^-).\]

The following theorem is an improvement of [6, Proposition 2.3] (also, see [7, Proposition 3.9] and [9, Proposition 2.3]).

**Theorem 2.23.** Assume that \(P \in \text{Spec}(L)\) and is countably \(\lor\)-complete in \(L\). We define
\[\hat{P} : C(L) \to \mathbb{R}, \quad \hat{P}(\alpha) = \inf A_P(\alpha).\]
Then \(\hat{P}\) is an \(f\)-algebra homomorphism; that is,
(a) \(\hat{P}(\alpha + \beta) = \hat{P}(\alpha) + \hat{P}(\beta)\) for every \(\alpha, \beta \in C(L)\).
(b) \(\hat{P}(\alpha\beta) = \hat{P}(\alpha)\hat{P}(\beta)\) for every \(\alpha, \beta \in C(L)\).
(c) \(\hat{P}(r\alpha) = r\hat{P}(\alpha)\) for every \(r \in \mathbb{R}\) and every \(\alpha \in C(L)\).
(d) \(\hat{P}(\alpha \lor \beta) = \hat{P}(\alpha) \lor \hat{P}(\beta)\) for every \(\alpha, \beta \in C(L)\).
(e) \(\hat{P}(\alpha \land \beta) = \hat{P}(\alpha) \land \hat{P}(\beta)\) for every \(\alpha, \beta \in C(L)\).

**Proof.** (a): Let \(x = \hat{P}(\alpha + \beta)\). Since \(P\) is countably \(\lor\)-complete, we have
\[(\alpha + \beta)(x, +\infty) \in P.\]
Therefore,
\[(\alpha + \beta)(x, +\infty) = \bigvee \{\alpha(r, s) \land \beta(t, u) : (r, s) + (t, u) \subseteq (x, +\infty)\}\]
\[= \bigvee \{\alpha(r, s) \land \beta(t, u) : r + t \geq x\}\]
\[= \bigvee \{\alpha(r, +\infty) \land \beta(t, +\infty) : r + t \geq x\}\]
\[= \bigvee \{\alpha(r, +\infty) \land \beta(x - r, +\infty) : r \in \mathbb{R}\} \in P.\]

Hence
\[\bigvee \{\alpha(r, +\infty) \land \beta(x - r, +\infty) : r < \hat{P}(\alpha), r \in \mathbb{Q}\} \in P.\]

Since \(\alpha(r, +\infty) \notin P\) for every \(r < \hat{P}(\alpha)\), it follows that \(\beta(x - r, +\infty) \in P\) for every rational \(r < \hat{P}(\alpha)\) and so, by countably \(\lor\)-completeness of \(P\), we can write
\[\beta(x - \hat{P}(\alpha), +\infty) = \bigvee \{\beta(x - r, +\infty) : r < \hat{P}(\alpha), r \in \mathbb{Q}\} \in P.\]
Thus,
\[ \hat{P}(\beta) \leq x - \hat{P}(\alpha) \implies \hat{P}(\alpha) + \hat{P}(\beta) \leq x. \] (1)

On the other hand, it is clear that for every \( s > \sup B_P(\alpha) = \hat{P}(\alpha) \), we have \( \alpha(-\infty, s) \notin P \). Therefore, similar to the above, it conclude that \( \beta(-\infty, x - s) \in P \) for every \( s > \hat{P}(\alpha) \). Consequently,
\[ \beta(-\infty, x - \hat{P}(\alpha)) = \bigvee \{ \beta(-\infty, x - s) : s > \hat{P}(\alpha), s \in \mathbb{Q} \} \in P. \]

Hence, we can write
\[ x - \hat{P}(\alpha) \leq \hat{P}(\beta) \implies x \leq \hat{P}(\alpha) + \hat{P}(\beta). \] (2)

The desired equality follows from (1) and (2).

(b): Case (1): \( \alpha, \beta \geq 0 \) and \( \hat{P}(\alpha \beta) = 0 \). In this case, we show that \( \hat{P}(\alpha) = 0 \) or \( \hat{P}(\beta) = 0 \). Since \( \hat{P}(\alpha \beta) = 0 \), \( (\alpha \beta)(0, +\infty) \in P \) and since \( \alpha(-\infty, 0) = 0, \beta(-\infty, 0) = 0 \), we can write
\[ (\alpha \beta)(0, +\infty) = \bigvee \{ \alpha(r, s) \land \beta(t, u) : (r, s)(t, u) \in (0, +\infty) \} \]
\[ = \bigvee \{ \alpha(r, s) \land \beta(t, u) : r, t \geq 0 \} \]
\[ = \bigvee \{ \alpha(r, +\infty) \land \beta(t, +\infty) : r, t \geq 0 \} \]
\[ = \alpha(0, +\infty) \land \beta(0, +\infty) \in P. \]

Therefore, \( \beta(\mathbb{R}_0) = \beta(0, +\infty) \in P \) or \( \alpha(\mathbb{R}_0) = \alpha(0, +\infty) \in P \). Thus, \( \hat{P}(\alpha) = 0 \) or \( \hat{P}(\beta) = 0 \).

Case (2): \( \alpha, \beta \geq 0 \) and \( \hat{P}(\alpha \beta) = x > 0 \). In this case
\[ \alpha \beta(x, +\infty) \in P \implies \alpha \beta(x, +\infty) = \bigvee_{r>0} \left( \alpha(r, +\infty) \land \beta\left(\frac{x}{r}, +\infty\right) \right) \in P. \]

Since \( \alpha(r, +\infty) \notin P \) for every \( 0 < r < \hat{P}(\alpha) \), it follows that \( \beta(\frac{x}{r}, +\infty) \in P \) for every \( 0 < r < \hat{P}(\alpha) \). Therefore, for every \( 0 < r < \hat{P}(\alpha) \), we have \( \frac{x}{r} \geq \hat{P}(\beta) \) and so \( \frac{x}{\hat{P}(\alpha)} \geq \hat{P}(\beta) \). This implies that
\[ x \geq \hat{P}(\alpha) \hat{P}(\beta). \] (3)
Since $\alpha(-\infty, s) \not\in P$ for every $s > \hat{P}(\alpha)$, similar to above, we conclude that $\beta(-\infty, \frac{x}{s}) \in P$ for every $s > \hat{P}(\alpha)$. Thus, $\frac{x}{s} \leq \hat{P}(\beta)$ for every $s > \hat{P}(\alpha)$ and consequently, $\frac{x}{\hat{P}(\alpha)} \leq \hat{P}(\beta)$. Hence,

$$x \leq \hat{P}(\alpha)\hat{P}(\beta). \quad (4)$$

From (3) and (4), it follows that $\hat{P}(\alpha\beta) = \hat{P}(\alpha)\hat{P}(\beta)$.

Final case: Let $\alpha, \beta \in C(L)$ be arbitrary. By previous cases, we can write

$$\hat{P}(\alpha\beta) = \hat{P}((\alpha^+ - \alpha^-)(\beta^+ - \beta^-))$$

$$= \hat{P}(\alpha^+)\hat{P}(\beta^+) - \hat{P}(\alpha^+)\hat{P}(\beta^-) - \hat{P}(\alpha^-)\hat{P}(\beta^+) + \hat{P}(\alpha^-)\hat{P}(\beta^-).$$

On the other hand, by Lemma 2.22, we have $\hat{P}(\alpha^-) = (\hat{P}(\alpha))^-$ and $\hat{P}(\alpha^+) = (\hat{P}(\alpha))^+$. Therefore

$$\hat{P}(\alpha\beta) = (\hat{P}(\alpha))^+(\hat{P}(\beta))^+ - (\hat{P}(\alpha))^+(\hat{P}(\beta))^-- (\hat{P}(\alpha))^-(\hat{P}(\beta))^+ + (\hat{P}(\alpha))^-(\hat{P}(\beta))^--$$

$$= (\hat{P}(\alpha)^+ - \hat{P}(\alpha)^-)(\hat{P}(\beta)^+ - \hat{P}(\beta)^-) = \hat{P}(\alpha)\hat{P}(\beta).$$

(c): If $r = 0$, the assertion is clear. If $r > 0$, then

$$\hat{P}(r\alpha) = \inf \{ x : r\alpha(x, +\infty) \in P \} = \inf \left\{ x : \alpha \left( \frac{x}{r}, +\infty \right) \in P \right\}$$

$$= \inf \{ ry : \alpha(y, +\infty) \in P \} = r\hat{P}(\alpha).$$

Finally, if $r < 0$, then

$$\hat{P}(r\alpha) = \inf \{ x : r\alpha(x, +\infty) \in P \} = \inf \{ x : -r\alpha(-\infty, -x) \in P \}$$

$$= \inf \left\{ x \in \mathbb{R} : \alpha(-\infty, \frac{x}{r}) \in P \right\} = \inf \{ ry : \alpha(-\infty, y) \in P \}$$

$$= r \sup \{ y : \alpha(-\infty, y) \in P \} = r\hat{P}(\alpha).$$

Therefore, $\hat{P}(r\alpha) = r\hat{P}(\alpha)$ for every $r \in \mathbb{R}$.

(d): Clearly, we can write

$$\hat{P}(\alpha \lor \beta) = \sup \{ x \in \mathbb{R} : (\alpha \lor \beta)(-\infty, x) \in P \}$$

$$= \sup \{ x : \alpha(-\infty, x) \cap \beta(-\infty, x) \in P \}$$

$$= \sup \left( \{ x : \alpha(-\infty, x) \in P \} \cup \{ x : \beta(-\infty, x) \in P \} \right)$$

$$= \sup \{ x : \alpha(-\infty, x) \in P \} \lor \sup \{ x : \beta(-\infty, x) \in P \}$$

$$= \hat{P}(\alpha) \lor \hat{P}(\beta).$$
(e): It is similar to the proof of the part (d).

Note that, by Lemma 2.20, we obtain the following result, clearly.

**Corollary 2.24.** Suppose that \( P \in \text{Spec}(L) \) is countably \( \lor \)-complete. Then \( \hat{P}(\alpha) = x \) if and only if \( \alpha(\mathbb{R}_x) \in P \).

**Corollary 2.25.** Assume that \( p \in \text{Sp}L \) and

\[
\hat{p} : C(L) \to \mathbb{R}, \quad \hat{p}(\alpha) = \inf\{x \in \mathbb{R} : \alpha(x, +\infty) \leq p\}.
\]

Then \( \hat{p} \) is an \( f \)-algebra homomorphism.

**Proof.** It suffices to put \( P = \downarrow p \), then, by Theorem 2.23, we are done.

We are now ready to answer the Question 3 which we raised earlier.

**Theorem 2.26.** Suppose that \( L \) is a frame in which every maximal ideal is countable \( \lor \)-complete. Then for every \( \alpha, \beta \in C(L) \), we have the following relations:

1. \( \text{pim}(\alpha + \beta) \subseteq \text{pim}(\alpha) + \text{pim}(\beta) \).
2. \( \text{pim}(\alpha\beta) \subseteq \text{pim}(\alpha)\text{pim}(\beta) \).
3. \( \text{pim}(\alpha \lor \beta) \subseteq \text{pim}(\alpha) \lor \text{pim}(\beta) \).
4. \( \text{pim}(\alpha \land \beta) \subseteq \text{pim}(\alpha) \land \text{pim}(\beta) \).

**Proof.** We only prove part (a); other parts are proved by the same manner. Suppose that \( x \in \text{pim}(\alpha + \beta) \). Thus, \( (\alpha + \beta)(\mathbb{R}_x) \neq \text{Top} \) and so there exists an element \( M \in \text{Max}(L) \) such that \( (\alpha + \beta)(\mathbb{R}_x) \in M \). Therefore, by Theorem 2.23 and Corollary 2.24, \( x = \hat{M}(\alpha + \beta) = \hat{M}(\alpha) + \hat{M}(\beta) \). Taking \( \hat{M}(\alpha) = a \) and \( \hat{M}(\beta) = b \), it is sufficient to show that \( a \in \text{pim}(\alpha) \) and \( b \in \text{pim}(\beta) \). To see this, by Corollary 2.24, \( \alpha(\mathbb{R}_a) \in M \) and \( \beta(\mathbb{R}_b) \in M \). Hence, \( \alpha(\mathbb{R}_a) \neq \text{Top} \neq \beta(\mathbb{R}_b) \), so \( a \in \text{pim}(\alpha) \) and \( b \in \text{pim}(\beta) \). Therefore, \( \text{pim}(\alpha + \beta) \subseteq \text{pim}(\alpha) + \text{pim}(\beta) \). \( \Box \)
3 Comparing $\text{pim}(\alpha)$ with images of two real functions $\overline{\alpha}$ and $\hat{\alpha}$

In this section, first, for any $\alpha \in C(L)$, we introduce two real functions $\overline{\alpha}$ and $\hat{\alpha}$ induced naturally by $\alpha$, then we compare $\text{pim}(\alpha)$ with the images of these two functions.

**Definition 3.1.** Suppose that $\alpha \in C(L)$. By Corollary 2.25, we can define $\overline{\alpha} : \text{Sp}L \to \mathbb{R}$ with $\overline{\alpha}(p) = \hat{\alpha}(\alpha)$. Also, supposing $\overline{\alpha} : \text{Sp}L \to \mathbb{R}$ with $\overline{\alpha}(p) = \hat{\alpha}(\alpha)$, we can define $\hat{\alpha} : X_\alpha \to \mathbb{R}$ with $\hat{\alpha}(P) = \hat{P}(\alpha)$.

Note that the mapping $p \to \downarrow p$ is an embedding from $\text{Sp}L$ to $\text{Spec}(L)$, where $\text{Spec}(L)$ is equipped with hall-kernel topology (that is, the Zariski topology). Therefore, we can suppose that $\text{Sp}L$ is a subspace of $\text{Spec}(L)$ and so $\hat{\alpha}|_{\text{Sp}L} = \overline{\alpha}$.

**Proposition 3.2.** For every $\alpha \in C(L)$, $\hat{\alpha}$ is continuous and so is $\overline{\alpha}$.

**Proof.** Assume that $(x, y)$ is an open interval in $\mathbb{R}$. Taking $a = \alpha(x, +\infty)$ and $b = \alpha(-\infty, y)$, it suffices to show that $(\hat{\alpha})^{-1}(x, y) = h^c_{X_\alpha}(a) \cap h^c_{X_\alpha}(b)$, where $h^c_{X_\alpha}(a) = X_\alpha \cap h^c(a)$. Too see this, for every $P \in X_\alpha$, we can write

\[
P \in (\hat{\alpha})^{-1}(x, y) \iff x < \hat{\alpha}(P) = \hat{P}(\alpha) < y \iff a = \alpha(x, +\infty) \notin P, b = \alpha(-\infty, y) \notin P \iff P \in h^c_{X_\alpha}(a) \cap h^c_{X_\alpha}(b).
\]

The following remark shows that $\overline{\alpha}$ is not a new concept.

**Remark 3.3.** Recall that $\text{Sp} \mathcal{O} \mathbb{R} = \{R_x : x \in \mathbb{R}\}$ and $g : \text{Sp} \mathcal{O} \mathbb{R} \to \mathbb{R}$ with $g(R_x) = x$ is a homeomorphism. For every continuous real function $\alpha \in C(L)$, we have $\text{Sp} \alpha : \text{Sp}L \to \text{Sp} \mathcal{O} \mathbb{R}$ with $(\text{Sp} \alpha)(p) = \alpha^*(p) = \bigvee\{w \in \mathcal{O} \mathbb{R} : \alpha(w) \leq p\}$. Since $\alpha^*(p) \in \text{Sp} \mathcal{O} \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $(\text{Sp} \alpha)(p) = \alpha^*(p) = R_x$. In fact, $(\text{Sp} \alpha)(p) = R_x$ if and only if $\alpha(R_x) \leq p$. Therefore, for every $\alpha \in C(L)$, we have a natural function $\overline{\alpha} = g \text{Sp} \alpha$ from $\text{Sp}L$ to $\mathbb{R}$ with $\overline{\alpha}(p) = x$ such that $\alpha(R_x) \leq p$. Also, according to this fact, for every $p \in \text{Sp}L$, we can define a function $\hat{\alpha} : C(L) \to \mathbb{R}$ with $\hat{\alpha}(\alpha) = \overline{\alpha}(p)$. 
Proposition 3.4. Assume that $\alpha \in C(L)$. Then $\text{Im}(\alpha) \subseteq \text{Im}(\hat{\alpha}) \subseteq \text{pim}(\alpha)$.

Proof. Clearly, $\text{Im}(\alpha) \subseteq \text{Im}(\hat{\alpha})$. Now, suppose that $x \in \text{Im}(\hat{\alpha})$. Thus, there exists a $P \in \text{Spec}(L)$ such that $\hat{\alpha}(P) = x$. Hence, $P(\alpha) = x$ and by Corollary 2.24, it follows that $\alpha(\mathbb{R}_x) \in P$. Therefore, $\alpha(\mathbb{R}_x) \neq \text{Top}$ and consequently $x \in \text{pim}(\alpha)$. □

The first inclusion in the above proposition may be strict. To see this, we need the following lemma.

Lemma 3.5. Suppose that $L$ has no non-trivial complemented element. Then for every $\alpha \in C(L)$, there exists an element $x \in \mathbb{R}$ such that $\alpha(\mathbb{R}_x) \neq \text{Top}$.

Proof. Let $\alpha \in C(L)$ and, on the contrary, for every $x \in \mathbb{R}$, we have $\alpha(\mathbb{R}_x) = \text{Top}$. By hypothesis, for every $x \in \mathbb{R}$, we $\alpha(-\infty, x) = \text{Top}$ and $\alpha(x, +\infty) = \bot$ or $\alpha(-\infty, x) = \bot$ and $\alpha(x, +\infty) = \text{Top}$. It is easy to see that there exists an element $c \in \mathbb{R}$ such that $\alpha(c, +\infty) = \bot$ and so $x_0 = \inf\{x \in \mathbb{R} : \alpha(x, +\infty) = \bot\}$ exists. Thus, $\alpha(x_0, +\infty) = \bot$ and $\alpha(t, +\infty) = \text{Top}$ for every $t < x_0$ and so $\alpha(-\infty, t) = \bot$ for every $t < x_0$. Therefore, $\alpha(-\infty, x_0) = \bigvee\{\alpha(-\infty, t) : t < x_0\} = \bot$. Hence, $\alpha(\mathbb{R}_{x_0}) = \bot$ and this is a contradiction. □

In the following example we introduce a frame $L$ such that $\text{Im}(\alpha) \subsetneq \text{pim}(\hat{\alpha})$ for every $\alpha \in C(L)$.

Example 3.6. Suppose $L = [0, 1) \times [0, 1) \oplus \text{Top}$. Clearly, $L$ is a frame, $\text{Top}$ is a $\lor$-prime element of $L$ and $\text{Sp}L = \emptyset$. Therefore, $L$ does not have any non-trivial complemented element and so, by Lemma 3.5, for every $\alpha \in C(L)$ we have $\alpha(\mathbb{R}_x) \neq \text{Top}$ for some $x \in \mathbb{R}$. We show that $C(L) = \{r : r \in \mathbb{R}\}$. To see this, assume that $\alpha \in C(L)$. Thus, there exists an element $r \in \mathbb{R}$ such that $\alpha(\mathbb{R}_r) \neq \text{Top}$. Now, for every $w \in \mathcal{O}_r$, since $\text{Top}$ is $\lor$-prime, we can write

$$\text{Top} = \alpha(\mathbb{R}) = \alpha(w \cup \mathbb{R}_r) = \alpha(w) \lor \alpha(\mathbb{R}_r) \Rightarrow \alpha(w) = \text{Top}.$$ 

This conclude that $\alpha = r$. On the other hand, it is clear that $\text{Im}(\mathbb{R}) = \emptyset$, ...
whereas
\[ x \in \text{Im}(\hat{r}) \iff \exists P \in \text{Spec}(L), \quad \hat{r}(P) = x \]
\[ \iff \exists P \in \text{Spec}(L), \quad \hat{P}(r) = x \]
\[ \iff \exists P \in \text{Spec}(L), \quad r(R_x) \in P \]
\[ \iff r = x. \]

Therefore, \( \text{Im}(\hat{r}) = \{ r \} \).

**Proposition 3.7.** Assume that \( \alpha \in C(L) \). Then the following statements hold:

- (a) If \( \text{Sp} L \) is cofinal in \( L \setminus \{ \text{Top} \} \), then \( \text{Im}(\overline{\alpha}) = \text{Im}(\hat{\alpha}) = \text{pim}(\alpha) \).
- (b) If \( \bigcup X_\alpha = L \setminus \{ \text{Top} \} \), then \( \text{Im}(\hat{\alpha}) = \text{pim}(\alpha) \).

**Proof.** (a): It is enough to prove that \( \text{pim}(\alpha) \subseteq \text{Im}(\overline{\alpha}) \). Suppose that \( x \in \text{pim}(\alpha) \). Thus, \( \alpha(R_x) \neq \text{Top} \) and by hypothesis, there exists an element \( p \in \text{Sp} L \) such that \( \alpha(R_x) \leq p \) and this is equivalent to \( \overline{\alpha}(p) = \check{p}(\alpha) = x \). Therefore, \( x \in \text{Im}(\overline{\alpha}) \).

(b): Suppose that \( x \in \text{pim}(\alpha) \). Thus, \( \alpha(R_x) \neq \text{Top} \) and by hypothesis, there exists an element \( P \in X_\alpha \) such that \( \alpha(R_x) \in P \) and this is equivalent to \( \hat{\alpha}(P) = \hat{P}(\alpha) = x \). Therefore, \( x \in \text{Im}(\hat{\alpha}) \).

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