



Cofree objects in the centralizer and the center categories

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Abstract. We study cocompleteness, co-wellpoweredness, and generators in the centralizer category of an object or morphism in a monoidal category, and the center or the weak center of a monoidal category. We explicitly give some answers for when colimits, cocompleteness, co-wellpoweredness, and generators in these monoidal categories can be inherited from their base monoidal categories. Most importantly, we investigate cofree objects of comonoids in these monoidal categories.

1 Introduction and Preliminaries

Universal properties are crucially considered as one of the most important concepts in mathematics. Indeed, they can be thought of as the skeleton of all mathematics concepts. They show how the objects and the morphisms nicely relate the whole category that they live in. Many influential concepts, such as kernels, cokernels, products, coproducts, limits, colimits, etc., are essentially involved with universal properties. Perhaps the most important

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notion concerned with them is the concept of adjoint functors. It is simply because “Adjoint functors arise everywhere” [16, p. (vii)]. Significantly, free and cofree objects play a crucial role in recasting the adjunctions of the forgetful functors in terms of comma categories. For fundamental concepts and examples of adjoint functors, we refer the reader to [15], [16], [3], [20], [19], or [18]. For the basic notions of comma categories, we refer to [15] and [16].

Let \mathfrak{X} be a category. A *concrete category* over \mathfrak{X} is a pair $(\mathfrak{A}, \mathfrak{U})$, where \mathfrak{A} is a category and $\mathfrak{U} : \mathfrak{A} \rightarrow \mathfrak{X}$ is a faithful functor [2, p. 61]. Let $(\mathfrak{A}, \mathfrak{U})$ be a concrete category over \mathfrak{X} . Following [2, p. 140-143], a *free object* over \mathfrak{X} -object X is an \mathfrak{A} -object A such that there exists a *universal arrow* (A, u) over X ; that is, $u : X \rightarrow \mathfrak{U}A$ such that for every arrow $f : X \rightarrow \mathfrak{U}B$, there exists a unique morphism $f' : A \rightarrow B$ in \mathfrak{A} such that $\mathfrak{U}f'u = f$. We also say that (A, u) is the free object over X . A concrete category $(\mathfrak{A}, \mathfrak{U})$ over \mathfrak{X} is said to *have free objects* provided that for each \mathfrak{X} -object X , there exists a universal arrow over X . For example, the category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over a field \mathbb{K} has free objects. So do the category \mathbf{Top} of topological spaces and the category \mathbf{Grp} of groups. However, some interesting categories do not have free objects [2, p. 142]).

Dually, *co-universal arrows*, *cofree objects*, and categories that *have cofree objects* can be defined. For the basic concepts of concrete categories, free objects, and cofree objects, we refer the readers to [14, p. 138-155]. It turns out that a concrete category $(\mathfrak{A}, \mathfrak{U})$ over \mathfrak{X} has (co)free objects if and only if the functor that constructs (co)free objects is a (right) left adjoint to the faithful functor $\mathfrak{U} : \mathfrak{A} \rightarrow \mathfrak{X}$.

Although cofree objects are the dual of free objects, the behavior of cofree objects is more complicated than the one of free objects. Furthermore, studying such behavior cannot be obtained by studying free objects, because “the categories considered are not selfdual generally” [14, p. 149]. In this paper, we are interested in investigating cofree objects in the centralizer category of an object or morphism in a monoidal category and the center or the weak center of a monoidal category. For the basic notions of monoidal categories, we refer the readers to [10], [4], and [8, Chapter 6].

More recently, these monoidal categories play a vibrant role in characterizing and identifying many of the interesting categories. For instance, to show that two finite tensor categories are Morita equivalent, it suffices

to show that their centers are equivalent as braided tensor categories [10, p. 222]. Another example is to show that a fusion category is group-theoretical, it is sufficient to show its center contains a Lagrangian subcategory [10, p. 313]. In addition, there is a special importance for the center of a finite tensor category in finding its Frobenius-Perron dimension. This comes from the fact that for any finite tensor category \mathcal{C} , we have $FPdim(\mathcal{Z}(\mathcal{C})) = FPdim(\mathcal{C})^2$ [10, p. 168]. We refer to [13] for basics on centralizer categories while we refer to [23, p. 76] and [10, p. 162] for basics on center categories.

Explicitly, the problem can be formulated as follows. Let \mathcal{C} be a monoidal category. Fix an object X and a morphism $h : A \rightarrow B$ in \mathcal{C} . For any $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$, let $\mathcal{U}_{\mathcal{A}} : CoMon(\mathcal{A}) \rightarrow \mathcal{A}$ be the forgetful functor corresponding to \mathcal{A} . Does $\mathcal{U}_{\mathcal{A}}$ have a right adjoint? A reasonably expected machinery for the answer of this question is the dual of Special Adjoint Functor Theorem (D-SAFT).

We start our inspection by studying cocompleteness in \mathcal{A} , and we give some answers for the question: under what conditions the colimits of diagrams in \mathcal{A} can be obtained from the corresponding construction of diagrams in \mathcal{C} . The later implicitly implies that the forgetful functor $\mathcal{U}_{\mathcal{A}}$ is cocontinuous. Next, we investigate conditions under which the category \mathcal{A} inherits the co-wellpoweredness of \mathcal{C} . We also show how the braiding forces the category \mathcal{A} to inherit generators from its base category \mathcal{C} . Finally, we apply the mechanism of D-SAFT for each case. Furthermore, we try to visualize some interesting consequences by studying the braid category.

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and for every $X \in \mathcal{C}$, let $\mathcal{P}_X, \mathcal{Q}_X$ be the functors defined by

$$\begin{aligned} \mathcal{P}_X &= X \otimes - : \mathcal{C} \rightarrow \mathcal{C}, M \mapsto X \otimes M, \\ \mathcal{Q}_X &= - \otimes X : \mathcal{C} \rightarrow \mathcal{C}, M \mapsto M \otimes X. \end{aligned}$$

If \mathcal{C} is a biclosed monoidal category, then both functors \mathcal{P}_X and \mathcal{Q}_X have right adjoints for every object $X \in \mathcal{C}$, hence they are cocontinuous for every object $X \in \mathcal{C}$. However, this is not true in general; that is, if \mathcal{C} is a cocomplete monoidal category, then the tensor product functors \mathcal{P}_X and \mathcal{Q}_X needs not be cocontinuous in each object X . Thus, one might need to consider them more carefully.

Theorem 1.1. [11, p. 148] *If \mathfrak{A} is cocomplete, co-wellpowered, and with a generating set, then every cocontinuous functor from \mathfrak{A} to a locally small category has a right adjoint.*

Definition 1.2. (i) [21, p. 284] The *(left) weak center* of \mathcal{C} denoted $\mathcal{Z}_\omega(\mathcal{C})$ is a category whose objects are pairs $(A, \sigma_{A,-})$, where $A \in \mathcal{C}$ and $\sigma_{A,-} : A \otimes - \rightarrow - \otimes A$ is a natural transformation such that the following conditions hold

$$\sigma_{A,I} = id_A \quad (1.1)$$

and

$$\sigma_{A,X \otimes Y} = (id_X \otimes \sigma_{A,Y})(\sigma_{A,X} \otimes id_Y) \quad (1.2)$$

for all $X, Y \in \mathcal{C}$.

An arrow $f : (A, \sigma_{A,-}) \rightarrow (B, \tau_{B,-})$ in $\mathcal{Z}_\omega(\mathcal{C})$ is an arrow $f : A \rightarrow B$ in \mathcal{C} such that, for all $X \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} A \otimes X & \xrightarrow{(f \otimes id_X)} & B \otimes X \\ \sigma_{A,X} \downarrow & & \downarrow \tau_{B,X} \\ X \otimes A & \xrightarrow{(id_X \otimes f)} & X \otimes B \end{array} \quad (1.3)$$

commutes.

The category $\mathcal{Z}_\omega(\mathcal{C})$ is monoidal with

$$(A, \sigma_{A,-}) \otimes (B, \tau_{B,-}) = (A \otimes B, \delta_{A \otimes B, -}), \quad (1.4)$$

where

$$\delta_{A \otimes B, X} : A \otimes B \otimes X \rightarrow X \otimes A \otimes B = (\sigma_{A,X} \otimes id_B)(id_A \otimes \tau_{B,X}). \quad (1.5)$$

(ii) [23, p. 76] The *center* of \mathcal{C} , denoted by $\mathcal{Z}(\mathcal{C})$, is a category whose objects are pairs $(A, \sigma_{A,-})$, where $A \in \mathcal{C}$ and $\sigma_{A,-} : A \otimes - \xrightarrow{\sim} - \otimes A$ is a natural isomorphism such that the following conditions hold:

$$\sigma_{A,I} = id_A \quad (1.6)$$

(more precisely, $\sigma_{A,I}$ is the composite of the canonical isomorphisms $A \otimes I \cong A \cong I \otimes A$), and

$$\sigma_{A,X \otimes Y} = (id_X \otimes \sigma_{A,Y})(\sigma_{A,X} \otimes id_Y) \quad (1.7)$$

for all $X, Y \in \mathcal{C}$.

An arrow $f : (A, \sigma_{A,-}) \rightarrow (B, \tau_{B,-})$ in $\mathcal{Z}(\mathcal{C})$ is an arrow $f : A \rightarrow B$ in \mathcal{C} such that, for all $X \in \mathcal{C}$, the following diagram is commutative

$$\begin{array}{ccc} A \otimes X & \xrightarrow{(f \otimes id_X)} & B \otimes X \\ \sigma_{A,X} \downarrow \sim & & \sim \downarrow \tau_{B,X} \\ X \otimes A & \xrightarrow{(id_X \otimes f)} & X \otimes B \end{array} \quad (1.8)$$

The category $\mathcal{Z}(\mathcal{C})$ is monoidal with

$$(A, \sigma_{A,-}) \otimes (B, \tau_{B,-}) = (A \otimes B, \delta_{A \otimes B,-}), \quad (1.9)$$

where

$$\delta_{A \otimes B, X} = (\sigma_{A, X} \otimes id_B)(id_A \otimes \tau_{B, X}). \quad (1.10)$$

The category $\mathcal{Z}(\mathcal{C})$ is braided via

$$\Psi_{(A, \sigma_{A,-}), (B, \tau_{B,-})} = \sigma_{A, B} : (A, \sigma_{A,-}) \otimes (B, \tau_{B,-}) \rightarrow (B, \tau_{B,-}) \otimes (A, \sigma_{A,-}). \quad (1.11)$$

Definition 1.3. [13, p. 46-47] The *centralizer* $\mathcal{Z}_X(\mathcal{C})$ of an object $X \in \mathcal{C}$ is the category whose objects are pairs (A, α) , where $A \in \mathcal{C}$ and $\alpha : A \otimes X \xrightarrow{\sim} X \otimes A$.

An arrow $f : (A, \alpha) \rightarrow (B, \beta)$ in $\mathcal{Z}_X(\mathcal{C})$ is an arrow $f : A \rightarrow B$ in \mathcal{C} such that the following diagram is commutative:

$$\begin{array}{ccc} A \otimes X & \xrightarrow{(f \otimes id_X)} & B \otimes X \\ \alpha \downarrow \sim & & \sim \downarrow \beta \\ X \otimes A & \xrightarrow{(id_X \otimes f)} & X \otimes B \end{array} \quad (1.12)$$

This becomes a monoidal category with

$$(A, \sigma) \otimes (B, \tau) = (A \otimes B, \gamma), \quad (1.13)$$

where

$$\gamma = (\alpha \otimes id_Y)(id_X \otimes \beta). \quad (1.14)$$

Definition 1.4. [13, p. 49] The *centralizer* $\mathcal{Z}_h(\mathcal{C})$ of an arrow $h : A \rightarrow B$ in \mathcal{C} is the category whose objects are triples (X, α, β) , where $X \in \mathcal{C}$ and $\alpha : A \otimes X \xrightarrow{\sim} X \otimes A, \beta : B \otimes X \xrightarrow{\sim} X \otimes B$ are isomorphisms such that the following diagram is commutative

$$\begin{array}{ccc} A \otimes X & \xrightarrow{(h \otimes id_X)} & B \otimes X \\ \alpha \downarrow \sim & & \sim \downarrow \beta \\ X \otimes A & \xrightarrow{(id_X \otimes h)} & X \otimes B \end{array} \quad (1.15)$$

An arrow $f : (X, \alpha, \beta) \rightarrow (Y, \alpha', \beta')$ in $\mathcal{Z}_h(\mathcal{C})$ is an arrow $f : X \rightarrow Y$ in \mathcal{C} such that the following diagrams are commutative

$$\begin{array}{ccc} A \otimes X & \xrightarrow{\sim \alpha} & X \otimes A \\ (id_A \otimes f) \downarrow & & \downarrow (f \otimes id_A) \\ A \otimes Y & \xrightarrow{\sim \alpha'} & Y \otimes A \end{array} \quad \begin{array}{ccc} B \otimes X & \xrightarrow{\sim \beta} & X \otimes B \\ (id_B \otimes f) \downarrow & & \downarrow (f \otimes id_B) \\ B \otimes Y & \xrightarrow{\sim \beta'} & Y \otimes B \end{array} \quad (1.16)$$

Remark 1.5. The category $\mathcal{Z}_h(\mathcal{C})$ was introduced in [13, p. 49] as an essential part of the proof of Lemma 7, and the authors implicitly indicated that it is a monoidal category. For convenience, we explicitly show that the category $\mathcal{Z}_h(\mathcal{C})$ is monoidal.

Proposition 1.6. *Let $h : A \rightarrow B$ be an arrow in \mathcal{C} . Then the category $\mathcal{Z}_h(\mathcal{C})$ is monoidal with*

$$(X, \alpha, \beta) \otimes (Y, \alpha', \beta') = (X \otimes Y, \bar{\alpha}, \bar{\beta}), \quad (1.17)$$

where $\bar{\alpha}, \bar{\beta}$ are given respectively by the compositions

$$A \otimes X \otimes Y \xrightarrow[\sim]{\alpha \otimes id_Y} X \otimes A \otimes Y \xrightarrow[\sim]{id_X \otimes \alpha'} X \otimes Y \otimes A \quad (1.18)$$

$$B \otimes X \otimes Y \xrightarrow[\sim]{\beta \otimes id_Y} X \otimes B \otimes Y \xrightarrow[\sim]{id_X \otimes \beta'} X \otimes Y \otimes B \quad (1.19)$$

Proof. We have

$$\begin{aligned}
\bar{\beta}(h \otimes id_{X \otimes Y}) &= (id_X \otimes \beta')(\beta \otimes id_Y)(h \otimes id_{X \otimes Y}) \text{ (definition of } \bar{\beta}) \\
&= (id_X \otimes \beta')(\beta \otimes id_Y)(h \otimes id_X \otimes id_Y) \text{ (}\otimes \text{ is a bifunctor)} \\
&= (id_X \otimes \beta')(\beta(h \otimes id_X) \otimes id_Y) \text{ (naturality of } \otimes) \\
&= (id_X \otimes \beta')((id_X \otimes h)\alpha \otimes id_Y) \text{ (since } X \in \mathcal{Z}_h(\mathcal{C})) \\
&= (id_X \otimes \beta')(id_X \otimes h \otimes id_Y)(\alpha \otimes id_Y) \text{ (naturality of } \otimes) \\
&= (id_X \otimes \beta'(h \otimes id_Y))(\alpha \otimes id_Y) \text{ (naturality of } \otimes) \\
&= (id_X \otimes (id_Y \otimes h)\alpha')(\alpha \otimes id_Y) \text{ (since } Y \in \mathcal{Z}_h(\mathcal{C})) \\
&= (id_X \otimes id_Y \otimes h)(id_X \otimes \alpha')(\alpha \otimes id_Y) \text{ (naturality of } \otimes) \\
&= (id_{X \otimes Y} \otimes h)(id_X \otimes \alpha')(\alpha \otimes id_Y) \text{ (}\otimes \text{ is a bifunctor)} \\
&= (id_{X \otimes Y} \otimes h)\bar{\alpha} \text{ (definition of } \bar{\alpha}).
\end{aligned}$$

Therefore, $X \otimes Y \in \mathcal{Z}_h(\mathcal{C})$ and, hence, the category $\mathcal{Z}_h(\mathcal{C})$ is monoidal. \square

Remark 1.7. (i) For all $X \in \mathcal{C}$, we have the following evaluation functor

$$\mathcal{Z}(\mathcal{C}) \xrightarrow{\mathcal{H}_X} \mathcal{Z}_X(\mathcal{C}), \quad (1.20)$$

where \mathcal{H}_X is defined by $(A, \sigma_{A,-}) \mapsto (A, \sigma_{A,X})$. It turns out that if $(A, \sigma_{A,-}) \in \mathcal{Z}(\mathcal{C})$, then $(A, \sigma_{A,X}) \in \mathcal{Z}_X(\mathcal{C})$, for every $X \in \mathcal{C}$. However, to show that an object $(A, \sigma_{A,-}) \in \mathcal{Z}(\mathcal{C})$, it suffices to show that $(A, \sigma_{A,X}) \in \mathcal{Z}_X(\mathcal{C})$ for every $X \in \mathcal{C}$, σ is a natural transformation and the condition 1.7 holds as well (since the condition 1.6 on objects of $\mathcal{Z}(\mathcal{C})$ is redundant [23, p. 76]).

(ii) For all $A \in \mathcal{C}$, if $(X, \alpha, \beta) \in \mathcal{Z}_{id_A}(\mathcal{C})$, then Definition 1.4 implies that $\alpha = \beta$. This gives rise to an isomorphism given by

$$\mathcal{Z}_A(\mathcal{C}) \begin{array}{c} \xrightarrow{\mathcal{S}_A} \\ \xleftarrow{\mathcal{T}_A} \end{array} \mathcal{Z}_{id_A}(\mathcal{C}) \quad (1.21)$$

where \mathcal{S}_A is defined by $(X, \alpha, \alpha) \mapsto (X, \alpha^{-1})$, and \mathcal{T}_A is defined by $(X, \alpha) \mapsto (X, \alpha^{-1}, \alpha^{-1})$.

Thus, we have $\mathcal{Z}_A(\mathcal{C}) \cong \mathcal{Z}_{id_A}(\mathcal{C})$. It turns out that the centralizer category of an object A in \mathcal{C} can be identified as the centralizer category of the identity morphism of A in \mathcal{C} . However, we will explicitly study the centralizer category of an object due to the discussion of part (i).

(iii) From Definitions 1.2, 1.3, $\mathcal{Z}(\mathcal{C})$ is a subcategory of $\mathcal{Z}_\omega(\mathcal{C})$.

2 Cocompleteness

Recall that a category \mathfrak{C} is cocomplete when every functor $\mathfrak{F} : \mathfrak{D} \rightarrow \mathfrak{C}$, with \mathfrak{D} a small category has a colimit [7]. For the basic notions of cocomplete categories and examples, we refer to [2], [7], or [22]. A functor is *cocontinuous* if it preserves all small colimits [11, p. 142].

Proposition 2.1. [1, p. 5] *Let $\text{CoMon}(\mathcal{C})$ be the category of comonoids of \mathcal{C} and $\mathcal{U} : \text{CoMon}(\mathcal{C}) \rightarrow \mathcal{C}$ the forgetful functor. If \mathcal{C} is cocomplete, then $\text{CoMon}(\mathcal{C})$ is cocomplete and \mathcal{U} is cocontinuous.*

Proposition 2.2. *Let \mathcal{C} be a cocomplete category and $h : A \rightarrow B$ an arrow in \mathcal{C} . If \mathcal{P}_J and \mathcal{Q}_J are cocontinuous $\forall J \in \{A, B\}$, then $\mathcal{Z}_h(\mathcal{C})$ is cocomplete and the forgetful functor $\mathcal{U} : \mathcal{Z}_h(\mathcal{C}) \rightarrow \mathcal{C}$ is cocontinuous. Furthermore, the colimits of diagrams in $\mathcal{Z}_h(\mathcal{C})$ can be obtained by the corresponding construction of diagrams in \mathcal{C} .*

Proof. Let \mathcal{D} be a small category, and let $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}_h(\mathcal{C})$ be a functor. Since \mathcal{C} is a cocomplete category, the functor $\mathcal{U}\mathcal{F}$ has a colimit $(C, (\phi_D)_{D \in \mathcal{D}})$. Since \mathcal{P}_A is cocontinuous, $\mathcal{P}_A \mathcal{U}\mathcal{F}$ has a colimit $(\mathcal{P}_A(C), (\mathcal{P}_A(\phi_D))_{D \in \mathcal{D}})$. Equivalently, $(A \otimes C, (id_A \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_A \mathcal{U}\mathcal{F}$.

First, we note that the functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}_h(\mathcal{C})$ assigns to each object $D \in \mathcal{D}$ an object $(\mathcal{F}D, \alpha_{\mathcal{F}D}, \beta_{\mathcal{F}D}) \in \mathcal{Z}_h(\mathcal{C})$. We also have $\mathcal{F}f : (\mathcal{F}D, \alpha_{\mathcal{F}D}, \beta_{\mathcal{F}D}) \rightarrow (\mathcal{F}D', \alpha_{\mathcal{F}D'}, \beta_{\mathcal{F}D'})$ is an arrow in $\mathcal{Z}_h(\mathcal{C})$ for every arrow $f : D \rightarrow D'$ in \mathcal{D} . Thus, we have the following commutative diagrams:

$$\begin{array}{ccc}
 A \otimes D & \xrightarrow[\sim]{\alpha_{\mathcal{F}D}} & D \otimes A \\
 (id_A \otimes \mathcal{F}f) \downarrow & & \downarrow (\mathcal{F}f \otimes id_A) \\
 A \otimes D' & \xrightarrow[\sim]{\alpha_{\mathcal{F}D'}} & D' \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 B \otimes D & \xrightarrow[\sim]{\beta_{\mathcal{F}D}} & D \otimes B \\
 (id_B \otimes \mathcal{F}f) \downarrow & & \downarrow (\mathcal{F}f \otimes id_B) \\
 B \otimes D' & \xrightarrow[\sim]{\beta_{\mathcal{F}D'}} & D' \otimes B
 \end{array}
 \tag{2.1}$$

Now consider the diagram

$$\begin{array}{ccccc}
 A \otimes \mathcal{F}D & \xrightarrow{id_A \otimes \mathcal{F}f} & A \otimes \mathcal{F}D' & & \\
 \downarrow \alpha_{\mathcal{F}D} \sim & \searrow id_A \otimes \phi_D & \swarrow id_A \otimes \phi_{D'} & & \downarrow \alpha_{\mathcal{F}D'} \sim \\
 & A \otimes C & & & \\
 & \downarrow \exists! \bar{\alpha} & & & \\
 \mathcal{F}D \otimes A & \xrightarrow{\phi_D \otimes id_A} & C \otimes A & \xleftarrow{\phi_{D'} \otimes id_A} & \mathcal{F}D' \otimes A \\
 & \searrow \mathcal{F}f \otimes id_A & & & \swarrow \\
 & & & &
 \end{array} \tag{2.2}$$

For any D in \mathcal{D} , we have

$$\begin{aligned}
 (\phi_{D'} \otimes id_A) \alpha_{\mathcal{F}D'} (id_A \otimes \mathcal{F}f) &= (\phi_{D'} \otimes id_A) (\mathcal{F}f \otimes id_A) \alpha_{\mathcal{F}D} \text{ (by 2.1)} \\
 &= (\phi_{D'} \mathcal{F}f \otimes id_A) \alpha_{\mathcal{F}D} \text{ (naturality of } \otimes) \\
 &= (\phi_D \otimes id_A) \alpha_{\mathcal{F}D}.
 \end{aligned}$$

The last equality comes from the fact that $(C, (\phi_D)_{D \in \mathcal{D}})$ is a cocone on $\mathcal{U}\mathcal{F}$. Therefore, $(C \otimes A, ((\phi_D \otimes id_A) \alpha_{\mathcal{F}D})_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_A \mathcal{U}\mathcal{F}$. Since $(A \otimes C, (id_A \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_A \mathcal{U}\mathcal{F}$, there exists a unique arrow $\bar{\alpha} : A \otimes C \rightarrow C \otimes A$ in \mathcal{C} with $\bar{\alpha}(id_A \otimes \phi_D) = (\phi_D \otimes id_A) \alpha_{\mathcal{F}D}$. Similarly, since \mathcal{Q}_A is cocontinuous, $\mathcal{Q}_A \mathcal{U}\mathcal{F}$ has a colimit of $(\mathcal{Q}_A(C), (\mathcal{Q}_A(\phi_D))_{D \in \mathcal{D}})$. So $(C \otimes A, (\phi_D \otimes id_A)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_A \mathcal{U}\mathcal{F}$.

Correspondingly, we have

$$\begin{aligned}
 (id_A \otimes \phi_{D'}) \alpha_{\mathcal{F}D'}^{-1} (\mathcal{F}f \otimes id_A) &= (id_A \otimes \phi_{D'}) (id_A \otimes \mathcal{F}f) \alpha_{\mathcal{F}D}^{-1} \text{ (by (2.1))} \\
 &= (id_A \otimes \phi_{D'} \mathcal{F}f) \alpha_{\mathcal{F}D}^{-1} \text{ (naturality of } \otimes) \\
 &= (id_A \otimes \phi_D) \alpha_{\mathcal{F}D}^{-1}
 \end{aligned}$$

for any D in \mathcal{D} . The last equality follows from the fact that $(C, (\phi_D)_{D \in \mathcal{D}})$ is a cocone on $\mathcal{U}\mathcal{F}$. Hence, $(A \otimes C, ((id_A \otimes \phi_D) \alpha_{\mathcal{F}D}^{-1})_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_A \mathcal{U}\mathcal{F}$. Since $(C \otimes A, (\phi_D \otimes id_A)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_A \mathcal{U}\mathcal{F}$, there exists a unique arrow $\bar{\alpha}' : C \otimes A \rightarrow A \otimes C$ in \mathcal{C} with $\bar{\alpha}'(\phi_D \otimes id_A) = (id_A \otimes \phi_D) \alpha_{\mathcal{F}D}^{-1}$.

Therefore, we get the commutative diagram

$$\begin{array}{ccc}
 \mathcal{F}D \otimes A & \xrightarrow{\mathcal{F}f \otimes id_A} & \mathcal{F}D' \otimes A \\
 \downarrow \alpha_{\mathcal{F}D}^{-1} \sim & \searrow \phi_D \otimes id_A & \swarrow \phi_{D'} \otimes id_A \\
 & C \otimes A & \\
 & \downarrow \exists! \bar{\alpha}' & \\
 A \otimes \mathcal{F}D & \xrightarrow{id_A \otimes \phi_D} & A \otimes C \xleftarrow{id_A \otimes \phi_{D'}} A \otimes \mathcal{F}D' \\
 \downarrow \alpha_{\mathcal{F}D}^{-1} \sim & \searrow id_A \otimes \phi_D & \swarrow id_A \otimes \phi_{D'} \\
 & A \otimes C & \\
 & \downarrow id_A \otimes \mathcal{F}f & \\
 & &
 \end{array} \tag{2.3}$$

Next, we show that $\bar{\alpha}$ is an invertible arrow. From the commutativity of the diagrams 2.2 and 2.3, we have

$$\bar{\alpha}(id_A \otimes \phi_D) = (\phi_D \otimes id_A)\alpha_{\mathcal{F}D} \Leftrightarrow \bar{\alpha}(id_A \otimes \phi_D)\alpha_{\mathcal{F}D}^{-1} = (\phi_D \otimes id_A) \Leftrightarrow \bar{\alpha}'(\phi_D \otimes id_A) = (\phi_D \otimes id_A).$$

Obviously, $(C \otimes A, (\bar{\alpha}'(\phi_D \otimes id_A))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{Q}_A \mathcal{U}\mathcal{F}$. Since $(C \otimes A, (\phi_D \otimes id_A)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_A \mathcal{U}\mathcal{F}$, we have $\bar{\alpha}' = id_{C \otimes A}$. From the commutativity of the diagrams 2.2 and 2.3, we have

$$(id_A \otimes \phi_D)\alpha_{\mathcal{F}D}^{-1} = \bar{\alpha}'(\phi_D \otimes id_A) \Leftrightarrow (id_A \otimes \phi_D) = \bar{\alpha}'(\phi_D \otimes id_A)\alpha_{\mathcal{F}D} \Leftrightarrow (id_A \otimes \phi_D) = \bar{\alpha}'\bar{\alpha}(id_A \otimes \phi_D).$$

Clearly, $(A \otimes C, (\bar{\alpha}'\bar{\alpha}(id_A \otimes \phi_D))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_A \mathcal{U}\mathcal{F}$. Since $(A \otimes C, (id_A \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_A \mathcal{U}\mathcal{F}$, we have $\bar{\alpha}'\bar{\alpha} = id_{A \otimes C}$. Therefore, the arrow $\bar{\alpha}$ is invertible and $\bar{\alpha}^{-1} = \bar{\alpha}'$. Replacing the object A by B and following the same strategy we did to get $\bar{\alpha}$, we can similarly get

an invertible arrow $\bar{\beta} : B \otimes C \xrightarrow{\sim} C \otimes B$ and the commutative diagrams

$$\begin{array}{ccc}
 B \otimes \mathcal{F}D & \xrightarrow{id_B \otimes \mathcal{F}f} & B \otimes \mathcal{F}D' \\
 \beta_{\mathcal{F}D} \downarrow \sim & \searrow id_B \otimes \phi_D & \swarrow id_B \otimes \phi_{D'} \\
 & B \otimes C & \\
 & \downarrow \exists! \bar{\beta} & \\
 \mathcal{F}D \otimes B & \xrightarrow{\phi_D \otimes id_B} & C \otimes B \xleftarrow{\phi_{D'} \otimes id_B} \mathcal{F}D' \otimes B \\
 & \searrow \mathcal{F}f \otimes id_B & \swarrow \\
 & &
 \end{array} \quad (2.4)$$

$$\begin{array}{ccc}
 \mathcal{F}D \otimes B & \xrightarrow{\mathcal{F}f \otimes id_B} & \mathcal{F}D' \otimes B \\
 \beta_{\mathcal{F}D}^{-1} \downarrow \sim & \searrow \phi_D \otimes id_B & \swarrow \phi_{D'} \otimes id_B \\
 & C \otimes B & \\
 & \downarrow \exists! \bar{\beta}' & \\
 B \otimes \mathcal{F}D & \xrightarrow{id_B \otimes \phi_D} & B \otimes C \xleftarrow{id_B \otimes \phi_{D'}} B \otimes \mathcal{F}D' \\
 & \searrow id_B \otimes \mathcal{F}f & \swarrow \\
 & &
 \end{array} \quad (2.5)$$

To show that $(C, \bar{\alpha}, \bar{\beta}) \in \mathcal{Z}_h(\mathcal{C})$, we need to show that the following diagram is commutative:

$$\begin{array}{ccc}
 A \otimes C & \xrightarrow{(h \otimes id_C)} & B \otimes C \\
 \bar{\alpha} \downarrow \sim & & \sim \downarrow \bar{\beta} \\
 C \otimes A & \xrightarrow{(id_C \otimes h)} & C \otimes B
 \end{array} \quad (2.6)$$

To show this, consider the following diagram

$$\begin{array}{ccccc}
 A \otimes \mathcal{F}D & \xrightarrow{id_A \otimes \mathcal{F}f} & A \otimes \mathcal{F}D' & & \\
 \downarrow \alpha_{\mathcal{F}D} \sim & \searrow id_A \otimes \phi_D & \swarrow id_A \otimes \phi_{D'} & & \downarrow \alpha_{\mathcal{F}D'} \sim \\
 & A \otimes C & & & \\
 \mathcal{F}D \otimes A & \xrightarrow{\bar{\alpha}} & C \otimes A & \xrightarrow{id_C \otimes h} & C \otimes B \\
 \downarrow \phi_D \otimes id_A & \searrow \mathcal{F}f \otimes id_A & \swarrow \phi_{D'} \otimes id_A & & \downarrow \phi_{D'} \otimes h \\
 & B \otimes C & & & \\
 \mathcal{F}D' \otimes A & \xrightarrow{\bar{\beta}} & B \otimes C & \xrightarrow{id_C \otimes h} & C \otimes B \\
 \downarrow \phi_{D'} \otimes h & \swarrow h \otimes id_C & \searrow \phi_{D'} \otimes h & & \downarrow \phi_{D'} \otimes h \\
 & C \otimes B & & &
 \end{array}
 \tag{2.7}$$

We note that $(C \otimes A, ((id_C \otimes h)\bar{\alpha}(id_A \otimes \phi_D))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_A \mathcal{U} \mathcal{F}$ since

$$\begin{aligned}
 (id_C \otimes h)\bar{\alpha}(id_A \otimes \phi_{D'}) (id_A \otimes \mathcal{F}f) &= (id_C \otimes h)\bar{\alpha}(id_A \otimes \phi_{D'} \mathcal{F}f) \\
 &= (id_C \otimes h)\bar{\alpha}(id_A \otimes \phi_D)
 \end{aligned}$$

for any D in \mathcal{D} . Furthermore, for any D in \mathcal{D} , we have

$$\begin{aligned}
 (id_C \otimes h)\bar{\alpha}(id_A \otimes \phi_D) &= (id_C \otimes h)(\phi_D \otimes id_A)\alpha_{\mathcal{F}D} \text{ (by 2.2)} \\
 &= (\phi_D \otimes h)\alpha_{\mathcal{F}D} \text{ (naturality of } \otimes) \\
 &= (\phi_D \otimes id_B)(id_{\mathcal{F}D} \otimes h)\alpha_{\mathcal{F}D} \text{ (naturality of } \otimes) \\
 &= (\phi_D \otimes id_B)\beta_{\mathcal{F}D}(h \otimes id_{\mathcal{F}D}) \text{ } ((\mathcal{F}D, \alpha_{\mathcal{F}D}, \beta_{\mathcal{F}D}) \in \mathcal{Z}_h(\mathcal{C})) \\
 &= \bar{\beta}(id_B \otimes \phi_D)(h \otimes id_{\mathcal{F}D}) \text{ (by 2.4)} \\
 &= \bar{\beta}(h \otimes \phi_D) \text{ (naturality of } \otimes) \\
 &= \bar{\beta}(h \otimes id_C)(id_A \otimes \phi_D) \text{ (naturality of } \otimes)
 \end{aligned}$$

Since $(A \otimes C, (id_A \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_A \mathcal{U} \mathcal{F}$, we must have $(id_C \otimes h)\bar{\alpha} = \bar{\beta}(h \otimes id_C)$. Therefore, $(C, \bar{\alpha}, \bar{\beta}) \in \mathcal{Z}_h(\mathcal{C})$ and ϕ_D is an arrow in $\mathcal{Z}_h(\mathcal{C}), \forall D \in \mathcal{D}$. Thus, $((C, \bar{\alpha}, \bar{\beta}), (\phi_D)_{D \in \mathcal{D}})$ is a cocone on \mathcal{F} .

To show that $((C, \bar{\alpha}, \bar{\beta}), (\phi_D)_{D \in \mathcal{D}})$ is a colimit of \mathcal{F} , let $((C', \lambda, \gamma), (\psi_D)_{D \in \mathcal{D}})$ be a cocone on \mathcal{F} . Since $(C, (\phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{U} \mathcal{F}$, there exists a unique morphism $g : C \rightarrow C'$ in \mathcal{C} with $g\phi_D = \psi_D$ for every $D \in \mathcal{D}$. The proof is complete whence we show that g is a morphism in $\mathcal{Z}_h(\mathcal{C})$. Explicitly,

we need to show that the diagrams

$$\begin{array}{ccc}
 A \otimes C & \xrightarrow{\bar{\alpha}} & C \otimes A \\
 (id_A \otimes g) \downarrow & & \downarrow (g \otimes id_A) \\
 A \otimes C' & \xrightarrow{\lambda} & C' \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 B \otimes C & \xrightarrow{\bar{\beta}} & C \otimes B \\
 (id_B \otimes g) \downarrow & & \downarrow (g \otimes id_B) \\
 B \otimes C' & \xrightarrow{\gamma} & C' \otimes B
 \end{array}
 \tag{2.8}$$

commute. Consider the diagram

$$\begin{array}{ccccc}
 A \otimes \mathcal{F}D & \xrightarrow{id_A \otimes \mathcal{F}f} & & & A \otimes \mathcal{F}D' \\
 \downarrow \alpha_{\mathcal{F}D} \sim & \searrow id_A \otimes \phi_D & & id_A \otimes \phi_{D'} \swarrow & \downarrow \alpha_{\mathcal{F}D'} \sim \\
 & & A \otimes C & & \\
 \mathcal{F}D \otimes A & \xrightarrow{\phi_D \otimes id_A} & C \otimes A & \xrightarrow{g \otimes id_A} & C' \otimes A \\
 & \searrow \bar{\alpha} & & \swarrow \lambda & \\
 & & & & A \otimes C' \\
 & & & & \swarrow \phi_{D'} \otimes id_A \\
 & & & & \mathcal{F}D' \otimes A \\
 & & & & \downarrow \psi_{D'} \otimes id_A \\
 & & & & C' \otimes A
 \end{array}
 \tag{2.9}$$

Notably, $(C' \otimes A, ((g \otimes id_A)\bar{\alpha}(id_A \otimes \phi_D))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_A \mathcal{U} \mathcal{F}$ since for every $D \in \mathcal{D}$, we have

$$\begin{aligned}
 (g \otimes id_A)\bar{\alpha}(id_A \otimes \phi_{D'}) &= (g \otimes id_A)\bar{\alpha}(id_A \otimes \phi_{D'}) \mathcal{F}f = \\
 &= (g \otimes id_A)\bar{\alpha}(id_A \otimes \phi_D).
 \end{aligned}$$

We also have

$$\begin{aligned}
 (g \otimes id_A)\bar{\alpha}(id_A \otimes \phi_D) &= (g \otimes id_A)(\phi_D \otimes id_A)\alpha_{\mathcal{F}D} \text{ (by 2.2)} \\
 &= (g\phi_D \otimes id_A)\alpha_{\mathcal{F}D} \text{ (naturality of } \otimes) \\
 &= (\psi_D \otimes id_A)\alpha_{\mathcal{F}D} \text{ (since } g\phi_D = \psi_D, \forall D \in \mathcal{D}) \\
 &= \lambda(id_A \otimes \psi_D) \text{ (}\psi_D \text{ is a morphism in } \mathcal{Z}_h(\mathcal{C}), \forall D \in \mathcal{D}) \\
 &= \lambda(id_A \otimes g\phi_D) \text{ (since } g\phi_D = \psi_D, \forall D \in \mathcal{D}) \\
 &= \lambda(id_A \otimes g)(id_A \otimes \phi_D) \text{ (naturality of } \otimes)
 \end{aligned}$$

for any D in \mathcal{D} . Since $(A \otimes C, (id_A \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_A \mathcal{U} \mathcal{F}$, it follows that

$$(g \otimes id_A)\bar{\alpha} = \lambda(id_A \otimes g). \tag{2.10}$$

Similarly, replacing A by B and considering the following diagram

$$\begin{array}{ccccc}
 B \otimes \mathcal{F}D & \xrightarrow{id_B \otimes \mathcal{F}f} & B \otimes \mathcal{F}D' & & \\
 \downarrow \beta_{\mathcal{F}D} \sim & \searrow id_B \otimes \phi_D & \swarrow id_B \otimes \phi_{D'} & & \downarrow \sim \beta_{\mathcal{F}D'} \\
 & & B \otimes C & & \\
 \mathcal{F}D \otimes B & \xrightarrow{\phi_D \otimes id_B} & C \otimes B & \xrightarrow{g \otimes id_B} & C' \otimes B \\
 \downarrow \psi_D \otimes id_B & \searrow \phi_D \otimes id_B & \swarrow g \otimes id_B & & \downarrow \psi_{D'} \otimes id_B \\
 & & B \otimes C' & \xrightarrow{\gamma} & \\
 \mathcal{F}D' \otimes B & \xrightarrow{\phi_{D'} \otimes id_B} & B \otimes C' & \xrightarrow{\gamma} & \\
 \downarrow \psi_{D'} \otimes id_B & \swarrow \phi_{D'} \otimes id_B & \searrow \gamma & & \\
 & & C' \otimes B & &
 \end{array}
 \tag{2.11}$$

give us

$$(g \otimes id_B) \bar{\beta} = \gamma(id_B \otimes g). \tag{2.12}$$

From (2.10) and (2.12), we have g is a morphism in $\mathcal{Z}_h(\mathcal{C})$, and thus $((C, \bar{\alpha}, \bar{\beta}), (\phi_D)_{D \in \mathcal{D}})$ is a colimit of \mathcal{F} , and the proof is complete. \square

Corollary 2.3. *Let \mathcal{C} be a cocomplete category and X an object in \mathcal{C} . If $\mathcal{P}_X, \mathcal{Q}_X$ are cocontinuous, then $\mathcal{Z}_X(\mathcal{C})$ is cocomplete and the forgetful functor $\mathcal{U} : \mathcal{Z}_X(\mathcal{C}) \rightarrow \mathcal{C}$ is cocontinuous. Moreover, the colimits of diagrams in $\mathcal{Z}_X(\mathcal{C})$ can be obtained by the corresponding construction of diagrams in \mathcal{C} .*

Proof. The proof follows from Remark 1.7 (ii) and Proposition 2.2 by setting $h = id_X$. \square

Proposition 2.4. *Let $(\mathcal{C}, \otimes, I)$ be monoidal category with \mathcal{C} cocomplete. If \mathcal{P}_X and \mathcal{Q}_X are cocontinuous $\forall X \in \mathcal{C}$, then $\mathcal{Z}(\mathcal{C})$ is cocomplete and the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ is cocontinuous. Further, the colimits of diagrams in $\mathcal{Z}(\mathcal{C})$ can be obtained by the corresponding construction of diagrams in \mathcal{C} .*

Proof. Let \mathcal{D} be a small category, and let $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{C})$ be a functor. For any $X \in \mathcal{C}$, let $\mathcal{U}_X : \mathcal{Z}_X(\mathcal{C}) \rightarrow \mathcal{C}$ be the corresponding forgetful functor. Since \mathcal{U} is cocontinuous, \mathcal{U}_X is cocontinuous for every $X \in \mathcal{C}$. Moreover,

we have $\mathcal{H}_X \mathcal{F} :: \mathcal{D} \rightarrow \mathcal{Z}_X(\mathcal{C})$ is a (small) functor and $\mathcal{U}_X \mathcal{H}_X = \mathcal{U}$ for any $X \in \mathcal{C}$, where \mathcal{H}_X is the functor defined in Remark (1.7). Since \mathcal{C} is a cocomplete category, the functor $\mathcal{U} \mathcal{F}$ has a colimit $(C, (\phi_D)_{D \in \mathcal{D}})$. For every $X \in \mathcal{C}$, $(\mathcal{P}_X(C), (\mathcal{P}_X(\phi_D))_{D \in \mathcal{D}})$ is a colimit of the functor $\mathcal{P}_X \mathcal{U} \mathcal{F}$ because \mathcal{P}_X is cocontinuous. Thus, for any $X \in \mathcal{C}$, $(X \otimes C, (id_X \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_X \mathcal{U} \mathcal{F}$, hence, $(X \otimes C, (id_X \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$ for every $X \in \mathcal{C}$.

Further, the functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{C})$ assigns to each object $D \in \mathcal{D}$ an object $(\mathcal{F}D, \alpha^{\mathcal{F}D}) \in \mathcal{Z}(\mathcal{C})$, and $\mathcal{F}f : (\mathcal{F}D, \alpha^{\mathcal{F}D}) \rightarrow (\mathcal{F}D', \alpha^{\mathcal{F}D'})$ is an arrow in $\mathcal{Z}(\mathcal{C})$ for every arrow $f : D \rightarrow D'$ in \mathcal{D} . Fix an object $X \in \mathcal{C}$, we have $\mathcal{H}_X \mathcal{F} :: \mathcal{D} \rightarrow \mathcal{Z}_X(\mathcal{C})$ is a (small) functor. Since \mathcal{C} is a cocomplete category, the category $\mathcal{Z}_X(\mathcal{C})$ is cocomplete by Corollary 2.3. Hence, $\mathcal{U}_X \mathcal{H}_X \mathcal{F}$ has a colimit. We will explicitly show that $((C, \mu_{C,X}), (\phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{H}_X \mathcal{F}$, where $\mu_{C,X} : C \otimes X \rightarrow X \otimes C$ is a unique invertible arrow in \mathcal{C} with $\mu_{C,X}(\phi_D \otimes id_X) = (id_X \otimes \phi_D) \alpha_{\mathcal{H}_X \mathcal{F}D}$.

First, we note that the functor $\mathcal{H}_X \mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}_X(\mathcal{C})$ assigns to each object $D \in \mathcal{D}$ an object $(\mathcal{H}_X \mathcal{F}D, \alpha^{\mathcal{H}_X \mathcal{F}D}) \in \mathcal{Z}_X(\mathcal{C})$. In addition, we have $\mathcal{H}_X \mathcal{F}f : (\mathcal{H}_X \mathcal{F}D, \alpha^{\mathcal{H}_X \mathcal{F}D}) \rightarrow (\mathcal{H}_X \mathcal{F}D', \alpha^{\mathcal{H}_X \mathcal{F}D'})$ is an arrow in $\mathcal{Z}_X(\mathcal{C})$ for every arrow $f : D \rightarrow D'$ in \mathcal{D} . By 1.12, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_X \mathcal{F}D \otimes X & \xrightarrow{(\mathcal{H}_X \mathcal{F}f \otimes id_X)} & \mathcal{H}_X \mathcal{F}D' \otimes X \\ \alpha^{\mathcal{H}_X \mathcal{F}D} \downarrow \sim & & \sim \downarrow \alpha^{\mathcal{H}_X \mathcal{F}D'} \\ X \otimes \mathcal{H}_X \mathcal{F}D & \xrightarrow{(id_X \otimes \mathcal{H}_X \mathcal{F}f)} & X \otimes \mathcal{H}_X \mathcal{F}D' \end{array} \quad (2.13)$$

Consider the diagram

$$\begin{array}{ccccc} \mathcal{H}_X \mathcal{F}D \otimes X & \xrightarrow{\mathcal{H}_X \mathcal{F}f \otimes id_X} & \mathcal{H}_X \mathcal{F}D' \otimes X & & \\ \alpha^{\mathcal{H}_X \mathcal{F}D} \downarrow \sim & \searrow \phi_D \otimes id_X & \swarrow \phi_{D'} \otimes id_X & & \downarrow \sim \alpha^{\mathcal{H}_X \mathcal{F}D'} \\ & C \otimes X & & & \\ & \downarrow \exists! \mu_{C,X} & & & \\ X \otimes \mathcal{H}_X \mathcal{F}D & \xrightarrow{id_X \otimes \phi_D} & X \otimes C & \xleftarrow{id_X \otimes \phi_{D'}} & X \otimes \mathcal{H}_X \mathcal{F}D' \\ & \searrow id_X \otimes \phi_D & \swarrow id_X \otimes \phi_{D'} & & \\ & X \otimes \mathcal{H}_X \mathcal{F}D' & & & \end{array} \quad (2.14)$$

Since \mathcal{Q}_X is cocontinuous, $(\mathcal{Q}_X(C), (\mathcal{Q}_X(\phi_D))_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. So $(C \otimes X, (\phi_D \otimes id_X)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. We also have

$$\begin{aligned} (id_X \otimes \phi_{D'}) \alpha^{\mathcal{H}_X \mathcal{F} D'} (\mathcal{H}_X \mathcal{F} f \otimes id_X) &= (id_X \otimes \phi_{D'}) (id_X \otimes \mathcal{H}_X \mathcal{F} f) \alpha^{\mathcal{H}_X \mathcal{F} D'} \quad (\text{by 2.13}) \\ &= (id_X \otimes \phi_{D'} \mathcal{H}_X \mathcal{F} f) \alpha^{\mathcal{H}_X \mathcal{F} D'} \quad (\text{naturality of } \otimes) \\ &= (id_X \otimes \phi_D) \alpha^{\mathcal{H}_X \mathcal{F} D} \end{aligned}$$

for any D in \mathcal{D} . The last equality comes from the fact that $(C, (\phi_D)_{D \in \mathcal{D}})$ is a cocone on $\mathcal{U}_X \mathcal{H}_X \mathcal{F}$. Hence, $(X \otimes C, ((id_X \otimes \phi_D) \alpha^{\mathcal{H}_X \mathcal{F} D})_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. Since $(C \otimes X, (\phi_D \otimes id_X)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$, there exists a unique arrow $\mu_{C,X} : C \otimes X \rightarrow X \otimes C$ in \mathcal{C} with $\mu_{C,X}(\phi_D \otimes id_X) = (id_X \otimes \phi_D) \alpha^{\mathcal{H}_X \mathcal{F} D}$. Similarly, we consider the diagram

$$\begin{array}{ccccc} X \otimes \mathcal{H}_X \mathcal{F} D & \xrightarrow{id_X \otimes \mathcal{H}_X \mathcal{F} f} & X \otimes \mathcal{H}_X \mathcal{F} D' & & \\ \downarrow (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1} \sim & \searrow id_X \otimes \phi_D & \swarrow id_X \otimes \phi_{D'} & & \downarrow \sim (\alpha^{\mathcal{H}_X \mathcal{F} D'})^{-1} \\ & X \otimes C & & & \\ & \downarrow \exists! \nu_{C,X} & & & \\ \mathcal{H}_X \mathcal{F} D \otimes X & \xrightarrow{\phi_D \otimes id_X} & C \otimes X & \xleftarrow{\phi_{D'} \otimes id_X} & \mathcal{H}_X \mathcal{F} D' \otimes X \\ & \searrow \mathcal{H}_X \mathcal{F} f \otimes id_X & & & \swarrow \end{array} \quad (2.15)$$

We notice that

$$\begin{aligned} (\phi_{D'} \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D'})^{-1} (id_X \otimes \mathcal{H}_X \mathcal{F} f) &= (\phi_D \otimes id_X) (\mathcal{H}_X \mathcal{F} f \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1} \quad (\text{by 2.13}) \\ &= (\phi_D \mathcal{H}_X \mathcal{F} f \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1} \quad (\text{naturality of } \otimes) \\ &= (\phi_D \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1} \end{aligned}$$

for any D in \mathcal{D} , where the last equality is coming from the fact that $(C, (\phi_D)_{D \in \mathcal{D}})$ is a cocone on $\mathcal{U}_X \mathcal{H}_X \mathcal{F}$.

Therefore, $(C \otimes X, ((\phi_D \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1})_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. Since $(X \otimes C, (id_X \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$, there exists a unique arrow $\nu_{C,X} : X \otimes C \rightarrow C \otimes X$ in \mathcal{C} with $\nu_{C,X}(id_X \otimes \phi_D) = (\phi_D \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F} D})^{-1}$.

Next, we show that $\mu_{C,X}$ is an invertible arrow. From the commutativity of the diagrams 2.14 and 2.15, we have

$$\begin{aligned} (id_X \otimes \phi_D) \alpha^{\mathcal{H}_X \mathcal{F}^D} &= \mu_{C,X}(\phi_D \otimes id_X) \Leftrightarrow (id_X \otimes \phi_D) = \\ \mu_{C,X}(\phi_D \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F}^D})^{-1} &\Leftrightarrow (id_X \otimes \phi_D) = \mu_{C,X} \nu_{C,X}(id_X \otimes \phi_D). \end{aligned}$$

Obviously, $(X \otimes C, (\mu_{C,X} \nu_{C,X}(id_X \otimes \phi_D))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. Since $(X \otimes C, (id_X \otimes \phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$, we have $\mu_{C,X} \nu_{C,X} = id_{X \otimes C}$. In a similar way, from the commutativity of the diagrams 2.14 and 2.15, we have

$$\begin{aligned} \nu_{C,X}(id_X \otimes \phi_D) &= (\phi_D \otimes id_X) (\alpha^{\mathcal{H}_X \mathcal{F}^D})^{-1} \Leftrightarrow \nu_{C,X}(id_X \otimes \phi_D) \alpha^{\mathcal{H}_X \mathcal{F}^D} = \\ (\phi_D \otimes id_X) &\Leftrightarrow \nu_{C,X} \mu_{C,X}(\phi_D \otimes id_X) = (\phi_D \otimes id_X). \end{aligned}$$

Clearly, $(C \otimes X, (\nu_{C,X} \mu_{C,X}(\phi_D \otimes id_X))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$. Since $(C \otimes X, (\phi_D \otimes id_X)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$, we have $\nu_{C,X} \mu_{C,X} = id_{C \otimes X}$. Therefore, the arrow $\mu_{C,X}$ is invertible and $\mu_{C,X}^{-1} = \nu_{C,X}$. It follows that $(C, \mu_{C,X}) \in \mathcal{Z}_X(\mathcal{C})$, and ϕ_D is an arrow in $\mathcal{Z}_X(\mathcal{C})$, $\forall D \in \mathcal{D}$. Hence, $((C, \mu_{C,X}), (\phi_D)_{D \in \mathcal{D}})$ is a cocone on $\mathcal{H}_X \mathcal{F}$.

It remains to show that $((C, \mu_{C,X}), (\phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{H}_X \mathcal{F}$. Let $((C', \eta), (\psi_D)_{D \in \mathcal{D}})$ be a cocone on $\mathcal{H}_X \mathcal{F}$. Since $(C, (\phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{U}_X \mathcal{H}_X \mathcal{F}$, there exists a unique morphism $g : C \rightarrow C'$ in \mathcal{C} with $g\phi_D = \psi_D$ for every $D \in \mathcal{D}$. Clearly, all we need is to show that g is a morphism in $\mathcal{Z}_X(\mathcal{C})$. Indeed, we need to show that the diagram

$$\begin{array}{ccc} C \otimes X & \xrightarrow{\mu_{C,X}} & X \otimes C \\ (g \otimes id_X) \downarrow & \sim & \downarrow (id_X \otimes g) \\ C' \otimes X & \xrightarrow{\eta_X} & X \otimes C' \end{array} \quad (2.16)$$

commutes. Consider the diagram

$$\begin{array}{ccccc}
 \mathcal{H}_X \mathcal{F} D \otimes X & \xrightarrow{\mathcal{H}_X \mathcal{F} f \otimes id_X} & \mathcal{H}_X \mathcal{F} D' \otimes X & & \\
 \downarrow \alpha^{\mathcal{H}_X \mathcal{F} D} \sim & \searrow \phi_D \otimes id_X & \swarrow \phi_{D'} \otimes id_X & & \downarrow \alpha^{\mathcal{H}_X \mathcal{F} D'} \sim \\
 & C \otimes X & & & \\
 X \otimes \mathcal{H}_X \mathcal{F} D & \xrightarrow{\mu_{C,X}} & C \otimes X & \xrightarrow{g \otimes id_X} & X \otimes \mathcal{H}_X \mathcal{F} D' \\
 \downarrow id_X \otimes \phi_D & \searrow id_X \otimes \mathcal{H}_X \mathcal{F} f & \swarrow id_X \otimes \phi_{D'} & & \downarrow id_X \otimes \psi_{D'} \\
 X \otimes C & & C' \otimes X & & \\
 \downarrow id_X \otimes g & \searrow id_X \otimes \phi_D & \swarrow \eta & & \downarrow id_X \otimes \psi_{D'} \\
 X \otimes C' & & X \otimes C' & & \\
 \downarrow id_X \otimes \psi_D & & \downarrow id_X \otimes \psi_{D'} & & \\
 & & & &
 \end{array} \tag{2.17}$$

Notably, $(X \otimes C', ((id_X \otimes g)\mu_{C,X}(\phi_D \otimes id_X))_{D \in \mathcal{D}})$ is a cocone on $\mathcal{P}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$ since

$$\begin{aligned}
 (id_X \otimes g)\mu_{C,X}(\phi_{D'} \otimes id_X)(\mathcal{H}_X \mathcal{F} f \otimes id_X) &= (id_X \otimes g)\mu_{C,X}(\phi_{D'} \mathcal{H}_X \mathcal{F} f \otimes id_X) \\
 &= (id_X \otimes g)\mu_{C,X}(\phi_D \otimes id_X)
 \end{aligned}$$

We also have

$$\begin{aligned}
 (id_X \otimes g)\mu_{C,X}(\phi_D \otimes id_X) &= (id_X \otimes g)(id_X \otimes \phi_D)\alpha^{\mathcal{H}_X \mathcal{F} D} \text{ (by 2.14)} \\
 &= (id_X \otimes g\phi_D)\alpha^{\mathcal{H}_X \mathcal{F} D} \text{ (naturality of } \otimes) \\
 &= (id_X \otimes \psi_D)\alpha^{\mathcal{H}_X \mathcal{F} D} \text{ (since } g\phi_D = \psi_D, \forall D \in \mathcal{D}) \\
 &= \eta(\psi_D \otimes id_X) \\
 &= \eta(g\phi_D \otimes id_X) \text{ (since } g\phi_D = \psi_D, \forall D \in \mathcal{D}) \\
 &= \eta(g \otimes id_X)(\phi_D \otimes id_X) \text{ (naturality of } \otimes)
 \end{aligned}$$

for any D in \mathcal{D} , where the forth equality follows from the fact that ψ_D is a morphism in $\mathcal{Z}_X(\mathcal{C})$, $\forall D \in \mathcal{D}$. Since $(C \otimes X, (\phi_D \otimes id_X)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_X \mathcal{U}_X \mathcal{H}_X \mathcal{F}$, we have

$$(id_X \otimes g)\mu_{C,X} = \eta(g \otimes id_X). \tag{2.18}$$

Thus, we obtain $((C, \mu_{C,X}), (\phi_D)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{H}_X \mathcal{F}$, $\forall X \in \mathcal{C}$, and thus, we obtain a family of invertible arrows $\{\mu_{C,X}\}_{X \in \mathcal{C}}$ in \mathcal{C} , where $\mu_{C,X}$ is the map in diagram 2.14 for all $X \in \mathcal{C}$. Therefore, the proof is complete

whence we show that $\{\mu_{C,X}\}_{X \in \mathcal{C}}$ are natural in X , for any $X \in \mathcal{C}$, and the conditions (1.6) and (1.7) hold.

To show that $\mu_{C,-} : C \otimes - \rightarrow - \otimes C$ is a natural transformation, let $\zeta : A \rightarrow B$ be an arrow in \mathcal{C} . We need to show that the following diagram is commutative:

$$\begin{array}{ccc}
 C \otimes A & \xrightarrow[\sim]{\nu_{C,A}} & A \otimes C \\
 (id_C \otimes \zeta) \downarrow & & \downarrow (\zeta \otimes id_C) \\
 C \otimes B & \xrightarrow[\sim]{\nu_{C,B}} & B \otimes C
 \end{array} \quad (2.19)$$

Since $(\mathcal{F}D, \alpha^{\mathcal{F}D}) \in \mathcal{Z}(\mathcal{C})$, $\forall D \in \mathcal{D}$, $\alpha^{\mathcal{F}D}$ is a natural transformation, $\forall D \in \mathcal{D}$. Thus, for any $D \in \mathcal{D}$, the diagram

$$\begin{array}{ccc}
 \mathcal{F}D \otimes A & \xrightarrow[\sim]{\alpha_A^{\mathcal{F}D}} & A \otimes \mathcal{F}D \\
 (id_{\mathcal{F}D} \otimes \zeta) \downarrow & & \downarrow (\zeta \otimes id_{\mathcal{F}D}) \\
 \mathcal{F}D \otimes B & \xrightarrow[\sim]{\alpha_B^{\mathcal{F}D}} & B \otimes \mathcal{F}D
 \end{array} \quad (2.20)$$

commutes. Now, consider the following diagram

$$\begin{array}{ccccccc}
 & \mathcal{F}D \otimes A & \xrightarrow{\mathcal{F}f \otimes id_A} & \mathcal{F}D' \otimes A & & & \\
 & \downarrow id_{\mathcal{F}D} \otimes \zeta & \searrow \phi_D \otimes id_A & \swarrow \phi_{D'} \otimes id_A & \downarrow id_{\mathcal{F}D'} \otimes \zeta & & \\
 \mathcal{F}D \otimes B & \downarrow \alpha_A^{\mathcal{F}D} & A \otimes \mathcal{F}D & \xrightarrow{\nu_{C,A}} & C \otimes A & \xrightarrow{id_C \otimes \zeta} & A \otimes \mathcal{F}D' \\
 & \downarrow \zeta \otimes id_{\mathcal{F}D} & \downarrow id_A \otimes \phi_D & \downarrow id_A \otimes \mathcal{F}f & \downarrow id_A \otimes \phi_{D'} & \downarrow \zeta \otimes id_{\mathcal{F}D'} & \\
 & B \otimes \mathcal{F}D & \downarrow id_A \otimes \phi_D & A \otimes C & \xrightarrow{id_A \otimes \mathcal{F}f} & C \otimes B & \downarrow id_A \otimes \phi_{D'} \\
 & \downarrow \alpha_B^{\mathcal{F}D} & \downarrow id_B \otimes \phi_D & \downarrow \zeta \otimes id_C & \downarrow id_B \otimes \phi_D & \downarrow \nu_{C,B} & \downarrow id_B \otimes \phi_D \\
 & B \otimes \mathcal{F}D & \xrightarrow{id_B \otimes \phi_D} & B \otimes C & \xrightarrow{id_B \otimes \phi_D} & B \otimes \mathcal{F}D' & \downarrow \alpha_B^{\mathcal{F}D'} \\
 & & & & & & \mathcal{F}D' \otimes B
 \end{array} \quad (2.21)$$

Clearly, $(B \otimes C, ((\zeta \otimes id_C) \nu_{C,A} (\phi_D \otimes id_A)))_{D \in \mathcal{D}}$ is a cocone on $\mathcal{Q}_A \mathcal{U} \mathcal{F}$ since

$$\begin{aligned}
(\zeta \otimes id_C)\nu_{C,A}(\phi_{D'} \otimes id_A)(\mathcal{F} \otimes id_A) &= (\zeta \otimes id_C)\nu_{C,A}(\phi_{D'} \mathcal{F} \otimes id_A) \\
&= (\zeta \otimes id_C)\nu_{C,A}(\phi_D \otimes id_A)
\end{aligned}$$

Furthermore, for any $D \in \mathcal{D}$, we have

$$\begin{aligned}
(\zeta \otimes id_C)\nu_{C,A}(\phi_D \otimes id_A) &= (\zeta \otimes id_C)(id_A \otimes \phi_D)\alpha_A^{\mathcal{F}D} \\
&\quad \text{(by (2.14) taking } A \text{ and } \alpha_A^{\mathcal{F}D} \text{ in place of } X \text{ and } \alpha_{\mathcal{F}D}, \text{ respectively)} \\
&= (\zeta id_A \otimes id_C \phi_D)\alpha_A^{\mathcal{F}D} \text{ (naturality of } \otimes) \\
&= (\zeta \otimes \phi_D)\alpha_A^{\mathcal{F}D} \\
&= (id_B \otimes \phi_D)(\zeta \otimes id_{\mathcal{F}D})\alpha_A^{\mathcal{F}D} \text{ (naturality of } \otimes) \\
&= (id_B \otimes \phi_D)\alpha_B^{\mathcal{F}D}(id_{\mathcal{F}D} \otimes \zeta)(\alpha^{\mathcal{F}D} \text{ is a natural transformation)} \\
&= \nu_{C,B}(\phi_D \otimes id_B)(id_{\mathcal{F}D} \otimes \zeta) \\
&\quad \text{(by (2.14) taking } B \text{ and } \alpha_B^{\mathcal{F}D} \text{ in place of } X \text{ and } \alpha_{\mathcal{F}D}, \text{ respectively.)} \\
&= \nu_{C,B}(\phi_D id_{\mathcal{F}D} \otimes id_B \zeta) \text{ (naturality of } \otimes) \\
&= \nu_{C,B}(\phi_D \otimes \zeta) \\
&= \nu_{C,B}(id_C \otimes \zeta)(\phi_D \otimes id_A) \text{ (naturality of } \otimes).
\end{aligned}$$

Since $(C \otimes A, (\phi_D \otimes id_A)_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_A \mathcal{U} \mathcal{F}$, it follows that $(\zeta \otimes id_C)\nu_{C,A} = \nu_{C,B}(id_C \otimes \zeta)$. Hence, $\mu_{C,-} : C \otimes - \rightarrow - \otimes C$ is a natural transformation. By Remark 1.7, it remains to show that the condition (1.7) holds. Consider the following diagram

$$\begin{array}{ccc}
\mathcal{F}D \otimes X \otimes Y & \xrightarrow{\mathcal{F}f \otimes id_{X \otimes Y}} & \mathcal{F}D' \otimes X \otimes Y \\
\downarrow \alpha_{X \otimes Y}^{\mathcal{F}D} \sim & \searrow \phi_D \otimes id_{X \otimes Y} & \swarrow \phi_{D'} \otimes id_{X \otimes Y} \\
& C \otimes X \otimes Y & \\
\downarrow id_{X \otimes Y} \otimes \phi_D & \searrow \mu_X \otimes id_Y & \swarrow \mu_X \otimes id_Y \\
X \otimes Y \otimes \mathcal{F}D & & X \otimes C \otimes Y \\
\downarrow id_{X \otimes Y} \otimes \phi_D & \searrow \mu_{X \otimes Y} & \swarrow id_X \otimes \mu_Y \\
& X \otimes Y \otimes C & \\
\downarrow id_{X \otimes Y} \otimes \phi_{D'} & & \\
X \otimes Y \otimes \mathcal{F}D' & & \\
\downarrow id_{X \otimes Y} \otimes \phi_{D'} & & \\
& X \otimes Y \otimes C &
\end{array}
\tag{2.22}$$

We have

$$\begin{aligned}
(id_X \otimes \mu_Y)(\mu_X \otimes id_Y)(\phi_D \otimes id_{X \otimes Y}) &= (id_X \otimes \mu_Y)(\mu_X \otimes id_Y)(\phi_D \otimes id_X \otimes id_Y) \quad (\otimes \text{ is a bifunctor}) \\
&= (id_X \otimes \mu_Y)(\mu_X(\phi_D \otimes id_X) \otimes id_Y id_Y) \quad (\text{naturality of } \otimes) \\
&= (id_X \otimes \mu_Y)((id_X \otimes \phi_D)\alpha_X^{\mathcal{F}^D} \otimes id_Y) \\
&\quad (\text{by (2.14) taking } \alpha_X^{\mathcal{F}^D} \text{ in place of } \alpha_{\mathcal{F}^D}) \\
&= (id_X \otimes \mu_Y)(id_X \otimes \phi_D \otimes id_Y)(\alpha_X^{\mathcal{F}^D} \otimes id_Y) \quad (\text{naturality of } \otimes) \\
&= (id_X id_X \otimes \mu_Y(\phi_D \otimes id_Y))(\alpha_X^{\mathcal{F}^D} \otimes id_Y) \quad (\text{naturality of } \otimes) \\
&= (id_X \otimes (id_Y \otimes \phi_D)\alpha_Y^{\mathcal{F}^D})(\alpha_X^{\mathcal{F}^D} \otimes id_Y) \\
&\quad (\text{taking } Y, \alpha_Y^{\mathcal{F}^D} \text{ in (2.14) in place of } X, \alpha_{\mathcal{F}^D} \text{ respectively}) \\
&= (id_X \otimes id_Y \otimes \phi_D)(id_X \otimes \alpha_Y^{\mathcal{F}^D})(\alpha_X^{\mathcal{F}^D} \otimes id_Y) \quad (\text{naturality of } \otimes) \\
&= (id_X \otimes id_Y \otimes \phi_D)\alpha_{X \otimes Y}^{\mathcal{F}^D} \quad (\text{since } (\mathcal{F}^D, \alpha^{\mathcal{F}^D}) \in \mathcal{Z}(\mathcal{C})) \\
&= (id_{X \otimes Y} \otimes \phi_D)\alpha_{X \otimes Y}^{\mathcal{F}^D} \quad (\text{naturality of } \otimes)
\end{aligned}$$

Since $(C \otimes X \otimes Y, (\phi_D \otimes id_{X \otimes Y})_{D \in \mathcal{D}})$ is a colimit of $\mathcal{Q}_{X \otimes Y} \mathcal{U} \mathcal{F}$, it follows that the condition (1.7) is satisfied. Therefore, $(C, \mu_{C, -})$ is a colimit of \mathcal{F} and the proof is complete. \square

The following is an immediate consequence of the proof of Proposition 2.4 and Remark 1.7.

Corollary 2.5. *Let \mathcal{C} be a cocomplete category. If $\mathcal{P}_X, \mathcal{Q}_X$ are cocontinuous $\forall X \in \mathcal{C}$, then $\mathcal{Z}_\omega(\mathcal{C})$ is cocomplete and the forgetful functor $\mathcal{U} : \mathcal{Z}_\omega(\mathcal{C}) \rightarrow \mathcal{C}$ is cocontinuous. Moreover, the colimits of diagrams in $\mathcal{Z}_\omega(\mathcal{C})$ can be obtained by the corresponding construction of diagrams in \mathcal{C} .*

The following theorem is well-known.

Theorem 2.6. [3, p. 225] *Right adjoints preserve limits, and left adjoints preserve colimits.*

Following [8, p. 293, 294], a monoidal category \mathcal{C} is *biclosed* when, for each object $X \in \mathcal{C}$, both functors $\mathcal{P}_X = X \otimes -$ and $\mathcal{Q}_X = - \otimes X$ have a right adjoint. A biclosed symmetric monoidal category is called a symmetric monoidal closed category. Since in a symmetric monoidal category, both functors $\mathcal{P}_X = X \otimes -$ and $\mathcal{Q}_X = - \otimes X$ are naturally isomorphic, it follows that a symmetric monoidal category \mathcal{C} is closed if and only if, for each object $X \in \mathcal{C}$, the functor $\mathcal{Q}_X = - \otimes X : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint [8, p. 294]. Therefore, by using Proposition 2.1 together with Proposition 2.2, Corollary 2.3, Proposition 2.4, Corollary 2.5 and Theorem 2.6, we have the following immediate consequence.

Proposition 2.7. *Let \mathcal{C} be a monoidal category. Fix an object X and a morphism $h : A \rightarrow B$ in \mathcal{C} . For any $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$, let*

$\mathcal{U}_{\mathcal{A}} : \text{CoMon}(\mathcal{A}) \rightarrow \mathcal{A}$ be the forgetful functor corresponding to \mathcal{A} . If \mathcal{C} is a cocomplete biclosed monoidal category, then \mathcal{A} is cocomplete and the forgetful functor $\mathcal{U}_{\mathcal{A}}$ is cocontinuous for any $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$. Furthermore, the colimits of diagrams in \mathcal{A} can be obtained by the corresponding construction of diagrams in \mathcal{C} .

Example 2.8.

(1) The category **Set** of sets and mappings is cocomplete [22, p. 66], and it can also be seen to be cartesian closed [8, p. 296], hence symmetric monoidal closed. By Proposition 2.7, $\mathcal{Z}_h(\mathbf{Set})$, $\mathcal{Z}_X(\mathbf{Set})$, $\mathcal{Z}(\mathbf{Set})$, and $\mathcal{Z}_\omega(\mathbf{Set})$ are cocomplete for any $X \in \mathbf{Set}$ and a morphism h in **Set**.

(2) The category **Cat** of small categories and functors is cocomplete [22, p. 66], and it can also be seen as cartesian closed [8, p. 296], hence symmetric monoidal closed. By Proposition 2.7, $\mathcal{Z}_h(\mathbf{Cat})$, $\mathcal{Z}_X(\mathbf{Cat})$, $\mathcal{Z}(\mathbf{Cat})$, and $\mathcal{Z}_\omega(\mathbf{Cat})$ are cocomplete for any $X \in \mathbf{Cat}$ and a morphism h in **Cat**.

(3) The category **Top** of topological spaces and continuous mappings is cocomplete [22, p. 66], and it can be provided with the structure of a symmetric monoidal closed category (See [8, p. 299]). By Proposition 2.7, $\mathcal{Z}_h(\mathbf{Top})$, $\mathcal{Z}_X(\mathbf{Top})$, $\mathcal{Z}(\mathbf{Top})$, and $\mathcal{Z}_\omega(\mathbf{Top})$ are cocomplete for any $X \in \mathbf{Top}$ and a morphism h in **Top**.

(4) If $\text{Mod}_{\mathcal{T}}$ is the category of models of a commutative algebraic theory \mathcal{T} . Then the category $\text{Mod}_{\mathcal{T}}$ is cocomplete [8, p. 138]. Further, $\text{Mod}_{\mathcal{T}}$ is symmetric monoidal closed [8, p. 297]. It follows by Proposition 2.7 that $\mathcal{Z}_h(\text{Mod}_{\mathcal{T}})$, $\mathcal{Z}_X(\text{Mod}_{\mathcal{T}})$, $\mathcal{Z}(\text{Mod}_{\mathcal{T}})$, and $\mathcal{Z}_\omega(\text{Mod}_{\mathcal{T}})$ are cocomplete for any $X \in \text{Mod}_{\mathcal{T}}$ and a morphism h in $\text{Mod}_{\mathcal{T}}$.

(5) The category Ban_1 of Banach spaces and linear contractions is symmetric monoidal closed. The tensor product $A \otimes B$ of two Banach spaces is the so-called “projective tensor product” of A, B . By Proposition 2.7, $\mathcal{Z}_h(\text{Ban}_1)$, $\mathcal{Z}_X(\text{Ban}_1)$, $\mathcal{Z}(\text{Ban}_1)$, and $\mathcal{Z}_\omega(\text{Ban}_1)$ are cocomplete.

(6) If \mathbb{R} is a ring, the category ${}_{L-R}\mathcal{M}_{\mathbb{R}-\mathbb{R}}$ of $\mathbb{R} - \mathbb{R}$ -bimodules and left-right- \mathbb{R} -linear mappings is monoidal biclosed, for the structure given by $L \otimes M \cong L \otimes_{\mathbb{R}} M$. The right adjoint to $- \otimes M$ is given by $[M, -]_r$ where

$$[M, N]_r = \{f : M \rightarrow N \mid f \text{ right } - \mathbb{R} - \text{linear}\}$$

while the right adjoint to $L \otimes -$ is given by $[M, -]_l$ where

$$[L, N]_l = \{f : L \rightarrow N \mid f \text{ left } - \mathbb{R} - \text{linear}\}$$

Proposition 2.7 implies that the categories $\mathcal{Z}_h(L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}})$, $\mathcal{Z}_X(L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}})$, $\mathcal{Z}(L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}})$, and $\mathcal{Z}_\omega(L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}})$ are cocomplete for any $X \in L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}}$ and a morphism h in $L\text{-}R\mathcal{M}_{\mathbb{R}-\mathbb{R}}$.

(7) The category C_{infSL} of complete inf semi-lattices is evidently complete and therefore cocomplete, and it can also be seen as a symmetric monoidal closed category [5]. By Proposition 2.7, $\mathcal{Z}_h(C_{infSL})$, $\mathcal{Z}_X(C_{infSL})$, $\mathcal{Z}(C_{infSL})$, and $\mathcal{Z}_\omega(C_{infSL})$ are cocomplete for any $X \in C_{infSL}$ and a morphism h in C_{infSL} .

(8) Grothendieck topos $GTYP$ is evidently complete and therefore cocomplete, and it is also a symmetric monoidal closed category [5]. By Proposition 2.7, $\mathcal{Z}_h(GTYP)$, $\mathcal{Z}_X(GTYP)$, $\mathcal{Z}(GTYP)$ and $\mathcal{Z}_\omega(GTYP)$ are cocomplete for any $X \in GTYP$ and a morphism h in $GTYP$.

(9) The category \mathbf{LCA} of locally compact abelian groups for which the duality reduces to the standard duality for those groups [6]. By Proposition 2.7, $\mathcal{Z}_h(\mathbf{LCA})$, $\mathcal{Z}_X(\mathbf{LCA})$, $\mathcal{Z}(\mathbf{LCA})$, and $\mathcal{Z}_\omega(\mathbf{LCA})$ are cocomplete for any $X \in \mathbf{LCA}$ and a morphism h in \mathbf{LCA} .

(10) If \mathcal{D} is a small category, the category $Fun(\mathcal{D}, \mathbf{Set})$ of functors and natural transformations is cartesian closed, hence a symmetric monoidal closed [8, p. 297]. The category $Fun(\mathcal{D}, \mathbf{Set})$ is also cocomplete [22, p. 66]. By Proposition 2.7, the categories $\mathcal{Z}_h(Fun(\mathcal{D}, \mathbf{Set}))$, $\mathcal{Z}_X(Fun(\mathcal{D}, \mathbf{Set}))$, $\mathcal{Z}(Fun(\mathcal{D}, \mathbf{Set}))$ and $\mathcal{Z}_\omega(Fun(\mathcal{D}, \mathbf{Set}))$ are cocomplete for any $X \in Fun(\mathcal{D}, \mathbf{Set})$ and a morphism h in $Fun(\mathcal{D}, \mathbf{Set})$.

(11) Let

$$\begin{array}{ccc}
 & 1 & \\
 & \searrow & \\
 Q : & & 3 \longleftarrow 4 \\
 & \nearrow & \\
 & 2 &
 \end{array} \tag{2.23}$$

be a quiver, and let Rep_Q be the category of representations of Q over \mathbb{K} . Let Rep_{Q_1} be the full subcategory of Rep_Q of all representations of Q (over \mathbb{K}) whose vector spaces at the vertices 2, 3 and 4 are zeros, and let $Rep_{Q_{13}}$ be the full subcategory of Rep_Q of all representations of Q (over \mathbb{K}) whose vector spaces at the vertices 2 and 4 are zeros.

Fix $X \in Rep_Q$ and a morphism $h : A \rightarrow B$ in Rep_Q . Every object in $\mathcal{Z}_h(Rep_{Q_{13}})$ can be viewed as a 7-tuple $(V_1, V_3, \varphi, \alpha_1, \alpha_3, \beta_1, \beta_3)$, where

$(V_1, \alpha_1, \beta_1), (V_3, \alpha_3, \beta_3) \in \mathcal{Z}_h(\text{Vec}_{\mathbb{K}})$ and $\varphi : V_1 \rightarrow V_2$ is a morphism in the category $\text{Vec}_{\mathbb{K}}$ of vector spaces over \mathbb{K} . Every object in $\mathcal{Z}(\text{Rep}_{Q13})$ can be viewed as a 5-tuple $(V_1, V_3, \varphi, \sigma_1, \sigma_3)$, where $(V_1, \sigma_1), (V_3, \sigma_3) \in \mathcal{Z}(\text{Vec}_{\mathbb{K}})$ and $\varphi : V_1 \rightarrow V_2$ is a morphism in $\text{Vec}_{\mathbb{K}}$. The categories $\mathcal{Z}_X(\text{Rep}_{Q13})$ and $\mathcal{Z}_\omega(\text{Rep}_{Q13})$ can be described similarly. The category Rep_Q is a cocomplete (and a complete) category (since it is equivalent to a category of modules over suitable algebra). Consequently, $\mathcal{Z}_h(\text{Rep}_{Q13}), \mathcal{Z}(\text{Rep}_{Q13}), \mathcal{Z}_X(\text{Rep}_{Q13})$ and $\mathcal{Z}_\omega(\text{Rep}_{Q13})$ are cocomplete (and complete) categories.

Note that $\text{Rep}_{Q1} \cong \text{Vec}_{\mathbb{K}}$ and hence we have $\mathcal{Z}_h(\text{Rep}_{Q1}) \cong \mathcal{Z}_h(\text{Vec}_{\mathbb{K}}), \mathcal{Z}(\text{Rep}_{Q1}) \cong \mathcal{Z}(\text{Vec}_{\mathbb{K}}), \mathcal{Z}_X(\text{Rep}_{Q1}) \cong \mathcal{Z}_X(\text{Vec}_{\mathbb{K}})$ and $\mathcal{Z}_\omega(\text{Rep}_{Q1}) \cong \mathcal{Z}_\omega(\text{Vec}_{\mathbb{K}})$. Thus, $\mathcal{Z}_h(\text{Rep}_{Q1}), \mathcal{Z}(\text{Rep}_{Q1}), \mathcal{Z}_X(\text{Rep}_{Q1})$ and $\mathcal{Z}_\omega(\text{Rep}_{Q1})$ are cocomplete (and complete) categories.

Definition 2.9. [10, p. 40] Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category.

(i) An object X^* in \mathcal{C} is said to be a *left dual* of X if there exist morphisms $ev_X : X^* \otimes X \rightarrow I$ and $coev_X : I \rightarrow X \otimes X^*$ called the *evaluation* and *coevaluation*, such that the compositions

$$\begin{aligned} X &\xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X, X^*, X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X \\ X^* &\xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a_{X^*, X, X^*}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^* \end{aligned}$$

are the identity morphisms.

An object *X in \mathcal{C} is said to be a *right dual* of X if there exist morphisms $ev'_X : X \otimes {}^*X \rightarrow I$ and $coev'_X : I \rightarrow {}^*X \otimes X$ such that the compositions

$$\begin{aligned} X &\xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a_{X, {}^*X, X}^{-1}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X \\ {}^*X &\xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X, X, {}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X \end{aligned}$$

are the identity morphisms.

(ii) An object in \mathcal{C} is called *rigid* if it has left and right duals. A monoidal category \mathcal{C} is called *rigid* if every object of \mathcal{C} is rigid.

Remark 2.10. Let \mathcal{C} be a monoidal category. Fix an object X and a morphism $h : A \rightarrow B$ in \mathcal{C} . Let $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$. If \mathcal{C} is rigid, then \mathcal{A} needs not be cocomplete as in the following examples.

Example 2.11.

(1) Let $Rep(H)$ be the category of finite dimensional representations of a finite dimensional Hopf algebra H . Then $Rep(H)$ is rigid [10, p. 113]. Since the category $\mathcal{Z}(Rep(H))$ is equivalent to as a braided tensor category to $Rep(D(H))$, the category of finite dimensional representations of the quantum double of H [10, p. 208], $\mathcal{Z}(Rep(H))$ is not cocomplete.

(2) Let G be a monoid (which we will usually take to be a group), and let A be an abelian group (with operation written multiplicatively). Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category whose objects δ_g are labeled by elements of G (so there is only one object in each isomorphism class), $Hom_{\mathcal{C}_G}(\delta_{g_1}, \delta_{g_2}) = \emptyset$ if $g_1 \neq g_2$, and $Hom_{\mathcal{C}_G}(\delta_g, \delta_g) = A$, with the functor \otimes defined by $\delta_g \otimes \delta_h = \delta_{gh}$, and the tensor product of morphisms defined by $a \otimes b = ab$. Then \mathcal{C}_G is a monoidal category with the associativity isomorphism being the identity, and the identity object $I =$ being the unit element of G .

This example has a “linear” version. Namely, let \mathbb{K} be a field and $\mathbb{K} - Vec_G$ denote the category of G -graded vector spaces over \mathbb{K} , that is, vector spaces V with a decomposition $V = \bigoplus_{g \in G} V_g$. Morphisms in this category are linear maps which preserve the grading. Define the tensor product on this category by the formula $(V \otimes W)_g = \bigoplus_{x,y \in G: xy=g} V_x \otimes W_y$, and the unit object I by $I_1 = \mathbb{K}$ and $I_g = 0$ for $g \neq 1$. Then, defining the associativity constraint and left and right unit constraints in an obvious way, we equip $\mathbb{K} - Vec_G$ with the structure of a monoidal category. Similarly one defines the monoidal category $f.d.\mathbb{K} - Vec_G$ of finite dimensional G -graded \mathbb{K} -vector spaces.

When no confusion is possible, we will denote the categories $\mathbb{K} - Vec_G$, $f.d.\mathbb{K} - Vec_G$ simply by Vec_G , $f.d.Vec_G$ [10, p. 27]. Then the category Vec_G is rigid if and only if the monoid G is a group [10, p. 43]. Furthermore, if G is a finitely generated infinite simple group (it is known that such groups exist), then $\mathcal{Z}(Vec_G)$ is equivalent to the category $f.d.Vec$ of finite dimensional spaces [10, p. 207]. Thus, $\mathcal{Z}(Vec_G)$ is not cocomplete.

3 Co-wellpoweredness

Let \mathfrak{E} be the class of all epimorphisms of a category \mathfrak{A} . Then \mathfrak{A} is called *co-wellpowered* provided that no \mathfrak{A} -object has a proper class of pairwise non-isomorphic quotients [2, p. 125]. In other words, for every object the

quotients form a set [22, p. 92, 95]. We refer the reader to [2] basics on quotients and co-wellpowered categories.

Proposition 3.1. [1, p. 5] *Let $\text{CoMon}(\mathcal{C})$ be the category of comonoids of \mathcal{C} and $U : \text{CoMon}(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor. If \mathcal{C} is co-wellpowered, then so is $\text{CoMon}(\mathcal{C})$.*

Proposition 3.2. *Let \mathcal{C} be a co-wellpowered category, and let $h : A \rightarrow B$ be an arrow in \mathcal{C} . If \mathcal{P}_J is cocontinuous $\forall J \in \{A, B\}$, then $\mathcal{Z}_h(\mathcal{C})$ is co-wellpowered.*

Proof. It is enough to show that if $p : (X, \alpha, \beta) \rightarrow (Y, \alpha', \beta')$ and $q : (X, \alpha, \beta) \rightarrow (Z, \alpha'', \beta'')$ are in $\mathcal{Z}_h(\mathcal{C})$ and equivalent as epimorphisms in \mathcal{C} , then they are equivalent (as epimorphisms) in $\mathcal{Z}_h(\mathcal{C})$. Let $\theta : Y \rightarrow Z$ be an isomorphism in \mathcal{C} for which $\theta p = q$. We show that θ is in fact an isomorphism in $\mathcal{Z}_h(\mathcal{C})$.

Since p and q are arrows in $\mathcal{Z}_h(\mathcal{C})$, the following diagrams are commutative.

$$\begin{array}{ccc}
 A \otimes X & \xrightarrow{\sim} & X \otimes A \\
 (id_A \otimes p) \downarrow & & \downarrow (p \otimes id_A) \\
 A \otimes Y & \xrightarrow{\sim} & Y \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 B \otimes X & \xrightarrow{\sim} & X \otimes B \\
 (id_B \otimes p) \downarrow & & \downarrow (p \otimes id_B) \\
 B \otimes Y & \xrightarrow{\sim} & Y \otimes B
 \end{array}
 \tag{3.1}$$

$$\begin{array}{ccc}
 A \otimes X & \xrightarrow{\sim} & X \otimes A \\
 (id_A \otimes q) \downarrow & & \downarrow (q \otimes id_A) \\
 A \otimes Z & \xrightarrow{\sim} & Z \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 B \otimes X & \xrightarrow{\sim} & X \otimes B \\
 (id_B \otimes q) \downarrow & & \downarrow (q \otimes id_B) \\
 B \otimes Z & \xrightarrow{\sim} & Z \otimes B
 \end{array}
 \tag{3.2}$$

Consider the following diagrams:

$$\begin{array}{ccc}
 A \otimes X & \xrightarrow[\sim]{\alpha} & X \otimes A \\
 \downarrow (id_A \otimes p) & & \downarrow (p \otimes id_A) \\
 A \otimes Y & \xrightarrow[\sim]{\alpha'} & Y \otimes A \\
 \downarrow (id_A \otimes \theta) & & \downarrow (\theta \otimes id_A) \\
 A \otimes Z & \xrightarrow[\sim]{\alpha''} & Z \otimes A
 \end{array}
 \quad (3.3)$$

$(id_A \otimes q)$ $(q \otimes id_A)$

$$\begin{array}{ccc}
 B \otimes X & \xrightarrow[\sim]{\alpha} & X \otimes B \\
 \downarrow (id_B \otimes p) & & \downarrow (p \otimes id_B) \\
 B \otimes Y & \xrightarrow[\sim]{\alpha'} & Y \otimes B \\
 \downarrow (id_B \otimes \theta) & & \downarrow (\theta \otimes id_B) \\
 B \otimes Z & \xrightarrow[\sim]{\alpha''} & Z \otimes B
 \end{array}
 \quad (3.4)$$

$(id_B \otimes q)$ $(q \otimes id_B)$

We have

$$\begin{aligned}
 \alpha''(id_A \otimes \theta)(id_A \otimes p) &= \alpha''(id_A \otimes \theta p) \text{ (naturality of } \otimes) \\
 &= \alpha''(id_A \otimes q) \text{ (since } \theta p = q) \\
 &= (q \otimes id_A)\alpha \text{ (by (3.2))} \\
 &= (\theta p \otimes id_A)\alpha \text{ (since } \theta p = q) \\
 &= (\theta \otimes id_A)(p \otimes id_A)\alpha \text{ (naturality of } \otimes) \\
 &= (\theta \otimes id_A)\alpha'(id_A \otimes p) \text{ (by (3.1))}
 \end{aligned}$$

Since P_A is cocontinuous, it preserves epimorphisms [16, p. 72]. Hence, $P_A(p) = (id_A \otimes p)$ is an epimorphism. Thus, $\alpha''(id_A \otimes \theta) = (\theta \otimes id_A)\alpha'$. Similarly, from diagram (3.4), we get $\alpha''(id_B \otimes \theta) = (\theta \otimes id_B)\alpha'$. Therefore, θ is an isomorphism in $\mathcal{Z}_h(\mathcal{C})$. \square

Proposition 3.3. *Let \mathcal{C} be a co-wellpowered category and X an object in \mathcal{C} . If \mathcal{Q}_X is cocontinuous, then $\mathcal{Z}_X(\mathcal{C})$ is co-wellpowered.*

Proof. As in Proposition 3.2, it suffices to show that if $p : (A, \alpha) \rightarrow (B, \beta)$ and $q : (A, \alpha) \rightarrow (B', \beta')$ are in $\mathcal{Z}_X(\mathcal{C})$ and equivalent as epimorphisms in \mathcal{C} , then they are equivalent (as epimorphisms) in $\mathcal{Z}_X(\mathcal{C})$. Let $\theta : B \rightarrow B'$ be an isomorphism in \mathcal{C} with $\theta p = q$. We show that θ is in fact an isomorphism in $\mathcal{Z}_X(\mathcal{C})$.

Since p and q are arrows in $\mathcal{Z}_X(\mathcal{C})$, the following diagrams are commutative:

$$\begin{array}{ccc}
 A \otimes X & \xrightarrow{\sim \alpha} & X \otimes A \\
 (p \otimes id_X) \downarrow & & \downarrow (id_X \otimes p) \\
 B \otimes X & \xrightarrow{\sim \beta} & X \otimes B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \otimes X & \xrightarrow{\sim \alpha} & X \otimes A \\
 (q \otimes id_X) \downarrow & & \downarrow (id_X \otimes q) \\
 B' \otimes X & \xrightarrow{\sim \beta'} & X \otimes B'
 \end{array}
 \tag{3.5}$$

Consider the following diagram

$$\begin{array}{ccc}
 A \otimes X & \xrightarrow{\sim \alpha} & X \otimes A \\
 (p \otimes id_X) \downarrow & & \downarrow (id_X \otimes p) \\
 B \otimes X & \xrightarrow{\sim \beta} & X \otimes B \\
 (\theta \otimes id_X) \downarrow & & \downarrow (id_X \otimes \theta) \\
 B' \otimes X & \xrightarrow{\sim \beta'} & X \otimes B'
 \end{array}
 \tag{3.6}$$

$(q \otimes id_X)$ } $(id_X \otimes q)$

We have

$$\begin{aligned}
 \beta'(\theta \otimes id_X)(p \otimes id_X) &= \beta'(\theta p \otimes id_X) \text{ (naturality of } \otimes) \\
 &= \beta'(q \otimes id_X) \text{ (since } \theta p = q) \\
 &= (id_X \otimes q)\alpha \text{ (by (3.5))} \\
 &= (id_X \otimes \theta p)\alpha \text{ (since } \theta p = q) \\
 &= (id_X \otimes \theta)(id_X \otimes p)\alpha \text{ (naturality of } \otimes) \\
 &= (id_X \otimes \theta)\beta(p \otimes id_X) \text{ (by (3.5))}
 \end{aligned}$$

Since Q_X is cocontinuous, it preserves epimorphisms. Hence, $Q_X(p) = (p \otimes id_X)$ is an epimorphism. Thus, $\beta'(\theta \otimes id_X) = (id_X \otimes \theta)\beta$. Therefore, θ is an isomorphism in $\mathcal{Z}_X(\mathcal{C})$, and the proof is complete. \square

Corollary 3.4. *Let \mathcal{C} be a co-wellpowered category. If Q_X is cocontinuous, $\forall X \in \mathcal{C}$, then $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}_\omega(\mathcal{C})$ are co-wellpowered.*

Proof. This immediately follows from the proof of Proposition 3.3 and Remark 1.7. \square

Using Theorem 2.6 implies the following immediate consequence.

Proposition 3.5. *Let \mathcal{C} be a monoidal category. Fix an object X and a morphism $h : A \rightarrow B$ in \mathcal{C} . Let $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$. If \mathcal{C} is a co-wellpowered biclosed monoidal category, then \mathcal{A} is co-wellpowered for any $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$.*

Example 3.6.

(1) The category **Set** of sets and mappings is co-wellpowered [22, p. 66], and it can also be seen as cartesian closed [8, p. 296], hence symmetric monoidal closed. By Proposition 3.5, the categories $\mathcal{Z}_h(\mathbf{Set})$, $\mathcal{Z}_X(\mathbf{Set})$, $\mathcal{Z}(\mathbf{Set})$, and $\mathcal{Z}_\omega(\mathbf{Set})$ are co-wellpowered for any $X \in \mathbf{Set}$ and a morphism h in **Set**.

(2) The category **Ab** of abelian groups with its tensor product of abelian groups is a biclosed monoidal category. By Proposition 3.5, $\mathcal{Z}_h(\mathbf{Ab})$, $\mathcal{Z}_X(\mathbf{Ab})$, $\mathcal{Z}(\mathbf{Ab})$, and $\mathcal{Z}_\omega(\mathbf{Ab})$ are co-wellpowered for any $X \in \mathbf{Ab}$ and a morphism h in **Ab**.

(3) The category **Top** of topological spaces and continuous mappings is wellpowered [22, p. 66], and it can be provided with the structure of a symmetric monoidal closed category (See [8, p. 299]). By Proposition 3.5, $\mathcal{Z}_h(\mathbf{Top})$, $\mathcal{Z}_X(\mathbf{Top})$, $\mathcal{Z}(\mathbf{Top})$, and $\mathcal{Z}_\omega(\mathbf{Top})$ are co-wellpowered for any $X \in \mathbf{Top}$ and a morphism h in **Top**.

(4) Consider Example 2.8(11). We have $\mathcal{Z}_h(\mathit{Rep}_{Q13})$, $\mathcal{Z}(\mathit{Rep}_{Q13})$, $\mathcal{Z}_X(\mathit{Rep}_{Q13})$ and $\mathcal{Z}_\omega(\mathit{Rep}_{Q13})$ (as well as the categories $\mathcal{Z}_h(\mathit{Rep}_{Q1})$, $\mathcal{Z}(\mathit{Rep}_{Q1})$, $\mathcal{Z}_X(\mathit{Rep}_{Q1})$ and $\mathcal{Z}_\omega(\mathit{Rep}_{Q1})$) are co-wellpowered categories.

4 Generators

Following [16, p. 127], a set \mathcal{G} of objects of the category \mathcal{C} is said to *generate* \mathcal{C} when any parallel pair $f, g : X \rightarrow Y$ of arrows of \mathcal{C} , $f \neq g$ implies that there is an $G \in \mathcal{G}$ and an arrow $\alpha : G \rightarrow X$ in \mathcal{C} with $f\alpha \neq g\alpha$ (the term “generates” is well established but poorly chosen; “*separates*” would have been better). For the basic concepts of generating sets, we refer to [16], [2], or [11].

Let \mathcal{C} be a monoidal category with a generating set \mathcal{G} , and let $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$. Fix an object X and a morphism $h : A \rightarrow B$

in \mathcal{C} . Our inspection in the previous sections gives rise to the following question. When can the category \mathcal{A} inherit a generating set involved with \mathcal{G} from \mathcal{C} ?

Under the assumption above, let $f, g : Z \rightarrow W$ be any parallel pair of morphisms in \mathcal{A} with $f \neq g$. Since \mathcal{G} is a generating set for \mathcal{C} , there is an $G \in \mathcal{G}$ and an arrow $\alpha : G \rightarrow X$ in \mathcal{C} with $f\alpha \neq g\alpha$. Now, if we want to show that \mathcal{A} has a generating set \mathcal{G} whose underlying is \mathcal{G} , we need to show that $G \in \mathcal{A}$ and the morphism $\alpha : G \rightarrow X$ is in \mathcal{A} . Although, this is not true in general, it perfectly works when \mathcal{C} is a braided. For the basic notions of braided monoidal categories, we refer to [23], [12] and [17]. It turns out that we have the following theorem.

Theorem 4.1. *Let \mathcal{C} be a braided monoidal category with a braiding Ψ , and let \mathcal{G} be a generating set for \mathcal{C} . Fix an object X and a morphism h in \mathcal{C} . Then the category \mathcal{A} has a generating set for any*

$$\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}.$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} & & \mathcal{Z}(\mathcal{C}) & & (4.1) \\ & & \uparrow \Phi_1 & & \\ & \mathcal{Z}_X(\mathcal{C}) & \xleftarrow{\Phi_2} & \mathcal{C} & \xrightarrow{\Phi_3} & \mathcal{Z}_h(\mathcal{C}) \\ & & \downarrow \Phi_4 & & \\ & & \mathcal{Z}_\omega(\mathcal{C}) & & \end{array}$$

It is well-known that there is an embedding $\Phi_1 : \mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$ via $W \mapsto (W, \Psi_{W,-})$ [9, p. 264]. Define the functors

$$\Phi_2 : \mathcal{C} \hookrightarrow \mathcal{Z}_X(\mathcal{C}), W \mapsto (W, \Psi_{W,X}),$$

$$\Phi_3 : \mathcal{C} \hookrightarrow \mathcal{Z}_h(\mathcal{C}), W \mapsto (W, \Psi_{A,W}, \Psi_{B,W}),$$

$$\Phi_4 : \mathcal{C} \hookrightarrow \mathcal{Z}_\omega(\mathcal{C}), W \mapsto (W, \Psi_{W,-}).$$

Clearly, Φ_i is embedding for all $i = 2, 3, 4$. Therefore, the category \mathcal{C} can be viewed as a subcategory of the category \mathcal{A} ,

$$\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}.$$

Now, let $f, g : Z \rightarrow W$ be any parallel pair of morphisms in \mathcal{A} with $f \neq g$. Since \mathcal{G} is a generating set for \mathcal{C} , there is an $G \in \mathcal{G}$ and an arrow $\alpha : G \rightarrow X$ in \mathcal{C} with $f\alpha \neq g\alpha$. From the diagram 4.1, we have $G \in \mathcal{A}$, and the morphism $\alpha : G \rightarrow X$ is in \mathcal{A} . Thus, for every $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$, \mathcal{A} has a generating set $\mathcal{G}_{\mathcal{A}}$ whose underlying is \mathcal{G} . \square

The following assertion is important in characterizing the cofree objects in $CoMon(\mathcal{A})$, for all $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$.

Corollary 4.2. *Let \mathcal{C} be a braided monoidal category. Fix an object X and a morphism h in \mathcal{C} , and let $\mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$. If the (monoidal) category $CoMon(\mathcal{C})$ has a generating set, then the category $CoMon(\mathcal{A})$ has a generating set.*

Example 4.3. Consider Example 2.8(11). We have $\mathcal{Z}_h(Rep_{Q13}), \mathcal{Z}(Rep_{Q13}), \mathcal{Z}_X(Rep_{Q13})$ and $\mathcal{Z}_\omega(Rep_{Q13})$ have generating sets, and $\mathcal{Z}_h(Rep_{Q1}), \mathcal{Z}(Rep_{Q1}), \mathcal{Z}_X(Rep_{Q1})$ and $\mathcal{Z}_\omega(Rep_{Q1})$ also have generating sets.

5 Investigating cofree objects

In this section, we use Theorem 1.1 and Propositions 2.1, 3.1, and the consequences we have to show that the concrete category $(CoMon(\mathcal{A}), \mathcal{U}_{\mathcal{A}})$ has cofree objects, $\forall \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$.

Theorem 5.1. *Let $\mathcal{U} : CoMon(\mathcal{Z}_h(\mathcal{C})) \rightarrow \mathcal{Z}_h(\mathcal{C})$ be the forgetful functor and $h : A \rightarrow B$ be an arrow in \mathcal{C} , and let \mathcal{P}_J and \mathcal{Q}_J be cocontinuous $\forall J \in \{A, B\}$. If \mathcal{C} is cocomplete, co-wellpowered and if $CoMon(\mathcal{Z}_h(\mathcal{C}))$ has a generating set, then \mathcal{U} has a right adjoint or, equivalently, the concrete category $(CoMon(\mathcal{Z}_h(\mathcal{C})), \mathcal{U})$ has cofree objects.*

Proof. This immediately follows from Propositions 2.1, 2.2, 3.2 and Theorem 1.1. \square

Corollary 5.2. *Let $(\mathcal{C}, \otimes, I)$ be a braided monoidal category and $h : A \rightarrow B$ an arrow in \mathcal{C} . Let $\mathcal{U} : CoMon(\mathcal{Z}_h(\mathcal{C})) \rightarrow \mathcal{Z}_h(\mathcal{C})$ be the forgetful functor and $\mathcal{P}_J, \mathcal{Q}_J$ be cocontinuous $\forall J \in \{A, B\}$. If \mathcal{C} is cocomplete, co-wellpowered and if $CoMon(\mathcal{C})$ has a generating set, then \mathcal{U} has a right adjoint, and hence the concrete category $(CoMon(\mathcal{Z}_h(\mathcal{C})), \mathcal{U})$ has cofree objects.*

Proof. It is immediate from Theorem 5.1 and Corollary 4.2. \square

Similarly, the following are immediate consequences of Proposition 2.1, Corollary 2.3, 3.3 and Theorem 1.1.

Theorem 5.3. *Let X be an object in \mathcal{C} . Let $\mathcal{U} : \text{CoMon}(\mathcal{Z}_X(\mathcal{C})) \rightarrow \mathcal{Z}_X(\mathcal{C})$ be the forgetful functor and $\mathcal{P}_X, \mathcal{Q}_X$ cocontinuous. If \mathcal{C} is cocomplete, co-wellpowered and if $\text{CoMon}(\mathcal{Z}_X(\mathcal{C}))$ has a generating set, then the functor \mathcal{U} has a right adjoint, hence, the concrete category $(\text{CoMon}(\mathcal{Z}_X(\mathcal{C})), \mathcal{U})$ has cofree objects.*

Corollary 5.4. *Let $(\mathcal{C}, \otimes, I)$ be a braided monoidal category and X an object in \mathcal{C} . Let $\mathcal{U} : \text{CoMon}(\mathcal{Z}_X(\mathcal{C})) \rightarrow \mathcal{Z}_X(\mathcal{C})$ be the forgetful functor and $\mathcal{P}_X, \mathcal{Q}_X$ be cocontinuous. If \mathcal{C} is cocomplete, co-wellpowered and if $\text{CoMon}(\mathcal{C})$ has a generating set, then \mathcal{U} has a right adjoint, hence, the concrete category $(\text{CoMon}(\mathcal{Z}_X(\mathcal{C})), \mathcal{U})$ has cofree objects.*

Proof. It follows immediately from Theorem 5.3 and Corollary 4.2. \square

By Propositions 2.1, 2.4, Corollary 3.4 and Theorem 1.1, we have the following version for the existence of cofree objects in the monoidal center.

Theorem 5.5. *Let*

$$\mathcal{U} : \text{CoMon}(\mathcal{Z}(\mathcal{C})) \rightarrow \mathcal{Z}(\mathcal{C})$$

(respectively $\mathcal{U}' : \text{CoMon}(\mathcal{Z}_\omega(\mathcal{C})) \rightarrow \mathcal{Z}(\mathcal{C}))$ be the forgetful functor, and let $\mathcal{P}_X, \mathcal{Q}_X$ be cocontinuous $\forall X \in \mathcal{C}$. If \mathcal{C} is cocomplete, co-wellpowered and if $\text{CoMon}(\mathcal{Z}(\mathcal{C}))$ (resp. $\text{CoMon}(\mathcal{Z}_\omega(\mathcal{C}))$) has a generating set, then \mathcal{U} (resp. \mathcal{U}') has a right adjoint. It turns out that, equivalently, the concrete category $(\text{CoMon}(\mathcal{Z}(\mathcal{C})), \mathcal{U})$ (resp. $(\text{CoMon}(\mathcal{Z}_\omega(\mathcal{C})), \mathcal{U}')$) has cofree objects.

Corollary 5.6. *Let $(\mathcal{C}, \otimes, I)$ be a braided monoidal category and*

$$\mathcal{U} : \text{CoMon}(\mathcal{Z}(\mathcal{C})) \rightarrow \mathcal{Z}(\mathcal{C})$$

(resp. $\mathcal{U}' : \text{CoMon}(\mathcal{Z}_\omega(\mathcal{C})) \rightarrow \mathcal{Z}(\mathcal{C}))$ the forgetful functor, and let $\mathcal{P}_X, \mathcal{Q}_X$ be cocontinuous $\forall X \in \mathcal{C}$. If \mathcal{C} is cocomplete, co-wellpowered and if $\text{CoMon}(\mathcal{C})$ has a generating set, then \mathcal{U} (resp. \mathcal{U}') has a right adjoint. It turns out that, equivalently, the concrete category $(\text{CoMon}(\mathcal{Z}(\mathcal{C})), \mathcal{U})$ (resp. $(\text{CoMon}(\mathcal{Z}_\omega(\mathcal{C})), \mathcal{U}')$) has cofree objects.

Proof. The required statement follows from Theorem 5.5 and Corollary 4.2. \square

Example 5.7.

(1) Consider Example 2.8(11). The categories $CoMon(\mathcal{Z}_h(Rep_{Q_{13}}))$, $CoMon(\mathcal{Z}(Rep_{Q_{13}}))$, $CoMon(\mathcal{Z}_X(Rep_{Q_{13}}))$ and $CoMon(\mathcal{Z}_\omega(Rep_{Q_{13}}))$ (as well as the categories $CoMon(\mathcal{Z}_h(Rep_{Q_1}))$, $CoMon(\mathcal{Z}(Rep_{Q_1}))$, $CoMon(\mathcal{Z}_X(Rep_{Q_1}))$ and $CoMon(\mathcal{Z}_\omega(Rep_{Q_1}))$) have cofree objects.

(2) Following [23, p. 69-70], the braid category \mathcal{B} has as objects the natural numbers $0, 1, 2, \dots$ and as arrows $\alpha : n \rightarrow n$ the braids on n strings; there are no arrows $n \rightarrow n$ for $m \neq n$. A braid α on n strings can be regarded as an element of the Artin braid group \mathcal{B}_n with generators s_1, \dots, s_{n-1} subject to the relations

$$\begin{aligned} s_i s_j &= s_j s_i, \text{ for } j < i - 1 \\ s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i. \end{aligned}$$

Composition of braids is just multiplication in this group, represented diagrammatically by vertical stacking of braids with the same number of strings. Tensor product of braids adds the number of strings by placing one braid next to the other longitudinally. This makes \mathcal{B} a strict monoidal category. A braiding $c_{m,n} : m + n \rightarrow n + m$ is given by crossing the first m strings over the remaining n . Then \mathcal{B} is braided monoidal category. Indeed, it is a balanced monoidal category. To see how the braid s_i , the composition of braids, tensor product of braids and the braiding $c_{m,n}$ can be depicted, we refer the reader to [23, p. 69-70].

Proposition 5.8. *The category \mathcal{B} is not cocomplete.*

Proof. Let \mathcal{D} be a small category, and let $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{B}$ be a functor. By the way of contradiction, let \mathcal{B} be a cocomplete category. It follows that \mathcal{F} has a colimit $(t, (\phi_D)_{D \in \mathcal{D}})$. The definition of \mathcal{B} implies that $\mathcal{F}D = t$, for all $D \in \mathcal{D}$. In particular, we have \mathcal{F} is a constant functor, for every functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{B}$ with \mathcal{D} a small category. It is clear that this is a contradiction because we can always define a nonconstant functor from a small category to \mathcal{B} . Therefore, the category \mathcal{B} is not cocomplete. \square

Theorem 5.9. Fix an object X and a morphism h in \mathcal{B} . We have

$$\text{CoMon}(\mathcal{A}) \cong \begin{array}{c} \curvearrowright \\ \bullet \end{array} \quad \text{for all } \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{B}), \mathcal{Z}_X(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}_\omega(\mathcal{B})\},$$

where $\begin{array}{c} \curvearrowright \\ \bullet \end{array}$ is the category with one object and one arrow.

Proof. We prove the theorem for $\mathcal{A} = \mathcal{Z}(\mathcal{B})$, and the rest can be proved similarly. Let $((m, \sigma), \Delta, \epsilon)$ be a comonoid in $\mathcal{Z}(\mathcal{B})$ with a comultiplication $\Delta : m \rightarrow m + m$ and a counit $\epsilon : m \rightarrow 0$. The definition of the category \mathcal{B} implies that $m = 0$, $\Delta = id_0 = \epsilon$, and $\sigma : 0 + - \rightarrow - + 0$ with $\sigma_n = id_n$, for every natural number n . Thus, the category $\text{CoMon}(\mathcal{Z}(\mathcal{B}))$ consists of one object $((0, \sigma), id_0, id_0)$, where $\sigma : 0 + - \rightarrow - + 0$ is the trivial natural isomorphism with $\sigma_n = id_n$, for every natural number n . \square

Theorem 5.10. Fix an object X and a morphism h in \mathcal{B} . The forgetful functor $\mathcal{U}_{\mathcal{A}} : \text{CoMon}(\mathcal{A}) \rightarrow \mathcal{A}$ has a right adjoint

$$\forall \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{B}), \mathcal{Z}_X(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}_\omega(\mathcal{B})\},$$

and thus, equivalently, the concrete category $(\text{CoMon}(\mathcal{A}), \mathcal{U}_{\mathcal{A}})$ has cofree objects.

Proof. It follows immediately from Theorem 5.9 that $\text{CoMon}(\mathcal{A})$ is cocomplete, co-wellpowered, and with a generating set. Thus, using Theorem 1.1 completes the proof. \square

Remark 5.11. Fix an object X and a morphism h in \mathcal{B} . It follows from Theorem 5.10 that the forgetful functor $\mathcal{U}_{\mathcal{A}} : \text{CoMon}(\mathcal{A}) \rightarrow \mathcal{A}$ has a right adjoint $\mathcal{V}_{\mathcal{A}} : \mathcal{A} \rightarrow \text{CoMon}(\mathcal{A})$ $\forall \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{B}), \mathcal{Z}_X(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}_\omega(\mathcal{B})\}$. Theorem 5.10, furthermore, implies that $\mathcal{V}_{\mathcal{A}} : \mathcal{A} \rightarrow \text{CoMon}(\mathcal{A})$ is a constant functor $\forall \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{B}), \mathcal{Z}_X(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}_\omega(\mathcal{B})\}$. Therefore, $\forall \mathcal{A} \in \{\mathcal{Z}_h(\mathcal{B}), \mathcal{Z}_X(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}_\omega(\mathcal{B})\}$, all the objects in the category \mathcal{A} have the same corresponding cofree object.

Example 5.12. Following [23, p. 74-75], the monoidal category $\tilde{\mathcal{B}}$ is defined similarly to \mathcal{B} , except that the arrows are braids on ribbons (instead of on strings) and it is permissible to twist the ribbons through full 2π turns. The homsets $\tilde{\mathcal{B}}(n, n) = \tilde{\mathcal{B}}_n$ are groups under composition. A presentation

of this group $\tilde{\mathcal{B}}_n$ is given by generators s_1, \dots, s_n where s_1, \dots, s_{n-1} satisfy the relations as for \mathcal{B}_n . These are depicted by thickened versions of the diagrams in Example 5.7, along with the extra relation

$$s_{n-1}s_n s_{n-1}s_n = s_n s_{n-1}s_n s_{n-1}.$$

Composition in $\tilde{\mathcal{B}}$ is vertical stacking of diagrams, and tensor product for $\tilde{\mathcal{B}}$ is horizontal placement of diagrams, much as for \mathcal{B} . The braiding $c_{m,n} : m + n \rightarrow n + m$ for $\tilde{\mathcal{B}}$ is obtained by placing the first m ribbons over the remaining n without introducing any twists. Then $\tilde{\mathcal{B}}$ is a braided monoidal category. Indeed, it is a balanced monoidal category. To see how s_n and the braiding $c_{m,n}$ can be visualized, we refer the reader to [23, p. 74-75].

The identification of $\tilde{\mathcal{B}}$ is similar to that of \mathcal{B} . Thus, for any (fixed) object X and an arrow h in $\tilde{\mathcal{B}}$, Proposition 5.8 and Theorems 5.9 and 5.10 imply the following consequences.

Proposition 5.13. *For all $\mathcal{A} \in \{\mathcal{Z}_h(\tilde{\mathcal{B}}), \mathcal{Z}_X(\tilde{\mathcal{B}}), \mathcal{Z}(\tilde{\mathcal{B}}), \mathcal{Z}_\omega(\tilde{\mathcal{B}})\}$, let $\mathcal{U}_{\mathcal{A}} : \text{CoMon}(\mathcal{A}) \rightarrow \mathcal{A}$ be the forgetful functor. We have the following:*

- (i) *The category $\tilde{\mathcal{B}}$ is not cocomplete.*
- (ii) *For any $\mathcal{A} \in \{\mathcal{Z}_h(\tilde{\mathcal{B}}), \mathcal{Z}_X(\tilde{\mathcal{B}}), \mathcal{Z}(\tilde{\mathcal{B}}), \mathcal{Z}_\omega(\tilde{\mathcal{B}})\}$, we have*

$$\text{CoMon}(\mathcal{A}) \cong \begin{array}{c} \curvearrowright \\ \bullet \end{array}.$$

- (iii) *For any $\mathcal{A} \in \{\mathcal{Z}_h(\tilde{\mathcal{B}}), \mathcal{Z}_X(\tilde{\mathcal{B}}), \mathcal{Z}(\tilde{\mathcal{B}}), \mathcal{Z}_\omega(\tilde{\mathcal{B}})\}$, the corresponding forgetful functor $\mathcal{U}_{\mathcal{A}}$ has a right adjoint, and hence the corresponding concrete category $(\text{CoMon}(\mathcal{A}), \mathcal{U}_{\mathcal{A}})$ has cofree objects).*

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