

Subpullbacks and coproducts of S -posets

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Abstract. In 2001, S. Bulman-Fleming et al. initiated the study of three flatness properties (weakly kernel flat, principally weakly kernel flat, translation kernel flat) of right acts A_S over a monoid S that can be described by means of when the functor $A_S \otimes -$ preserves pullbacks. In this paper, we extend these results to S -posets and present equivalent descriptions of weakly kernel po-flat, principally weakly kernel po-flat and translation kernel po-flat S -posets. Moreover, we show that most of flatness properties of S -posets can be transferred to their coproducts and vice versa.

1 Introduction and preliminaries

Let S be a pomonoid. A poset A is called a *right S -poset* (denoted by A_S) if there exists a right action $A \times S \rightarrow A$, $(a, s) \mapsto as$, which satisfies (i) the action is monotone in each variable, (ii) $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left S -posets are defined analogously. The notation A_S (respectively, ${}_S A$) will often be used to denote a right (respectively, left) S -poset, and $\Theta_S = \{\theta\}$ is the one-element right S -poset. All right (respectively, left) S -posets form a category, denoted **Pos- S** (respectively, **S -Pos**) (see [4]), whose *morphisms* are the functions that preserve both

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the action and the order. In these categories, as in the category **Pos** of posets, the monomorphisms and epimorphisms are the injective and surjective morphisms, respectively. In **Pos- S** and **S -Pos**, a morphism $g : A \rightarrow B$ is called an *order embedding* if $g(a) \leq g(a')$ implies $a \leq a'$ for all $a, a' \in A$. A surjective order embedding is called an *order isomorphism*.

A nonempty subset I of a pomonoid S is called an *ordered right ideal* of S if (i) $IS \subseteq I$ and (ii) $a \leq b \in I$ implies $a \in I$ for all $a, b \in S$. An S -subposet B_S of a right S -poset A_S is called *strongly convex* if $a \leq b$ implies $a \in B_S$ for any $a \in A_S$ and $b \in B_S$. Clearly, if I is an ordered right ideal of a pomonoid S , then I is a strongly convex S -subposet of the S -poset S . A pomonoid S is called *weakly right reversible* if for any $s, s' \in S$, there exist $u, v \in S$ such that $us \leq vs'$.

Preliminary work on flatness properties of S -posets, was done by Fakhrudin in [6, 7], and continued in recent papers [1, 3, 9, 12] etc.

To define the tensor product $A \otimes_S B$ of a right S -poset A_S and a left S -poset ${}_S B$ (see [12]), we first equip the Cartesian product $A \times B$ with component-wise order. Let $A \otimes_S B = (A \times B)/\rho$, where ρ is the order-congruence on the right S -poset $A \times B$ (on which S acts trivially) generated by the relation $H = \{((as, b), (a, sb)) \mid a \in A_S, b \in {}_S B, s \in S\}$. The equivalence class of (a, b) in $A \otimes_S B$ is denoted $a \otimes b$. The order relation on $A \otimes_S B$ will be described in Lemma 2.1. Similar to S -acts, it is easy to see that $A \otimes_S S$ can be equipped with a natural right S -action, and $A \otimes_S S \cong A$ for all S -posets A_S . It can be seen that $a \otimes s \leq a' \otimes t$ in $A \otimes_S S$ if and only if $as \leq a't$ in A_S .

Subpullbacks and subequalizers in the category **S -Pos** are defined in [1]. The categories **S -Pos** and **Pos** are poset-enriched concrete categories, where the order relation on morphism sets is defined pointwise (i.e. $f \leq g$ for $f, g : A \rightarrow B$ if and only if $f(a) \leq g(a)$ for every $a \in A$). In such categories, a diagram

$$\begin{array}{ccc} {}_S P & \xrightarrow{p_2} & {}_S N \\ p_1 \downarrow & & \downarrow g \\ {}_S M & \xrightarrow{f} & {}_S Q \end{array}$$

(P₁)

is called *the subpullback diagram for f and g* if

- (1) the diagram (\mathbf{P}_1) is subcommutative (i.e. $fp_1 \leq gp_2$), and
 (2) if

$$\begin{array}{ccc} {}_S P' & \xrightarrow{p'_2} & {}_S N \\ p'_1 \downarrow & & \downarrow g \\ {}_S M & \xrightarrow{f} & {}_S Q \end{array}$$

is a diagram in $S\text{-Pos}$ such that $fp'_1 \leq gp'_2$, then there exists a unique morphism $\varphi : {}_S P' \rightarrow {}_S P$ such that $p_1\varphi = p'_1$ and $p_2\varphi = p'_2$.

In $S\text{-Pos}$ or Pos , ${}_S P$ may in fact be realized as

$$P = \{(m, n) \in M \times N \mid f(m) \leq g(n)\}$$

with restrictions p_1 and p_2 of the projections of $M \times N$ onto ${}_S M$ and ${}_S N$ (note that P is possibly empty). The subpullback diagram (\mathbf{P}_1) is denoted by $P(M, N, f, g, Q)$ and tensoring it by any right S -poset A_S one gets the subcommutative diagram

$$\begin{array}{ccc} A \otimes_S P & \xrightarrow{id_A \otimes p_2} & A \otimes_S N \\ id_A \otimes p_1 \downarrow & & \downarrow id_A \otimes g \\ A \otimes_S M & \xrightarrow{id_A \otimes f} & A \otimes_S Q \end{array}$$

in Pos . For the subpullback of mappings $id_A \otimes f$ and $id_A \otimes g$, we may take

$$P' = \{(a \otimes m, a' \otimes n) \in (A \otimes_S M) \times (A \otimes_S N) \mid a \otimes f(m) \leq a' \otimes g(n)\}$$

with p'_1, p'_2 being the restrictions of the projections.

From the definition of subpullbacks it follows that there exists a unique monotonic mapping $\phi : A \otimes_S P \rightarrow P'$ such that, in the diagram

$$\begin{array}{ccccc}
A \otimes_S P & & & & \\
\downarrow \phi & \searrow id_A \otimes p_2 & & & \\
& P' & \xrightarrow{p'_2} & A \otimes_S N & \\
id_A \otimes p_1 \downarrow & \downarrow p'_1 & & \downarrow id_A \otimes g & \\
& A \otimes_S M & \xrightarrow{id_A \otimes f} & A \otimes_S Q & \\
& & & & \\
& & & & (\mathbf{P}_2)
\end{array}$$

we have $p'_i \phi = id_A \otimes p_i$ for $i = 1, 2$. This mapping is called *the ϕ corresponding to the subpullback diagram $P(M, N, f, g, Q)$* for A_S . It can be seen that the mapping ϕ in diagram (\mathbf{P}_2) is given by

$$\phi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m, n) \in_S P$. Note that for ϕ to be surjective requires

$$\begin{aligned}
& (\forall a, a' \in A_S)(\forall m \in_S M)(\forall n \in_S N)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow \\
& \quad (\exists a'' \in A_S)(\exists m' \in_S M)(\exists n' \in_S N) \\
& \quad (f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n')],
\end{aligned}$$

and for ϕ to be order embeddable requires

$$\begin{aligned}
& (\forall a, a' \in A_S)(\forall m, m' \in_S M)(\forall n, n' \in_S N) \\
& [f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m \leq a' \otimes m' \wedge a \otimes n \leq a' \otimes n' \Rightarrow \\
& \quad a \otimes (m, n) \leq a' \otimes (m', n') \text{ in } A \otimes_S P].
\end{aligned}$$

Moreover, if the mapping ϕ is both a surjection and an order embedding, then ϕ is an order isomorphism.

Similar to S -acts, coproducts of S -posets are disjoint unions, with S -action and order defined componentwise.

If $A = \bigcup_{i \in I} A_i$, where A_i are strongly convex right S -subposets of A_S , then by the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_i, i \in I$, we mean the unique monotonic mapping ϕ_i which makes, in the diagram

$$\begin{array}{ccccc}
 A_i \otimes_S P & & & & \\
 \searrow^{\phi_i} & \searrow^{id_{A_i} \otimes p_2} & & & \\
 & P'_i & \xrightarrow{p'_{2i}} & A_i \otimes_S N & \\
 \searrow^{id_{A_i} \otimes p_1} & \downarrow p'_{1i} & & \downarrow id_{A_i} \otimes g & \\
 & A_i \otimes_S M & \xrightarrow{id_{A_i} \otimes f} & A_i \otimes_S Q &
 \end{array}$$

(P_{2i})

we have $p'_{ji}\phi_i = id_{A_i} \otimes p_j$ for $j = 1, 2$, where

$$P'_i = \{(a \otimes m, a' \otimes n) \in (A_i \otimes_S M) \times (A_i \otimes_S N) \mid a \otimes f(m) \leq a' \otimes g(n)\}$$

and p'_{1i}, p'_{2i} are the restrictions of projections to P'_i .

It is shown in [2, 10] that, if we require either bijectivity or surjectivity of ϕ for pullback diagram of certain types, we not only recover most of the well-known forms of flatness, but obtain some new properties of acts as well. Furthermore, some of these results are extended to S -posets, and the classes of right S -posets corresponding to all of the cells in the first and second columns of Figure 1 are considered in [9]. This paper continues the investigation of the classes of right S -posets A_S over S for which the functor $A_S \otimes -$ has certain subpullback preservation properties. The variations of the types of subpullbacks considered in [9] and this paper are of the following types:

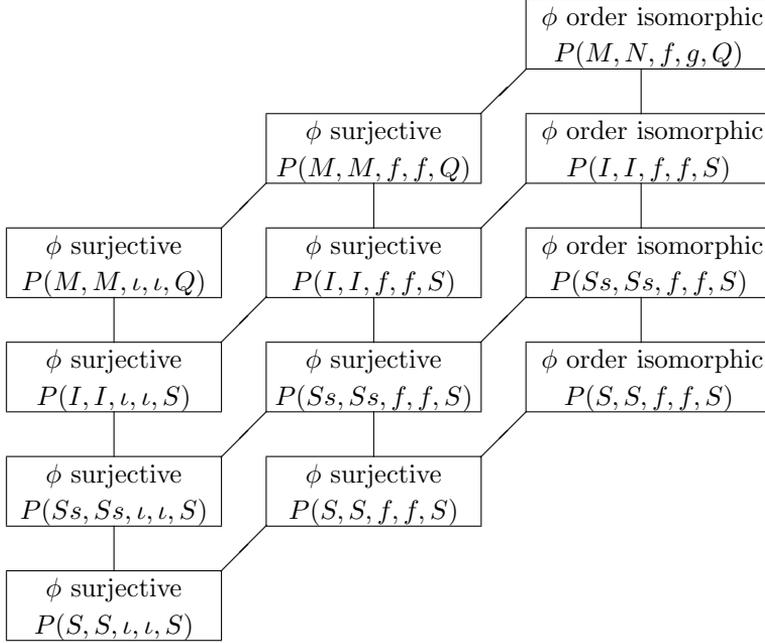


Figure 1.

Where I (Ss) stands for a (principal) left ideal of S , and ι for a monomorphism of left S -posets. Every rectangle stands for a class of right S -posets that is defined by the property it contains. In the second and third columns, for instance, a rectangle with the text “ ϕ order isomorphic $P(M, N, f, g, Q)$ ” denotes the class of all right S -posets A_S such that the mapping ϕ is order isomorphic corresponding to every subpullback diagram $P(M, N, f, g, Q)$. But in the first column, for instance, a rectangle with the text “ ϕ surjective $P(Ss, Ss, \iota, \iota, S)$ ” denotes the class of all right S -posets A_S such that the mapping ϕ is surjective corresponding to every pullback diagram $P(Ss, Ss, \iota, \iota, S)$. A line between two rectangles indicates that the class of right S -posets corresponding to the rectangle at the upper end of the line is contained in the class corresponding to the rectangle at the lower end.

An S -poset A_S is called *subpullback flat* (respectively, *subequalizer flat*) if the functor $A_S \otimes -$ takes subpullbacks (respectively, subequalizers) in $S\text{-Pos}$ to subpullbacks (respectively, subequalizers) in \mathbf{Pos} . Clearly, A_S is subpullback flat if and only if the mapping ϕ is order isomorphic corresponding to every subpullback diagram $P(M, N, f, g, Q)$ in $S\text{-Pos}$.

It is proved in [1] that an S -poset A_S is subpullback flat and subequalizer flat if and only if A_S satisfies the following conditions:

$$(P): (\forall a, a' \in A_S)(\forall u, v \in S)(au \leq a'v)$$

$$\Rightarrow (\exists a'' \in A_S)(\exists s, t \in S)(a = a''s \wedge a' = a''t \wedge su \leq tv));$$

$$(E): (\forall a \in A_S)(\forall u, v \in S)(au \leq av \\ \Rightarrow (\exists a' \in A_S)(\exists s \in S)(a = a's \wedge su \leq sv)).$$

It is shown in [9] that an S -poset A_S satisfies condition (P) if and only if the mapping ϕ is surjective corresponding to every subpullback diagram $P(M, N, f, g, Q)$. Conditions (WP) and (PWP) are also introduced in [9]. An S -poset A_S is said to satisfy *condition (WP)* if the mapping ϕ is surjective corresponding to every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S . An S -poset A_S is said to satisfy *condition (PWP)* if the mapping ϕ is surjective corresponding to every subpullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$.

In Section 2 of this paper, we introduce three additional flatness properties (weakly kernel po-flat, principally weakly kernel po-flat, translation kernel po-flat) of S -posets by means of subpullback preservation, and present equivalent descriptions of them (both for arbitrary and for cyclic S -posets).

It is shown in [10] that most of flatness properties of acts over a monoid S are equivalent to the surjectivity or bijectivity of mappings corresponding to the pullback diagrams in special cases. Furthermore, it is shown in [8] that these flatness properties can be transferred to their coproducts. The purpose of Section 3 of this paper is to carry over these results to the setting of S -posets, and we show that flatness properties introduced in [9] can be transferred from S -posets over a pomonoid S to their coproducts.

Although much of our work follows directly from the unordered case, some care is needed. Moreover, the results need to be stated and justified, which is the aim of this article.

2 Subpullbacks and flatness

In this section, we discuss the classes of right S -posets A_S corresponding to the three lowest cells in the third column of Figure 1. We give an alternative description of a right (cyclic, one-element) S -poset having the corresponding property.

We begin with the following result used by many authors and formulated in [12, Theorem 5.2].

Lemma 2.1. *Let A_S be a right S -poset, ${}_S B$ a left S -poset, $a, a' \in A_S$, $b, b' \in {}_S B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, a_2, \dots, a_n \in A_S$,*

$b_2, \dots, b_n \in {}_S B$ and $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{array}{rcl} a & \leq & a_1 s_1 \\ a_1 t_1 & \leq & a_2 s_2 \quad s_1 b \leq t_1 b_2 \\ a_2 t_2 & \leq & a_3 s_3 \quad s_2 b_2 \leq t_2 b_3 \\ & \vdots & \vdots \\ a_n t_n & \leq & a' \quad s_n b_n \leq t_n b'. \end{array}$$

Definition 2.2. A right S -poset A_S is called

- (i) *weakly kernel po-flat* if the mapping ϕ is order isomorphic corresponding to every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S ;
- (ii) *principally weakly kernel po-flat* if the mapping ϕ is order isomorphic corresponding to every subpullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$;
- (iii) *translation kernel po-flat* if the mapping ϕ is order isomorphic corresponding to every subpullback diagram $P(S, S, f, f, S)$.

From Figure 1 and Theorems 2.1, 2.2, 2.3, 2.4, 3.2, 4.1, 5.3 of [9], we see that the new properties just defined are related to properties already studied as shown in Figure 2.

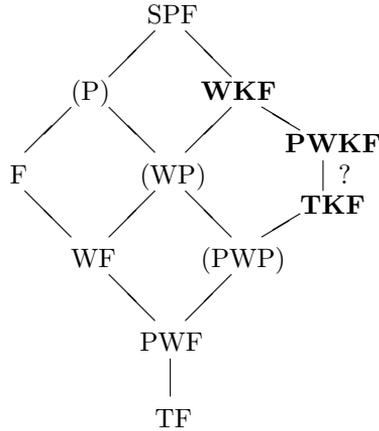


Figure 2.

Note 2.3. SPF = subpullback flatness, F = flatness, WF = weak flatness, PWF = principal weak flatness, WKF = weak kernel po-flatness, PWKF = principal weak kernel po-flatness, TKF = translation kernel po-flatness, TF = torsion freeness.

If S is a pomonoid and $t \in S$, then $\rho_t : S \rightarrow S$ will denote the *right translation* by t , that is, $\rho_t(s) = st$ for every $s \in S$.

Recall that a binary relation σ on an S -poset A_S is called a *compatible quasi-order* on A_S if it is transitive, compatible with the S -action, and contains the relation \leq on A_S . The relationship between order-congruences and compatible quasi-order on A_S is given in [13].

Suppose that ρ is a right order congruence on a pomonoid S . Define a relation $\widehat{\rho}$ by

$$s \widehat{\rho} t \Leftrightarrow [s]_\rho \leq [t]_\rho \quad \text{in } S/\rho.$$

It is clear that $\widehat{\rho}$ is a compatible quasi-order on A_S .

The *subkernel* or *directed kernel* of an S -poset morphism $f : A_S \rightarrow B_S$ is defined by $\overrightarrow{\ker} f = \{(a, a') \in A \times A \mid f(a) \leq f(a')\}$ (see [5]). It is shown in [13] that $\overrightarrow{\ker} f$ is a compatible quasi-order on A_S . Furthermore, we first give equivalent characterizations of weak kernel flatness, principal weak kernel flatness and translation kernel flatness, both for arbitrary and for cyclic S -posets. If ρ is an equivalence relation on S and $s \in S$, then \bar{s} denotes the equivalence class of s modulo ρ .

Proposition 2.4. *A right S -poset A_S is weakly kernel po-flat if and only if A_S satisfies condition (WP) and for every left ideal I of S and every morphism $f : {}_S I \rightarrow_S S$ the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t' \in I)$$

$$\left. \begin{array}{l} a \otimes s \leq a' \otimes s' \text{ in } A \otimes I, \quad f(s) \leq f(t) \\ a \otimes t \leq a' \otimes t' \text{ in } A \otimes I, \quad f(s') \leq f(t') \end{array} \right\} \Longrightarrow$$

$$a \otimes (s, t) \leq a' \otimes (s', t') \text{ in } A \otimes_S \overrightarrow{\ker} f.$$

Proof. It follows from Lemma 5.1 of [9] and Definition 2.2. □

As a direct consequence, we have

Corollary 2.5. *A cyclic right S -poset S/ρ is weakly kernel po-flat if and only if S/ρ satisfies condition (WP) and for every left ideal I of S and every morphism $f : {}_S I \rightarrow_S S$ the following condition holds:*

$$(\forall s, s', t, t' \in I)$$

$$\left. \begin{array}{l} \bar{1} \otimes s \leq \bar{1} \otimes s' \text{ in } S/\rho \otimes I, \quad f(s) \leq f(t) \\ \bar{1} \otimes t \leq \bar{1} \otimes t' \text{ in } S/\rho \otimes I, \quad f(s') \leq f(t') \end{array} \right\} \Longrightarrow$$

$$\bar{1} \otimes (s, t) \leq \bar{1} \otimes (s', t') \text{ in } S/\rho \otimes_S \overrightarrow{\ker} f.$$

Proposition 2.6. *A right S -poset A_S is principally weakly kernel po-flat if and only if A_S satisfies condition (PWP) and the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z, x \in S \text{ such that } \overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z})$$

$$\left. \begin{array}{l} asx \leq a's'x, \quad sz \leq tz \\ atx \leq a't'x, \quad s'z \leq t'z \end{array} \right\} \implies a \otimes (sx, tx) \leq a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,$$

where ${}_S P = \{(ux, vx) \mid u, v \in S, uz \leq vz\}$.

Proof. Necessity. Let A_S be principally weakly kernel po-flat. Then A_S satisfies condition (PWP). Suppose that

$$\begin{array}{l} asx \leq a's'x, \quad sz \leq tz, \\ atx \leq a't'x, \quad s'z \leq t'z \end{array}$$

for some $a, a' \in A_S$ and $s, s', t, t', z, x \in S$ such that $\overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z}$. Define a mapping $f : {}_S(Sx) \rightarrow {}_S S$ by $f(x) := z$. Since $\overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z}$, f is well-defined. Clearly, f is a morphism of left S -posets. Using Theorem 4.1 of [9], from the inequality $asx \leq a's'x$ we obtain that there exist $b \in A_S$ and $u, v \in S$ such that $as = bu$, $a's' = bv$ and $ux \leq vx$. Hence we have

$$\begin{aligned} a \otimes sx &= as \otimes x = bu \otimes x = b \otimes ux \leq b \otimes vx \\ &= bv \otimes x = a's' \otimes x = a' \otimes s'x \end{aligned}$$

in $A \otimes_S Sx$. Analogously, $a \otimes tx \leq a' \otimes t'x$ in $A \otimes_S Sx$. Because A_S is principally weakly kernel po-flat, the mapping ϕ is an order embedding corresponding to the subpullback diagram $P(Sx, Sx, f, f, S)$. Then the inequalities

$$\begin{array}{l} a \otimes sx \leq a' \otimes s'x, \quad f(sx) \leq f(tx), \\ a \otimes tx \leq a' \otimes t'x, \quad f(s'x) \leq f(t'x) \end{array}$$

imply

$$a \otimes (sx, tx) \leq a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,$$

where

$$\begin{aligned} P &= \{(ux, vx) \in Sx \times Sx \mid f(ux) \leq f(vx)\} \\ &= \{(ux, vx) \mid u, v \in S, uz \leq vz\}. \end{aligned}$$

Sufficiency. Let ϕ be the canonical mapping corresponding to the subpullback diagram $P(Sx, Sx, f, f, S)$ for A_S , where $s \in S$ and $f : {}_S(Sx) \rightarrow {}_S S$ is a morphism. Because A_S satisfies condition (PWP), the mapping ϕ is surjective corresponding to every subpullback diagram $P(Sx, Sx, f, f, S)$ by Theorem 4.1 of [9]. Next we

show that ϕ is also an order embedding corresponding to every subpullback diagram $P(Sx, Sx, f, f, S)$. Suppose that

$$\begin{aligned} a \otimes sx &\leq a' \otimes s'x \text{ in } A \otimes_S Sx, & f(sx) &\leq f(tx), \\ a \otimes tx &\leq a' \otimes t'x \text{ in } A \otimes_S Sx, & f(s'x) &\leq f(t'x) \end{aligned}$$

for some $a, a' \in A_S$ and $s, t, s', t', x \in S$. Then

$$\begin{aligned} asx &\leq a's'x, & sz &\leq tz, \\ atx &\leq a't'x, & s'z &\leq t'z, \end{aligned}$$

where $z = f(x)$. By assumption

$$a \otimes (sx, tx) \leq a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,$$

where ${}_S P = \{(ux, vx) \mid u, v \in S, uz \leq vz\} = \overrightarrow{\ker f}$. Hence ϕ is an order embedding, and so A_S is principally weakly kernel po-flat. \square

Using Proposition 2.6, we have the following description for a principally weakly kernel po-flat cyclic S -poset.

Corollary 2.7. *A cyclic right S -poset S/ρ is principally weakly kernel po-flat if and only if S/ρ satisfies condition (PWP) and the following condition holds:*

$$(\forall s, s', t, t', z, x \in S \text{ such that } \overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z})$$

$$\left. \begin{array}{l} sx \widehat{\rho} s'x, \quad sz \leq tz \\ tx \widehat{\rho} t'x, \quad s'z \leq t'z \end{array} \right\} \implies \bar{1} \otimes (s, t) = \bar{1} \otimes (s', t') \text{ in } S/\rho \otimes_S P,$$

where ${}_S P = \{(ux, vx) \mid u, v \in S, uz \leq vz\}$.

Proposition 2.8. *A right S -poset A_S is translation kernel po-flat if and only if A_S satisfies condition (PWP) and the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z \in S)$$

$$\left. \begin{array}{l} as \leq a's', \quad sz \leq tz \\ at \leq a't', \quad s'z \leq t'z \end{array} \right\} \implies a \otimes (s, t) \leq a' \otimes (s', t') \text{ in } A \otimes_S \overrightarrow{\ker \rho_z}.$$

Proof. It is similar to that of Proposition 2.6. \square

For a cyclic right S -poset, Proposition 2.8 yields the following

Corollary 2.9. *A cyclic right S -poset S/ρ is translation kernel po-flat if and only if S/ρ satisfies condition (PWP) and the following condition holds:*

$$(\forall s, s', t, t', z \in S)$$

$$\left. \begin{array}{l} s \widehat{\rho} s', \quad sz \leq tz \\ t \widehat{\rho} t', \quad s'z \leq t'z \end{array} \right\} \implies \bar{1} \otimes (s, t) \leq \bar{1} \otimes (s', t') \text{ in } S/\rho \otimes_S \overrightarrow{\ker \rho_z}.$$

We now consider whether a one-element S -poset $\Theta_S = \{\theta\}$ satisfies each of our new properties. In preparation, we need to give the definition of connectedness for S -posets.

Definition 2.10. An S -poset ${}_S B$ is called *connected* if for all $b, b' \in {}_S B$ there exist elements $s_1, t_1, \dots, s_n, t_n \in S$ and $b_2, \dots, b_n \in {}_S B$ such that

$$\begin{aligned} s_1 b &\leq t_1 b_2 \\ s_2 b_2 &\leq t_2 b_3 \\ &\vdots \\ s_n b_n &\leq t_n b'. \end{aligned}$$

The foregoing sequence of inequalities will be called a *scheme of length n connecting b and b'* .

Proposition 2.11. For any pomonoid S , the following statements are equivalent:

- (i) Θ_S is principally weakly kernel po-flat;
- (ii) Θ_S is translation kernel po-flat;
- (iii) For every $z \in S$, $\overrightarrow{\ker \rho_z}$ is connected as a left S -poset.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). Take $(s, t), (s', t') \in \overrightarrow{\ker \rho_z}$. Using translation kernel po-flatness,

$$\begin{aligned} \theta s &\leq \theta s', & sz &\leq tz, \\ \theta t &\leq \theta t', & s'z &\leq t'z \end{aligned}$$

imply $\theta \otimes (s, t) \leq \theta \otimes (s', t')$ in $\Theta \otimes_S \overrightarrow{\ker \rho_z}$. By Lemma 2.1, there exist $s_1, t_1, \dots, s_n, t_n \in S$ and $b_2, \dots, b_n \in \overrightarrow{\ker \rho_z}$ such that

$$\begin{array}{lll} \theta & \leq \theta s_1 & \\ \theta t_1 & \leq \theta s_2 & s_1(s, t) \leq t_1 b_2 \\ \theta t_2 & \leq \theta s_3 & s_2 b_2 \leq t_2 b_3 \\ & \vdots & \vdots \\ \theta t_n & \leq \theta & s_n b_n \leq t_n(s', t'). \end{array}$$

The right hand part of a scheme corresponding to the latter inequality shows that $\overrightarrow{\ker \rho_z}$ is connected.

(iii) \Rightarrow (i). Note first from Theorem 4.1 of [9] that Θ_S always satisfies condition (PWP). Consider any $x, z \in S$ such that $\overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z}$. Because $\overrightarrow{\ker \rho_z}$ is connected, there exists a scheme corresponding to the inequality $\theta \otimes (s, t) \leq \theta \otimes (s', t')$

in $\Theta \otimes_S \overrightarrow{\ker \rho_z}$ for any $(s, t), (s', t') \in \overrightarrow{\ker \rho_z}$. Multiplying each inequality in the right hand column of this scheme (on the right) by x establishes

$$\theta \otimes (sx, tx) \leq \theta \otimes (s'x, t'x) \text{ in } \Theta \otimes_S P,$$

where ${}_S P = \{(ux, vx) \mid u, v \in S, uz \leq vz\}$, and so, by Proposition 2.6, Θ_S is principally weakly kernel po-flat and the proof is complete. \square

From Corollary 5.4 of [9], it follows that Θ_S satisfies condition (WP) if and only if S is weakly right reversible. So, using Proposition 2.4, we have

Proposition 2.12. *Θ_S is weakly kernel po-flat if and only if S is weakly right reversible, and for every left ideal I of S , $\overrightarrow{\ker f}$ is connected for every homomorphism $f : {}_S I \rightarrow {}_S S$.*

The following example from [2, Proposition 26] illustrates that principal weak kernel flatness does not imply weak kernel flatness.

Example 2.13. Let S be a right zero semigroup K with 1 adjoined and $|K| > 1$. The order of S is discrete. Then S is not weakly right reversible, and so by Proposition 2.12, Θ_S is not weakly kernel po-flat. Now we show that $\overrightarrow{\Theta_S}$ is principally weakly kernel po-flat. By Proposition 2.11, we need to check that $\overrightarrow{\ker \rho_z}$ is connected as a left S -poset for every $z \in S$. Since the order of S is discrete, $\overrightarrow{\ker \rho_z} = \ker \rho_z$ and connectedness only involves equalities. Thus, we could directly apply Proposition 26 from [2] and obtain the result.

Note 2.14. From the preceding example we obtain that there exists a principally weakly kernel po-flat right S -poset, but does not satisfy conditions (WP) , $(WP)_w$, (P) or $(P)_w$, and is not subpullback flat, flat, po-flat, weakly flat, or weakly po-flat by Theorem 6.2 of [9].

We have been unable so far to answer the question of whether principally weakly kernel po-flat and translation kernel po-flat are equivalent, we also have not yet been able to provide a suitable example to distinguish them. But, if S is an ordered lpp monoid, then all translation kernel po-flat S -posets are principally weakly kernel po-flat.

Recall that a pomonoid S is called an *ordered lpp monoid* if the S -subposet Sx is projective for all $x \in S$. By Proposition 4.8 of [12], a pomonoid S is an ordered lpp monoid if and only if for every $a \in S$ there exists an idempotent e of S such that $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$. These pomonoids comprise quite an extensive class, including all I -regular pomonoids and all right po-cancellable pomonoids (See [12], for more information).

Theorem 2.15. *If S is an ordered lpp monoid, then all translation kernel po-flat S -posets are principally weakly kernel po-flat.*

Proof. Suppose S is an ordered lpp monoid and A_S is translation kernel po-flat. To show that A_S is principally weakly kernel po-flat, we check the condition of Proposition 2.6. Suppose that $a, a' \in A_S$ and $s, s', t, t', z, x \in S$ are such that $\overrightarrow{\ker\rho_x} \subseteq \overrightarrow{\ker\rho_z}$ and

$$\begin{aligned} asx &\leq a's'x, & sz &\leq tz, \\ atx &\leq a't'x, & s'z &\leq t'z. \end{aligned}$$

Because S is an ordered lpp monoid, there exists $e \in E(S)$ such that $ex = x$ (and hence $ez = z$), and $px \leq qx$ implies $pe \leq qe$ for all $p, q \in S$. Because A_S satisfies condition (PWP), from $(as)x \leq (a's')x$, we obtain $c \in A_S$ and $p, q \in S$ such that $as = cp, a's' = cq$ and $px \leq qx$ by Theorem 4.1 of [9]. From $(at)x \leq (a't')x$, we obtain $d \in A_S$ and $g, h \in S$ such that $at = dg, a't' = dh$ and $gx \leq hx$. Because S is an ordered lpp monoid, we have $pe \leq qe$ and $ge \leq he$. We can now calculate

$$\begin{aligned} ase &= cpe \leq cqe = a's'e, \\ ate &= dge \leq dhe = a't'e. \end{aligned}$$

Therefore, we have $ase \leq a's'e$ and $ate \leq a't'e$. Moreover,

$$sez = sz \leq tz = tez \text{ and } s'ez = s'z \leq t'z = t'ez.$$

Using translation kernel po-flatness of A_S , we know that $a \otimes (se, te) \leq a' \otimes (s'e, t'e)$ in $A \otimes_S \overrightarrow{\ker\rho_z}$ by Proposition 2.8. Using Lemma 2.1, there exist $a_1, \dots, a_n \in A_S, (x_2, y_2), \dots, (x_n, y_n) \in \overrightarrow{\ker\rho_z}$, and $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{array}{rcl} a & \leq & a_1 s_1 \\ a_1 t_1 & \leq & a_2 s_2 \quad s_1(se, te) \leq t_1(x_2, y_2) \\ a_2 t_2 & \leq & a_3 s_3 \quad s_2(x_2, y_2) \leq t_2(x_3, y_3) \\ & \vdots & \vdots \\ a_n t_n & \leq & a' \quad s_n(x_n, y_n) \leq t_n(s'e, t'e). \end{array}$$

Multiplication of each inequality in the right-hand column on the right by x produces the scheme

$$\begin{array}{rcl} a & \leq & a_1 s_1 \\ a_1 t_1 & \leq & a_2 s_2 \quad s_1(sx, tx) \leq t_1(x_2x, y_2x) \\ a_2 t_2 & \leq & a_3 s_3 \quad s_2(x_2x, y_2x) \leq t_2(x_3x, y_3x) \\ & \vdots & \vdots \\ a_n t_n & \leq & a' \quad s_n(x_nx, y_nx) \leq t_n(s'x, t'x), \end{array}$$

where each $(x_i x, y_i x)$, $i = 2, \dots, n$, belongs to $P = \{(ux, vx) \mid u, v \in S, uz \leq vz\}$. In other words,

$$a \otimes (sx, tx) \leq a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,$$

as was to be shown. \square

3 Flatness and coproducts

In this section, we will show that most of flatness properties of S -posets over a pomonoid S can be transferred to their coproducts.

Recall from [12] that an S -poset A_S is called *decomposable* if there exist nonempty strongly convex S -subposets $A_1, A_2 \subseteq A$ such that $A = A_1 \dot{\cup} A_2$ (i.e. $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$). Otherwise A_S is called *indecomposable*.

Lemma 3.1. *Let $A = \bigcup_{i \in I} A_i$, where A_i , $i \in I$, are right S -subposets of A_S . Let ${}_S B$ be a left S -poset. If $a \otimes b \leq a' \otimes b'$ in $A_i \otimes_S B$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$.*

Proof. It is obvious. \square

The next result will be useful for the remainder of this section.

Lemma 3.2. *Let $A = \bigcup_{i \in I} A_i$, where A_i , $i \in I$, are right strongly convex S -subposets of A_S . Let ${}_S B$ be a left S -poset and suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$. Then $a \in A_i$ for some $i \in I$, if and only if $a' \in A_i$.*

Proof. Necessity. Let $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$. Using Lemma 2.1, there exist $a_1, \dots, a_n \in A_S$, $b_2, \dots, b_n \in {}_S B$ and $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{array}{rcl} a & \leq & a_1 s_1 \\ a_1 t_1 & \leq & a_2 s_2 \quad s_1 b \leq t_1 b_2 \\ a_2 t_2 & \leq & a_3 s_3 \quad s_2 b_2 \leq t_2 b_3 \\ & \vdots & \vdots \\ a_n t_n & \leq & a' \quad s_n b_n \leq t_n b'. \end{array} \quad (*)$$

Since $a \in A_i$, we have $a_1 \in A_i$. Otherwise there exists $j \neq i \in I$ such that $a_1 \in A_j$, and so $a_1 s_1 \in A_j$. The inequality $a \leq a_1 s_1$ and the fact that A_j is strongly convex imply $a \in A_j$ which is a contradiction. Thus $a_1 \in A_i$ and $a_1 t_1 \in A_i$. Again the inequality $a_1 t_1 \leq a_2 s_2$ implies $a_2 \in A_i$. Otherwise there exists $j \neq i \in I$ such that $a_2 \in A_j$, and so $a_2 s_2 \in A_j$. Applying strong convexity of A_j to the inequality $a_1 t_1 \leq a_2 s_2$, we obtain $a_1 t_1 \in A_j$, and so this implies that $a_1 t_1 \in A_i \cap A_j = \emptyset$

which is again a contradiction. By continuing this process we get $a_n \in A_i$. Since $a_n t_n \leq a'$ and A_i is strongly convex, we have $a' \in A_i$, as required.

Sufficiency. Suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$. By Lemma 2.1, we get the above system of inequalities (*). Since $a' \in A_i$ and A_i is strongly convex, we have $a_n t_n \in A_i$, and so $a_n \in A_i$. Applying strong convexity of A_i to the inequality $a_n t_n \leq a_n s_n$, we obtain $a_n t_{n-1} \in A_i$, and so $a_{n-1} \in A_i$. By continuing this process we get $a_1 s_1 \in A_i$. Again applying strong convexity of A_i to the inequality $a \leq a_1 s_1$, we have $a \in A_i$, as required. \square

Using Lemmas 3.1 and 3.2, we immediately get the following

Corollary 3.3. *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$, are right strongly convex S -subposets of A_S . Let ${}_S B$ be a left S -poset. If $a \in A_i$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if $a \otimes b \leq a' \otimes b'$ in $A_i \otimes_S B$.*

Lemma 3.4. *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$, are right strongly convex S -subposets of A_S . Let ${}_S B$ be a left S -poset and $a \otimes b \in A \otimes_S B$ for $a \in A_S$ and $b \in {}_S B$. If $a \in A_i$ for some $i \in I$, then $a \otimes b \in A_i \otimes_S B$.*

Proof. If there exists $j \neq i$ such that $a \otimes b \in A_j \otimes_S B$, then $a \otimes b = a' \otimes b'$ for some $a' \in A_j$ and $b' \in {}_S B$. By Lemma 3.2, $a' \in A_i$ which is a contradiction. Hence $a \otimes b \in A_i \otimes_S B$, as required. \square

Corollary 3.5. *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$, are right strongly convex S -subposets of A_S . Let $\phi : A \otimes_S P \rightarrow P'$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S . If $\phi_i = \phi|_{A_i \otimes_S P}$, then $\phi_i : A_i \otimes_S P \rightarrow P'_i$.*

Proof. Because $\phi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$ for all $a \in A_S$ and $(m, n) \in {}_S P$, it suffices to show that $a \otimes m \in A_i \otimes_S M$ and $a \otimes n \in A_i \otimes_S N$ for $a \in A_i, m \in {}_S M$ and $n \in {}_S N$. But these are true by Lemma 3.4. \square

Lemma 3.6. *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$, are right strongly convex S -subposets of A_S . Let $\phi : A \otimes_S P \rightarrow P'$ be a mapping and $\phi_i = \phi|_{A_i \otimes_S P}$. Then ϕ is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S , if and only if ϕ_i is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_i .*

Proof. Necessity. Suppose that ϕ is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S . We need to prove that $p'_{1i}\phi_i = id_{A_i} \otimes p_1$ and $p'_{2i}\phi_i = id_{A_i} \otimes p_2$ in the diagram (\mathbf{P}_{2i}) . By assumption, the lower square

$$\begin{array}{ccc} P'_i & \xrightarrow{p'_{2i}} & A_i \otimes_S N \\ p'_{1i} \downarrow & & \downarrow id_{A_i} \otimes g \\ A_i \otimes_S M & \xrightarrow{id_{A_i} \otimes f} & A_i \otimes_S Q \end{array}$$

in diagram (\mathbf{P}_{2i}) is subcommutative. Next to show $p'_{1i}\phi_i = id_{A_i} \otimes p_1$. Let $a_i \otimes (m, n) \in A_i \otimes_S P$. Then

$$\begin{aligned} p'_{1i}\phi_i(a_i \otimes (m, n)) &= p'_{1i}(a_i \otimes m, a_i \otimes n) = a_i \otimes m \\ &= id_{A_i}(a_i) \otimes p_1(m, n) = (id_{A_i} \otimes p_1)(a_i \otimes (m, n)). \end{aligned}$$

It can also be seen that $p'_{2i}\phi_i = id_{A_i} \otimes p_2$. Since ϕ_i makes $p'_{ji}\phi_i = id_{A_i} \otimes p_j$ for $j = 1, 2$, in the diagram (\mathbf{P}_{2i}) , then by uniqueness, ϕ_i is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_i .

Sufficiency. Let ϕ_i be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_i, i \in I$. Since the lower square

$$\begin{array}{ccc} P' & \xrightarrow{p'_2} & A \otimes_S N \\ p'_1 \downarrow & & \downarrow id_A \otimes g \\ A \otimes_S M & \xrightarrow{id_A \otimes f} & A \otimes_S Q \end{array}$$

in the diagram (\mathbf{P}_2) is subcommutative, it suffices to show that $p'_1\phi = id_A \otimes p_1$ and $p'_2\phi = id_A \otimes p_2$. Let $(a \otimes (m, n)) \in A \otimes_S P$. Then there exists $i \in I$ such that $a \in A_i$. Thus we have

$$\begin{aligned} p'_1\phi(a \otimes (m, n)) &= p'_1\phi_i(a \otimes (m, n)) = p'_1(a \otimes m, a \otimes n) = a \otimes m \\ &= id_{A_i}(a) \otimes p_1(m, n) = id_A(a) \otimes p_1(m, n) = (id_A \otimes p_1)(a \otimes (m, n)). \end{aligned}$$

The same argument shows that $p'_2\phi = id_A \otimes p_2$. \square

The following two theorems are our main results in this section.

Theorem 3.7. *Let ϕ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S and let $\phi_i, i \in I$, be as in Lemma 3.6. Then ϕ is surjective if and only if ϕ_i is surjective for every $i \in I$.*

Proof. Necessity. Let ϕ be surjective. Since $a_i, a'_i \in A_i$, we have $a_i, a'_i \in A_S$. By Lemma 3.1, $a_i \otimes f(m) \leq a'_i \otimes g(n)$ in $A_i \otimes_S Q$ implies $a_i \otimes f(m) \leq a'_i \otimes g(n)$ in $A \otimes_S Q$. Using surjectivity of ϕ ,

$$(\exists a''_i \in A_S)(\exists m' \in {}_S M)(\exists n' \in {}_S N)$$

$$(f(m') \leq g(n') \wedge a_i \otimes m = a''_i \otimes m' \wedge a'_i \otimes n = a''_i \otimes n').$$

By Lemma 3.2, $a_i \otimes m = a''_i \otimes m'$ in $A \otimes_S M$ and $a_i \in A_i$ imply $a''_i \in A_i$. Hence ϕ_i is surjective.

Sufficiency. Let ϕ_i be surjective for every $i \in I$ and suppose that $a \otimes f(m) \leq a' \otimes g(n)$ in $A \otimes_S Q$ for $a, a' \in A_S$, $m \in {}_S M$, $n \in {}_S N$. Since $a \in A_S$, there exists $i \in I$ such that $a \in A_i$. By Corollary 3.3, we have $a \otimes f(m) \leq a' \otimes g(n)$ in $A_i \otimes_S Q$. Using surjectivity of ϕ_i ,

$$(\exists a'' \in A_i)(\exists m' \in {}_S M)(\exists n' \in {}_S N)$$

$$(f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n').$$

By Lemma 3.1, $a \otimes m = a'' \otimes m'$ in $A_i \otimes_S M$ and $a' \otimes n = a'' \otimes n'$ in $A_i \otimes_S N$ imply that $a \otimes m = a'' \otimes m'$ in $A \otimes_S M$ and $a' \otimes n = a'' \otimes n'$ in $A \otimes_S N$, respectively. Thus ϕ is surjective, as required. \square

Theorem 3.8. *Let ϕ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S and let $\phi_i, i \in I$, be as in Lemma 3.6. Then ϕ is an order embedding if and only if ϕ_i is an order embedding for every $i \in I$.*

Proof. Necessity. Let ϕ be an order embedding and suppose that

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge (a \otimes m \leq a' \otimes m') \wedge (a \otimes n \leq a' \otimes n'),$$

where $a \otimes m \leq a' \otimes m'$ in $A_i \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A_i \otimes_S N$ for $i \in I$. Because $a, a' \in A_i$, and by Lemma 3.1, we have $a \otimes m \leq a' \otimes m'$ in $A \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S N$. Using order embeddability of ϕ , we obtain $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A \otimes_S P$. But $a \in A_i$ and so $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A_i \otimes_S P$ by Corollary 3.3. Hence ϕ_i is an order embedding.

Sufficiency. Let ϕ_i is an order embedding for every $i \in I$ and suppose that

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge (a \otimes m \leq a' \otimes m') \wedge (a \otimes n \leq a' \otimes n'),$$

where $a \otimes m \leq a' \otimes m'$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S M$ and $A \otimes_S N$, respectively. Because $a \in A_S$, there exists $i \in I$ such that $a \in A_i$. By Corollary 3.3, $a \otimes m \leq a' \otimes m'$ in $A \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S N$ imply $a \otimes m \leq a' \otimes m'$ in $A_i \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A_i \otimes_S N$, respectively. Using order embeddability of ϕ_i , we obtain $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A_i \otimes_S P$. By Lemma 3.1, we have $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A \otimes_S P$, and so ϕ is an order embedding. The proof is complete. \square

For every subpullback diagram, the corresponding ϕ is a surjection or an order embedding if and only if the corresponding ϕ_i is a surjection or an order embedding, and surjectivity or order isomorphism of ϕ for a special subpullback diagram is equivalent to certain kind of flatness property. It follows from Theorems 2.1, 2.2, 2.3, 2.4, 3.2, 4.1, 5.3 of [9] and Definition 2.2 that

Proposition 3.9. *Let S be a pomonoid and $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$, are right strongly convex S -subposets of A_S . Then A_S is torsion free, principally weakly flat, weakly flat, flat, pullback flat, subpullback flat, principally weakly kernel po-flat, weakly kernel po-flat, translation kernel po-flat, and satisfies conditions (P), (WP), (PWP) if and only if A_i has these properties for every $i \in I$.*

From Proposition 3.9 and Theorem 2.3 of [12], we have

Corollary 3.10. *Let S be a pomonoid. Then a right S -poset A_S is torsion free, principally weakly flat, weakly flat, flat, pullback flat, subpullback flat, principally weakly kernel po-flat, weakly kernel po-flat, translation kernel po-flat, and satisfies conditions (P), (WP), (PWP) if and only if its strongly convex indecomposable components have these properties.*

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