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Constructing the Banaschewski compactification through the functionally countable subalgebra of $C(X)$

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Abstract. Let X be a zero-dimensional space and $C_c(X)$ denote the functionally countable subalgebra of $C(X)$. It is well known that $\beta_0 X$ (the Banaschewski compactfication of X) is a quotient space of βX. In this article, we investigate a construction of $\beta_0 X$ via βX by using $C_c(X)$ which determines the quotient space of βX homeomorphic to $\beta_0 X$. Moreover, the construction of $\upsilon_0 X$ via $\upsilon_{C_c} X$ (the subspace $\{p \in \beta X : \forall f \in C_c(X), f^*(p) < \infty\}$ of βX) is also investigated.

1 Introduction

Throughout this article all topological spaces are assumed to be zero-dimensional (that is, are Hausdorff and contain a base of clopen sets). For a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on X, and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. Also, as we recalled earlier, $C_c(X)$ denotes the sub-

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algebra of $C(X)$ consisting of functions with countable image. It should be emphasized that this subalgebra is first introduced in [9] and later studied in various papers such as [4], [6], [10] and [12]. The reader is referred to [11] and [9] for undefined terms and notations concerning $C(X)$ and $C_c(X)$, respectively. In [9, Theorem 4.6], it is proved that when we are dealing with $C_c(X)$, where X is any space (not necessarily even completely regular) we may take X to be zero-dimensional, by showing that, there always exists a zero-dimensional space Y which is a continuous image of X and $C_c(X) \cong C_c(Y)$. This also means that the topological property of zero-dimensionality is an algebraic property (in the sense that if for any space X, $C(X) \cong C_c(Y)$, where Y is any space, then X is zero-dimensional, see also [10]). It is well-known that every zero-dimensional space X has a zero-dimensional compactification, which is called the Banaschewski compactification of X and is denoted by $\beta_0 X$, such that every continuous map $f : X \longrightarrow Y$, where Y is a zero-dimensional compact space, has an extension to a continuous map $F : \beta_0 X \longrightarrow Y$. It is shown in [21] that $\beta X = \beta_0 X$ if and only if X is a strong zero-dimensional space. Note that a Tychonoff space X is called strongly zero-dimensional if every two disjoint zero-sets in X are separated by disjoint clopen sets. It is evident that every strongly zero-dimensional space is zero-dimensional. However, the converse of this fact is not true, in general, see [24, 3.39] or [14, Example 2]. A topological space X is called $\mathbb{N}$-compact if it can be embedded as a closed subset in the product space $\mathbb{N}^\kappa$, for some cardinal number κ. $\mathbb{N}$-compact spaces were first introduced by S. Mrowka in [13]. It is well-known that, for every zero-dimensional space X, there exists an $\mathbb{N}$-compact space $\upsilon_0 X$ such that X is dense in it and every continuous function $f : X \longrightarrow Y$, with Y an $\mathbb{N}$-compact space, has a unique extension $F : \upsilon_0 X \longrightarrow Y$. It is shown in [21, 4.7] that the structure of $\beta_0 X$ is related to the clopen ultrafilters defined on X. Also, in [21, Exercise 5E], an outline for recovering $\upsilon_0 X$ as a subspace of all clopen ultrafilters on X which have countable intersection property is given. Therefore we have $X \subseteq \upsilon_0 X \subseteq \beta_0 X$. Note that, by a clopen ultrafilter, we mean an ultrafilter of the Boolean algebra of clopen subsets of X. The authors in [4] have used $\beta_0 X$ for studying $C_c(X)$. In particular in [4, Remark 3.6], they have shown that the structure space of $C_c(X)$ (that is, the space of maximal ideals of $C_c(X)$ equipped with the hull-kernel topology) is homeomorphic to $\beta_0 X$. In Theorem 4.2 of the same paper, maximal ideals of $C_c(X)$ are also characterized via $\beta_0 X$.

The aim of the present paper is to investigate a construction of $\beta_0 X$ and $\upsilon_0 X$ via βX and $\upsilon_{C_c} X$ $(= \{p \in \beta X : \forall f \in C_c(X) : f^*(p) < \infty\})$ by using $C_c(X)$ and establishing an outline for recovering these spaces as the spaces of all the z_c-ultrafilters on X and all the z_c-ultrafilters with the countable intersection property, respectively. This paper consists of three sections. In Section 2, using $C_c(X)$, we define a topology on βX and the induced space is denoted by $\beta_c X$. Also, the equivalence relation $\sim_c$ is defined on $\beta_c X$. It is shown that $\beta_0 X$ is homeomorphic to the quotient space $\frac{\beta_c X}{\sim_c}$ and the latter space is homeomorphic to the structure space of $C_c(X)$. It follows that $\beta_0 X$ is homeomorphic to the quotient space $\frac{\beta X}{\sim_c}$ of βX. Using these, a different approach to some basic results of [4] follows. In Section 3, the subspace $\upsilon_{C_c} X$ is introduced of βX is considered. We use $\upsilon_c X$ to denote $\upsilon_{C_c} X$ as a subspace of $\beta_c X$. It is proved that $\frac{\upsilon_c X}{\sim_c}$ is homeomorphic to the structure space of real maximal ideals of $C_c(X)$. Moreover, z_c-ultrafilters on X are characterized.

2 $\beta_0 X$ as a quotient of βX by using $C_c(X)$

We recall that an ideal I of a subring R of $C(X)$ is called a z-ideal if whenever $f \in I$, then $M_f(R) \subseteq I$, in which $M_f(R)$ denotes the intersection of all the maximal ideals of R containing f. It is well-known that an ideal I in $C(X)$ is a z-ideal if and only if $g \in I$ whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$. Moreover, an ideal I of a subring R of $C(X)$ is called a z_R-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in R$ imply that $g \in I$. Evidently, z_C-ideals of $C(X)$ coincide with z-ideals. However, the same fact does not hold for subrings of $C(X)$, especially, for subrings containing $C^*(X)$, see [5, Theorem 2.2]. Following [2], we call a subring R of $C(X)$ an invertible subring, if f is a unit of R whenever $f \in R$ and $Z(f) = \emptyset$. It has been shown in [2, Proposition 3.7] that in an invertible subalgebra R of $C(X)$, the two notions of z-ideals and z_R-ideals coincide. From [2, Proposition 3.7] and [3, Proposition 4.2] it follows that every maximal ideal of an invertible subring R is of the form $M^p \cap R$ for some $p \in \beta X$. Also, by [9, Remarks 2.2], $C_c(X)$ is an invertible subalgebra of $C(X)$. Thus, z-ideals of $C_c(X)$ coincide with z_{C_c}-ideals, and every maximal ideal in $C_c(X)$ is of the form $M_c^p = M^p \cap C_c(X)$ for some $p \in \beta X$. Moreover, an ideal J in $C_c(X)$ is a

z-ideal if and only if it is a contraction of a z-ideal of $C(X)$ [4, Proposition 4.3. (a)].

Let $Z_c(X)$ denote the collection of all zero-sets of elements of $C_c(X)$. It is easy to observe that the collection $\mathrm{cl}_{\beta X} Z_c(X) = \{\mathrm{cl}_{\beta X} Z : Z \in Z_c(X)\}$ constitutes a base for the closed sets of a topology on βX which we denote by τ_c. We denote by $\beta_c X$ the topological space $(\beta X, \tau_c)$ and by τ the usual topology on βX. Evidently, $\tau_c \subseteq \tau$ and the containment may be proper, see Example 2.4 in below. As $\tau_c \subseteq \tau$, $\beta_c X$ is compact. Moreover, as X is a zero-dimensional space, X is dense in $\beta_c X$, and $\beta_c X = \beta X$ if and only if X is strongly zero-dimensional. Thus, whenever X is not strongly zero-dimensional, $\beta_c X$ is not Hausdorff. Now, define the relation $\sim_c$ on $\beta_c X$ as follows: for $p, q \in \beta_c X$, $p \sim_c q$ if and only if $M_c^p = M_c^q$. It easily follows that $\sim_c$ is an equivalence relation on $\beta_c X$ which does not identify points of X due to zero-dimensionality of X. We use $[p]$ to denote the equivalence class of $p \in \beta_c X$ and $[\mathrm{cl}_{\beta X} Z(f)]$ to denote the set $\{[p] : p \in \mathrm{cl}_{\beta_X} Z(f)\}$ for each $f \in C_c(X)$. It is easy to prove that each equivalence class $[p]$ is a connected subset of βX and the equivalence relation $\sim_c$ separates βX into connected components. The next lemma gives some connections between βX, $\beta_c X$, and $\frac{\beta_c X}{\sim_c}$.

Lemma 2.1. *The following statements hold for each* $f, g \in C_c(X)$*.*

(i) $cl_{\beta_c X} Z(f) = cl_{\beta X} Z(f)$.

(ii) *For each* $p \in \beta_c X$, $p \in cl_{\beta X} Z(f)$ *if and only if* $[p] \subseteq cl_{\beta X} Z(f)$, *if and only if* $[p] \in [cl_{\beta X} Z(f)]$.

(iii) $cl_{\beta_c X} Z(f) \cap cl_{\beta_c X} Z(g) = \emptyset$ *if and only if* $[cl_{\beta X} Z(f)] \cap [cl_{\beta X} Z(g)] = \emptyset$.

(iv) $[cl_{\beta X} Z(f) \cap cl_{\beta X} Z(g)] = [cl_{\beta X} Z(f)] \cap [cl_{\beta X} Z(g)]$.

(v) U *is a clopen set in* X *if and only if* $cl_{\beta_c X} U$ *is clopen in* $\beta_c X$.

Proof. (i) As $\mathrm{cl}_{\beta X} Z(f)$ is a closed set in $\beta_c X$, we have $\mathrm{cl}_{\beta_c X} Z(f) \subseteq \mathrm{cl}_{\beta_c X} \mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} Z(f)$. The reverse inclusion is also clear, since $\tau_c \subseteq \tau$.

(ii) Evident.

(iii) This is clear by (2).

(iv) Let $[p] \notin [\mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(g)]$. Thus, there exists $q \in [p]$ such that $q \notin \mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(g)$. Let for example $q \notin \mathrm{cl}_{\beta X} Z(f)$. Hence, there exists $h \in C_c(X)$ such that $q \in \mathrm{cl}_{\beta X} Z(h)$ and $\mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(h) = \emptyset$. It follows that $[p] \in [\mathrm{cl}_{\beta X} Z(h)]$ and $[\mathrm{cl}_{\beta X} Z(f)] \cap [\mathrm{cl}_{\beta X} Z(h)] = \emptyset$. This clearly implies that $[p] \notin [\mathrm{cl}_{\beta X} Z(f)] \cap [\mathrm{cl}_{\beta X} Z(g)]$; that is, $[\mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(g)] \subseteq$

$[\mathrm{cl}_{\beta X} Z(f)] \cap [\mathrm{cl}_{\beta X} Z(g)]$. The reverse inclusion is evident and thus the equality follows.

(v) Let U be a clopen set in X. Clearly, $U = Z(g)$, where g is the characteristic function of U. Thus, there exists $h \in C_c(X)$ such that $Z(g) \cap Z(h) = \emptyset$ and $Z(g) \cup Z(h) = X$. It follows that $\mathrm{cl}_{\beta_c X} Z(g) \cup \mathrm{cl}_{\beta_c X} Z(h) = \beta_c X$. Also, if $p \in \mathrm{cl}_{\beta_c X} Z(g)$, then $p \in \beta_c X \setminus \mathrm{cl}_{\beta_c X} Z(h) \subseteq \mathrm{cl}_{\beta_c X} Z(g)$, which means $p \in \mathrm{int}_{\beta_c X} \mathrm{cl}_{\beta_c X} Z(g)$. Thus, $\mathrm{cl}_{\beta_c X} Z(g)$ is a clopen set. The converse is evident, as X is zero-dimensional. □

By the next example, we investigate the construction of a topological space X for which $\beta_c X \neq \beta X$ from which it follows that τ_c may be properly contained in τ.

Example 2.2. Let M be the space introduced by Dowker example in [24, 3.39]. We claim that $\beta_c M$ is not a Hausdorff space, which means that τ_c is properly contained in τ. As stated in [24, 3.39], $\{\omega_1\} \times I \subseteq \beta M$ in which ω_1 denotes the first uncountable ordinal and I denotes the unit interval $[0, 1]$ in $\mathbb{R}$. We show that the two points $(\omega_1, 0)$ and $(\omega_1, 1)$ could not be separated by elements of τ_c. Assume, on the contrary, that there exists $f \in C_c(M)$ such that $(\omega_1, 0) \in \mathrm{cl}_{\beta M} Z(f)$ and $(\omega_1, 1) \notin \mathrm{cl}_{\beta M} Z(f)$. As $(\omega_1, 1) \notin \mathrm{cl}_{\beta M} Z(f)$, there exists some $g \in C_c(M)$ such that $(\omega_1, 1) \in \mathrm{cl}_{\beta M} Z(g)$ and $\mathrm{cl}_{\beta M} Z(f) \cap \mathrm{cl}_{\beta M} Z(g) = \emptyset$. It follows that $Z(f) \cap Z(g) = \emptyset$ and hence, there exists $h \in C_c(M)$ such that $Z(f) = Z(h)$ and $Z(g) = h^{-1}(1)$. Let $0 < r < 1$ be such that $r \notin h(M)$. It follows that $U = h^{-1}((-\infty, r))$ is a clopen set in M such that $Z(f) \subseteq U$ and $Z(g) \subseteq M \setminus U$. By part (5) of Lemma 2.2, $\mathrm{cl}_{\beta X} U$ is a clopen set in $\beta_c M$. Moreover, $(\omega_1, 0) \in \mathrm{cl}_{\beta M} Z(f) \subseteq \mathrm{cl}_{\beta M} U$ and $(\omega_1, 1) \notin \mathrm{cl}_{\beta X} U$ and this clearly contradicts the connectedness of $\{\omega_1\} \times I$.

The next statement, which is the main result of this section, investigates the construction of $\beta_0 X$ via βX and the relation $\sim_c$ which leads to the existence of a homeomorphism between $\beta_0 X$ and the quotient space $\frac{\beta X}{\sim_c}$ of βX. Recall that, whenever X is a dense subspace of a space Y, then X is said to be 2-embedded in Y if each continuous two-valued function $f : X \longrightarrow \{0, 1\}$ has a continuous extension $F : Y \longrightarrow \{0, 1\}$.

Theorem 2.3. *For any zero-dimensional space X, $\beta_0 X$ is homeomorphic to the quotient space $\frac{\beta X}{\sim_c}$ of βX.*

Proof. We first show that $\beta_0 X$ is homeomorphic to $\frac{\beta_c X}{\sim_c}$ and then show that the latter space is homeomorphic to $\frac{\beta X}{\sim_c}$. It is easy to prove that $\frac{\beta_c X}{\sim_c}$ is a compact zero-dimensional space. Thus, for showing that $\frac{\beta_c X}{\sim_c}$ is homeomorphic to $\beta_0 X$, by [21, 4.7 (e)], it suffices to show that X is 2-embedded in $\frac{\beta_c X}{\sim_c}$. Let $f : X \longrightarrow \{0,1\}$ be a continuous function. As $\{1,2\}$ is a compact space, f has a continuous extension $F : \beta X \longrightarrow \{0,1\}$. It follows that $Z(F) \cap X = Z(f)$ and, as $Z(F) = F^{-1}(0)$ is a clopen set in βX, we have $\mathrm{cl}_{\beta X} Z(f) = Z(F)$. Let $g : \frac{\beta_c X}{\sim_c} \longrightarrow \{0,1\}$ be defined by $g([p]) = F(p)$. We show that g is a continuous extension of f. It is clear that $g|_X = f$, since, $[p] = \{p\}$ for each $p \in X$. We claim that $F(p) = F(q)$ for each $p, q \in \beta_c X$ with $p \sim_c q$. Let $p, q \in \beta_c X$ and $p \sim_c q$. As we could consider f as an element of $C_c(X)$, thus $p, q \in \mathrm{cl}_{\beta X} Z(f) = Z(f)$ or $p, q \notin \mathrm{cl}_{\beta X} Z(f) = Z(F)$, which in any case implies that $F(p) = F(q)$. This means that g is well-defined. Moreover, $g^{-1}(0) = \{[p] : F(p) = 0\} = [Z(F)] = [cl_{\beta X} Z(f)]$, which is clearly a clopen set in $\frac{\beta_c X}{\sim_c}$, which implies the continuity of g. Now, let $i : \beta X \longrightarrow \beta_c X$ be the identity mapping. As $\tau_c \subseteq \tau$, the mapping i is continuous. It follows that the induced mapping $\hat{i} : \frac{\beta X}{\sim_c} \longrightarrow \frac{\beta_c X}{\sim_c}$ is continuous. Hence, $\hat{i}$ is a continuous bijection from the compact space $\frac{\beta X}{\sim_c}$ to the Hausdorff space $\frac{\beta_c X}{\sim_c}$, which implies that $\hat{i}$ is a homeomorphism. □

It evidently follows from Lemma 2.1 that maximal ideals of $C_c(X)$ are precisely the ideals $\{f \in C_c(X) : [p] \subseteq \mathrm{cl}_{\beta X} Z(f)\}$, for $p \in \beta X$, which we denote by $M_c^{[p]}$ for each $p \in \beta X$. Thus, the mapping $\varphi : \frac{\beta_c X}{\sim_c} \longrightarrow Max(C_c(X))$ defined by $\varphi([p]) = M_c^{[p]}$ is a homeomorphism and thus the structure space of $C_c(X)$ is homeomorphic to $\beta_0 X$.

Let ξ be the homeomorphism from $\frac{\beta_c X}{\sim_c}$ onto $\beta_0 X$ whose existence just proved in the proof of Theorem 3.3. The next proposition shows that ξ preserves closures.

Proposition 2.4. *Let X be a zero-dimensional space. Then we would have $\xi([cl_{\beta X} Z(f)]) = cl_{\beta_0 X} Z(f)$ for each $f \in C_c(X)$.*

Proof. Let $f \in C_c(X)$ be given. It is clear that $Z(f) \subseteq \xi([\mathrm{cl}_{\beta X} Z(f)])$ and thus $\mathrm{cl}_{\beta_0 X} Z(f) \subseteq \xi([\mathrm{cl}_{\beta X} Z(f)])$, since $\xi([\mathrm{cl}_{\beta X} Z(f)])$ is a closed set in $\beta_0 X$. Now, let $\xi([p]) \notin \mathrm{cl}_{\beta_0 X} Z(f)$. Thus, there exists $g \in C_c(X)$ such that $\xi([p]) \in \mathrm{cl}_{\beta_0 X} Z(g)$ and $\mathrm{cl}_{\beta_0 X} Z(f) \cap \mathrm{cl}_{\beta_0 X} Z(g) = \emptyset$. It follows that $Z(f) \cap Z(g) = \emptyset$

and hence $\mathrm{cl}_{\beta X}Z(f)\cap\mathrm{cl}_{\beta X}Z(g)=\emptyset$. As $f,g\in C_c(X)$, we have $[\mathrm{cl}_{\beta X}Z(f)]\cap[\mathrm{cl}_{\beta X}Z(g)]=\emptyset$. But $[p]\in[\mathrm{cl}_{\beta X}Z(g)]$, since, as $\mathrm{cl}_{\beta_0 X}Z(g)\subseteq\xi([\mathrm{cl}_{\beta X}Z(g)])$, we have $\xi^{-1}(\mathrm{cl}_{\beta_0 X}Z(g))\subseteq[\mathrm{cl}_{\beta X}Z(g)]$. Therefore, $[p]\notin[\mathrm{cl}_{\beta X}Z(f)])$, which implies that $\xi([p])\notin\xi([\mathrm{cl}_{\beta X}Z(f)])$; that is, $\xi([\mathrm{cl}_{\beta X}Z(f)])\subseteq\mathrm{cl}_{\beta_0 X}Z(f)$ and thus the equality follows. □

Corollary 2.5. *[4, Theorem 4.2] Maximal ideals of $C_c(X)$ are precisely the ideals $M_c^p=\{f\in C_c(X):p\in cl_{\beta_0 X}Z(f)\}$ for $p\in\beta_0 X$.*

Corollary 2.6. *[16, Lemma 2.1] For each $f,g\in C_c(X)$, we have $cl_{\beta_0 X}(Z(f)\cap Z(g))=cl_{\beta_0 X}Z(f)\cap cl_{\beta_0 X}Z(g)$.*

The ideals $O_c^p=\{f\in C_c(X):p\in\mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X}Z(f)\}$ for $p\in\beta_0 X$ are introduced and studied in [4] as a model for the ideals $O^p=\{f\in C(X):p\in\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)\}$. It could be easily proved that, for each $f\in C_c(X)$ and each $p\in\beta X$, we have $p\in\mathrm{int}_{\beta_c X}\mathrm{cl}_{\beta_c X}Z(f)$ if and only if $\xi([p])\in\mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X}Z(f)$. It follows that $O_c^{\xi([p])}=\{f\in C_c(X):p\in\mathrm{int}_{\beta_c X}\mathrm{cl}_{\beta_c X}Z(f)\}$ for each $p\in\beta X$. From this fact, a different approach to [4, Lemma 4.11 and Remark 4.12] follows.

3 Constructing $\upsilon_0 X$ via $C_c(X)$

We recall that a maximal ideal M in a subring R of $C(X)$ is said to be a real-maximal ideal, if $\frac{R}{M}$ is isomorphic to $\mathbb{R}$. Whenever the residue class field $\frac{R}{M}$ properly contains a copy of $\mathbb{R}$, then M is called a hyper-real. It is well-known that a maximal ideal M^p in $C(X)$ is real if and only if $p\in\upsilon X$. For a subset $A(X)$ of $C(X)$, we set $\upsilon_A X=\{p\in\beta X:\forall f\in A, f^*(p)<\infty\}$. It follows from [1, Theorem 1.6] that whenever $A(X)$ is a subring of $C(X)$ containing $C^*(X)$, then the structure space of real maximal ideals of $A(X)$ is homeomorphic to $\upsilon_A X$. However, the same statement does not hold for arbitrary subrings of $C(X)$, in general. For example, consider the subring $M^p+\mathbb{R}$, where $p\in\beta\mathbb{R}\setminus\mathbb{R}$, of $C(\mathbb{R})$. By [22, Remark 1.8], the structure space of real maximal ideals of $M^p+\mathbb{R}$ is homeomorphic to $\mathbb{R}\cup\{\alpha\}$, where neighborhoods of α are of the form $U\cup\{\alpha\}$ in which U is an open subset of $\mathbb{R}$. Also, by [18, Proposition 4.7], $\upsilon_{M^p+\mathbb{R}}X=\mathbb{R}\cup\{p\}$, which clearly is not homeomorphic to $\mathbb{R}\cup\{\alpha\}$. It is clear that $\upsilon_{C^*(X)}X=\beta X$ and $\upsilon_{C(X)}X=\upsilon X$. Also, $\upsilon X\subseteq\upsilon_A X\subseteq\beta X$ for each $A(X)\subseteq C(X)$

and thus $\beta(\upsilon_A X) = \beta X$. Moreover, by [11, 8B.3], a subset K of βX is a realcompactification of X (that is, a realcompact space containing X as a dense subspace) if and only if $K = \upsilon_A X$ for some subset $A(X)$ of $C(X)$. The next statement investigates the relation between real maximal ideals of $C_c(X)$ and the space $\upsilon_{C_c} X$.

Proposition 3.1. *For a zero-dimensional space X, the following statements are equivalent.*

(i) $p \in \upsilon_{C_c} X$.
(ii) M_c^p *is a real maximal ideal in* $C_c(X)$.
(iii) $\mathcal{U}_c^p$ *is closed under countable intersection.*
(iv) $\mathcal{U}_c^p$ *has the countable intersection property.*

Proof. We only prove the equivalence of (i) and (ii). The equivalence of other parts follows from [12, Proposition 2.15].

(i) $\Rightarrow$ (ii) If M_c^p is a hyper-real maximal ideal in $C_c(X)$, then there exists $f \in C_c(X)$ such that $|M_c^p(f)|$ is infinitely large. Hence, $|M_c^p(|f| - n)| \geq 0$ for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $Z_n \in Z[M_c^p]$ such that $|f| - n$ is non-negative on Z_n. Evidently, $p \in \mathrm{cl}_{\beta X} Z_n$ which implies that there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in Z_n such that $x_\lambda \to p$. Also, $f^*(x_\lambda) \to f^*(p)$. Clearly, $f^*(x_\lambda) = f(x_\lambda)$ and thus $|f^*(x_\lambda)| \geq n$ for each $x_\lambda \in Z_n$. It follows that $|f^*(p)| \geq n$ for each $n \in \mathbb{N}$ and therefore $f^*(p) = \infty$, which contradicts the hypothesis.

(ii) $\Rightarrow$ (i) Let $f^*(p) = \infty$, for some $f \in C_c(X)$. We show that, for each $n \in \mathbb{N}$, $p \in \mathrm{cl}_{\beta X} Z_n$, where $Z_n = \{x \in X : |f(x)| \geq n\}$. Let U be an open set in βX containing p. Hence, there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in X such that $x_\lambda \to p$. Thus, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for each $\lambda \geq \lambda_0$. It follows that $(f^*(x_\lambda))_{\lambda \in \Lambda}$ converges to $f^*(p)$. Thus, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \Lambda$ such that $f^*(x_\lambda) \in (-\infty, -n) \cup (n, \infty) \cup \{\infty\}$ for each $\lambda \geq \lambda_n$. As $f^*(x_\lambda) = f(x_\lambda)$, for each $\lambda \in \Lambda$, $|f(x_\lambda)| > n$, for $\lambda \geq \lambda_n$. Choose some $\lambda_2 \in \Lambda$ such that $\lambda_2 \geq \lambda_1, \lambda_2$. Thus, for each $\lambda \geq \lambda_2$ we have $x_\lambda \in U \cap Z_n \neq \emptyset$. Hence, $p \in \mathrm{cl}_{\beta X} Z_n$. Now, for each $n \in \mathbb{N}$, define $h_n : X \longrightarrow \mathbb{R}$ by $h_n(x) = (|f(x)| - n) \wedge 0$. Evidently, $h_n \in C_c(X)$ and $Z(h_n) = Z_n$. As $|f| - n$ is non-negative on Z_n, we have $M_c^p(|f| - n) \geq 0$. This means that $M_c^p(f) \geq n$ and thus $|M_c^p(f)|$ is infinitely large. This implies that M_c^p is hyper-real. □

Note that a topological space X is said to be c-realcompact (in the sense of [12]), if every real maximal ideal in $C_c(X)$ is fixed. By Proposition 3.1, X is c-realcompact if and only if $\upsilon_{C_c}X = X$. Evidently, every c-realcompact space is realcompact. Moreover, $\upsilon_{C_c}X \subseteq \upsilon_{C_c}(\upsilon_{C_c}X) \subseteq \beta X$, since, if $p \in \upsilon_{C_c}X$, then for each $f \in C_c(\upsilon_{C_c}X)$ we have $f|_X \in C_c(X)$ and thus $f^*(p) < \infty$, which means $p \in \upsilon_{C_c}(\upsilon_{C_c}X)$. Also, $\upsilon_{C_c}X$ is the largest subspace of βX for which elements of $C_c(X)$ could be extended continuously; that is, $\upsilon_{C_c}(\upsilon_{C_c}X) \subseteq \upsilon_{C_c}X$. Therefore, $\upsilon_{C_c}X = \upsilon_{C_c}(\upsilon_{C_c}X)$, which means that $\upsilon_{C_c}X$ is a c-realcompact space. However, $\upsilon_{C_c}X$ may not be the smallest c-realcompact space in which X is embedded. We next investigate the construction of the smallest c-realcompact space in which X is dense. We use $\upsilon_c X$ to denote $\upsilon_{C_c}X$ as a subspace of $\beta_c X$. It clearly follows that $\{\mathrm{cl}_{\upsilon_c X}Z(f) : f \in C_c(X)\}$ constitutes a base for the closed sets of $\upsilon_c X$. We denote by $Max_r(C_c(X))$, the space of real maximal ideals of $C_c(X)$ endowed with the hull-kernel topology. We need the following lemma.

Lemma 3.2. *Let X be a zero-dimensional space. Then for each $f \in C_c(X)$, we have $cl_{\upsilon_c X}Z(f) = Z(f^{\upsilon C_c})$.*

Proof. Let $p \notin \mathrm{cl}_{\upsilon_c X}Z(f)$. Thus, there exists some $g \in C_c(X)$ such that $p \in \mathrm{cl}_{\upsilon_c X}Z(g)$ and $\mathrm{cl}_{\upsilon_c X}Z(f) \cap \mathrm{cl}_{\upsilon_c X}Z(f) = \emptyset$ and hence $Z(f) \cap Z(g) = \emptyset$. Therefore, there exists $h \in C_c(X)$ such that $(f^2 + g^2)h = 1$. It follows that $((f^2 + g^2)h)^{\upsilon C_c}(p) = (f^2 h)^{\upsilon C_c}(p) = 1$. Hence, $p \notin Z(f^{\upsilon C_c})$ and thus $Z(f^{\upsilon C_c}) \subseteq \mathrm{cl}_{\upsilon_c X}Z(f)$. The reverse inclusion is evident. □

From Proposition 3.1 and Lemma 3.2, it follows that the mapping $\psi : \frac{\upsilon_c X}{\sim_c} \longrightarrow Max_r(C_c(X))$ defined by $\psi([p]) = M_c^{[p]}$ is a homeomorphism.

Remark 3.3. We denote by $\mathcal{U}^p$ the unique z-ultrafilter $\{Z \in Z(X) : p \in \mathrm{cl}_{\beta X}Z\}$ on X converging to p, for each $p \in \beta X$, and by $\mathcal{U}_c^{[p]}$ the set $\{Z \in Z_c(X) : [p] \subseteq \mathrm{cl}_{\beta X}Z\}$ for each $[p] \in \frac{\beta_c X}{\sim_c}$. Also, the collection of all z_c-ultrafilters on X is denoted by $\mathcal{U}(C_c(X))$. It is straightforward to prove that the mapping $\varphi : \frac{\beta_c X}{\sim_c} \longrightarrow \mathcal{U}(C_c(X))$, defined by $\varphi([p]) = \mathcal{U}^{[p]}$, is a homeomorphism. It follows that $\beta_0 X$ is homeomorphic to $\mathcal{U}(C_c(X))$. This means that $\beta_0 X$ could be recovered as the space of z_c-ultrafilters $\mathcal{U}_c^{[p]}$ on X. Let $B(X)$ denotes the Boolean algebra of all clopen sets of X and $\mathcal{CU}(X)$ denotes the space of all clopen ultrafilters on X equipped with the Stone topology. Moreover, let $\mathcal{CU}_c(X)$ and $\mathcal{CCU}(X)$ denote the subspaces of $\mathcal{U}_c(X)$ and $\mathcal{CU}(X)$ consisting of all the z_c-ultrafilters and all the clopen ultrafilters with the countable intersection property, respectively. One can easily prove that the mapping $\eta : \mathcal{U}_c(X) \longrightarrow \mathcal{CU}(X)$ defined by $\eta(\mathcal{U}_c^{[p]}) = \mathcal{U}_c^{[p]} \cap B(X)$ is a homeomorphism and its restriction to $\mathcal{CU}_c(X)$ is a homeomorphism onto $\mathcal{CCU}(X)$.

Theorem 3.4. *Let X be a zero-dimensional space X. Then $\upsilon_0 X$ is homeomorphic to the quotient space $\frac{\upsilon_{C_c}X}{\sim_c}$ of $\upsilon_{C_c}X$.*

Proof. For $p \in \beta_0 X$, let A^p be the unique clopen ultrafilter converging to p. By Remark 3.3, there exists $[q]_p \in \frac{\beta_c X}{\sim_c}$ such that $A^p = \mathcal{U}_c^{[q]_p} \cap B(X)$. Moreover, for each $t \in \upsilon_0 X$ there exists $[s]_t \in \frac{\upsilon_c X}{\sim_c}$ such that $A^t = \mathcal{U}_c^{[s]_t} \cap B(X)$. It thus follows that the mapping $\lambda : \beta_0 X \longrightarrow \frac{\beta_c X}{\sim_c}$ defined by $\lambda(p) = [q]_p$ is a homeomorphism and its restriction to $\upsilon_0 X$ is a homeomorphism onto $\frac{\upsilon_c X}{\sim_c}$. Also, from Remark 3.3, it follows that $\hat{i}|_{\frac{\upsilon_{C_c}X}{\sim_c}} : \frac{\upsilon_{C_c}X}{\sim_c} \longrightarrow \frac{\upsilon_c X}{\sim_c}$ is a homeomorphism. Thus, the composite of the two mappings $\lambda|_{\upsilon_0 X}$ and $\hat{i}|_{\frac{\upsilon_{C_c}X}{\sim_c}}$ generates a homeomorphism from $\upsilon_0 X$ onto $\frac{\upsilon_{C_c}X}{\sim_c}$. □

The next statement determines conditions equivalent to coincidence of $\upsilon_{C_c}X$ and $\upsilon_c X$.

Proposition 3.5. *For a zero-dimensional space X, the following statements are equivalent:*

(i) $\upsilon_{C_c}X = \upsilon_c X$.

(ii) $\upsilon_{C_c}X$ *is* $\mathbb{N}$-*compact.*

(iii) $\upsilon_{C_c}X = \upsilon_0 X$.

Proof. (i) ⇒ (ii) By our hypothesis and the fact that $\upsilon_{C_c}X$ is a Hausdorff space, we have $\upsilon_{C_c}X = \frac{\upsilon_c X}{\sim_c}$ which, by Remark 3.3, implies $\upsilon_{C_c}X = \upsilon_0 X$.

(ii) ⇒ (iii) By the hypothesis and the fact X is dense in $\upsilon_{C_c}X$, the identity mapping of X to $\upsilon_{C_c}X$ has a continuous extension to $\upsilon_0 X$ and hence, by Remark 3.6, has a continuous extension to $\frac{\upsilon_c X}{\sim_c}$. It follows that $\upsilon_{C_c}X = \upsilon_0 X$.

(iii) ⇒ (i) An easy consequence of Remark 3.6. □

It follows, from Remark 3.6 and [7, Theorem *A*], that X is c-realcompact if and only if $\upsilon_{C_c}X = X$ if and only if $\frac{\upsilon_c X}{\sim_c} = X$, if and only if $\upsilon_0 X = X$ if and only if X is $\mathbb{N}$-compact. It is easy to see that Theorem 5.2, Proposition

5.8, Theorem 5.14, and Proposition 5.20 of [12] are consequences of the above mentioned facts and the corollary of [15, Theorem 2].

We recall that a subalgebra $A(X)$ of $C(X)$ is said to be closed under local bounded inversion, briefly, an LBI-subalgebra, if for each element f in $A(X)$, which is bounded away from zero on some cozero-set E, there exists $g \in A(X)$ such that $fg|_E = 1$. These subalgebras were first introduced in [23] and further studied in [17]. By the next statement, we show that $C_c^*(X)$ is an LBI-subalgebra of $C(X)$.

Lemma 3.6. *Let X be a zero-dimensional space X, then $C_c^*(X)$ is an LBI-subalgebra of $C(X)$.*

Proof. Let $f \in C_c^*(X)$ and $f(x) \geq c > 0$ for each $x \in E = Coz(h)$, where $c \in \mathbb{R}$ is positive, and $h \in C(X)$. It follows that $E \subseteq f^{-1}[c, +\infty)$. As $f \in C_c(X)$, there exists some $0 < c_0 < c$ such that $c_0 \notin f(X)$. Thus, $f^{-1}(-\infty, c_0]$ is a clopen subset of X and $Z(f) \subseteq f^{-1}(-\infty, c_0)$. Let $A = f^{-1}(-\infty, c_0)$ and $B = X \setminus A$. It follows that A and B are closed subsets of X and $E \subseteq B$. We define $g(x) = 0$ for each $x \in A$ and $g(x) = \frac{1}{f(x)}$ for each $x \in B$. It clearly follows that $g \in C_c^*(X)$ and $fg|_E = 1$. $\square$

Following [20], we set $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$ for each element f of a subalgebra $A(X)$ of $C(X)$. It is easy to see that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2 + g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$ for each $f, g \in A(X)$ and each $n \in \mathbb{N}$. Furthermore, $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$. Also, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ for each $f \in C(X)$ and $S_{C^*}(f) = Z(f^\beta)$ for each $f \in C^*(X)$. From Lemma 3.6 and [20, Proposition 2.7], it easily follows that every maximal ideal of $C_c^*(X)$ is of the form $M_c^{*p} = M^{*p} \cap C_c^*(X)$ for some $p \in \beta X$. This implies that every maximal ideal of $C_c^*(X)$ is a contraction of some maximal ideal in $C^*(X)$ ([4, Proposition 4.9] and [12, Corollary 2.10, 2.11]).

Remark 3.7. From Lemma 3.6 and [17, Proposition 2.7] it follows that an ideal I in $C_c^*(X)$ is a z-ideal if and only if $g \in I$ whenever $Z(f^\beta) \subseteq Z(g^\beta)$ with $f \in I$ and $g \in C_c^*(X)$. Therefore, if I is a z-ideal in $C_c^*(X)$, then $J = \{f \in C^*(X) : \exists g \in I, Z(g^\beta) \subseteq Z(f^\beta)\}$ is a z-ideal in $C^*(X)$ and $I = J \cap C_c^*(X)$. Therefore, an ideal I in $C_c^*(X)$ is a z-ideal if and only if is a contraction of some z-ideal of $C^*(X)$.

It is well-known that every maximal ideal in $C^*(X)$ is real. Thus, every maximal ideal in $C_c^*(X)$ is real ([12, Theorem 2.6 (1)]).

Theorem 3.8. *Let X be a zero-dimensional space. Then the following statements are equivalent:*

(i) $M_c^{*p} = M_c^p$ *for each* $p \in \beta X$.

(ii) $\beta_c X = \upsilon_c X$.

(iii) $\frac{\beta_c X}{\sim_c} = \frac{\upsilon_c X}{\sim_c}$.

(iv) $\beta_0 X = \upsilon_0 X$.

(v) X *is a pseudocompact space.*

Proof. (i) $\Rightarrow$ (ii) It clearly follows from the hypothesis that $\beta X = \upsilon_{C_c} X$ and thus $\beta_c X = \upsilon_c X$.

(ii) $\Rightarrow$ (i) Let $f \in C_c^*(X)$ and $f \notin M_c^p$. Thus, $p \notin \mathrm{cl}_{\beta_c X} Z(f)$ and hence there exists some $g \in C_c(X)$ such that $p \in \mathrm{cl}_{\beta X} Z(g)$ and $\mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(g) = \emptyset$. Using our hypothesis, we would have $\mathrm{cl}_{\upsilon_c X} Z(f) \cap \mathrm{cl}_{\upsilon_c X} Z(g) = \emptyset$. Therefore, by Lemma 3.2, $p \notin \mathrm{cl}_{\upsilon_{C_c} X} Z(f) = Z(f^{\upsilon_{C_c}})$ which implies that $p \notin Z(f^\beta)$ and thus $f \notin M_c^{*p}$.

(ii) $\Rightarrow$ (iii) Evident.

(iii) $\Rightarrow$ (iv) This is clear by Remark 3.3 and Theorem 3.4.

(iv) $\Rightarrow$ (ii) Let $p \in \beta_c X$. Then, $\xi([p]) \in \beta_0 X = \upsilon_0 X$ which implies that $[p] \in \xi^{-1}(\upsilon_0 X) = \frac{\upsilon_c X}{\sim_c}$. Therefore, $p \in \upsilon_c X$; i.e., $\beta_c X \subseteq \upsilon_c X$.

(i) $\Rightarrow$ (iii) If X is not pseudocompact, then, by [19, Lemma 1.9.3], there exists a continuous onto mapping $f : X \longrightarrow \mathbb{N}$. Clearly, we could consider f as an element of $C_c(X)$. Also, there exists $p \in \beta X$ such that $f^*(p) = \infty$. Set $g = \frac{1}{1+|f|}$. It follows that $g \in C_c^*(X)$, $Z(g) = \emptyset$ and $g^\beta(p) = 0$. This means that $g \in M_c^{*p} \setminus M_c^p$, which contradicts the hypothesis.

(iii) $\Rightarrow$ (i) If $f \in M_c^{*p}$, then, by the hypothesis and Lemma 3.2, $p \in Z(g^\beta) = Z(g^{\upsilon_c}) = \mathrm{cl}_{\upsilon_c X} Z(g) = \mathrm{cl}_{\beta X} Z(g)$ and thus $f \in M_c^p$; i.e., $M^{*p} \subseteq M_c^p$. The reverse inclusion is obvious. □

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