



Distributive lattices with strong endomorphism kernel property as direct sums

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Abstract. Unbounded distributive lattices which have strong endomorphism kernel property (SEKP) introduced by Blyth and Silva in [3] were fully characterized in [11] using Priestley duality (see Theorem 2.8). We shall determine the structure of special elements (which are introduced after Theorem 2.8 under the name strong elements) and show that these lattices can be considered as a direct product of three lattices, a lattice with exactly one strong element, a lattice which is a direct sum of 2 element lattices with distinguished elements 1 and a lattice which is a direct sum of 2 element lattices with distinguished elements 0, and the sublattice of strong elements is isomorphic to a product of last two mentioned lattices.

1 Introduction

The concept of the (strong) endomorphism kernel property for a universal algebra has been introduced by Blyth, Fang and Silva in [1] and [3] as

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follows.

Definition 1.1. An algebra A has the *endomorphism kernel property* (EKP) if every congruence relation θ on A different from the universal congruence $\iota_A = A \times A$ is the kernel of an endomorphism on A .

Let $\theta \in \text{Con}(A)$ be a congruence on A . A mapping $f: A \rightarrow A$ is said to be *compatible* with θ if $a \equiv b(\theta)$ implies $f(a) \equiv f(b)(\theta)$, it means if it preserves the congruence θ . An endomorphism of A is called *strong*, if it is compatible with every congruence $\theta \in \text{Con}(A)$.

The notion of compatibility of functions with congruences has been studied in various contexts by many authors. We refer to the monograph [13] for an overview. Compatible functions are sometimes called “congruence preserving functions” or “functions with substitution property”.

Definition 1.2. An algebra A has the *strong endomorphism kernel property* (SEKP) if every congruence relation θ on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A .

If the algebra A has two or more nullary operations and corresponding elements are different in A , the universal congruence ι_A can not be the kernel of an endomorphism and that is the reason why the universal congruence ι_A is excluded from the definition of both EKP and SEKP. It is not necessary to exclude it for algebras with one-element subalgebras.

Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras in [3]. For instance, a Boolean algebra has SEKP if and only if it has exactly two elements. A full characterization of MS-algebras having SEKP is provided in this paper. A full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP was proved by Blyth, J. Fang and Wang in [2]. SEKP for distributive p-algebras and Stone algebras has been studied and fully characterized by G. Fang and J. Fang in [6]. Semilattices with SEKP were fully described by J. Fang and Z.-J. Sun in [7]. Guričan and Ploščica described unbounded distributive lattices with SEKP in [11]. Halušková described monounary algebras with SEKP in [12]. Double MS-algebras with SEKP were described by J. Fang in [8].

2 Preliminary results

The following notion is from [10]. Let V be a variety. Let A_i , $i \in I$ be algebras from V such that they all have one element subalgebra and we have chosen (distinguished) elements $e_{A_i} \in A_i$ such that $\{e_{A_i}\}$ is one element subalgebra of A_i . (The situation is easier if the one element algebra is given by a nullary operation in V — no choice is needed in this case.) We denote $\text{supp}(f) = \{i; f(i) \neq e_{A_i}\}$ for $f \in \prod(A_i, i \in I)$. Now let us consider the following subset B of $\prod(A_i, i \in I)$: $B = \{f \in \prod(A_i, i \in I); \text{supp}(f) \text{ is finite}\}$.

It is easy to check that B is a subalgebra of a direct product $\prod(A_i, i \in I)$. We shall denote it as $\sum((A_i, e_{A_i}); i \in I)$ and call it the direct sum of A_i 's with distinguished elements e_{A_i} . All other notions and results in this section are from [11].

First, let us recall the full characterization of distributive $\{1\}$ -lattices (it means distributive lattices in which only top element is considered as a part of its signature, bottom element need not exist and if it exists, it need not be preserved by endomorphisms/homomorphisms) and $\{0\}$ -lattices (defined analogously to $\{1\}$ -lattices) which have SEKP, dual result works for a distributive $\{0\}$ -lattice:

Theorem 2.1. *Let L be a distributive $\{1\}$ -lattice. Then L has SEKP if and only if it is isomorphic to the lattice of all cofinite subsets of some set Z .*

It is clear that these lattices are isomorphic to the sublattices of $\{0, 1\}^Z$ consisting of all $(x_i)_{i \in Z}$ with $\{i \in Z; x_i \neq 1\}$ finite (a direct sum of Z copies of $\{0, 1\}$ with the distinguished elements 1).

For the bounded case we have the following theorem.

Theorem 2.2. *Let L be a bounded distributive lattice. Then L has SEKP if and only if it is a 1- or 2-element chain.*

In this note we shall deal with distributive lattices considered as unbounded lattices (that is, the top and/or bottom elements - if they exist - are not a part of the signature and therefore need not be preserved by homomorphisms). Let L be an unbounded distributive lattice in what follows.

The main tool which we will use is Priestley duality for unbounded distributive lattices. We follow [4, Section 1.2] to introduce its basic elements.

The bounded Priestley space assigned to a distributive lattice L is

$$\mathbf{D}(L) = (\text{Spec}(L); 0, 1, \subseteq, \tau),$$

where $\text{Spec}(L)$ is the set of all prime ideals of L , including \emptyset and L , $0 = \emptyset$, $1 = L$, the set inclusion \subseteq is the order relation on $\text{Spec}(L)$ and τ is the topology on $\text{Spec}(L)$ with the subbasis consisting of all sets $A_x = \{P \in \text{Spec}(L); x \notin P\}$ and their complements $B_y = \{P \in \text{Spec}(L); y \in P\}$ for $x, y \in L$. This means that $\mathbf{D}(L)$ is an ordered topological space. This space as an ordered set is bounded, it is a compact topological space and it is totally order-disconnected. In general, a bounded Priestley space is $\mathbf{X} = (X, 0_X, 1_X, \leq_X, \tau_X)$ with the mentioned structures (X being a nonempty set) which have just mentioned properties.

Let $\mathcal{O}(\mathbf{D}(L))$ be the set all nonempty proper clopen down sets of $\mathbf{D}(L)$, ordered by the set inclusion (a set $U \subseteq \text{Spec}(L)$ is a down set if $x \in U$, $y \in \text{Spec}(L)$ and $y \subseteq x$ implies $y \in U$, up sets are defined dually). The representation theorem is

Theorem 2.3. *An unbounded distributive lattice L is isomorphic to $\mathcal{O}(\mathbf{D}(L))$. The isomorphism $e_L: L \rightarrow \mathcal{O}(\mathbf{D}(L))$ is defined as $e_L(x) = A_x$.*

The Priestley duality between the variety of (unbounded) distributive lattices and the category of bounded Priestley spaces - $\mathbb{IS}_c\mathbb{P}^+\widetilde{\mathbf{D}}$, where $\widetilde{\mathbf{D}} = (\{0, 1\}; 0, 1, \leq, \tau)$ is, by [4, Theorem 6.3.2], a strong duality. For strong dualities, products of two members of a variety (of distributive lattices in our case) correspond to coproducts in the dual category (it also means that the dual category - the category of bounded Priestley spaces in the case, is closed under coproducts).

The object $\widetilde{\mathbf{D}} = (\{0, 1\}; 0, 1, \leq, \tau)$ satisfies one of equivalent conditions of “Unary structure theorem” [4, Theorem 6.2.2] and using [4, Lemma 6.3.1] there is a constructive description of coproducts in the category of bounded Priestley spaces by direct unions \mathbf{Z} of two objects $\mathbf{X} = (X, 0_X, 1_X, \leq_X, \tau_X)$ and $\mathbf{Y} = (Y, 0_Y, 1_Y, \leq_Y, \tau_Y)$.

Here $\mathbf{Z} = (Z, 0_Z, 1_Z, \leq_Z, \tau_Z)$ is defined as follows: $Z = (X \cup Y)/R$ (we assume that X, Y are disjoint) - a (disjoint) union of X, Y factorised by an equivalence relation R which identifies 0_X with 0_Y and 1_X with 1_Y . The topology τ_Z is given as a factor topology, that is, unique, strongest

topology which makes inclusion maps of X and Y into Z embeddings, 0_Z is the block $\{0_X, 0_Y\}$ of equivalence relation R , 1_Z is the block $\{1_X, 1_Y\}$ and the ordering \leq_Z is defined by a “union” of \leq_X and \leq_Y , that is, $x \leq_Z y$ if and only if $x, y \in X$ and $x \leq_X y$ or $x, y \in Y$ and $x \leq_Y y$, and block $0_Z = \{0_X, 0_Y\}$ is a bottom element, block $1_Z = \{1_X, 1_Y\}$ is a top element.

It is known, that the category of bounded Priestley spaces is closed under direct unions and therefore, by [4, Lemma 6.3.2], coproducts are given exactly by direct unions (in spite of the fact that there is a binary relation in \mathbf{D}). We are mostly interested in clopen down sets of \mathbf{Z} (and induced clopen down sets of “components” \mathbf{X}, \mathbf{Y}). As τ_Z is a factor topology, A is a nonempty proper clopen down set of \mathbf{Z} if and only if $(A \cap X) \cup \{0_X\}$ is a nonempty proper clopen down set in \mathbf{X} , and $(A \cap Y) \cup \{0_Y\}$ is a nonempty proper clopen down set in \mathbf{Y} .

We shall summarize most important properties of Priestley spaces of distributive lattices with SEKP, see [11]. Let us start with the description of an order relation \subseteq of $\text{Spec}(L)$.

Lemma 2.4. *If L has SEKP, then $X = \text{Spec}(L) \setminus \{\emptyset, L\}$ is a disjoint union of three antichains $A_0 \cup A_1 \cup A_2$, where $A_1 = \{a \in X; (\exists b \in X)(a < b)\}$ (“bottom” elements), $A_2 = \{b \in X; (\exists a \in X)(a < b)\}$ (“top” elements) and $A_0 = X \setminus (A_1 \cup A_2)$ (“incomparable” elements).*

We shall keep the notation A_0, A_1, A_2 in the paper. Here are most important properties of $\mathbf{D}(L)$ and of $\mathcal{O}(\mathbf{D}(L))$.

Lemma 2.5. *Let L be any distributive lattice. Let L have SEKP, $P \in \text{Spec}(L)$, $P \neq \emptyset, P \neq L$. Then P is a discrete point in the topology τ .*

Lemma 2.6. *Let L have SEKP. Then for every $A \in \mathcal{O}(\mathbf{D}(L))$ the sets $A \cap A_2$ and $A_1 \setminus A$ are finite.*

Lemma 2.7. *Let L have SEKP. Then there exists a clopen down set $C \in \mathcal{O}(\mathbf{D}(L))$ such that $A_1 \subseteq C$ and $C \cap A_2 = \emptyset$. Moreover, for any such C and for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $A \subseteq C$ the interval $[A, C]$ of $\mathcal{O}(\mathbf{D}(L))$ is a (finite) Boolean lattice and also for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $C \subseteq A$ the interval $[C, A]$ of $\mathcal{O}(\mathbf{D}(L))$ is a (finite) Boolean lattice.*

The element $C \in \mathcal{O}(\mathbf{D}(L))$ with this property is not determined uniquely and can be obtained from any $A \in \mathcal{O}(\mathbf{D}(L))$ as $C = [A \cup (A_1 \setminus A)] \setminus (A \cap A_2)$.

Theorem 2.8. *Let L be an unbounded distributive lattice. Then the following conditions are equivalent:*

(i) L has SEKP.

(ii) L is locally finite, meaning that every closed interval $[a, b]$ of the lattice L is finite, and there exists $c \in L$ such that for every $x < c$ or $x > c$ intervals $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are (finite) Boolean lattices.

The existence of c from the item 2 of this theorem follows from Lemma 2.7, the clopen down set $C = A_c = \{P \in \text{Spec}(L); c \notin P\} \in \mathcal{O}(\mathbf{D}(L))$, which is the $\mathcal{O}(\mathbf{D}(L))$ “form” of such c can be fully characterized by the fact that it is a clopen down set and that $A_1 \subseteq C$ and $A_2 \cap C = \emptyset$ (one implication of this characterization also follows from Lemma 2.7 and the second one is proved in [11] within the proof of Theorem 2.8).

Definition 2.9. Let L be an unbounded distributive lattice which has SEKP. The element $c \in L$ is called *strong* if for every $x < c$ or $x > c$ intervals $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are (finite) Boolean lattices.

Equivalently, clopen down set $C \in \mathcal{O}(\mathbf{D}(L))$ is called *strong* element if $A_1 \subseteq C$ and $C \cap A_2 = \emptyset$.

Every strong element C (the $\mathcal{O}(\mathbf{D}(L))$ form) can be written as $C = A_1 \cup (C \cap A_0) \cup \{\emptyset\}$ and therefore it is uniquely determined by its intersection with the set A_0 .

3 Strong elements and direct sums

We shall describe the structure of strong elements of an unbounded distributive lattice with SEKP and show that an unbounded distributive lattice with SEKP can be considered as a direct product of three lattices, a lattice with exactly one strong element, a lattice which is a direct sum of 2 element lattices with distinguished elements 1 and a lattice which is a direct sum of 2 element lattices with distinguished elements 0. All lattices in this section are considered as unbounded lattices.

Lemma 3.1. *Let L be a distributive lattice with SEKP. Then the set of all strong elements form a convex sublattice of L .*

Proof. By the definition of a strong element c , for

$$C = A_c = \{P \in \text{Spec}(L); c \notin P\} \in \mathcal{O}(\mathbf{D}(L))$$

we know that $A_1 \subseteq C$ and $A_2 \cap C = \emptyset$. These properties are clearly preserved by the union and the intersection and they also ensure the convexity in $\mathcal{O}(\mathbf{D}(L))$.

□

Lemma 3.2. *Let L be a distributive lattice with SEKP. Let $B, C \in \mathcal{O}(\mathbf{D}(L))$. Denote $B_0 = B \cap A_0$, $C_0 = C \cap A_0$. Then sets B, C and also B_0, C_0 differ only in a finite number of elements (that is, the symmetrical differences $B\Delta C$ and $B_0\Delta C_0$ are finite).*

Proof. We know that $(B\Delta C) \cap A_0 = B_0\Delta C_0$, therefore it is enough to prove that $B\Delta C$ is finite. Now, $B\Delta C = (B \cup C) \setminus (B \cap C)$, so that $B\Delta C$ is the difference of two clopen sets, therefore it is clopen, and hence compact. By Lemma 2.5, it consists of discrete points and by compactness it is finite. □

Now we can formulate the first “decomposition” theorem

Theorem 3.3. *Let L be a distributive lattice which has SEKP. Then there are distributive lattices L_1 and L' such that*

- (i) L is isomorphic to $L_1 \times L'$.
- (ii) L_1 and L' have SEKP.
- (iii) L_1 has exactly one strong element c .
- (iv) $\{c\} \times L'$ correspond to all strong elements of L (L' is isomorphic to the sublattice of all strong elements of L).

Proof. Let us use the description of the order from Lemma 2.4. Let $\mathbf{S} = (A_1 \cup A_2 \cup \{\emptyset, L\}, \emptyset, L, \subseteq, \tau_S)$, $\mathbf{T} = (A_0 \cup \{\emptyset, L\}, \emptyset, L, \subseteq, \tau_T)$ with topologies induced from $\mathbf{D}(L)$. The elements of A_0 are incomparable with elements of $A_1 \cup A_2$ and therefore $\mathbf{D}(L)$ is a direct union of \mathbf{S} and \mathbf{T} , it means that it is a coproduct in the category of bounded Priestley spaces. (We do not use indices like $\emptyset_S, \emptyset_T, \dots$)

Let us denote by L_1 the lattice corresponding to the bounded Priestley space \mathbf{S} , L' the lattice corresponding to the bounded Priestley space \mathbf{T} . As coproducts in the category of bounded Priestley spaces correspond to products in the variety of (unbounded) distributive lattices, L is isomorphic to $L_1 \times L'$.

\mathbf{S} and \mathbf{T} have orders and topologies corresponding to what is described in Lemmas 2.4 - 2.7 and therefore both L_1, L' have SEKP. Bounded Priestley

space \mathbf{S} does not contain incomparable elements and therefore the only strong element of \mathbf{S} is given by $C = A_1 \cup \{\emptyset\}$. Let us denote c the element of L_1 with $A_c = C$.

By the definition of strong elements of $\mathcal{O}(\mathbf{D}(L))$ and the description of topologies of \mathbf{S}, \mathbf{T} , every proper clopen down set D of \mathbf{T} is the “intersection” of some strong element C' of $\mathcal{O}(\mathbf{D}(L))$ (of L) with the set $A_0 \cup \{\emptyset\}$, more precisely $D = (C' \cap A_0) \cup \{\emptyset\}$, and vice versa. Therefore $\{c\} \times L'$ correspond exactly to all strong elements of L . \square

Now we shall describe a decomposition of L' .

Theorem 3.4. *Let L' be the distributive lattice from Theorem 3.3. Then there are sets U and V and distributive lattices L_2 and L_3 such that*

- (i) L' is isomorphic to $L_2 \times L_3$.
- (ii) L_2 is a direct sum of U copies of $\{0, 1\}$ with the distinguished elements 1 for some set U .
- (iii) L_3 is a direct sum of V copies of $\{0, 1\}$ with the distinguished elements 0 for some set V .

Sets U, V can be chosen in such a way that one of the following holds:

- (a) both U, V are infinite, (b) one of U, V is empty.

Proof. By the proof of Theorem 3.3, we know that $A_1 \cup A_2$ is empty for L' , i.e. $\mathcal{O}(\mathbf{D}(L')) = \mathbf{T}$ of Theorem 3.3. As the order on A_0 is trivial, we can, for example, decompose \mathbf{T} as follows.

Take C a nonempty proper clopen down set of \mathbf{T} . Let us denote $\mathbf{X}' = ((A_0 \cap C) \cup \{\emptyset, L'\}, \emptyset, L', \subseteq, \tau_{X'})$ and $\mathbf{Y}' = ((A_0 \setminus C) \cup \{\emptyset, L'\}, \emptyset, L', \subseteq, \tau_{Y'})$ with topologies induced from \mathbf{T} .

It is clear that \mathbf{T} is the direct union of \mathbf{X}' and \mathbf{Y}' , so that it is a coproduct and denoting L_2 a lattice corresponding to \mathbf{X}' , L_3 a lattice corresponding to \mathbf{Y}' , we see that L' is isomorphic to $L_2 \times L_3$.

\mathbf{X}' and \mathbf{Y}' have orders and topologies corresponding to what is described in Lemmas 2.4 - 2.7 (with $A_1 = A_2 = \emptyset$) and therefore both have SEKP. Let us discuss at first the most general case, when both $U = C \cap A_0$ and $V = A_0 \setminus C$ are infinite.

Applying Lemma 2.5 - by removing a finite number of discrete points from a clopen set we obtain a clopen set - and Lemma 3.2 to $\mathcal{O}(\mathbf{D}(L_2))$ we see that L_2 is isomorphic to the lattice of all cofinite subsets of the set

U , which is isomorphic to the direct sum of U copies of $\{0, 1\}$ with the distinguished elements 1.

Applying Lemma 2.5 - by adding a finite number of discrete points to a clopen set we obtain a clopen set - and Lemma 3.2 to $\mathcal{O}(\mathbf{D}(L_3))$, we see that L_3 is isomorphic to the lattice of all finite subsets of the set V , which is isomorphic to the direct sum of V copies of $\{0, 1\}$ with the distinguished elements 0.

If one of U, V is finite and the other one is infinite, the finite one can be “made” empty by Lemma 2.5, because by removing/adding finite number of discrete points from/to a clopen set we get a clopen set.

If both U, V are finite, we can make one of them empty. □

Summarizing these results we get

Theorem 3.5. *Let L be an unbounded distributive lattice with SEKP. Then there are sets U, V and lattices L_1, L_2, L_3 such that*

- (i) $L \cong L_1 \times L_2 \times L_3$.
- (ii) L_1 has SEKP and contains exactly one strong element.
- (iii) L_2 is the direct sum $\Sigma(\{0, 1\}; i \in U)$ with distinguished elements 1.
- (iv) L_3 is the direct sum $\Sigma(\{0, 1\}; i \in V)$ with distinguished elements 0.
- (v) the product $L_2 \times L_3$ is isomorphic to the sublattice of all strong elements of L .

Sets U, V can be chosen in such a way that one of the following holds:

- (a) both U, V are infinite, (b) one of U, V is empty.

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