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Crossed squares, crossed modules over groupoids and $\text{cat}^{1-2}-$groupoids

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Abstract. The aim of this paper is to introduce the notion of $\text{cat}^{\mathbf{1}}-$groupoids which are the groupoid version of $\text{cat}^{\mathbf{1}}-$groups and to prove the categorical equivalence between crossed modules over groupoids and $\text{cat}^{\mathbf{1}}-$groupoids. In section 4 we introduce the notions of crossed squares over groupoids and of $\text{cat}^{\mathbf{2}}-$groupoids, and then we show their categories are equivalent. These equivalences enable us to obtain more examples of groupoids.

1 Introduction

Crossed modules over groups which are defined by Whitehead in [25, 26] as algebraic models for homotopy 2-types are equivalent to several algebraic and combinatorial categories such as the categories of group-groupoids (or alternatively named $\mathcal{G}$-groupoids in [5] or 2-groups in [3]) and of $\text{cat}^{\mathbf{1}}$-groups (or categorical groups) [6, 17]. A crossed module can be thought as the case n=1 of a crossed n-cube which should be the 'algebraic core' of a $\text{cat}^{\boldsymbol{n}}-$group (or n-cat-group) [11]. One can find some applications of crossed modules in homotopy theory [6], homological algebra [14] and algebraic K-

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theory [16]. The equivalence between $\mathbf{cat^1}$-groups and crossed modules is useful for extension of crossed modules to higher dimensions, see [17]. A crossed square was first defined by Guin-Walery and Loday [13] in their investigation of applications of some problems in algebraic K-theory. In [17] it was proved that the category of $\mathbf{cat^2}$-groups is isomorphic to the category of crossed squares.

For the groupoid case of crossed modules, basic references are Brown-Higgins [7, 8] and Brown-Icen [9]. The categorical equivalence between crossed modules over groupoids and 2-groupoids is given in [15]. Recently normal and quotient structures in the category of crossed modules over groupoids and of 2-groupoids were compared and the corresponding structures related 2-groupoids were characterized in [24] (see also [20]) using the categorical equivalence between 2-groupoids and crossed modules over groupoids. The definition of 2- and 3-crossed modules over groupoids were introduced in [2] by extending the definition of 2- and 3-crossed modules over groups to the notion of groupoids.

There are several but useful 2-dimensional concepts of groupoids such as double groupoids, 2-groupoids and crossed modules of groupoids. However there is a gap in this context and so we investigate a new 2-dimensional version of a groupoid which we called *cat*$^{\mathbf{1}}$*-groupoid* since it is the groupoid case of a $\mathbf{cat^1}$-group and prove the categorical equivalence between $\mathbf{cat^1}$-groupoids and crossed modules over groupoids. Morover we introduce the notion of crossed squares over groupoids and prove that the category of them is equivalent to the category of $\mathbf{cat^2}$-groupoids which are the groupoid version of $\mathbf{cat^2}$-groups.

2 Preliminaries

A category $\mathcal{C} = (X, C, d_0, d_1, \varepsilon)$ consists of the set of objects X, the set of morphisms $C = \cup_{x,y \in X} C(x, y)$ where $C(x, y)$ is the set of morphisms in $\mathcal{C}$ from x to y as follows

$$x \xrightarrow{a} y$$

with the source and the target maps $d_0, d_1 \colon C \to X$, respectively, such that $d_0(a) = x, d_1(a) = y$, the associative composition map $m \colon C(y, z) \times C(x, y) \to C(x, z)$, $m(b, a) = b \circ a$ and the unit map $\varepsilon \colon X \to C$ sending each object x of $\mathcal{C}$ to its identity morphism $1_x \in C(x) := C(x, x)$ such that

$a \circ 1_x = a$ and $1_x \circ a' = a'$ where $a' \in C(w,x)$. A groupoid $\mathcal{G} = (X, G)$ is a category with the inversion map $\eta\colon G \to G$, $\eta(a) = a^{-1} \in G(y,x)$ such that $a \circ a^{-1} = 1_y$, $a^{-1} \circ a = 1_x$. For further details, see [4, 19]. Since all categories in this paper are over a fixed base set, namely X, we use the notation X for the base set of all categories and groupoids in whole of the paper.

Example 2.1. Let X be a set and G be a group. Then $\mathcal{G} = (X, X \times G \times X)$ is a groupoid called *trivial groupoid*, see [18]. Recall that $d_0(x,g,y) = x, d_1(x,g,y) = y, \varepsilon(x) = (x,e,x)$, where e is the identity element of G and $\eta(x,g,y) = (y, g^{-1}, x)$ where the composition of morphisms is defined by $(y, g', z) \circ (x, g, y) = (x, g'g, z)$.

Recall that a morphism is a functor between groupoids which is identity on the base set X, on the other hand the base preserving morphisms are morphisms in $\mathcal{G} = (X, G)$. Furthermore, $\mathcal{H} = (X, H)$ is a (wide) subgroupoid of $\mathcal{G} = (X, G)$ if H is closed under composition and inversion.

Let $\mathcal{G}$ be a groupoid and $\mathcal{N}$ be a wide subgroupoid of $\mathcal{G}$. Then $\mathcal{N}$ is called *normal* if

$$g \circ h \circ g^{-1} \in N(y)$$

for all $h \in N$, $g \in G$ such that $d_0(h) = d_1(h) = d_1(g)$ [4].

By a crossed module over groups we mean a pair of groups M, N with an action $\bullet\colon N \times M \to M$ of N on M and a morphisms $\partial\colon M \to N$ of groups satisfies the conditions $\partial(n \bullet m) = n\partial(m)n^{-1}$ and $\partial(m) \bullet m_1 = mm_1m^{-1}$, for $m, m_1 \in M$ and $n \in N$ [25, 26]. The well-known equivalence between crossed modules over groups and 2-groups (or group-groupoids) was proved by Brown and Spencer [5].

Let G be a group. We recall that from [17] and [6] a cat^1-group is a triple (G, s, t) with group homomorphisms $s, t\colon G \to G$ satisfying following conditions

[CG 1] $st = t$ and $ts = s$,

[CG 2] $[\mathrm{Kers}\, s, \mathrm{Ker}\, t] = 1$.

The following theorem is proved in [6]:

Theorem 2.2. *The categories of crossed modules over groups and of cat*^{1}*-groups are equivalent.*

Remark that this theorem is widely extended for some other algebraic categories and also was proved for semi-abelian categories, but they don't cover our work properly.

A crossed square as defined in [10] (see also [23]) is a commutative diagram of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

together with four morphisms $\lambda, \lambda', \mu, \nu$ of groups, actions of the group P on L, M, N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h\colon M \times N \to L$ satisfy the following axioms

[CS 1] λ, λ' preserves the actions of P and μ, ν and $\kappa = \mu\lambda = \nu\lambda'$ are crossed modules of groups,

[CS 2] $\lambda h(m,n) = m(n \bullet m^{-1})$, $\lambda' h(m,n) = (m \bullet n)n^{-1}$,

[CS 3] $h(\lambda(l), n) = l(n \bullet l^{-1})$, $h(m, \lambda'(l)) = (m \bullet l)l^{-1}$,

[CS 4] $h(mm', n) = \big(m \bullet h(m', n)\big)h(m,n)$,
$h(m, nn') = h(m,n)(n \bullet h(m, n'))$,

[CS 5] $h(p \bullet m, p \bullet n) = p \bullet h(m,n)$,

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

We will give the definition of a cat$^{\mathbf{2}}$-group (or 2-cat-group) from [17] in terms of our notation. A cat$^{\mathbf{2}}$-group is a 5-tuple (G, s_1, t_1, s_2, t_2) where G is a group with four homomorphisms $s_1, t_1, s_2, t_2\colon G \to G$ such that

[C2G 1] $s_i t_i = t_i$ and $t_i s_i = s_i$,

[C2G 2] $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$ and $s_i t_j = t_j s_i$,

[C2G 3] $[\operatorname{Ker} s_i, \operatorname{Ker} t_i] = 1$,

for $i, j \in \{1, 2\}$, $i \neq j$. The following theorem was proved in [17].

Theorem 2.3. *The category of cat$^{\mathbf{2}}$-groups is isomorphic to the category of crossed squares.*

3 Crossed modules over groupoids and cat$^{\mathbf{1}}$-groupoids

In this section we define cat$^{\mathbf{1}}$-groupoids by extending the definition of cat$^{\mathbf{1}}$-groups to the notion of groupoids and give some examples using cat$^{\mathbf{1}}$-groups. Then we prove that there is a categorical equivalence between cat$^{\mathbf{1}}$-groupoids and crossed modules over groupoids.

Definition 3.1. Let $\mathcal{G} = (X, G)$ be a groupoid, $\sigma, \tau \colon \mathcal{G} \to \mathcal{G}$ be functors which are identities on objects. A *cat*$^{\mathbf{1}}$*-groupoid* is a triple $(\mathcal{G}, \sigma, \tau)$ satisfying

[C1Gd 1] $\sigma\tau = \tau$ and $\tau\sigma = \sigma$,

[C1Gd 2] $h \circ k \circ h^{-1} \circ k^{-1} = \varepsilon d_0(h)$, for all $h \in \mathrm{Ker}(\sigma), k \in \mathrm{Ker}(\tau)$ where $d_0(h) = d_0(k)$.

Here $\mathrm{Ker}(\sigma) = \{g \in G | \sigma(g) = \varepsilon d_0(g)\}$ and $\mathrm{Ker}(\tau) = \{g \in G | \tau(g) = \varepsilon d_0(g)\}$ are wide subgroupoids of $\mathcal{G}$ on the base set X. Also these subgroupoids are totally disconnected groupoids. Now, since $\mathrm{Ker}(\sigma)$ is a subgroupoid, we have also

$$\text{[C1Gd 2']} \qquad h \circ h_1 \circ h^{-1} \circ h_1^{-1} = \varepsilon d_0(h)$$

for $h, h_1 \in \mathrm{Ker}(\sigma)$.

Example 3.2. Since every group is a groupoid with a unique object, every cat$^{\mathbf{1}}$-group can be regarded as a cat$^{\mathbf{1}}$-groupoid with a single object.

Example 3.3. Let (G, s, t) be a cat$^{\mathbf{1}}$-group, X be a set and $\mathcal{G} = (X, X \times G \times X)$ be the trivial groupoid. Then $(\mathcal{G}, \sigma, \tau)$ is a cat$^{\mathbf{1}}$-groupoid where $\sigma(x, g, y) = (x, s(g), y)$ and $\tau(x, g, y) = (x, t(g), y)$.

Proposition 3.4. *Given any cat*$^{\mathbf{1}}$*-groupoid* $(\mathcal{G}, \sigma, \tau)$*, we have*
(i) $\sigma(G) = \tau(G)$*,*
(ii) σ *and* τ *are identities on* $\sigma(G)$ *and* $\tau(G)$*,*
(iii) $\sigma^2 = \sigma$ *and* $\tau^2 = \tau$*.*

Definition 3.5. A *morphism* $f \colon (\mathcal{G}, \sigma, \tau) \to (\mathcal{G}', \sigma', \tau')$ of cat$^{\mathbf{1}}$-groupoids is a commutative diagram of groupoids

$$\begin{array}{ccc} \mathcal{G} & \overset{\sigma}{\underset{\tau}{\rightrightarrows}} & \mathcal{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{G}' & \overset{\sigma'}{\underset{\tau'}{\rightrightarrows}} & \mathcal{G}' \end{array}$$

Therefore we construct the category CAT^1-GPD of $\mathbf{cat^1}$-groupoids.

We recall the definition of crossed modules over groupoids as given in [9]. Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids over the same object set X such that $\mathcal{H}$ is totally disconnected. A crossed module $K = (\mathcal{H}, \mathcal{G}, \partial)$ over groupoids consists of a morphism $\partial\colon \mathcal{H} \to \mathcal{G}$ of groupoids which is identity on objects together with a partial action $\bullet\colon G \times H \to H$ of groupoids satisfying

[CMG 1] $\partial(g \bullet h) = g \circ \partial(h) \circ g^{-1}$,

[CMG 2] $\partial(h) \bullet h_1 = h \circ h_1 \circ h^{-1}$, for $h, h_1 \in H(x)$ and $g \in G(x, y)$.

It is a fair remark that if $(\mathcal{H}, \mathcal{G}, \partial)$ is a crossed module over groupoids, then $\operatorname{Im}(\partial)$ is a normal subgroupoid and also $\operatorname{Ker}(\partial)$ lies in the center of G, where the center of G is a wide subgroupoid in which the morphisms are

$$\{g \in G\colon g \circ h = h \circ g, d_0(g) = d_1(g) = d_0(h) = d_1(h)\}.$$

Let $K = (\mathcal{H}, \mathcal{G}, \partial)$ and $K' = (\mathcal{H}', \mathcal{G}', \partial')$ be crossed modules over groupoids. A morphism $\lambda = (\lambda_2, \lambda_1, \lambda_0)\colon K \to K'$ is called a *morphism of crossed modules over groupoids* if $(\lambda_0, \lambda_1)\colon \mathcal{H} \to \mathcal{H}'$ and $(\lambda_0, \lambda_2)\colon \mathcal{G} \to \mathcal{G}'$ are morphisms of groupoids such that $\lambda_2\partial = \partial'\lambda_1$ and $\lambda_1(g \bullet h) = \lambda_2(g) \bullet' \lambda_1(h)$. Hence the category of crossed modules over groupoids can be defined which we denoted by CMG.

Theorem 3.6. *The category of $\mathbf{cat^1}$-groupoids is equivalent to the category of crossed modules over groupoids.*

Proof. A functor $\psi\colon \text{CMG} \to \text{CAT}^1\text{-GPD}$ is an equivalence of categories. If $(\mathcal{A}, \mathcal{B}, \partial)$ is a crossed module over groupoids, then $\psi(\mathcal{A}, \mathcal{B}, \partial) = (\mathcal{G}, \sigma, \tau)$ is a $\mathbf{cat^1}$-groupoid over the same object set where the set of morphisms of $\mathcal{G}$ is defined by $B \ltimes A = \{(b, a) | b \in B, a \in A, d_1(b) = d_0(a) = d_1(a)\}$, $\sigma(b, a) = (b, \varepsilon d_0(a))$ and $\tau(b, a) = (\partial(a) \circ b, \varepsilon d_0(a))$. Here, if $x \xrightarrow{b} y$ and $y \xrightarrow{a} y$ are morphisms of B and A, respectively, then (b, a) is a morphism of $\mathcal{G}$ as follows

$$x \xrightarrow{(b,a)} y\ .$$

where the structure maps are defined by $d_0(b,a) = d_0(b), d_1(b,a) = d_1(a), \varepsilon(x) = (1_x, 1_x)$, $\eta(b,a) = (b^{-1}, b^{-1} \bullet a^{-1})$ and the composition of morphisms is defined by

$$(b_1, a_1) \circ (b, a) = (b_1 \circ b, a_1 \circ (b_1 \bullet a))$$

when $y \xrightarrow{b_1} z \xrightarrow{a_1} z$.

Now we define a functor γ: $\text{CAT}^1\text{-GPD} \to \text{CMG}$ as a weak inverse of ψ. Given a $\text{cat}^{\mathbf{1}}$-groupoid $(\mathcal{G}, \sigma, \tau)$, then $\gamma(\mathcal{G}, \sigma, \tau) = (\text{Ker}(\sigma), \sigma(\mathcal{G}), \tau)$ is a crossed module over groupoids where an action of $\sigma(\mathcal{G})$ on $\text{Ker}(\sigma)$ is defined by $g \bullet h = g \circ h \circ g^{-1}$, for all $g \in \sigma(G), h \in \text{Ker}(\sigma)$.

[CMG 1] Since $g \in \sigma(G)$, by Proposition 3.4 we get $\tau(g) = g$ and so

$$\tau(g \bullet h) = \tau(g \circ h \circ g^{-1}) = \tau(g) \circ \tau(h) \circ \tau(g^{-1}) = g \circ \tau(h) \circ g^{-1}$$

[CMG 2] $\tau(h) \bullet h_1 = \tau(h) \circ h_1 \circ \tau(h)^{-1} = \tau(h) \circ h_1 \circ \tau(h^{-1}) \circ h \circ h^{-1}$
Since $\tau(h^{-1}) \circ h \in \text{Ker}(\tau)$ and $h_1 \in \text{Ker}(\sigma)$, they commute, and so

$$\tau(h) \bullet h_1 = \tau(h) \circ \tau(h^{-1}) \circ h \circ h_1 \circ h^{-1} = h \circ h_1 \circ h^{-1}.$$

A natural equivalence $S\colon 1_{\text{CAT}^1\text{-GPD}} \to \psi\gamma$ is defined via a mapping

$$S_{\mathcal{G}}(\mathcal{G}, \sigma, \tau) = ((X, \sigma(G) \ltimes \text{Ker}(\sigma)), \sigma', \tau')$$

which is defined such that to be identity on objects and

$$S_{\mathcal{G}}(g) = (\sigma(g), g \circ \sigma(g^{-1}))$$

for $g \in G$ where $\sigma'(g,h) = (g, \varepsilon d_0(h))$, $\tau'(g,h) = (\tau(h) \circ g, \varepsilon d_0(h))$. We will verify that $S_{\mathcal{G}}$ preserves composition.

$$S_{\mathcal{G}}(g_1 \circ g) = (\sigma(g_1 \circ g), g_1 \circ g \circ \sigma(g_1 \circ g)^{-1}) = (\sigma(g_1) \circ \sigma(g), g_1 \circ g \circ \sigma(g^{-1}) \circ \sigma(g_1^{-1}))$$

$$\begin{aligned} S_{\mathcal{G}}(g_1) \circ S_{\mathcal{G}}(g) &= \Big(\sigma(g_1), g_1 \circ \sigma(g_1^{-1})\Big) \circ \Big(\sigma(g), g \circ \sigma(g^{-1})\Big) \\ &= \Big(\sigma(g_1) \circ \sigma(g), g_1 \circ \sigma(g_1^{-1}) \circ (\sigma(g_1) \bullet (g \circ \sigma(g^{-1}))\Big) \\ &= \Big(\sigma(g_1) \circ \sigma(g), g_1 \circ \sigma(g_1^{-1}) \circ \sigma(g_1) \circ g \circ \sigma(g^{-1}) \circ \sigma(g_1^{-1})\Big) \\ &= (\sigma(g_1) \circ \sigma(g), g_1 \circ g \circ \sigma(g^{-1}) \circ \sigma(g_1^{-1})) \end{aligned}$$

Conversely, a natural equivalence $T\colon 1_{\mathrm{CMG}} \to \gamma\psi$ is defined such that

$$T_{\mathcal{C}}(b) = (b, \varepsilon d_1(b)), \qquad T_{\mathcal{C}}(a) = (\varepsilon d_1(a), a),$$

for $\mathcal{C} = (\mathcal{B}, \mathcal{A}, \partial)$. $T_{\mathcal{C}}$ preserves compositions as follows:

$$T_{\mathcal{C}}(b_1)\circ T_{\mathcal{C}}(b) = (b_1, 1_z)\circ(b, 1_y) = (b_1\circ b, 1_z\circ(b_1\bullet 1_y)) = (b_1\circ b, 1_z\circ 1_z) = T_{\mathcal{C}}(b_1\circ b)$$

for $x \xrightarrow{b} y \xrightarrow{b_1} z$ and

$$T_{\mathcal{C}}(a_1)\circ T_{\mathcal{C}}(a) = (1_x, a_1)\circ(1_x, a) = (1_x\circ 1_x, a_1\circ(1_x\bullet a)) = (1_x, a_1\circ a) = T_{\mathcal{C}}(a_1\circ a)$$

for $x \xrightarrow{a} x \xrightarrow{a_1} x$. □

As an application of this result, we compare normal objects of $\mathrm{cat}^{\mathbf{1}}-$groupoids and of crossed modules over groupoids. First we recall the definitions of subcrossed modules and of normal crossed modules over groupoids from [24].

Definition 3.7. Let $\mathcal{A} = (X, A)$, $\mathcal{B} = (X, B)$ be groupoids, $\mathcal{A}$ be totally disconnected and $(\mathcal{A}, \mathcal{B}, \partial)$ be a crossed module over groupoids. A crossed module $(\mathcal{M}, \mathcal{N}, \sigma)$ over groupoids is called a *subcrossed module* of $(\mathcal{A}, \mathcal{B}, \partial)$ if

[SCMG 1] $\mathcal{M} = (Y, M)$ is a subgroupoid of $\mathcal{A} = (X, A)$,

[SCMG 2] $\mathcal{N} = (Y, N)$ is a subgroupoid of $\mathcal{B} = (X, B)$,

[SCMG 3] σ is the restriction of ∂ to M,

[SCMG 4] the action of $\mathcal{N}$ on $\mathcal{M}$ is the restriction of the action of $\mathcal{B}$ on $\mathcal{A}$.

If $X = Y$ then $(\mathcal{M}, \mathcal{N}, \sigma)$ is called a wide subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$.

Definition 3.8. A *normal subcrossed module* over groupoids is a subcrossed module $(\mathcal{M}, \mathcal{N}, \sigma)$ of $(\mathcal{A}, \mathcal{B}, \partial)$ which satisfies

[NCMG 1] $\mathcal{N}$ is normal subgroupoid of $\mathcal{B}$,

[NCMG 2] $b \bullet m \in M(y)$, for all $b \in B(x, y), m \in M(x)$,

[NCMG 3] $(n \bullet a) \circ a^{-1} \in M(x)$, for all $n \in N(x), a \in A(x)$.

From [NCMG2] we have that $\partial(a) \bullet m = a \circ m \circ a^{-1} \in M$ and so $\mathcal{M}$ is normal subgroupoid of $\mathcal{A}$.

Now we introduce the notions of subcat$^{\mathbf{1}}-$groupoids and normal cat$^{\mathbf{1}}-$groupoids.

Definition 3.9. A *subcat$^{\mathbf{1}}-$groupoid* $(\mathcal{G}', \sigma', \tau')$ of a cat$^{\mathbf{1}}-$groupoid $(\mathcal{G}, \sigma, \tau)$ is a subgroupoid $\mathcal{G}' = (X', G')$ of $\mathcal{G} = (X, G)$ such that σ', τ' are restriction of σ, τ to $\mathcal{G}'$, respectively. We say $\mathcal{G}'$ is wide if $X' = X$. If $\mathcal{G}'$ is normal subgroupoid of $\mathcal{G}$, then $(\mathcal{G}', \sigma', \tau')$ is called normal subcat$^{\mathbf{1}}-$groupoid of $(\mathcal{G}, \sigma, \tau)$.

According the proof of the Theorem 3.6, we give following results.

Proposition 3.10. *Let $(\mathcal{M}, \mathcal{N}, \sigma)$ be a normal subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$ over the same object set X. Then the cat$^{\mathbf{1}}-$groupoid corresponding to $(\mathcal{M}, \mathcal{N}, \sigma)$ is a normal subcat$^{\mathbf{1}}-$groupoid of the one corresponding to $(\mathcal{A}, \mathcal{B}, \partial)$.*

Proof. We only need to show that the groupoid $(X, N \ltimes M)$ is a normal subgroupoid of $(X, B \ltimes A)$. For $b \in B(x, y)$, $a \in A(y)$, $(n_x, m_x) \in (N \ltimes M)(x) = N(x) \ltimes M(x)$,

$$\begin{aligned} (b,a) \circ (n_x, m_x) \circ (b,a)^{-1} &= \Big(b \circ n_x, a \circ (b \bullet m_x)\Big) \circ (b^{-1}, b^{-1} \bullet a^{-1}) \\ &= \Big(b \circ n_x \circ b^{-1}, a \circ (b \bullet m_x) \circ ((b \circ n_x) \bullet (b^{-1} \bullet a^{-1}))\Big) \\ &= \Big(b \circ n_x \circ b^{-1}, a \circ (b \bullet m_x) \circ ((b \circ n_x \circ b^{-1}) \bullet a^{-1})\Big) \end{aligned}$$

Let $b \bullet m_x = m_y$ and $b \circ n_x \circ b^{-1} = n_y$. Then, from [NCMG1] $n_y \in N(y)$, from [NCMG2] $m_y \in M(y)$ and from [NCMG3] $(n_y \bullet a^{-1}) \circ a = m'_y \in M(y)$. Now we have

$$\begin{aligned} (b,a) \circ (n_x, m_x) \circ (b,a)^{-1} &= \Big(n_y, a \circ m_y \circ (n_y \bullet a^{-1}) \circ a \circ a^{-1}\Big) \\ &= \Big(n_y, a \circ m_y \circ m'_y \circ a^{-1}\Big) \in (N \ltimes M)(y). \end{aligned}$$

□

Proposition 3.11. *Let $(\mathcal{G}', \sigma', \tau')$ be a normal subcat$^{\mathbf{1}}-$groupoid of $(\mathcal{G}, \sigma, \tau)$. Then the crossed module corresponding to $\mathcal{G}'$ is a normal subcrossed module of the one corresponding to $\mathcal{G}$.*

Proof. [NCMG 1] Clearly $\sigma(\mathcal{G}')$ is a normal subgroupoid of $\sigma(\mathcal{G})$.

[NCMG 2] Let $g \in G(x,y), g' \in G'(x)$. Then $g \bullet g' = g \circ g' \circ g^{-1} \in G'(y)$.

[NCMG 3] Let $g' \in \sigma(G')(x)$ and $g \in G(x)$. Then $(g' \bullet g) \circ g^{-1} = g' \circ g \circ g'^{-1} \circ g^{-1}$.

Since $\sigma(\mathcal{G}')$ is a normal subgroupoid of $\sigma(\mathcal{G})$, then $g \circ g'^{-1} \circ g^{-1} \in G'(x)$. Since $g' \in G'(x)$, it implies that $(g' \bullet g) \circ g^{-1} \in G'(x)$. □

Corollary 3.12. *Let $(\mathcal{G}, \sigma, \tau)$ be a cat$^{\mathbf{1}}$−groupoid and $(\mathcal{A}, \mathcal{B}, \partial)$ be the crossed module over groupoids corresponding to $\mathcal{G}$. Then the category* NC1GD/$(\mathcal{G}, \sigma, \tau)$ *of normal subcat$^{\mathbf{1}}$− groupoids of $(\mathcal{G}, \sigma, \tau)$ is equivalent to the category* NCMG/$(\mathcal{A}, \mathcal{B}, \partial)$ *of normal subcrossed modules of $(\mathcal{A}, \mathcal{B}, \partial)$.*

4 Crossed squares over groupoids and cat$^{\mathbf{2}}$−groupoids

In this section first we give the definition of crossed squares over groupoids as the groupoid case of crossed squares over groups. Then we define cat$^{\mathbf{2}}$-groupoids and prove that the category of cat$^{\mathbf{2}}$-groupoids is equivalent to the category of crossed squares over groupoids.

Definition 4.1. Let $\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}$ be groupoids over the same object set X and let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ be totally disconnected groupoids. A *crossed square* of groupoids is a commutative diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\lambda} & \mathcal{M} \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{N} & \xrightarrow[\nu]{} & \mathcal{P} \end{array}$$

together with groupoid morphisms $\lambda, \lambda', \mu, \nu$ which are identities on objects and actions of $\mathcal{P}$ on $\mathcal{L}, \mathcal{M}, \mathcal{N}$, (and therefore actions of $\mathcal{M}$ on $\mathcal{L}$ and $\mathcal{N}$ via μ and of $\mathcal{N}$ on $\mathcal{L}$ and $\mathcal{M}$ via ν) and a functor $h\colon \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ which is identity on X and satisfies the following conditions

[CSG 1] λ, λ' preserves the actions of P and $(\mathcal{M}, \mathcal{P}, \mu)$, $(\mathcal{N}, \mathcal{P}, \nu)$ and $(\mathcal{L}, \mathcal{P}, \kappa)$ are crossed modules over groupoids where $\kappa = \mu\lambda = \nu\lambda'$, [CSG 2] $\lambda h(m,n) = m \circ (n \bullet m^{-1}),\ \ \lambda' h(m,n) = (m \bullet n) \circ n^{-1}$,

[CSG 3] $h(\lambda(l),n) = l \circ (n \bullet l^{-1}),\ \ h(m,\lambda'(l)) = (m \bullet l) \circ l^{-1}$,

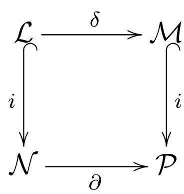
[CSG 4] $h(m \circ m', n) = \big(m \bullet h(m',n)\big) \circ h(m,n)$,
$h(m, n \circ n') = h(m,n) \circ (n \bullet h(m,n'))$,

[CSG 5] $h(p \bullet m, p \bullet n) = p \bullet h(m,n)$,

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$, whenever all compositions and actions are defined.

Using the definition of normal and wide subcrossed module over groupoids as defined in [20] and [24], we give following example.

Example 4.2. Let $(\mathcal{N}, \mathcal{P}, \partial)$ be a crossed module over groupoids and $(\mathcal{L}, \mathcal{M}, \delta)$ be a normal and wide subcrossed module of $(\mathcal{N}, \mathcal{P}, \partial)$ such that $\mathcal{M}$ is totally disconnected. Then the diagram

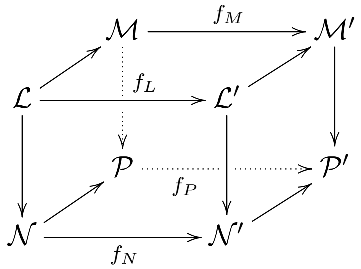


forms a crossed square of groupoids where the action of $\mathcal{P}$ on $\mathcal{L}$ is induced action from the action of $\mathcal{P}$ on $\mathcal{N}$ and the action of $\mathcal{P}$ on $\mathcal{M}$ is conjugation. Here the morphism h is defined on morphisms by

$$h(m,n) = (m \bullet n) \circ n^{-1}$$

for all $m \in M$ and $n \in N$.

Definition 4.3. A *morphism* $f = (f_L, f_M, f_N, f_P)\colon (\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}) \to (\mathcal{L}', \mathcal{M}', \mathcal{N}', \mathcal{P}')$ of crossed squares over groupoids consists of morphisms $f_L\colon \mathcal{L} \to \mathcal{L}'$, $f_M\colon \mathcal{M} \to \mathcal{M}'$, $f_N\colon \mathcal{N} \to \mathcal{N}'$, $f_P\colon \mathcal{P} \to \mathcal{P}'$ morphisms of groupoids which are identities on objects and compatible with the actions and the functors h and h'.



Then we construct the category CSG of crossed squares over groupoids.

Definition 4.4. Let $\mathcal{G} = (X, G)$ be a groupoid, $\sigma_i, \tau_i \colon \mathcal{G} \to \mathcal{G}$ be functors which are identities on objects. A *cat*$^{\mathbf{2}}$*-groupoid* $(\mathcal{G}, \sigma_i, \tau_i)$ is a groupoid satisfying

[C2Gd 1] $\sigma_i \tau_i = \tau_i$, $\tau_i \sigma_i = \sigma_i$,

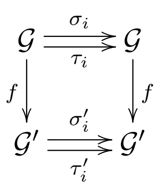
[C2Gd 2] $\sigma_i \sigma_j = \sigma_j \sigma_i$, $\tau_i \tau_j = \tau_j \tau_i$, $\sigma_i \tau_j = \tau_j \sigma_i$,

[C2Gd 3] $h_i \circ k_i \circ h_i^{-1} \circ k_i^{-1} = \varepsilon d_0(h_i)$, for all $h_i \in \mathrm{Ker}(\sigma_i), k_i \in \mathrm{Ker}(\tau_i)$ $i, j \in \{1, 2\}$ and $i \neq j$ where $d_0(h_i) = d_0(k_i)$.

Example 4.5. Let (G, s_1, t_1, s_2, t_2) be a cat$^{\mathbf{2}}$-group and X be a set. Using the trivial groupoid $\mathcal{G} = (X, X \times G \times X)$ we get a cat$^{\mathbf{2}}$-groupoid $(\mathcal{G}, \sigma_i, \tau_i)$ where $\sigma_i(x, g, y) = (x, s_i(g), y)$ and $\tau_i(x, g, y) = (x, t_i(g), y)$, for $i \in \{1, 2\}$.

Proposition 4.6. *Given any cat*$^{\mathbf{2}}$*-groupoid* $(G, \sigma_1, \tau_1, \sigma_2, \tau_2)$*, we have*
(i) $\sigma_i(G) = \tau_i(G)$,
(ii) σ_i *and* τ_i *are identities on* $\sigma_i(G)$ *and* $\tau_i(G)$,
(iii) $\sigma_i^2 = \sigma_i$ *and* $\tau_i^2 = \tau_i$*, for* $i \in \{1, 2\}$.

Definition 4.7. A *morphism* $f \colon (G, \sigma_1, \tau_1, \sigma_2, \tau_2) \to (G', \sigma'_1, t'_1, \sigma'_2, t'_2)$ of cat$^{\mathbf{2}}$-groupoids is a morphism of groupoids such that $\sigma'_i f = f \sigma_i$ and $\tau'_i f = f \tau_i$, for $i \in \{1, 2\}$.



Then we have the category CAT^2-GPD of cat$^{\mathbf{2}}$-groupoids.

Theorem 4.8. *The category of cat*$^{\mathbf{2}}$*-groupoids is equivalent to the category of crossed squares over groupoids.*

Proof. A functor $\psi\colon$ CSG $\to$ CAT^2-GPD can be defined by

$$\psi(\mathcal{L},\mathcal{M},\mathcal{N},\mathcal{P}) = (\mathcal{G},\sigma_1,\tau_1,\sigma_2,\tau_2)$$

to construct a cat^2−groupoid from a crossed square $(\mathcal{L},\mathcal{M},\mathcal{N},\mathcal{P})$ of groupoids. First, here are semi-direct products $P \ltimes M$ and $N \ltimes L$ and then an action of $P \ltimes M$ on $N \ltimes L$ is defined by

$$(p,m)\bullet(n,l) = (p\bullet n,(m\bullet(p\bullet l))\circ h(m,p\bullet n))$$

where

$$x \underset{l}{\overset{n}{\rightrightarrows}} x \xrightarrow{\;p\;} y \xrightarrow{\;m\;} y.$$

Hence the set of objects of $\mathcal{G}$ is the same set of objects of $\mathcal{L},\mathcal{M},\mathcal{N}$ and $\mathcal{P}$ and the set of morphisms of $\mathcal{G}$ is $(P \ltimes M)\ltimes(N \ltimes L)$. If

$$x \xrightarrow{\;p\;} y \xrightarrow{\;m\;} y \underset{l}{\overset{n}{\rightrightarrows}} y,$$

then (p,m,n,l) is a morphism of $\mathcal{G}$ from x to y where the structure maps are defined by

$$\sigma_1(p,m,n,l) = (p,m,1_y,1_y),$$

$$\sigma_2(p,m,n,l) = (p,1_y,n,1_y),$$

$$\tau_1(p,m,n,l) = (\nu(n)\circ p,\lambda(l)\circ(\nu(n)\bullet m),1_y,1_y),$$

$$\tau_2(p,m,n,l) = (\mu(m)\circ p,1_y,\lambda'(l)\circ n,1_y).$$

Let $y \xrightarrow{\;p'\;} z \xrightarrow{\;m'\;} z \underset{l'}{\overset{n'}{\rightrightarrows}} z$. Then composite of (p,m,n,l) and (p',m',n',l') is defined by

$$(p',m',n',l')\circ(p,m,n,l) = \Big((p',m')\circ(p,m),(n',l')\circ((p',m')\bullet(n,l))\Big)$$

Given a cat^2−groupoid $(\mathcal{G},\sigma_1,\tau_1,\sigma_2,\tau_2)$ we obtain a crossed square of groupoids via the functor $\gamma\colon$ CAT^2-GPD $\to$ CSG, $\gamma(\mathcal{G}) = (\mathcal{L},\mathcal{M},\mathcal{N},\mathcal{P})$ as a weak inverse for ψ where the sets of morphisms $L = \mathrm{Ker}(\sigma_1) \cap \mathrm{Ker}(\sigma_2)$, $M = \sigma_1(G) \cap \mathrm{Ker}(\sigma_2)$, $N = \mathrm{Ker}(\sigma_1) \cap \sigma_2(G)$, $P = \sigma_1(G) \cap \sigma_2(G)$ and restrictions $\lambda = \tau_1|_{\mathcal{L}}$, $\lambda' = \tau_2|_{\mathcal{L}}$, $\mu = \tau_2|_{\mathcal{M}}$ and $\nu = \tau_1|_{\mathcal{N}}$. The functor $h\colon \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ is defined by $h(m,n) = m \circ n \circ m^{-1} \circ n^{-1}$. Since $\tau_1\tau_2 = \tau_2\tau_1$, we have $\mu\lambda = \nu\lambda'$. The other axioms are easily satisfied where all the actions are defined by conjugation.

A natural equivalence $S\colon 1_{\mathrm{CAT}^2\text{-}\mathrm{GPD}} \to \psi\gamma$ is defined by a mapping

$$S_{\mathcal{G}}(\mathcal{G}, \sigma_1, \tau_1, \sigma_2, \tau_2) = \left(\mathcal{G}', \sigma_1', \tau_1', \sigma_2', \tau_2'\right)$$

which is defined to be identity on objects, on morphisms is given by

$$S_{\mathcal{G}}(g) = (\sigma_1\sigma_2(g), \sigma_1(g) \circ \sigma_1\sigma_2(g^{-1}), \sigma_2(g) \circ \sigma_1\sigma_2(g^{-1}), g \circ \sigma_1(g^{-1}) \circ \sigma_1\sigma_2(g) \circ \sigma_2(g^{-1}))$$

where

$$\sigma_1'(g_1, h_1, g_2, h_2) = (g_1, h_1, \varepsilon d_0(g_2), \varepsilon d_0(h_2)),$$

$$\sigma_2'(g_1, h_1, g_2, h_2) = (g_1, \varepsilon d_0(h_1), g_2, \varepsilon d_0(h_2)),$$

$$\tau_1'(g_1, h_1, g_2, h_2) = (\tau_1(g_2) \circ g_1, \tau_1(h_2 \circ g_2) \circ h_1 \circ \tau_1(g_2^{-1}), \varepsilon d_0(g_2), \varepsilon d_0(h_2)),$$

$$\tau_2'(g_1, h_1, g_2, h_2) = (\tau_2(h_1) \circ g_1, \varepsilon d_0(h_1), \tau_2(h_2) \circ g_2, \varepsilon d_0(h_2)).$$

On the other hand, a natural equivalence $T\colon 1_{\mathrm{CSG}} \to \gamma\psi$ is defined such that

$$T_{\mathcal{K}}(p) = (p, \varepsilon d_1(p), \varepsilon d_1(p), \varepsilon d_1(p)),$$

$$T_{\mathcal{K}}(m) = (\varepsilon d_1(m), m, \varepsilon d_1(m), \varepsilon d_1(m)),$$

$$T_{\mathcal{K}}(n) = (\varepsilon d_1(n), \varepsilon d_1(n), n, \varepsilon d_1(n)),$$

$$T_{\mathcal{K}}(l) = (\varepsilon d_1(l), \varepsilon d_1(l), \varepsilon d_1(l), l)$$

for $\mathcal{K} = (\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P})$. $\square$

5 Conclusion

There is need to investigate existence of epimorphisms and central extensions in the category of $\text{cat}^{\mathbf{1}}-$groupoids. Using the results of the paper [24], it could be possible to develop qoutient notions of $\text{cat}^{\mathbf{1}}-$groupoids. As an application of the equivalence given in [5, Theorem 1], the notions in one of these categories were interpreted in the other such as actor [12], normality, quotients [22], covering [1] and action [21]. So it would be interesting to explore similar notions in the categories introduced in this paper.

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