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The symmetric monoidal closed category of cpo M-sets

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Abstract. In this paper, we show that the category of directed complete posets with bottom elements (cpos) endowed with an action of a monoid M on them forms a monoidal category. It is also proved that this category is symmetric closed.

1 Introduction and preliminaries

The category **Dcpo** of directed complete partially ordered sets plays an important role in theoretical computer science, discrete mathematics, and specially in Domain Theory (see [1]). This category is complete, cocomplete, and closed (see [1, 7, 8]). It has also been shown that the category **Cpo** of directed complete partially ordered sets with bottom elements and strict continuous maps between them is monoidal closed, complete, and cocomplete (see [1, 8]).

In [10], we have studied the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ of cpo-acts; that is

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cpos with an action of a monoid M on them. In other words, we study M-sets in the category **Cpo**. Among other things, we have shown that this category is not cartesian closed. In this paper, we show that the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is a symmetric monoidal closed category. In fact, it is well-know that the category **Cpo** is a symmetric monoidal category. Hence and since $\mathbf{Cpo}_{\mathbf{Act}\text{-}M} \cong \mathbf{Cpo}^{M^{op}}$, it follows that the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is a symmetric monoidal category. Furthermore, we can not deduce the closedness of $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ from the closedness of **Cpo**. Therefore, in the final section we prove that this category is closed. Because of the constructive proofs and descriptions through our manuscript, and the important role of domain theory in denotational semantics, we think that our results would be useful and interesting for theoretical computer scientists, as well as for algebraists and order theorists. More precisely, the action of a monoid on a set would always be an important concept for computer scientists, where they use this concept in automata theory. Moreover, the subject of finding a mathematical model for programming languages would also be an interesting and helpful tools for computer scientists. From this point of view, the domain theory was introduced as a mathematical model for semantics of programming languages [1]. Furthermore, there are many models for semantics of the programming language PCF (Programming Computable Functionals) which one of them is Domain models, that is, cpos with a family of actions of the natural numbers, where such cpos called SFP in the contexts, see [17]. By knowing SFP cpos, one can have, for example, any finite cpo $\mathbb{N}$-act with the identity actions, as a SFP cpo, where $\mathbb{N}$ is considered with the binary operation min. On the other hands, an important problem in domain theory is the modelling of non-deterministic features of programming languages. There have been found some models in the literature, see, for example, [14–16]. In fact, to find such models, Plotkin and Smyth introduced the concept of a powerdomain, that is, a subset of a cpo, see [18]. In fact, they use the concept of a *d-cone* which is a commutative dcpo-monoid (in the sense of [11]) with an action of the monoid $\mathbb{R}^+$, of positive real numbers.

In the following we give some preliminaries needed in the sequel.

Dcpos and cpos. First of all, we recall some basic concepts of posets, dcpos, and cpos. For more information one can see [1, 4, 7, 8].

A *partially ordered set* (or a *poset*, for short) is a pair $(A, \leq)$, where A is a set and $\leq$ is a binary relation on A which is also reflexive, antisymmetric, and transitive.

Let $(A, \leq)$ be a poset and $S \subseteq A$. An element $a \in A$ is said to be an *upper bound* of S if $s \leq a$ for each $s \in S$. Moreover, it is said to be the *supremum* or the *join* of S, denoted by $\bigvee S$, if it is an upper bound of S and $a \leq b$ for each upper bound b of S.

A non-empty subset D of a partially ordered set $(A, \leq)$ is called *directed*, denoted by $D \subseteq^d A$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and A is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d A$, the supremum of D, denoted by $\bigvee^d D$ (read the directed join of D), exists in A. A dcpo which has a bottom (least) element $\perp$ is said to be a *cpo*.

A *dcpo map* or a *continuous map* $f\colon A \to B$ between dcpos is a map with the property that for every $D \subseteq^d A$, $f(D)$ is a directed subset of B and $f(\bigvee^d D) = \bigvee^d f(D)$. A dcpo map $f\colon A \to B$ between cpos is called *strict* if $f(\perp) = \perp$. Thus we have the categories **Dcpo** and **Cpo**, of all dcpos and cpos with (strict) continuous maps between them, respectively.

The following lemmas are frequently used in this paper.

Lemma 1.1. [4, 8] *Let $\{A_i \colon i \in I\}$ be a family of dcpos. Then the directed join of a directed subset $D \subseteq^d \prod_{i\in I} A_i$ is calculated as $\bigvee^d D = (\bigvee^d D_i)_{i\in I}$, where*

$$D_i = \{a \in A_i \colon \exists d = (d_k)_{k\in I} \in D, a = d_i\}$$

for all $i \in I$.

Lemma 1.2. [10] *Let A be a dcpo. Then $D \subseteq A_\perp = \perp \oplus A$ is directed if and only if $D \subseteq^d A$ or $D = \{\perp\} \cup D'$, where $D' = \emptyset$ or $D' \subseteq^d A$.*

M-sets and cpo M-sets. Now, we recall the preliminary notions of the action of a monoid. For more information, see [6, 9, 10, 12].

A *monoid* is a triple $(M, *, 1)$, where M is a set, $*$ is an associative binary operation on M, and 1 is an element of M called its *identity element* with the property that $m * 1 = m = 1 * m$, for all $m \in M$. From now on, whenever there is no confusion, we will write M for $(M, *, 1)$ and also write $m * n$ simply as mn.

Let M be a monoid with the identity 1. An *M-set* (or *M-act*) is a pair $(A; (\lambda_m)_{m\in M})$ where A is a set and for each $m \in M$, $\lambda_m \colon A \to A$, $\lambda_m(a) :=$

am is a map, called *an action*, such that $\lambda_1 = id_A$ and $\lambda_m \circ \lambda_n = \lambda_{nm}$, for all $m, n \in M$. That is, $a(nm) = (an)m, a1 = a$, for all $a \in A$. A map $f\colon A \to B$ between M-sets A and B is said to be *action-preserving* or an *M-set map*, if $f(am) = f(a)m$ for all $a \in A$ and $m \in M$. The category of all M-sets with action-preserving maps between them is denoted by **Act**-M.

Also, we recall from [10] that a *cpo M-act* is an M-act in the category **Cpo**. In other words, an M-set $(A; (\lambda_m)_{m\in M})$ is called a *cpo M-act* if A is a cpo and $\lambda_m\colon A \to A$ is a strict continuous map, for each $m \in M$.

Also, by a *cpo M-set map* between cpo M-sets, we mean a strict continuous map which is also an M-set map. We denote the category of all cpo M-sets and cpo M-set maps between them by $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$. Recall (see [10]) that this category is both complete and cocomplete.

Category Theory. Now, we recall from [2] the definitions of category and functor, for those who are not familiar with the subject. A *category* $\mathcal{A}$ consists of a class, also denoted by $\mathcal{A}$, whose elements will be called *objects* of the category and for every pair A, B of objects, a set $\mathcal{A}(A, B)$, whose elements will be called *morphisms* or *arrows* from A to B, and also for every triple A, B, C of objects, there exists a composition law $\mathcal{A}(A, B) \times \mathcal{A}(B, C) \to \mathcal{A}(B, C)$, the composite of the pair (f, g) will be written $g \circ f$ or just gf, which also satisfies the associativity axiom, that is, for arbitrary morphisms $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, $h \in \mathcal{A}(C, D)$, the equality $h\circ(g\circ f) = (h \circ g) \circ f$ holds. Moreover, for every object A there exists a morphism $\mathrm{Id}_A \in \mathcal{A}(A, A)$, called the *identity* on A, which satisfies the usual identity axiom, that is, for every pair of morphisms $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ the equalities $\mathrm{Id}_B \circ f = f$ and $g \circ \mathrm{Id}_B = g$ hold.

Also, a *functor* F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of a mapping $\mathcal{A} \to \mathcal{B}$ between the classes of objects of $\mathcal{A}$ and $\mathcal{B}$; the image of $A \in \mathcal{A}$ is written FA, and a mapping $\mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$, for every pair of objects A, A' of $\mathcal{A}$; the image of $f \in \mathcal{A}(A, A')$ is written Ff. Moreover, F must preserve the monoid structure on arrows, that is, for every pair of morphisms $f \in \mathcal{A}(A, A')$, $g \in \mathcal{A}(A', A'')$, $F(g \circ f) = Ff \circ Fg$, and for every object $A \in \mathcal{A}$, $F(\mathrm{Id}_A) = \mathrm{Id}_{FA}$.

Monoidal closed category. Finally we recall the definition of a monoidal category from [3]. A *monoidal category* $\mathcal{A}$ is a category together with a bifunctor $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(A, B) \mapsto A \otimes B$, called the *tensor product*, an object $\mathrm{I} \in \mathcal{A}$, and three natural isomorphisms $a_{ABC}\colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $r_A\colon A \otimes \mathrm{I} \cong A$, $l_A\colon \mathrm{I} \otimes A \cong A$, for all $A, B, C \in \mathrm{Obj}(\mathcal{A})$, satisfying the usual coherence axioms for a monoidal category (see axioms 4-5 in Definition 6.1.1 of [3]). If, furthermore, both $- \otimes A$ and $A \otimes -$ have right adjoints for each $A \in \mathcal{A}$, then $\mathcal{A}$ is called a *biclosed category*. A monoidal category $(\mathcal{A}, \otimes, \mathrm{I}, (l_A)_{A\in\mathrm{Obj}(\mathcal{A})}, (r_A)_{A\in\mathrm{Obj}(\mathcal{A})})$ is *symmetric* if, for each pair $A, B \in \mathrm{Obj}(\mathcal{A})$, there exists a natural isomorphism $\delta_{AB}\colon A \otimes B \cong B \otimes A$ satisfying the coherence axioms (see 2-4 in Definition 6.1.2 of [3]). A biclosed symmetric monoidal category is called a *symmetric monoidal closed category*.

2 The category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is a symmetric monoidal category

In this section, using the categorical properties of the category **Cpo**, we show that the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is a symmetric monoidal category.

First, we notice the following lemma with a sketch of its proof.

Lemma 2.1. *The category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is isomorphic to the functor category $\mathbf{Cpo}^{M^{op}}$, where M is considered as a one object category with elements of M as arrows.*

Proof. Define the functor $\Phi\colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}^{M^{op}}$ as follows. For a cpo M-set $(A; (\lambda_m)_{m\in M})$, define $\Phi(A)\colon M^{op} \to \mathbf{Cpo}$ to be the functor given by $M^{op} \rightsquigarrow A$ and $(m\colon M^{op} \to M^{op}) \rightsquigarrow (\Phi(m)\colon A \to A)$ with $\Phi(m)(a) = \lambda_m(a)$. Also, for each cpo M-set map $f\colon A \to B$, let $\Phi(f)\colon \Phi(A) \to \Phi(B)$ be the natural transformation whose only component is f.

Conversely, define the functor $\Psi\colon \mathbf{Cpo}^{M^{op}} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ as follows. Let, for each functor $F\colon M^{op} \to \mathbf{Cpo}$, $\Psi(F) = FM^{op}$ with the actions $\lambda_m := F(m)$ for $m \in M^{op}$. Moreover, for any natural transformation $\eta\colon F \to G$ in $\mathbf{Cpo}^{M^{op}}$, define $\Psi(\eta)$ to be the only component $\eta_{M^{op}}$ of η which is a strict continuous map. By the natural property of η, one can see that $\eta_{M^{op}}$ is also an M-set map, then so it is a cpo M-set map. Now, one can easily check that Φ and Ψ are actually functors and $\Phi \circ \Psi = \mathrm{Id}$, $\Psi \circ \Phi = \mathrm{Id}$. This proves the lemma. $\square$

Now, we recall the following two propositions.

Proposition 2.2. [5, 13] *Let $\mathcal{A}$ be a monoidal category and $\mathcal{B}$ any small category. Then the functor category $\mathcal{A}^{\mathcal{B}}$ is monoidal with the tensor product $F \otimes G$ given by $F \otimes G(B) = F(B) \otimes G(B)$, for every pair (F, G) of functors from $\mathcal{B}$ to $\mathcal{A}$ and every $B \in \mathrm{Obj}(\mathcal{B})$, and the constant functor $\mathrm{I} \colon \mathcal{B} \to \mathcal{A}$ as its unit which takes any object in $\mathcal{B}$ to the unit object I of $\mathcal{A}$.*

Proposition 2.3. [5] *Let $\mathcal{A}$ be a symmetric monoidal category and $\mathcal{B}$ any small category, then so is the functor category $\mathcal{A}^{B}$.*

Recall from [1] that the category **Cpo** is a symmetric monoidal category, in which the tensor product of two cpos A and B, which is also called *smash product*, is the cpo $A \otimes B = \perp \oplus (A \setminus \{\perp_A\}) \times (B \setminus \{\perp_B\})$. Furthermore, for two cpo maps $f \colon A \to B$ and $g \colon C \to D$, the tensor map $f \otimes g \colon A \otimes C \to B \otimes D$ defined by $(f \otimes g)(\perp) = \perp$ and $(f \otimes g)((a, c)) = (f(a), g(c))$ if $f(a) \neq \perp_B$ and $g(c) \neq \perp_D$, and otherwise $(f \otimes g)((a, c)) = \perp$ for all $(a, c) \in (A \setminus \{\perp_A\}) \times (C \setminus \{\perp_C\})$.

Therefore, applying Lemma 2.1 and Propositions 2.2, 2.3, we obtain the following theorem.

Theorem 2.4. *The category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is a symmetric monoidal category in which the tensor product of two cpo M-sets A and B is the cpo $A \otimes B = \perp \oplus (A \setminus \{\perp_A\}) \times (B \setminus \{\perp_B\})$ with the actions defined by*

$$(a, b) \cdot m = \begin{cases} (am, bm) & \text{if } am \neq \perp_A \text{ and } bm \neq \perp_B \\ \perp & \text{otherwise} \end{cases}$$

and $\perp \cdot m = \perp$, for all $m \in M$ and $(a, b) \in A \otimes B$. The two element chain $\mathrm{I} = \mathbf{2} = \{\perp, \top\}$ with the identity actions plays the role of the identity for tensor product in the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$. Moreover, the left and the right unit isomorphisms are given by the cpo M-act maps $l_A \colon \mathrm{I} \otimes A \to A$ defined by $(\top_{\mathrm{I}}, a) \mapsto a$, $\perp \mapsto \perp_A$ and $r_A \colon A \otimes \mathrm{I} \to A$ defined by $(a, \top_{\mathrm{I}}) \mapsto a$, $\perp \mapsto \perp_A$.

3 The closedness of the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$

In this section, we show that the symmetric category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is closed. First, we recall from [3] and [2] the following two propositions.

Proposition 3.1. [Prop. 6.1.4 of [3]] *A symmetric monoidal category $\mathcal{A}$ is closed if and only if, for each object $A \in \mathcal{A}$, the functor $A \otimes \text{-}\colon \mathcal{A} \to \mathcal{A}$ has a right adjoint.*

Proposition 3.2. [Prop. 3.2.4 of [2]] *Consider a functor $F\colon \mathcal{A} \to \mathcal{B}$ with a left adjoint $G\colon \mathcal{B} \to \mathcal{A}$. If $\mathcal{C}$ is any small category, then $G_*\colon \mathcal{B}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{C}}$ is itself a left adjoint to $F_*\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{B}^{\mathcal{C}}$, where $F_*\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{B}^{\mathcal{C}},\ H \mapsto F \circ H$.*

We recall from [1] that the functor $A \otimes \text{-}\colon \mathbf{Cpo} \to \mathbf{Cpo}$ is the left adjoint to the functor $(\text{-})^A\colon \mathbf{Cpo} \to \mathbf{Cpo}$, where for a cpo B, B^A denotes the set of all strict continuous maps from A to B. Hence, and by Proposition 3.1, the symmetric monoidal category $\mathbf{Cpo}$ is closed. Furthermore, by the above Proposition, $(A \otimes \text{-})_*\colon \mathbf{Cpo}^{M^{op}} \to \mathbf{Cpo}^{M^{op}}$ is a left adjoint to the functor $((\text{-})^A)_*\colon \mathbf{Cpo}^{M^{op}} \to \mathbf{Cpo}^{M^{op}}$.

Notice that, $(A \otimes \text{-})_*\colon \mathbf{Cpo}^{M^{op}} \to \mathbf{Cpo}^{M^{op}}$, $H \mapsto (A \otimes \text{-}) \circ H$, where

$$(A \otimes \text{-}) \circ H\colon M^{op} \to \mathbf{Cpo}$$

$$\begin{tikzcd}
M^{op} \arrow[r, mapsto] \arrow[d, "m"'] & A \otimes HM^{op} \arrow[d, "id_A \otimes H_m"] \\
M^{op} \arrow[r, mapsto] & A \otimes HM^{op}
\end{tikzcd}$$

and $H_m\colon HM^{op} \to HM^{op},\ x \mapsto H_m(x) := xm$ are actions and

$$id_A \otimes H_m((a,b)) = \begin{cases} (a, bm) & \text{if } am \neq \perp_{HM^{op}} \\ \perp & \text{otherwise} \end{cases}$$

also $id_A \otimes H_m(\perp) = \perp$, for all $m \in M$ and $(a,b) \in A \otimes HM^{op}$.

Next, consider the functor $(\text{-})^A_*\colon \mathbf{Cpo}^{M^{op}} \to \mathbf{Cpo}^{M^{op}}$, $H \mapsto (\text{-})^A_* \circ H$, where

$$(\text{-})^A_* \circ H\colon M^{op} \to \mathbf{Cpo}$$

$$\begin{tikzcd}
M^{op} \arrow[r, mapsto] \arrow[d, "m"'] & (HM^{op})^A \arrow[d, "{H_m}^A"] \\
M^{op} \arrow[r, mapsto] & (HM^{op})^A
\end{tikzcd}$$

and $H_m\colon HM^{op} \to HM^{op},\ x \mapsto H_m(x) := xm$, are actions and ${H_m}^A\colon (HM^{op})^A \to (HM^{op})^A$ given by ${H_m}^A(f) = H_m \circ f$, for all $f \in$

$(HM^{op})^A$, where $H_m \circ f \colon A \to (HM^{op})$ defined by $(H_m \circ f)(a) = H_m(f(a)) := (f(a))m$, for all $a \in A$.

In other words, we have an adjoint pair

$$(A \otimes \text{-})_* \colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$$

and

$$(\text{-})^A_* \colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M},$$

where for any cpo M-set B, $(A \otimes \text{-})_*(B) = A \otimes B$ is a cpo M-set with the action

$$(a, b) \cdot m = \begin{cases} (a, bm) & \text{if} \quad am \neq \perp_B \\ \perp & \text{otherwise} \end{cases}$$

and $\perp \cdot m = \perp$, for all $m \in M$ and $(a, b) \in A \otimes B$. Also, for any cpo M-set B, the cpo M-set $(\text{-})^A_*(B) = B^A = \mathrm{Hom}_{\mathbf{Cpo}}(A, B)$ with the action defined by $(fm)(a) = f(a)m$, for all $f \in B^A$ and $m \in M$. Consequently,

(i) for any cpo M-set A with the trivial actions, the functor $(A \otimes \text{-})_*$ is the same as the functor $(A \otimes \text{-})$ given in Theorem 2.4, and the functor $(\text{-})^A_*$ is its right adjoint.

(ii) for a cpo M-set A with a non-trivial actions the functor

$$(A \otimes \text{-})_* \colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$$

is different from the functor $A \otimes \text{-} \colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ given in Theorem 2.4. In Fact, while both of these functors take a cpo M-set B to the cpo $A \otimes B$, the actions defined on it are different.

In the following, unlike the above remark, we find a right adjoint for the functor $A \otimes \text{-} \colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ given in Theorem 2.4, for a general cpo M-set A. This proves that the symmetric monoidal category $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ is closed.

Theorem 3.3. *For any cpo M-set A, $\mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, \text{-})$ is an endofunctor on $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$.*

Proof. First notice that the $(M \times (A \setminus \{\perp_A\}))_\perp$ with the pointwise order and action and the zero element $\perp$ is a cpo M-set (see [10]). Now, we show that for a cpo M-set B, $\mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$ with the pointwise order and the actions defined by $(fm)(n, a) = f(mn, a)$ and $(fm)(\perp) = \perp_B$, for all $(n, a) \in M \times (A \setminus \{\perp_A\})$, $m \in M$ and $f \in \mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times$

$(A \setminus \{\perp_A\}))_\perp, B)$, is a cpo M-set. It is a cpo with the bottom element $f_\perp\colon (M \times (A \setminus \{\perp_A\}))_\perp \to B$, $x \mapsto \perp_B$. In fact, for each directed subset $F \subseteq^d \mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$, $\bigvee^d F$ is calculated pointwise. Recall from [1, 12] that $\bigvee^d F$ is a cpo map. Furthermore, $\bigvee^d F$ is action-preserving. To see this, let $m \in M$ and $(n, a) \in M \times (A \setminus \{\perp_A\})$. Then

$$\begin{aligned}
&\bigvee^d F((n,a)\cdot m) = \bigvee^d_{f\in F} f((n,a)\cdot m) = \bigvee^d_{f\in F} (f(n,a))m \\
&= (\bigvee^d_{f\in F} f(n,a))m = (\bigvee^d F(n,a))m,
\end{aligned}$$

where the third equality is true because B is a cpo M-set. Moreover,

$$\begin{aligned}
&\bigvee^d F(\perp\cdot m) = \bigvee^d F(\perp) = \bigvee^d_{f\in F} f(\perp) = \bigvee^d_{f\in F} \perp_B = \perp_B \\
&= (\bigvee^d_{f\in F} f(\perp))m = (\bigvee^d F(\perp))m.
\end{aligned}$$

Now, we show that the actions are strict continuous. The actions are strict, because

$$(f_\perp m)(n,a) = f_\perp(mn,a) = \perp_B = f_\perp(n,a)$$

for all $(n, a) \in M \times (A \setminus \{\perp_A\})$ and $m \in M$, also $(f_\perp m)(\perp) = \perp_B = f_\perp(\perp)$. To prove continuity, let $F \subseteq^d \mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$ and $m \in M$, then

$$((\bigvee^d F)m)(n,a) = (\bigvee^d F)(mn,a) = \bigvee^d_{f\in F} (f(mn,a)) = \bigvee^d_{f\in F} (fm)(n,a)$$

for all $(n, a) \in M \times (A \setminus \{\perp_A\})$, also

$$((\bigvee^d F)m)(\perp) = \perp_B = \bigvee^d_{f\in F} fm(\perp) = (\bigvee^d Fm)(\perp).$$

Consequently, $\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$ is a cpo M-set. $\square$

Theorem 3.4. *The functor* $A\otimes$-: $\mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ *is a left adjoint to the functor* $\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, \text{-})\colon \mathbf{Cpo}_{\mathbf{Act}\text{-}M} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}M}$.

Proof. We do this in four steps:
Step (I). Defining counit: We show that for every cpo M-set B, the map

$$\eta_B : A \otimes \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B) \to B$$

defined by $\eta_B(a,f) = f(1,a)$ and $\eta_B(\bot) = \bot_B$, for all $(a,f) \in A \otimes \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B)$, is a couniversal cpo M-set map from B to the functor $A \otimes -$. First, we show that η_B is a cpo M-set map. It is strict by its definition. It is also action-preserving. In fact, for every $m \in M$, $\eta_B(\bot \cdot m) = \eta_B(\bot) = \bot_B = \bot_B m = \eta_B(\bot)m$, and for $(a,g) \in A \otimes \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B)$, we consider two cases:

Case (1): If $am \neq \bot_A$ and $gm \neq f_\bot$, then

$$\begin{aligned}\eta_B((a,g)\cdot m) &= \eta_B(am, gm) = gm(1, am)\\ &= g(m, am) = g((1,a)\cdot m) = g((1,a))m = \eta_B(a,g)m.\end{aligned}$$

Case (2): If $am = \bot_A$ or $gm = f_\bot$, then

$$\eta_B((a,g)\cdot m) = \eta_B(\bot) = \bot_B = g((1,a)\cdot m) = g((1,a))m = \eta_B((a,g))m,$$

where the third equality is true because if $am = \bot_A$, then $g((1,a)\cdot m) = g(\bot) = \bot_B$. Also if $am \neq \bot_A$, then by hypothesis $gm = f_\bot$. Thus

$$g((1,a)\cdot m) = g((m, am)) = gm((1, am)) = f_\bot((1, am)) = \bot_B.$$

To prove continuity, let $D \subseteq^d A \otimes \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B)$. Then, by Lemma 1.2, we consider two cases:

Case (1): Let

$$D \subseteq^d (A \setminus \{\bot_A\}) \times (\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\})_\bot, B) \setminus \{f_\bot\}).$$

Then, by Lemma 1.1, $\bigvee^d D = (\bigvee^d D', \bigvee^d F)$, where $D' = \mathrm{Dom}D$ and $F = \mathrm{Cod}D$. So

$$\begin{aligned}\eta_B(\bigvee^d D) &= \eta_B((\bigvee^d D', \bigvee^d F)) = \bigvee^d F(1, \bigvee^d D') = \bigvee^d F(\bigvee^d_{x\in D'} (1,x))\\ &= \bigvee^d_{g\in F} \bigvee^d_{x\in D'} g(1,x) = \bigvee^d_{(x,g)\in D} g(1,x) = \bigvee^d_{(x,g)\in D} \eta_B((x,g)),\end{aligned}$$

where the forth equality is true because $\bigvee^d F$ is calculated pointwise and each $g \in F$ is continuous. Moreover, the fifth equality is true because $\bigvee^d_{(x,g)\in D} g(1,x) \leq \bigvee^d_{g\in F} \bigvee^d_{x\in D'} g(1,x)$; and if $g(1,x) \leq b$, for all $(x,g) \in D$ and some $b \in B$, then $\bigvee^d_{g\in F} \bigvee^d_{x\in D'} g(1,x) \leq b$. To see this, let $g \in F$ and $x \in D'$, then there exist $y \in A \setminus \{\perp_A\}$ and $h \in \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B) \setminus \{f_\perp\}$ such that $(y,g) \in D$ and $(x,h) \in D$. Since D is directed, there exists $(z,k) \in D$ with $(y,g),(x,h) \leq (z,k)$. This gives $y, x \leq z$ and $g, h \leq k$. So $g(1,x) \leq g(1,z) \leq k(1,z) \leq b$ for all $g \in F$ and $x \in D'$, as required.

Case (2): Let $D = D' \cup \{\perp\}$ where $D' \subseteq (A \setminus \{\perp_A\}) \times (\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\})_\perp, B)) \setminus \{f_\perp\})$ is directed. We also have $\bigvee^d D = \bigvee^d D'$ and then using the Case (1),

$$\eta_B(\bigvee^d D) = \eta_B(\bigvee^d D') = \bigvee^d_{x\in D'} \eta_B(x) = \bigvee^d_{x\in D'} \eta_B(x) \vee \eta_B(\perp) = \bigvee^d_{x\in D} \eta_B(x).$$

Therefore, η_B is a cpo M-set map.
Step (II). Universal property of the counit: Let C be a cpo M-set and $h\colon A \otimes C \to B$ be a cpo M-set map. Then the map $\hat{h}\colon C \to \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$, defined by

$$\hat{h}(x)(m,a) = \begin{cases} h(a, xm) & \text{if} \quad xm \neq \perp_C \\ \perp_B & \text{if} \quad xm = \perp_C \end{cases},$$

$\hat{h}(\perp_C) = f_\perp$, and $\hat{h}(x)(\perp) = \perp_B$ for $x \in C$, $a \in A \setminus \{\perp_A\}$, and $m \in M$, is a unique cpo M-set map satisfying $\eta_B \circ Id_A \otimes \hat{h} = h$. First, we see that $\hat{h}$ is well-defined. In fact, we show that $\hat{h}(x)$ is a cpo M-set map, for all $x \in C$. Notice that $\hat{h}(x)$ is strict by its definition. To prove continuity, let $D \subseteq^d (M \times (A \setminus \{\perp_A\}))_\perp$, then, by Lemma 1.2, we consider two cases:

Case (1): $D \subseteq^d M \times (A \setminus \{\perp_A\})$. Then $D = \{n\} \times D'$ where $D' \subseteq^d A$ and $n \in M$. Also $\bigvee^d D = (n, \bigvee^d D')$. Now we consider two subcases:

Subcase (1a): $xn \neq \perp_C$: Then

$$\begin{aligned}\hat{h}(x)(\bigvee^{d} D) &= \hat{h}(x)((n, \bigvee^{d} D')) = h((\bigvee^{d} D', xn)) \\ &= h(\bigvee^{d}_{y \in D'} (y, xn)) = \bigvee^{d}_{y \in D'} h(y, xn) \\ &= \bigvee^{d}_{y \in D'} \hat{h}(x)(n, y) = \bigvee^{d}_{(n,y) \in D} \hat{h}(x)(n, y).\end{aligned}$$

Subcase (1b): $xn = \perp_C$: Then

$$\hat{h}(x)(\bigvee^{d} D) = \hat{h}(x)((n, \bigvee^{d} D')) = \perp_B = \bigvee^{d}_{(n,y) \in D} \hat{h}(x)(n, y).$$

Case (2): $D = D' \cup \{\perp\}$ where $D' = \{n\} \times D''$ in which $D'' \subseteq^d A$ and $n \in M$. Then by Case (1),

$$\begin{aligned}\hat{h}(x)(\bigvee^{d} D) = \hat{h}(x)(\bigvee^{d} D') &= \bigvee^{d}_{(n,y) \in D'} \hat{h}(x)(n, y) \\ &= (\bigvee^{d}_{(n,y) \in D'} \hat{h}(x)(n, y)) \vee \perp_B \\ &= (\bigvee^{d}_{(n,y) \in D'} \hat{h}(x)(n, y)) \vee \hat{h}(x)(\perp) \\ &= \bigvee^{d}(\hat{h}(x)(D)).\end{aligned}$$

Now, we show that $\hat{h}(x)$ is action-preserving, for all $x \in C$. First notice that

$$\hat{h}(x)(\perp \cdot m) = \hat{h}(x)(\perp) = \perp_B = \perp_B m = (\hat{h}(x)(\perp))m.$$

Also for $(n, a) \in M \times (A \setminus \{\perp_A\})$ and $m \in M$, we consider two cases:

Case (1): $am \neq \perp_A$, then we consider two subcases:

Subcase $(1a')$: $xnm \neq \perp_C$, then

$$\hat{h}(x)((n, a) \cdot m) = \hat{h}(x)((nm, am)) = h(am, xnm) =$$

$$h((a, xn) \cdot m) = (h((a, xn)))m = (\hat{h}(x)((n, a)))m,$$

where the last equality is true because $xn \neq \perp_C$ (otherwise, we get $xnm = \perp_C m = \perp_C$, which is a contradiction).
Subcase $(1b')$: $xnm = \perp_C$: Then we consider two subcases:
(1) $xn = \perp_C$: Then

$$\hat{h}(x)((n, a) \cdot m) = \hat{h}(x)((nm, am)) = \perp_B = \perp_B m = (\hat{h}(x)(n, a))m.$$

(2) $xn \neq \perp_C$: Then

$$\hat{h}(x)((n, a) \cdot m) = \hat{h}(x)((nm, am)) = \perp_B = h(\perp) =$$

$$h((a, xn) \cdot m) = (h((a, xn)))m = (\hat{h}(x)((n, a)))m,$$

as required.

Case (2): $am = \perp_A$. Then $\hat{h}(x)((n, a) \cdot m) = \hat{h}(x)(\perp) = \perp_B$. Also $(\hat{h}(x)((n, a)))m = \perp_B$. In fact, if $xn = \perp_C$, then $(\hat{h}(x)((n, a)))m = \perp_B m = \perp_B$. If $xn \neq \perp_C$, then $(\hat{h}(x)((n, a)))m = (h(a, xn))m = h((a, xn) \cdot m) = h(\perp) = \perp_B$ (the third equality is true by the definition of actions on $A \otimes C$ and the fact $am = \perp_A$). Consequently, $\hat{h}(x)$ is action-preserving and so it is a cpo M-set map.

Now, we show that $\hat{h}$ is a cpo M-set map, first note that it is strict by its definition. To prove continuity, let $D \subseteq^d C$. Then $\hat{h}(\bigvee^d D) = \bigvee^d_{x \in D} \hat{h}(x)$, because $\hat{h}(\bigvee^d D)(\perp) = \perp_B = \bigvee^d_{x \in D} \hat{h}(x)(\perp) = (\bigvee^d_{x \in D} \hat{h}(x))(\perp)$, and for $(m, a) \in M \times (A \setminus \{\perp_A\})$ we consider two cases:

Case (1): $(\bigvee^d D)m = \perp_C$, then for all $x \in D$, $xm = \perp_C$. Thus

$$\hat{h}(\bigvee^d D)(m, a) = \perp_B = \bigvee^d_{x \in D} \hat{h}(x)(m, a) = (\bigvee^d_{x \in D} \hat{h}(x))(m, a).$$

Case (2): $(\bigvee^d D)m \neq \bot_C$. Take $K = \{x \in D \mid xm \neq \bot_C\}$, then

$$\begin{aligned}
\hat{h}(\bigvee^d D)(m,a) &= h(a,(\bigvee^d D)m) = h(a,\bigvee^d_{x\in D} xm)\\
&= h(a,\bigvee^d_{x\in K} xm) = \bigvee^d_{x\in K} h(a,xm)\\
&= \bigvee^d_{x\in K} \hat{h}(x)(m,a) = \bigvee^d_{x\in K} \hat{h}(x)(m,a) \vee \bot_B\\
&= \bigvee^d_{x\in K} \hat{h}(x)(m,a) \vee \bigvee^d_{x\in D\setminus K} \hat{h}(x)(m,a)\\
&= \bigvee^d_{x\in D} \hat{h}(x)(m,a) = (\bigvee^d_{x\in D} \hat{h}(x))(m,a).
\end{aligned}$$

Now, we show that $\hat{h}$ is action-preserving. To see this, let $x \in C$ and $m \in M$. We must prove that $\hat{h}(xm) = \hat{h}(x)m$. First notice that $\hat{h}(xm)(\bot) = \bot_B = (\hat{h}(x)m)(\bot)$ (the last equality is true by the definition of actions on $\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B))$, also for each $(n,a) \in M \times (A \setminus \{\bot_A\})$ we consider two cases:

Case (1): $xmn = \bot_C$, then

$$\hat{h}(xm)(n,a) = \bot_B = \hat{h}(x)(mn,a) = \hat{h}(x)m(n,a)$$

Case (2): $xmn \neq \bot_C$, then

$$\hat{h}(xm)(n,a) = h(a,xmn) = \hat{h}(x)(mn,a) = (\hat{h}(x)m)(n,a)$$

(the last equality is true by the definition of action on the cpo M-set $\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\bot_A\}))_\bot, B))$. Therefore, $\hat{h}$ is a cpo M-set map.

In this part, we show that $\eta_B \circ (id \otimes \hat{h}) = h$. First notice that

$$(\eta_B \circ (id \otimes \hat{h}))(\bot) = \eta_B((id \otimes \hat{h})(\bot)) = \eta_B(\bot) = \bot_B = h(\bot).$$

Also, for $(a,x) \in A \otimes C$, we consider two cases:

Case (1): $\hat{h}(x) \neq f_\perp$, then

$$(\eta_B \circ (id \otimes \hat{h}))((a,x)) = \eta_B((id \otimes \hat{h}))((a,x))$$
$$= \eta_B(a, \hat{h}(x)) = \hat{h}(x)(1,a) = h(a,x),$$

where the last equality is true because $x1 = x \neq \perp_C$.

Case (2): $\hat{h}(x) = f_\perp$, then

$$(\eta_B \circ (id \otimes \hat{h}))((a,x)) = \eta_B((id \otimes \hat{h}))((a,x)) = \eta_B(\perp) = \perp_B = h(a,x),$$

where the last equality is true because, we have $x \neq \perp_C$ and $\hat{h}(x)((1,a)) = \perp_B$. This gives $\hat{h}(x)((1,a)) = h(a,x1) = h(a,x) = \perp_B$.

Finally, we prove the uniqueness of $\hat{h}$. To see this, let $g\colon C \to \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$ be a cpo M-set map with $\eta_B \circ (id \otimes g) = h$. First notice that $g(\perp_C) = \perp = \hat{h}(\perp_C)$. We also show that $g(x) = \hat{h}(x)$, for all $x \in C \setminus \{\perp_C\}$. We have $g(x)(\perp) = \perp_B = \hat{h}(x)(\perp)$. Now for all $(m,a) \in M \times (A \setminus \{\perp_A\})$ we consider two cases:

Case (1): If $xm \neq \perp_C$, then we consider two subcases:
Subcase (1m): $g(xm) \neq f_\perp$, then

$$g(x)((m,a)) = (g(x)m)((1,a)) = g(xm)((1,a)) = \eta_B(a, g(xm)) =$$

$$\eta_B((id \otimes g)(a,xm)) = (\eta_B \circ (id \otimes g))(a,xm) = h(a,xm) = \hat{h}(x)((m,a)).$$

Subcase (1n): $g(xm) = f_\perp$, then $g(x)((m,a)) = (g(x)m)((1,a)) = f_\perp(1,a) = \perp_B$. Also we have $(\eta_B \circ (id \otimes g))((a,xm)) = \eta_B(id \otimes g((a,xm))) = h(a,xm)$, then $\perp_B = \eta_B(\perp) = h(a,xm)$. This gives $\hat{h}(x)((m,a)) = h(a,xm) = \perp_B$ (notice that $xm \neq \perp_C$), as required.

Case (2): If $xm = \perp_C$, then

$$g(x)((m,a)) = g(x)m((1,a)) = g(xm)((1,a))$$
$$= f_\perp((1,a)) = \perp_B = \hat{h}(x)((m,a)),$$

as required. $\square$

In the following, we see that, as one expects, for a cpo M-set A with the trivial actions, the two right adjoints $\mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, \text{-})$ and $(\text{-})^A_\star$ to the functor $A \otimes \text{-}$ are isomorphic.

Lemma 3.5. *Let A be a cpo M-set with the trivial action. Then*

$$\mathrm{Hom}_{\mathbf{Cpo}_{\mathbf{Act}\text{-}\mathrm{M}}}((M \times (A \setminus \{\perp_A\}))_\perp, B) \cong \mathrm{Hom}_{\mathbf{Cpo}}(A, B).$$

Proof. Define the map

$$\alpha\colon \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B) \to \mathrm{Hom}_{\mathbf{Cpo}}(A, B)$$

by $\alpha(f) = f_1$, where $f_1\colon A \to B$ is given by $f_1(a) = f(1, a)$ and $f_1(\perp_A) = \perp_B$, for all $f \in \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$ and $a \in A \setminus \{\perp_A\}$. First notice that, since f is a dcpo map from $M \times (A \setminus \{\perp_A\})$ to B, it is continuous in each variable (see [1]). This gives that f_1 is continuous and strict by its definition. Hence $f_1 \in \mathrm{Hom}_{\mathbf{Cpo}}(A, B)$ and α is well-defined. Now, we show that α is a cpo M-set map. To prove that it is action-preserving take $m \in M$ and $f \in \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$. Then we have $\alpha(fm)(a) = (fm)_1(a) = (fm)(1, a) = f(m, a) = f(m, am) = f(1, a)m = (f_1(a))m = (\alpha(f)(a))m = (\alpha(f)m)(a)$. This implies that $\alpha(fm) = \alpha(f)m$, for all $m \in M$ and $f \in \mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$, as required. Furthermore, α is continuous. To prove this, let F be a directed subset of $\mathrm{Hom}_{\mathbf{Cpo}}((M \times (A \setminus \{\perp_A\}))_\perp, B)$. Then

$$\alpha(\bigvee^d F)(\perp_A) = (\bigvee^d F)_1(\perp_A) = \perp_B = \bigvee^d_{f \in F} (f_1(\perp_A))$$

$$= (\bigvee^d_{f \in F} f_1)(\perp_A) = (\bigvee^d_{f \in F} \alpha(f))(\perp_A) = \bigvee^d \alpha(F)(\perp_A).$$

Also for all $\perp_A \neq a \in A$,

$$\alpha(\bigvee^d F)(a) = (\bigvee^d F)_1(a) = (\bigvee^d F)(1, a) = \bigvee^d_{f \in F} f(1, a) = \bigvee^d_{f \in F} (f_1(a))$$

$$= (\bigvee^d_{f \in F} f_1)(a) = (\bigvee^d_{f \in F} \alpha(f))(a) = \bigvee^d \alpha(F)(a).$$

This implies that $\alpha(\bigvee^d F) = \bigvee^d \alpha(F)$, as required. Consequently, α is a cpo M-set map.

Now, define the map

$$\beta\colon \mathrm{Hom}_{\mathbf{Cpo}}(A,B) \to \mathrm{Hom}_{\mathbf{Cpo}}((M\times(A\setminus\{\perp_A\}))_\perp, B)$$

by $\beta(f)=f_\star$, where $f_\star(\perp)=\perp_B$ and $f_\star(m,a)=f(a)m$, for all $m\in M$ and $a\in A\setminus\{\perp_A\}$. We show that β is a cpo M-set map. First notice that it is action-preserving. To see this, we show that $\beta(fm)=\beta(f)m$ or equivalently $(fm)_\star=f_\star m$. for all $m\in M$ and $f\in \mathrm{Hom}_{\mathbf{Cpo}}(A,B)$. First notice that $(fm)_\star(\perp)=\perp_B=(f_\star m)(\perp)$, where the last equality is true by the definition of action on $\mathrm{Hom}_{\mathbf{Cpo}}((M\times(A\setminus\{\perp_A\}))_\perp,B)$. Also, for all $t\in M$ and $a\in A\setminus\{\perp_A\}$, we have $(fm)_\star(t,a)=((fm)(a))t=(f(a)m)t=f(a)(mt)=f_\star(mt,a)=(f_\star m)(t,a)$. So $(fm)_\star=f_\star m$, as required. Moreover, β is trivially strict and also continuous. To see this, take the directed subset F of $\mathrm{Hom}_{\mathbf{Cpo}}(A,B)$. We show that $\beta(\bigvee^d F)=\bigvee^d \beta(F)$ or, equivalently, $(\bigvee^d F)_\star=\bigvee^d_{f\in F} f_\star$. First notice that $(\bigvee^d F)_\star(\perp)=\perp_B=\bigvee^d_{f\in F} f_\star(\perp)$. Also, for all $t\in M$ and $\perp_A\neq a\in A\setminus\{\perp_A\}$ we have $(\bigvee^d F)_\star(t,a)=((\bigvee^d F)(a))t=(\bigvee^d_{f\in F} f(a))t=\bigvee^d_{f\in F}(f(a)t)=\bigvee^d_{f\in F} f_\star(t,a)=(\bigvee^d_{f\in F} f_\star)(t,m)$. Hence we obtain $(\bigvee^d F)_\star=\bigvee^d_{f\in F} f_\star$, as required.

Finally, we prove $\alpha\circ\beta=\mathrm{Id}$ and $\beta\circ\alpha=\mathrm{Id}$. First notice that for all $f\in\mathrm{Hom}_{\mathbf{Cpo}}(A,B)$ and $a\in A\setminus\{\perp_A\}$ we have $(f_\star)_1(a)=f_\star(1,a)=f(a)1=f(a)$. This gives $(f_\star)_1=f$ or equivalently $\alpha(\beta(f))=\alpha(f_\star)=(f_\star)_1=f=\mathrm{Id}(f)$, for all $f\in\mathrm{Hom}_{\mathbf{Cpo}}(A,B)$, as required. Second, for all $f\in\mathrm{Hom}_{\mathbf{Cpo}}((M\times(A\setminus\{\perp_A\}))_\perp,B)$, we have $(f_1)_\star(\perp)=\perp_B=f(\perp)$. Also, for all $t\in M$ and $\perp_A\neq a\in A$, we have $(f_1)_\star(t,a)=(f_1(a))t=(f(1,a))m=f(m,am)=f(m,a)$. This gives $(f_1)_\star=f$ or equivalently $\beta(\alpha(f))=\beta(f_1)=(f_1)_\star=f=\mathrm{Id}(f)$, as required.

Consequently, α is a cpo M-set isomorphism and so we get

$$\mathrm{Hom}_{\mathbf{Cpo}}((M\times(A\setminus\{\perp_A\}))_\perp,B)\cong\mathrm{Hom}_{\mathbf{Cpo}}(A,B),$$

where the action on A is the trivial action. □

As a consequence of Theorems 2.4 and 3.4, we have

Theorem 3.6. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}M}$ *is a symmetric monoidal closed category.*

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