Abstract. In this paper, first we introduce the notions of an \((m, n)\)-hyperideal and a generalized \((m, n)\)-hyperideal in an ordered semihypergroup, and then, some properties of these hyperideals are studied. Thereafter, we characterize \((m, n)\)-regularity, \((m, 0)\)-regularity, and \((0, n)\)-regularity of an ordered semihypergroup in terms of its \((m, n)\)-hyperideals, \((m, 0)\)-hyperideals and \((0, n)\)-hyperideals, respectively. The relations \(mI, I_n, \mathcal{H}_m^n,\) and \(B_m^n\) on an ordered semihypergroup are, then, introduced. We prove that \(B_m^n \subseteq \mathcal{H}_m^n\) on an ordered semihypergroup and provide a condition under which equality holds in the above inclusion. We also show that the \((m, 0)\)-regularity \([(0, n)\)-regularity\] of an element induce the \((m, 0)\)-regularity \([(0, n)\)-regularity\] of the whole \(\mathcal{H}_m^n\)-class containing that element as well as the fact that \((m, n)\)-regularity and \((m, n)\)-right weakly regularity of an element induce the \((m, n)\)-regularity and \((m, n)\)-right weakly regularity of the whole \(B_m^n\)-class and \(\mathcal{H}_m^n\)-class containing that element, respectively.

* Corresponding author

Keywords: Ordered semihypergroups, \((m, n)\)-hyperideals, \((m, 0)\)-hyperideals, \((0, n)\)-hyperideals.

Mathematics Subject Classification [2010]: 20N20.

Received: 5 April 2019, Accepted: 12 June 2019.

ISSN: Print 2345-5853, Online 2345-5861.

© Shahid Beheshti University
1 Introduction and preliminaries

By an ordered semigroup, we mean an algebraic structure \((S, \cdot \leq)\), which satisfies the following conditions: (1) \(S\) is a semigroup with respect to the multiplication \(\cdot\); (2) \(S\) is a partially ordered set by \(\leq\); (3) if \(a\) and \(b\) are elements of \(S\) such that \(a \leq b\), then \(ac \leq bc\) and \(ca \leq cb\) for all \(c \in S\). Many authors, especially Alimov [1], Clifford [2–4], Hion [13], Conrad [5], and Kehayopulu [15] studied such semigroups with some restrictions.

In 1934, Marty [21] introduced the concept of a hyperstructure and defined hypergroup. Later on several authors studied hyperstructure in various algebraic structures such as rings, semirings, semigroups, ordered semigroups, \(\Gamma\)-semigroups and Ternary semigroups, etc. The concept of a semi-hypergroup is a generalization of the concept of a semigroup and many classical notions such as of ideals, quasi-ideals and bi-ideals defined in semigroups and regular semigroups have been generalized to semi-hypergroups (see [8, 9] for other related notions and results on semi-hypergroups). In [14], Heidari and Davvaz introduced the notion of an ordered semi-hypergroup as a generalization of the notion of an ordered semigroup. Davvaz et al. in [6, 7, 14, 22, 23, 25, 26] studied some properties of hyperideals and bi-hyperideals in ordered semi-hypergroups. Lajos [16] introduced the concept of \((m, n)\)-ideals in semigroups (see also [17–19]). In [12], the authors defined the notion of an \((m, n)\)-quasi-hyperideal in a semi-hypergroup and investigated several properties of these \((m, n)\)-quasi-hyperideals.

A hyperoperation on a non-empty set \(H\) is a map \(\circ : H \times H \to \mathcal{P}^*(H)\) where \(\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}\) (the set of all non-empty subsets of \(H\)). In such a case, \(H\) is called a hypergroupoid. Let \(H\) be a hypergroupoid and \(A, B\) be any non-empty subsets of \(H\). Then

\[
A \circ B = \bigcup_{a \in A, b \in B} a \circ b.
\]

We shall write, in whatever follows, \(A \circ x\) instead of \(A \circ \{x\}\) and \(x \circ A\) instead of \(\{x\} \circ A\), for any \(x \in H\). Also, for simplicity, throughout the paper, we shall write \(A^n\) for \(A \circ A \circ \cdots \circ A\) (\(n\) – copies of \(A\)) for any \(n \in \mathbb{Z}^+\). Also the integers \(m, n\) will stand for positive integers throughout the paper until and unless otherwise specified. Moreover, the hypergroupoid \(H\) is called a semi-hypergroup if, for all \(x, y, z \in H\),

\[
(x \circ y) \circ z = x \circ (y \circ z)
\]
that is,
\[ \bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v. \]

A non-empty subset \( T \) of a semihypergroup \( H \) is called a \textit{subsemihypergroup} of \( H \) if \( T \circ T \subseteq T \).

Let \( H \) be a non-empty set, the triplet \((H, \circ, \leq)\) is called an \textit{ordered semihypergroup} if \((H, \circ)\) is a semihypergroup and \((H, \leq)\) is a partially ordered set such that
\[ x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y \]
for all \( x, y, z \in H \). Here, if \( A \) and \( B \) are non-empty subsets of \( H \), then we say that \( A \leq B \) if for every \( a \in A \) there exists \( b \in B \) such that \( a \leq b \).

Let \( H \) be an ordered semihypergroup. For a non-empty subset \( A \) of \( H \), we denote \((A) = \{x \in H \mid x \leq a \text{ for some } a \in A \}\). A non-empty subset \( A \) of \( H \) is called \textit{idempotent} if \( A = (A \circ A) \). A non-empty subset \( A \) of \( H \) is called \textit{left (right)-hyperideal} [7] of \( H \) if \( H \circ A \subseteq A \) \((A \circ H \subseteq A)\) and \((A) \subseteq A \). A non-empty subset \( J \) of \( H \) is called a \textit{hyperideal} of \( H \) if \( J \) is both a left hyperideal and a right hyperideal of \( H \). A subsemihypergroup (non-empty subset) \( B \) of an ordered semihypergroup \( H \) is called a \textit{bi-hyperideal (generalized bi-hyperideal)} of \( H \) if \( B \circ H \circ B \subseteq B \) \((B \circ B \subseteq B)\). An ordered semihypergroup \( H \) is called \textit{regular (left-regular, right-regular)} [7] if for each \( x \in H \), \( x \in (x \circ H \circ x) \)(\( x \in (H \circ x \circ x) \)), \( x \in (x \circ x \circ H) \).

\textbf{Lemma 1.1.} [7] Let \( H \) be an ordered semihypergroup and \( A, B \) be any non-empty subsets of \( H \). Then the following conditions hold:

\begin{enumerate}
  \item \( A \subseteq (A); \)
  \item \( A \subseteq B \Rightarrow (A) \subseteq (B); \)
  \item \( (A) \circ (B) \subseteq (A \circ B); \)
  \item \( ((A) \circ (B)) = (A \circ B); \)
  \item \( (A) \cup (B) = (A \cup B). \)
\end{enumerate}

\section{(m, n)-Hyperideals, (0, n)-Hyperideals and (m, n)-Hyperideals in ordered semihypergroups}

In this section, the notions of \((m, n)\)-hyperideals and generalized \((m, n)\)-hyperideals in ordered semihypergroups are introduced. Moreover, important some properties of these hyperideals are studied.
Definition 2.1. Let $H$ be an ordered semihypergroup and $m, n$ be the positive integers. Then a subsemihypergroup (respectively, non-empty subset) $A$ of $H$ is called an (respectively, generalized) $(m, n)$-hyperideal of $H$ if

(i) $A^m \circ H \circ A^n \subseteq A$; and

(ii) $(A] \subseteq A$.

Note that in Definition 2.1, if $m = 1 = n$, then $A$ is called a (generalized) bi-hyperideal of $H$. Moreover, a (generalized) bi-hyperideal of an ordered semihypergroup $H$ is an (generalized) $(m, n)$-hyperideal of $H$ for all positive integers $m$ and $n$. It is clear that, for positive integers $m$ and $n$, the notion of (generalized) $(m, n)$-hyperideal of $H$ is a generalization of the notion of (generalized) bi-hyperideal of $H$. The following example shows that a generalized $(m, n)$-hyperideal of $H$ need not be an $(m, n)$-hyperideal and generalized bi-hyperideal of $H$.

Example 2.2. Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a, b}$</td>
</tr>
<tr>
<td>$d$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a, b}$</td>
<td>${a, b, c}$</td>
</tr>
</tbody>
</table>

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

The covering relation $\prec$ and the figure of $H$ are as follows:

$\prec := \{(a, b)\}$

Then $H$ is an ordered semihypergroup. The subset $\{a, d\}$ of $H$ is a generalized $(m, n)$-hyperideal of $H$ for all integers $m, n \geq 2$ which is neither an $(m, n)$-hyperideal nor a generalized bi-hyperideal of $H$. 
Definition 2.3. [20] Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. Then a subsemihypergroup $A$ of $H$ is called an $(m, 0)$-hyperideal (respectively, $(0, n)$-hyperideal) of $H$ if

(i) $A^m \circ H \subseteq A$ (respectively, $H \circ A^n \subseteq A$); and

(ii) $(A) \subseteq A$.

In Definition 2.3, if $m = 1 = n$, then $A$ is called a right hyperideal (left hyperideal) of $H$. Clearly, each right hyperideal (respectively, left hyperideal) of $H$ is an $(m, 0)$-hyperideal for each positive integer $m$ (respectively, $(0, n)$-hyperideal for each positive integer $n$), that is, the notion of an $(m, 0)$-hyperideal ($(0, n)$-hyperideal) of $H$ is a generalization of the notion of a right hyperideal (respectively, left hyperideal) of $H$. Conversely, an $(m, 0)$-hyperideal (respectively, $(0, n)$-hyperideal) of $H$ need not be a right hyperideal (respectively, left hyperideal) of $H$. We illustrate it by the following example.

Example 2.4. Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a, b}$</td>
<td>${a, b}$</td>
</tr>
<tr>
<td>$d$</td>
<td>${a}$</td>
<td>${a}$</td>
<td>${a, b}$</td>
<td>${a}$</td>
</tr>
</tbody>
</table>

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}$.

The covering relation $\prec$ and the figure of $H$ are as follows:

$\prec := \{(a, b), (a, c)\}$

\[ \begin{array}{c}
 a \\
 b \\
 c \\
 d \\
 a \end{array} \]
Then $H$ is an ordered semihypergroup. It is easy to verify that the subset $A = \{a, d\}$ of $H$ is an $(m, 0)$-hyperideal and a $(0, n)$-hyperideal of $H$ for all integers $m, n \geq 2$, but it is neither a right hyperideal nor a left hyperideal of $H$.

**Remark 2.5.** Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be any non-empty subset of $H$. Then $(B^m \cup B \circ H \circ B^m)$ is a (generalized) bi-hyperideal of $H$. Indeed, $(B^m \cup B \circ H \circ B^m) \circ (B^m \cup B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m \circ (B^m \cup B \circ H \circ B^m) = (B^m \circ B^m \cup B^m \circ B \circ H \circ B^m \cup B \circ H \circ B^m \cup B \circ B^m \circ B \circ H \circ B^m \subseteq (B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m)$ and $(B^m \cup B \circ H \circ B^m) \circ H \circ (B^m \cup B \circ H \circ B^m) \subseteq (B^m \circ H \circ B^m \cup B \circ B^m \circ H \circ B^m \cup B \circ H \circ B^m \cup B \circ H \circ B^m \subseteq (B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m)$.

Note that in Remark 2.5, if $m = 1$, then $(B \cup B \circ H \circ B)$ is a generalized bi-hyperideal of $H$ which is not a bi-hyperideal of $H$. Thus $(B^m \cup B \circ H \circ B^m)$ is a generalized bi-hyperideal of $H$ for each positive integer $m$.

**Theorem 2.6.** Let $B$ be a non-empty subset of an ordered semihypergroup $H$ and let $m \geq 2$ be any positive integer. Then the following are equivalent:

(i) $B$ is a $(1, m)$-hyperideal of $H$;

(ii) $B$ is a left hyperideal of some bi-hyperideals of $H$;

(iii) $B$ is a bi-hyperideal of some left hyperideals of $H$;

(iv) $B$ is a $(0, m)$-hyperideal of some right hyperideals of $H$;

(v) $B$ is a right hyperideal of some $(0, m)$-hyperideals of $H$.

**Proof.** (i) $\Rightarrow$ (ii) Let $B$ be a $(1, m)$-hyperideal of $H$. So $B \circ B \subseteq B, (B) \subseteq B$ and $B \circ H \circ B^m \subseteq B$. Therefore, $(B^m \cup B \circ H \circ B^m) \circ B = (B^m \cup B \circ H \circ B^m) \circ (B) \subseteq (B^m \cup B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m) \subseteq (B \circ B \circ H \circ B^m) \circ B \subseteq (B \circ B \circ H \circ B^m) \circ B \subseteq (B^2 \circ B \circ H \circ B^m) \subseteq (B^2 \circ B \circ A \circ A \circ H \circ A \circ B) \subseteq (A \circ B \circ A \circ A \circ B) \subseteq (B \circ A \circ B) \subseteq B$. Let $b \in B, h \in (B \cup H \circ B)$ such that $h \leq b$. As $b \in B \subseteq A, b \in A$. So $h \in A$. Thus $h \in B$. Hence, $B$ is a bi-hyperideal of the left hyperideal $(B \cup H \circ B)$ of $H$. 

(ii) $\Rightarrow$ (iii) Let $B$ be a left hyperideal of a bi-hyperideal $A$ of $H$. So $B \subseteq A, A \circ B \subseteq B$ and $A \circ H \circ A \subseteq A$. Therefore, $B \circ B \subseteq A \circ B \subseteq B$ and $B \circ (B \cup H \circ B) \circ B = (B) \circ (B \cup H \circ B) \circ (B) \subseteq (B \circ (B \cup H \circ B) \circ B) \subseteq (B \circ B \circ H \circ B) \circ B \subseteq (B^2 \circ B \circ H \circ B) \subseteq (B^2 \circ B \circ A \circ A \circ H \circ A \circ B) \subseteq (A \circ B \circ A \circ A \circ B) \subseteq (B \circ A \circ B) \subseteq B$. Let $b \in B, h \in (B \cup H \circ B)$ such that $h \leq b$. As $b \in B \subseteq A, b \in A$. So $h \in A$. Thus $h \in B$. Hence, $B$ is a bi-hyperideal of the left hyperideal $(B \cup H \circ B)$ of $H$. 


(iii) ⇒ (iv) Let $B$ be a bi-hyperideal of a left hyperideal $L$ of $H$. Then $B \subseteq L$, $B \circ L \circ B \subseteq B$ and $H \circ L \subseteq L$. Therefore, $(B \cup B \circ H] \circ B^m \subseteq (B \cup B \circ H] \circ B^m \subseteq (B^{m+1} \cup B \circ H \circ B^m] \subseteq (B \cup B \circ (H \circ B^{m-2}) \circ L \circ B] \subseteq (B \cup B \circ H \circ L \circ B] = (B \cup B \circ (H \circ L) \circ B] \subseteq (B \cup B \circ L \circ B] = (B] = B$. Let $b \in B$, $h \in (B \cup B \circ H]$ such that $h \leq b$. As $B \subseteq L$, $b \in L$. So $h \in L$ and, thus, $h \in B$. Hence, $B$ is a $(0, m)$-hyperideal of the right hyperideal $(B \cup B \circ H]$ of $H$.

(iv) ⇒ (v) Let $B$ be a $(0, m)$-hyperideal of a right hyperideal $R$ of $H$. So $B \subseteq R$, $R \circ B^m \subseteq B$ and $R \circ H \subseteq R$. Therefore, $B \circ (B \cup H \circ B^m] \subseteq (B \cup H \circ B^m)] = (B] = B$. Let $b \in B$, $h \in (B \cup H \circ B^m]$ be such that $h \leq b$. As $B \subseteq R$, $b \in R$. So $h \in R$ which implies that $h \in B$. Hence, $B$ is a right hyperideal of the $(0, m)$-hyperideal $(B \cup H \circ B^m]$ of $H$.

(v) ⇒ (i) Let $B$ be a right hyperideal of a $(0, m)$-hyperideal $A$ of $H$. Thus $B \subseteq A$, $B \circ A \subseteq B$ and $H \circ A^m \subseteq A$. Therefore, $B \circ H \circ B^m \subseteq B \circ H \circ A^m \subseteq B \circ A \subseteq B$. Let $b \in B$, $h \in H$ be such that $h \leq b$. As $B \subseteq R$, we have $b \in R$. Therefore, $h \in R$ and, thus, $h \in B$. Hence, $B$ is a $(1, m)$-hyperideal of $H$.

**Definition 2.7.** Let $H$ be an ordered semihypergroup, $m, n$ be positive integers and $A$ be any (generalized) $(m, n)$-hyperideal of $H$. Then $A$ is said to be a minimal (generalized) $(m, n)$-hyperideal of $H$ if for every (generalized) $(m, n)$-hyperideal $B$ of $H$, $B \subseteq A$ implies $B = A$.

Similarly, a minimal $(m, 0)$-hyperideal and a minimal $(0, n)$-hyperideal of $H$ may be defined.

**Lemma 2.8.** Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be a non-empty subset of $H$. Then $B$ is a minimal (generalized) $(m, m - 1)$-hyperideal of $H$ if and only if $B$ is a minimal (generalized) bi-hyperideal of $H$.

**Proof.** Let $H$ be an ordered semihypergroup and $B$ be a minimal $(m, m - 1)$-hyperideal of $H$. Since $(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ (B^m \circ H \circ B^{m-1}] \circ H \circ ((B^m \circ H \circ B^{m-1}] \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$ and $((B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$. Therefore, $(B^m \circ H \circ B^{m-1}]$ is a $(m, m - 1)$-hyperideal of $H$ such that $(B^m \circ H \circ B^{m-1}] \subseteq B$. So by minimality of $(m, m - 1)$-hyperideal $B$ of $H$, $(B^m \circ H \circ B^{m-1}] = B$. Now $B \circ B = \ldots$
\[(B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1}) \subseteq ((B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1})) \subseteq (B^m \circ H \circ B^{m-1}) = Bm \circ B = \cdots \]

Therefore, \( B \) is bi-hyperideal of \( H \). It remains to show that \( B \) is a minimal bi-hyperideal of \( H \), so assume that \( A \) is any bi-hyperideal of \( H \) contained in \( B \). Therefore, \( A \) is \((m, m-1)\)-hyperideal of \( H \). Since \( B \) is a minimal \((m, m-1)\)-hyperideal of \( H \), \( B = A \). Hence, \( B \) is a minimal bi-hyperideal of \( H \). For the converse, assume that \( B \) is a minimal bi-hyperideal of \( H \). As \( B^m \circ H \circ B^{m-1} = B \circ (B^{m-1} \circ H \circ B^m) \subseteq B \circ H \circ B \subseteq B \), \( B \) is a minimal \((m, m-1)\)-hyperideal of \( H \). To show that \( B \) is a minimal \((m, m-1)\)-hyperideal of \( H \), let \( A \) be any \((m, m-1)\)-hyperideal of \( H \) such that \( A \subseteq B \).

\[
(A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1}) \subseteq ((A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1})) \subseteq (A^m \circ H \circ A^{m-1}) \quad \text{and} \quad (A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1}) \subseteq ((A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1})) \subseteq (A^m \circ H \circ A^{m-1}) \quad \text{(}\quad \text{is a bi-hyperideal of } H \quad \text{)}
\]

Since \( B \) is a minimal bi-hyperideal of \( H \) and \( (A^m \circ H \circ A^{m-1}) \subseteq B \), \( (A^m \circ H \circ A^{m-1}) = B \). As \( (A^m \circ H \circ A^{m-1}) \subseteq A \), \( B \subseteq A \). Now, as \( A \subseteq B \), we have \( A = B \). Hence, \( B \) is a minimal \((m, m-1)\)-hyperideal of \( H \).

\[\square\]

**Theorem 2.9.** Let \( H \) be an ordered semihypergroup and \( \{A_i \mid i \in I\} \) be a set of \((m, n)\)-hyperideals of \( H \). If \( \bigcap_{i \in I} A_i \neq \emptyset \), then \( \bigcap_{i \in I} A_i \) is an \((m, n)\)-hyperideal of \( H \).

**Proof.** Assume that \( \bigcap_{i \in I} A_i \neq \emptyset \). Let \( x, y \in \bigcap_{i \in I} A_i \). Then, \( x, y \in A_i \) for each \( i \in I \). As for each \( i \in I \), \( A_i \) is an \((m, n)\)-hyperideal, \( x \circ y \subseteq A_i \). Therefore, \( x \circ y \subseteq \bigcap_{i \in I} A_i \). Thus, \( \bigcap_{i \in I} A_i \) is a subsemihypergroup of \( H \). Next we show that

\[
(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i.
\]

We have

\[
(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq (A_i)^m \circ H \circ (A_i)^n \quad \text{(as \( \bigcap_{i \in I} A_i \subseteq A_i, \forall i \in I \))}
\]

\[
\subseteq A_i \quad \text{(as } A_i \text{'s are } (m, n)\text{-hyperideals)}.
\]

Thus \( (\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i \). Finally, we show that \( \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i \).

Let \( a \in \bigcap_{i \in I} A_i, h \in H \) such that \( h \leq a \). As \( a \in A_i \) for each \( i \in I \) and \( A_i \)'s
are \((m,n)\)-hyperideals, \(h \in A_i\) for each \(i \in I\). Therefore, \(h \in \bigcap_{i \in I} A_i\), as required. \(\square\)

**Theorem 2.10.** \cite{20} Let \(H\) be an ordered semihypergroup. Then the following conditions hold:

(i) Let \(\{L_i \mid i \in I\}\) be a set of \((m,0)\)-hyperideals of \(H\). If \(\bigcap_{i \in I} L_i \neq \emptyset\), then \(\bigcap_{i \in I} L_i\) is an \((m,0)\)-hyperideal of \(H\).

(ii) Let \(\{R_i \mid i \in I\}\) be a set of \((0,n)\)-hyperideals of \(H\). If \(\bigcap_{i \in I} R_i \neq \emptyset\), then \(\bigcap_{i \in I} R_i\) is a \((0,n)\)-hyperideal of \(H\).

Let \(H\) be an ordered semihypergroup and \(A\) be any non-empty subset of \(H\). We denote \(\mathcal{P} = \{J \mid J\) is an \((m,n)\)-hyperideal of \(H\) containing \(A\}\). Clearly, \(\mathcal{P} \neq \emptyset\) since \(H \in \mathcal{P}\). Let \([A]_{m,n} = \bigcap_{J \in \mathcal{P}} J\). As \(A \subseteq J\) for each \(J \in \mathcal{P}\), \([A]_{m,n} \neq \emptyset\). By Theorem 2.10, \([A]_{m,n}\) is an \((m,n)\)-hyperideal of \(H\) containing \(A\). The \((m,n)\)-hyperideal \([A]_{m,n}\) is called the \((m,n)\)-hyperideal of \(H\) generated by \(A\). Similarly, \([A]_{m,0}\) and \([A]_{0,n}\) are called \((m,0)\)-hyperideal and \((0,n)\)-hyperideal of \(H\) generated by \(A\), respectively.

**Theorem 2.11.** Let \(H\) be an ordered semihypergroup and \(A\) be a non-empty subset of \(H\). Then

\[ [A]_{m,n} = \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \]

for any positive integers \(m, n\).

**Proof.** Clearly \(\left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \neq \emptyset\). Now we have

\[
\begin{align*}
   &\left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \\
   \subseteq &\left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \right) \\
   = &\left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \cup \left( A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \right) \\
   \subseteq &\left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \right) \\
   \subseteq &\left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \right) \\
   \subseteq &\left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \right) \\
   = &\left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right)
\end{align*}
\]

(1)

Let \(x \in \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right)\). Then, \(x \in z_1 \circ z_2\) for some \(z_1, z_2 \in \cdots\)
\[ \bigcup_{i=1}^{m+n} A^i. \] Then, \( z_1 = A^p, z_2 = A^q \) for some \( 1 < p, q \leq m + n \). There are two cases arising. If \( p + q \leq m + n \), then \( z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \). If \( m + n \leq p + q \), then \( z_1 \circ z_2 \subseteq A^m \circ H \circ A^n \). Therefore, in both cases \( z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \). As \( x \in z_1 \circ z_2 \), \( x \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \).

Thus, \((\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \). Therefore, from (1), \((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n \). Hence, \((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \) is a subsemihypergroup of \( H \) containing \( A \). We have

\[
((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \circ H \\
= (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{n-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ H \\
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{n-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \circ H) \\
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{n-1} \circ (A \circ H) \\
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{n-1} \circ (A^2 \circ H) \\
\vdots \\
= (A^m \circ H)
\]

Similarly, \( H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \subseteq (H \circ A^n) \). Therefore, we have

\[
((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \circ H \\
\subseteq (A^m \circ H \circ A^n) \\
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n
\]

Also \((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n \). Therefore, \((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n \) is an \((m, n)\)-hyperideal of \( H \) containing \( A \). It follows that \([a]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n \). For the reverse inclusion, let \( x \in (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \), that is, there exist \( z \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \)
such that \( x \leq z \). If \( z \in \bigcup_{i=1}^{m+n} A^i \), then \( z = A^p \) for some \( 1 \leq p \leq m + n \). Therefore, \( x \in [A]_{m,n} \). If \( z \in A^m \circ H \circ A^n \), then

\[
A^m \circ H \circ A^n \subseteq ([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n \subseteq [A]_{m,n}.
\]

Therefore, \( z \in [A]_{m,n} \) implies \( x \in [A]_{m,n} \). Hence, \( [A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \), as required.

**Theorem 2.12.** [20] Let \( H \) be an ordered semihypergroup and \( A \) be any non-empty subset of \( H \). Then:

(i) \( [A]_{m,0} = (\bigcup_{i=1}^{m} A^i \cup A^m \circ H) \);

(ii) \( [A]_{0,n} = (\bigcup_{i=1}^{n} A^i \cup H \circ A^n) \).

**Theorem 2.13.** Let \( H \) be an ordered semihypergroup and \( A \) be a non-empty subset of \( H \). Then

\[
(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n)
\]

for any positive integers \( m, n \).

**Proof.** We have

\[
([A]_{m,n})^m \circ H
= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n))^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ H
\]

\[
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A \circ H)
\]

\[
= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (A \circ H)
\]

\[
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (A \circ H \cup A^m \circ H \circ A^n \circ A \circ H)
\]

\[
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (A^2 \circ H)
\]
Similarly, $H \circ ([A]_{m,n})^n \subseteq H \circ A^n$. Therefore, $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) \subseteq (A^m \circ H \circ A^n)$. The reverse inclusion is obvious, that is, $(A^m \circ H \circ A^n) \subseteq (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n)$. Hence, $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n)$.

**Theorem 2.14.** [20] Let $H$ be an ordered semihypergroup and $A$ be a non-empty subset of $H$. Then

(i) $(([A]_{m,0})^m \circ H) = (A^m \circ H)$ for any positive integer $m$.

(ii) $(H \circ ([A]_{0,n})^n) = (H \circ A^n)$ for any positive integer $n$.

3 $(m,n)$-regularity in ordered semihypergroups

In this section, we characterize $(m,n)$-regular, $(m,0)$-regular and $(0,n)$-regular ordered semihypergroup in terms of its $(m,n)$-hyperideals, $(m,0)$-hyperideals and $(0,n)$-hyperideals.

**Definition 3.1.** Let $H$ be an ordered semihypergroup and $m,n$ be non-negative integers. An element $a$ of $H$ is said to be an $(m,n)$-regular element if $a \in (a^m \circ H \circ a^n)$. The ordered semihypergroup $H$ is said to be $(m,n)$-regular if each element of $H$ is $(m,n)$-regular, equivalently, for each subset $A$ of $H$ we have $A \subseteq (A^m \circ H \circ A^n)$. Here, $A^0 \circ H = H \circ A^0 = H$.

It is clear from Definition 3.1 that, for each non-negative integers $m$ and $n$ every $(m,n)$-regular ordered semihypergroup is $(r,s)$-regular ($r \leq m, s \leq n$ are non-negative integers). In particular, for any positive integers $m$ and $n$, an $(m,n)$-regular ordered semihypergroup is regular. Indeed, $a \in (a^m \circ H \circ a^n) \subseteq (a \circ H \circ a)$. On the other hand, for each positive integer $m$, an $(m,0)$-regular ordered semihypergroup need not be a regular ordered semihypergroup.

**Proposition 3.2.** Let $H$ be an $(m,n)$-regular ordered semihypergroup and $A$ be a generalized $(m,n)$-hyperideal of $H$ for any positive integers $m, n$. Then $A$ is an $(m,n)$-hyperideal of $H$. 
Proof. Let \( a, b \in A \). Since \( H \) is an \((m, n)\)-regular ordered semihypergroup, there exist \( x, y \in H \) such that \( a \leq a^m \circ x \circ a^n, b \leq b^m \circ y \circ b^n \). Therefore, \( a \circ b \leq a^m \circ x \circ a^n \circ b^m \circ y \circ b^n = a^m \circ (x \circ a^n \circ b^m \circ y) \circ b^n \subseteq A^n \circ H \circ A^m \subseteq A \) whence \( a \circ b \subseteq (A) = A \). Thus \( A \) is a subsemihypergroup of \( H \). Hence, \( A \) is an \((m, n)\)-hyperideal of \( H \).

\( \square \)

**Theorem 3.3.** Let \( H \) be an ordered semihypergroup and \( m, n \) be non-negative integers. The set of all \((m, 0)\)-hyperideals, \((0, n)\)-hyperideals, and \((m, n)\)-hyperideals will be denoted by \( I_{(m, 0)}, I_{(0, n)} \) and \( I_{(m, n)} \), respectively. Then, we have

(i) \( H \) is \((m, 0)\)-regular if and only if \( I_{(m, 0)} \) is \((m, 0)\)-regular;

(ii) \( H \) is \((0, n)\)-regular if and only if \( I_{(0, n)} \) is \((0, n)\)-regular;

(iii) \( H \) is \((m, n)\)-regular if and only if \( I_{(m, n)} \) is \((m, n)\)-regular.

Proof. (i) When \( m = 0 \), the statement holds trivially because \( H \) is the only \((0, 0)\)-hyperideal of \( H \). So, let \( m \neq 0 \) and \( A \in I_{(m, 0)} \). Therefore \( (A^m \circ H) \subseteq A \). As \( S \) is \((m, 0)\)-regular, \( A \subseteq (A^m \circ H) \). Thus, \( A = (A^m \circ H) \). Since \( H \in I_{(m, 0)} \), \( A \) is a \((m, 0)\)-regular element of \( I_{(m, 0)} \). Hence \( I_{(m, 0)} \) is \((m, 0)\)-regular. For the converse, assume that \( I_{(m, 0)} \) is \((m, 0)\)-regular. Take any \( a \in S \). As \([a]_{m, 0} \in I_{(m, 0)} \) and \( I_{(m, 0)} \) is \((m, 0)\)-regular, there exists \( B \in I_{(m, 0)} \) such that \([a]_{m, 0} = ([a]_{m, 0})^m \circ B \subseteq ([a]_{m, 0})^m \circ H \subseteq ([a]_{m, 0})^m \circ H \). By Theorem 2.14, \(([a]_{m, 0})^m \circ H = (a^m \circ H) \). As \( \{a\} \subseteq [a]_{m, 0} \), we have \( a \in (a^m \circ H) \). Hence \( H \) is \((m, 0)\)-regular.

(ii) On the similar lines to (i), we may prove (ii).

(iii) If \( m = n = 0 \), then the statement is true because \( I_{(0, 0)} = \{H\} \). If \( m \neq 0 \) and \( n = 0 \) or \( m = 0 \) and \( n \neq 0 \), then the statement follows by (i) and (ii), respectively. So, let \( m \neq 0, n \neq 0 \) and \( A \in I_{(m, n)} \). Therefore \( (A^m \circ H \circ A^n) \subseteq A \). As \( H \) is \((m, n)\)-regular, \( A \subseteq (A^m \circ H \circ A^n) \). Thus, \( A = (A^m \circ H \circ A^n) \). Since \( H \in I_{(m, n)} \), \( A \) is an \((m, n)\)-regular element of \( I_{(m, n)} \). Hence, \( I_{(m, n)} \) is \((m, n)\)-regular. For the converse, assume that \( I_{(m, n)} \) is \((m, n)\)-regular and \( a \in H \). As \([a]_{m, n} \in I_{(m, n)} \) and \( I_{(m, n)} \) is \((m, n)\)-regular, there exists \( B \in I_{(m, n)} \) such that \([a]_{m, n} = ([a]_{m, n})^m \circ B \circ ([a]_{m, n})^n \subseteq ([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n \subseteq ([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n \). By Theorem 4.1, we have \(([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n = (a^m \circ H \circ a^n) \). As \( \{a\} \subseteq [a]_{m, n} \), \( a \in (a^m \circ H \circ a^n) \). This implies that \( a \) is an \((m, n)\)-regular element of \( H \). Hence, \( H \) is \((m, n)\)-regular. \( \square \)
Lemma 3.4. [20] Let \( H \) be an ordered semihypergroup. If the sets of all \((m,0)\)-hyperideals and \((0,n)\)-hyperideals are denoted by \( I_{(m,0)} \) and \( I_{(0,n)} \) respectively, then

(i) \( H \) is \((m,0)\)-regular if and only if \( R = (R^m \circ H) \) (\( \forall R \in I_{(m,0)} \)), where \( m \) is any positive integer;

(ii) \( H \) is \((0,n)\)-regular if and only if \( L = (H \circ L^n) \) (\( \forall L \in I_{(0,n)} \)), where \( n \) is any positive integer.

Theorem 3.5. Let \( H \) be an ordered semihypergroup and \( m, n \) be non-negative integers. The set of all \((m,n)\)-hyperideals will be denoted by \( I_{(m,n)} \). Then \( H \) is \((m,n)\)-regular if and only if \( A = (A^m \circ H \circ A^n) \) for all \( A \in I_{(m,n)} \).

Proof. If \( m = n = 0 \), then the statement is true because \( I_{(0,0)} = \{H\} \). If \( m \neq 0 \) and \( n = 0 \) or \( m = 0 \) and \( n \neq 0 \), then the statement follows by Lemma 3.4. So, let \( m \neq 0, n \neq 0 \) and \( A \in I_{(m,n)} \). Then, by definition of \((m,n)\)-regularity, we have \( A \subseteq (A^m \circ H \circ A^n) \) and, by definition of \((m,n)\)-hyperideal, we have \((A^m \circ H \circ A^n) \subseteq (A) = A \). Hence, \( A = (A^m \circ H \circ A^n) \).

For the converse, assume that \( A = (A^m \circ H \circ A^n) \) for each \( A \in I_{(m,n)} \). Take any \( a \in H \), so \([a]_{m,n} \in I_{(m,n)} \). From Theorem 4.1 and by the assumption, \([a]_{m,n} = ([a]_{m,n})^m \circ H \circ [a]_{m,n}) = (a^m \circ H \circ a^n) \). As \( \{a\} \subseteq [a]_{m,n}, a \in (a^m \circ H \circ a^n) \). Hence, \( H \) is \((m,n)\)-regular.

Theorem 3.6. Let \( H \) be an ordered semihypergroup and \( m, n \) be non-negative integers. Then, \( H \) is \((m,n)\)-regular if and only if \( L \cap R = (R^m \circ L^n) \) for each \((m,0)\)-hyperideal \( R \) and for each \((0,n)\)-hyperideal \( L \) of \( H \).

Proof. The statement is trivially true for \( m = 0 = n \). If \( m = 0 \) and \( n \neq 0 \) or \( m \neq 0 \) and \( n = 0 \), then the result follows by Lemma 3.4. So, let \( m \neq 0, n \neq 0 \). \( R \) be any \((m,0)\)-hyperideal and \( L \) be any \((0,n)\)-hyperideal of \( H \). Therefore \((R^m \circ L^n) \subseteq (R^m \circ H) \subseteq (R) = R \) and \((R^m \circ L^n) \subseteq (H \circ L^n) \subseteq (L) = L \). Therefore, \((R^m \circ L^n) \subseteq R \cap L \). As \( H \) is \((m,n)\)-regular, we have

\[
\begin{align*}
(R \cap L) & \subseteq ((R \cap L)^m \circ H \circ (R \cap L)^n) \\
& \subseteq (R^m \circ H \circ L^n) \\
& \subseteq (R^m \circ H \circ L^{n-1} \circ (L^m \circ H \circ L^n)) \quad \text{(as \( H \) is \((m,n)\)-regular)} \\
& = (R^m \circ H \circ L^{n-1} \circ L^m \circ H \circ L^n) \quad \text{(by Lemma 1.1)} \\
& \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ H \circ L^n) \quad \text{(as \( H \) is \((m,n)\)-regular)} \\
& \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ (H \circ L^n)) \quad \text{(as \( H \circ L^n \subseteq (H \circ L^n) \)}}
\end{align*}
\]
\( (m, n) \)-Hyperideals in ordered semihypergroups

\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n \circ H \circ L^n)) \) (by Lemma 1.1)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n) \) (by Lemma 1.1)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n \circ H \circ L^n)) \)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n) \)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ \cdots \circ L^{m-1} \circ (L^m \circ H \circ L^n)) \)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ \cdots \circ L^{m-1} \circ (L^m \circ H \circ L^n)) \)
\( \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ \cdots \circ H \circ L^n) \)
\( = (R^m \circ H \circ L^{n-1} \circ (L^{m-1})^{n-1} \circ L^m \circ (H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n)) \)
\( = (R^m \circ H \circ (L^{n-1} \circ L^{mn-m-n+1} \circ L^m) \circ (H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n)) \)
\( = (R^m \circ (H \circ L^{mn}) \circ (H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n)) \)
\( \subseteq (R^m \circ (H \circ L^n)^{n-1} \circ (H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n)) \)
\( \subseteq (R^m \circ (H \circ L^n)^{n-1} \circ \cdots \circ H \circ L^n)) \)
\( \subseteq (R^m \circ (H \circ L^n)^n) \)
\( \subseteq (R^m \circ L^n) \).

Therefore, \( L \cap R = (R^m \circ L^n) \).

Conversely, assume that \( L \cap R = (R^m \circ L^n) \) for each \((m, 0)\)-hyperideal \( R \) and for each \((0, n)\)-hyperideal \( L \) of \( H \). Let \( a \in S \). As \([a]_{m,0}\) is an \((m, 0)\)-hyperideal and \( H \) is a \((0, n)\)-hyperideal of \( H \), we have

\[
[a]_{m,0} = [a]_{m,0} \cap H = (((a)_{m,0})^m \circ H^n)
\subseteq (\{a\}_{m,0}^m \circ H) = (a^m \circ H) \quad \text{(by Theorem 2.14)}
\]

Similarly, \([a]_{0,n} \subseteq (H \circ a^n)\). As \((a^m \circ H)\) and \((H \circ a^n)\) are an \((m, 0)\)-hyperideal and \((0, n)\)-hyperideal of \( H \), by hypothesis we get

\[
\{a\} \subseteq [a]_{m,0} \cap [a]_{0,n} \subseteq (a^m \circ H) \cap (H \circ a^n)
= (((a^m \circ H))^n \circ ((H \circ a^n)^n)) \quad \text{(by hypothesis)}
\subseteq (a^m \circ H \circ a^n).
\]

Hence, \( H \) is \((m, n)\)-regular.

\[\square\]

**Theorem 3.7.** Let \( H \) be an ordered semihypergroup and \( m, n \) be positive integers (either \( m \geq 2 \) or \( n \geq 2 \)). Then, the following are equivalent:
(i) Each \((m,n)\)-hyperideal of \(H\) is idempotent;
(ii) For each \((m,n)\)-hyperideals \(A, B\) of \(H\), \(A \cap B \subseteq (A^m \circ B^n)\);
(iii) \([a]_{m,n} \cap [b]_{m,n} \subseteq \{([a]_{m,n})^m \circ ([b]_{m,n})^n\}\ \forall a, b \in H;\)
(iv) \([a]_{m,n} \subseteq \{([a]_{m,n})^m \circ ([a]_{m,n})^n\}\ \forall a \in H;\)
(v) \(H\) is \((m,n)\)-regular.

Proof. (i) \(\Rightarrow\) (ii) Assume that each \((m,n)\)-hyperideal of \(H\) is idempotent. Let \(A\) and \(B\) be any \((m,n)\)-hyperideals of \(H\). As \(A \cap B\) is an \((m,n)\)-hyperideal of \(H\), we have

\[A \cap B = ((A \cap B)^2) = ((A \cap B) \circ ((A \cap B))^2)\]
\[= ((A \cap B)^3) = \cdots = ((A \cap B)^{m+n}]\]
\[= ((A \cap B)^m \circ (A \cap B)^n] \subseteq (A^m \circ B^n].\]

(ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (iv) are obvious.

(iv) \(\Rightarrow\) (v) Take any \((m,n)\)-hyperideal \(A\) of \(H\). As \(H\) is \((m,n)\)-regular \(1), (v)\) and \(A\) is an \((m,n)\)-hyperideal, \(A = (A^m \circ H \circ A^n]\). Now

\[(A \circ A] = ([A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n])) \subseteq (A^m \circ H \circ A^n] = A\]

and

\[A = (A^m \circ H \circ A^n] = (((A^m \circ H \circ A^n])^m \circ H \circ A^n]\]
\[= (A^m \circ H \circ A^n] \circ \cdots \circ (A^m \circ H \circ A^n] \circ H \circ A^n]\]
\[= (A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n] \circ \cdots \circ (A^m \circ H \circ A^n] \circ H \circ A^n] \circ \cdots \circ (A^m \circ H \circ A^n] \circ H \circ A^n].\]
\[(A_m \circ H \circ A^n) \circ \cdots \circ (A_m \circ H \circ A^n) \circ H \circ A^n \]

\[(m-2)\text{-times}\]
\[\subseteq ((A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ H \circ A^n] \]
\[\subseteq ((A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (H \circ A^n]] \]
\[\subseteq ((A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (H \circ A^n)] \]
\[\subseteq ((A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n)] \]
\[\subseteq (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (H \circ A^n)] \]
\[= (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n) \circ (A_m \circ H \circ A^n)] \]

Therefore, \( A = (A \circ A) \). Hence, each \((m, n)\)-hyperideal of \( H \) is an idempotent.

The following example shows that the condition \( m \geq 2 \) or \( n \geq 2 \) in Theorem 3.7 is necessary.

**Example 3.8.** [24] Let \( H = \{a, b, c, d, e\} \). Define a hyperoperation \( \circ \) on \( H \) by the table

\begin{tabular}{c|ccccc}
\( \circ \) & \( a \) & \( b \) & \( c \) & \( d \) & \( e \) \\
\hline
\( a \) & \{a\} & \{a\} & \{a\} & \{a\} & \{a\} \\
\( b \) & \{a\} & \{a, b\} & \{a\} & \{a, d\} & \{a\} \\
\( c \) & \{a\} & \{a, e\} & \{a, c\} & \{a, c\} & \{a, e\} \\
\( d \) & \{a\} & \{a, b\} & \{a, d\} & \{a, d\} & \{a, b\} \\
\( e \) & \{a\} & \{a, e\} & \{a\} & \{a, c\} & \{a\} \\
\end{tabular}

and the order \( \leq \) on \( H \) as \( \leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e)\} \). The covering relation \( \prec \) and the figure of \( H \) are as

\( \prec := \{(a, b), (a, c), (a, d), (a, e)\} \)

Now, \((H, \circ, \leq)\) is a regular ordered semihypergroup. One may easily check that \( A = \{a, e\} \) is a bi-hyperideal of \( H \), but \( A \neq (A^2) \).
4 Relations $I_n, mI, H_m^n$ and $B^n_m$ on ordered semihypergroups

In this section, the relations $I_n, mI, H_m^n$ and $B^n_m$ on an ordered semihypergroup are introduced. Then, some related properties of these relations are studied.

**Definition 4.1.** Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. We define the relations $I_n, mI, H_m^n$ and $B^n_m$ as

$$
I_n = \{(a, b) \in S \times S \mid [a]_{0,n} = [b]_{0,n}\};
$$
$$
mI = \{(a, b) \in S \times S \mid [a]_{m,0} = [b]_{m,0}\};
$$
$$
H_m^n = mI \cap I_n;
$$
$$
B^n_m = \{(a, b) \in S \times S \mid [a]_{m,n} = [b]_{m,n}\}.
$$

Clearly, all the relations defined above are equivalence relations on $H$.

**Lemma 4.2.** Let $H$ be an ordered semihypergroup and $a, b \in H$ be $mI$-related (respectively, $I_n$-related). Then, $(a^m \circ H) = (b^m \circ H)$ (respectively, $(H \circ a^n) = (H \circ b^n)$).

*Proof.* Suppose that $(a, b) \in mI$. Then, by definition, $[a]_{m,0} = [b]_{m,0}$, i.e. $(\bigcup_{i=1}^m a^i \cup b^m \circ H) = (\bigcup_{i=1}^m b^i \cup b^m \circ H)$. Therefore, $\{a\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H)$ and $\{b\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H)$. Thus, $(a^m \circ H) \subseteq ((\bigcup_{i=1}^m b^i \cup a^m \circ H)) \circ H = (a^m \circ H)$. Similarly, we may show that $(a, b) \in I_n$ implies $(H \circ a^n) = (H \circ b^n)$. \hfill \Box

**Lemma 4.3.** Let $H$ be an ordered semihypergroup and $a, b \in H$ be $H_m^n$-related. Then, $(a^m \circ H) = (b^m \circ H)$, $(H \circ a^n) = (H \circ b^n)$ and $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$.

*Proof.* Suppose that $(a, b) \in H_m^n$. Then, by definition, $(a, b) \in mI$ and $(a, b) \in I_n$. By Lemma 4.1, $(a^m \circ H) = (b^m \circ H)$ and $(H \circ a^n) = (H \circ b^n)$. Therefore, we have $(a^m \circ H \circ a^n) = ((a^m \circ H) \circ a^n) = ((b^m \circ H) \circ a^n) = (b^m \circ H \circ a^n) = (b^m \circ (H \circ a^n)) = (b^m \circ (H \circ b^n)) = (b^m \circ H \circ b^n)$. \hfill \Box

**Lemma 4.4.** Let $H$ be an ordered semihypergroup. Then, $B^n_m \subseteq H_m^n$. 

Proof. Let \((a, b) \in B^m_n\). Then, \([a]_{m,n} = [b]_{m,n}\), i.e. \(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\) = \((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\)
. So \(a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) and \(b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\) for each \(i \in \{1, 2, \ldots, m + n\}\). It follows that
\[
\bigcup_{i=1}^{m} a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\]
and
\[
\bigcup_{i=1}^{m} b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\).
\]
Now \((a^m \circ H) \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\))\(^m\circ H\) and \((b^m \circ H) \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\))\(^m\circ H\). Therefore, by Theorem 4.1, \((a^m \circ H) \subseteq (b^m \circ H)\) and \((b^m \circ H) \subseteq (a^m \circ H)\).

Now
\[
[a]_{m,0} = (\bigcup_{i=1}^{m} a^i \cup a^m \circ H]
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) \cup a^m \circ H)] (since \(\bigcup_{i=1}^{m+n} a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\)])
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) \cup (a^m \circ H)) (as \(a^m \circ H \subseteq (a^m \circ H)\))
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) \cup (b^m \circ H)) (as \(a^m \circ H \subseteq (b^m \circ H)\))

\[
= ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) \cup (b^m \circ H)) \quad \text{(by Lemma 1.1)}
\]
\[
= (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) \cup b^m \circ H) \quad \text{(by Lemma 1.1)}
\]
\[
\subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \cup b^m \circ H \circ b^n\) \cup b^m \circ H)] (since \(\bigcup_{i=1}^{m+n} b^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\)])
\]
\[
= (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \cup b^m \circ H \circ b^n\) \cup b^m \circ H) \quad \text{(as \(b^m \circ H \circ b^n \subseteq b^m \circ H\)}
\]
\[
= [b]_{m,0};
\]
and
\[
[b]_{m,0} = (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H]
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \cup b^m \circ H)] (since \(\bigcup_{i=1}^{m+n} b^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\)])
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \cup b^m \circ H)] (as \(b^m \circ H \subseteq (b^m \circ H)\))
\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \cup b^m \circ H) \cup (b^m \circ H)] (as \(b^m \circ H \subseteq (a^m \circ H)\))
\[= ((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H)] \text{ (by Lemma 1.1)}
\]
\[= (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H] \text{ (by Lemma 1.1)}
\]
\[\subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \cup a^m \circ H \circ a^n \cup a^m \circ H) \text{ (since } \bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H)\]
\[\subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H) \text{ (as } a^m \circ H \circ a^n \subseteq a^m \circ H)\]
\[= [a]_{m,0}\].

Therefore, \([a]_{m,0} = [b]_{m,0}\). Similarly, one can show that \([a]_{0,n} = [b]_{0,n}\). Thus, \((a, b) \in \mathcal{H}_m\). Hence, \(\mathcal{B}_m^n \subseteq \mathcal{H}_m^n\). □

**Theorem 4.5.** Let \(H\) be an \((m, n)\)-regular ordered semihypergroup. Then, \(\mathcal{B}_m^n = \mathcal{H}_m^n\).

**Proof.** Let \((a, b) \in \mathcal{H}_m^n\). Therefore, by Lemma 4.2, \((a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)\). As \(S\) is \((m, n)\)-regular, \(a \in (a^m \circ H \circ a^m)\) and \(b \in (b^m \circ H \circ b^m)\). So \(a^i \subseteq (a^m \circ H \circ a^m)\) for each \(i \in \{1, 2, \ldots, m + n\}\), it follows that \(\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^m)\). Thus, \([a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m) = (a^m \circ H \circ a^m)\) and similarly \([b]_{m,n} = (b^m \circ H \circ b^m)\). Thus, \([a]_{m,n} = [b]_{m,n}\), i.e. \((a, b) \in \mathcal{B}_m^n\). This implies that \(\mathcal{H}_m^n \subseteq \mathcal{B}_m^n\). Hence, by Lemma 4.3, \(\mathcal{B}_m^n = \mathcal{H}_m^n\). □

**Lemma 4.6.** If \(B_x\) and \(B_y\) are two \((m, n)\)-regular \(\mathcal{B}_m^n\)-classes contained in the same \(\mathcal{H}_m^n\)-class of ordered semihypergroup \(H\), then \(B_x = B_y\).

**Proof.** As \(x\) and \(y\) are \((m, n)\)-regular elements of \(H\), \(x \in (x^m \circ H \circ x^n)\) and \(y \in (y^m \circ H \circ y^n)\), \(\{x\}^i \subseteq (x^m \circ H \circ x^n)\) and \(\{y\}^i \subseteq (y^m \circ H \circ y^n)\) for each \(i \in \{1, 2, \ldots, m + n\}\). It follows that \(\bigcup_{i=1}^{m+n} x^i \subseteq (x^m \circ H \circ x^m)\) and \(\bigcup_{i=1}^{m+n} y^i \subseteq (y^m \circ H \circ y^m)\). Therefore, \([x]_{m,n} = (x^m \circ H \circ x^n)\) and \([y]_{m,n} = (y^m \circ H \circ y^n)\). Since \(x\) and \(y\) are contained in the same \(\mathcal{H}_m^n\)-class, by Lemma 4.2, \((x^m \circ H \circ x^n) = (y^m \circ H \circ y^n)\). So \([x]_{m,n} = [y]_{m,n}\). Therefore, \(xB_{m,y}\). Hence, \(B_x = B_y\). □

5 \((m, 0)\)-regularity [(0, n)-regularity] and \((m, n)\)-right weakly regularity of a \(\mathcal{B}_m^n\)-class, \(\mathcal{Q}_m^n\)-class and \(\mathcal{H}_m^n\)-class

In this section, the \((m, 0)\)-regular, \((0, n)\)-regular, \((m, n)\)-regular and \((m, n)\)-right weakly regular class of the relations \(\mathcal{H}_m^n\) and \(\mathcal{B}_m^n\) are studied.
Lemma 5.1. An $H_{m,n}$-class $H$ of an ordered semihypergroup is $(m,0)$-regular $[(0,n)$-regular] if it contains an $(m,0)$-regular $[(0,n)$-regular] element.

Proof. Let $a$ be an $(m,0)$-regular element and $c$ be an element of $H_{m,n}$-class $H$. This implies $[b]_{m,0} = [a]_{m,0}$ and $a \in (a^m \circ H)$. Therefore, $\{a\}^i \subseteq (a^m \circ H)$ for each $i \in \{1,2,\ldots,m\}$. Then $\bigcup_{i=1}^m a^i \subseteq (a^m \circ H)$ implies $(\bigcup_{i=1}^m a^i) \subseteq (a^m \circ H)$. Thus, $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^m a^i \cup a^m \circ H) = (\bigcup_{i=1}^m a^i) \cup (a^m \circ H) = (a^m \circ H)$. By Lemma 4.2, $(a^m \circ H) = (b^m \circ H)$. This implies that $[b]_{m,0} \subseteq (b^m \circ H)$. Hence, $b \in (b^m \circ H)$. So $b$ is an $(m,0)$-regular element of $H_{m,n}$-class $H$. Hence, the $H_{m,n}$-class $H$ is $(m,0)$-regular. The dual statement follows on the similar lines.

Lemma 5.2. An $H_{m,n}$-class $H$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.

Proof. The proof is similar to the proof of Lemma 5.1.

Lemma 5.3. A $B_{m,n}$-class $B$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.

Proof. Let $a \in B$ be an $(m,n)$-regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n)$ so that $\{a\}^i \subseteq (a^m \circ H \circ a^n)$ for each $i \in \{1,2,\ldots,m+n\}$, so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n)$ implies $(\bigcup_{i=1}^m a^i) \subseteq (a^m \circ H \circ a^n)$. Since $a,b \in B$, $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n) = (\bigcup_{i=1}^m a^i \cup (a^m \circ H \circ a^n) = (a^m \circ H \circ a^n)$. By Lemmas 4.3 and 4.2, we have $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$. This implies that $[b]_{m,n} \subseteq (b^m \circ H \circ b^n)$. So $b \in (b^m \circ H \circ b^n)$. Thus, $b$ is an $(m,n)$-regular element of $B$. Hence, $B$ is $(m,n)$-regular.

Definition 5.4. Let $H$ be an ordered semihypergroup and $m,n$ be positive integers. An element $a$ of $H$ is said to be an $(m,n)$-right weakly regular element if $a \in (a^m \circ H \circ a^n \circ H)$. The ordered semihypergroup $H$ is said to be $(m,n)$-right weakly regular if each element of $H$ is $(m,n)$-right weakly regular, equivalently, for each subset $A$ of $H$, $A \subseteq (A^m \circ H \circ A^n \circ H)$.

Lemma 5.5. A $B_{m,n}$-class $B$ of an ordered semihypergroup $H$ is $(m,n)$-right weakly regular if it contains an $(m,n)$-right weakly regular element.

Proof. Let $a \in B$ be an $(m,n)$-right weakly regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n \circ H)$. This implies that $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H)$
for each \( i \in \{1, 2, \ldots, m + n\} \), so \( \bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H) \) implies 
(\( \bigcup_{i=1}^{m+n} a^i \subseteq ((a^m \circ H \circ a^n \circ H) H \circ ((a^m \circ H \circ a^n \circ H)) \) \( \subseteq (a^m \circ H \circ a^n \circ H) \)). 
So, \( (a^m \circ H \circ a^n) \subseteq ((a^m \circ H \circ a^n \circ H) \circ H \circ ((a^m \circ H \circ a^n \circ H))) \subseteq (a^m \circ H \circ a^n \circ H) \). 
Since \( a, b \in B \), [\( b \)]\( m,n \) = \([a]_{m,n} = ([\bigcup_{i=1}^{m+n} a^i] \cup a^m \circ H \circ a^n) = ([\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n) \subseteq (\bigcup_{i=1}^{m+n} a^i) \cup (a^m \circ H \circ a^n) = (a^m \circ H \circ a^n \circ H) \) (since \( \bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H) \)). 
By Lemmas 4.3 and 4.2, \( (a^m \circ H \circ a^n) = (b^m \circ H \circ b^n) \). 
This implies that \( [b]_{m,n} \subseteq (a^m \circ H \circ a^n \circ H) = ((a^m \circ H \circ a^n) \circ H) = (b^m \circ H \circ b^n \circ H) \). 
So \( b \in (b^m \circ H \circ b^n \circ H) \). Thus, \( b \) is an \((m,n)\)-right weakly regular element of \( B \). Hence, \( B \) is \((m,n)\)-right weakly regular.

\[ \square \]

**Corollary 5.6.** An ordered semihypergroup \( H \) is \((m,n)\)-regular \((\text{(m,n)-right weakly regular}) \) if and only if each \( B^m_n \)-class of \( H \) contains an \((m,n)\)-regular \((\text{(m,n)-right weakly regular}) \) element.

**Lemma 5.7.** An \( \mathcal{H}^n_m \)-class \( H \) of an ordered semihypergroup is \((m,n)\)-right weakly regular if it contains an \((m,n)\)-right weakly regular element.

**Proof.** Let \( a \) be an \((m,n)\)-right weakly regular element and \( b \) be an element of \( \mathcal{H}^n_m \)-class \( H \). Then, \( (a^m \circ H \circ a^n \circ H) \) for each \( i \in \{1, 2, \ldots, m+n\} \), and so \( \bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H) \) implies 
(\( \bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H) \subseteq (a^m \circ H \circ a^n \circ H) \circ H \subseteq (a^m \circ H \circ a^n \circ H) \)). 
Since \( a, b \in H \), \([b]_{m,0} = [a]_{m,0} = ([\bigcup_{i=1}^{m+n} a^i] \cup a^m \circ H) = ([\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H) = (a^m \circ H) \subseteq (a^m \circ H \circ a^n \circ H) \). 
So, by Lemma 4.2, \( (a^m \circ H \circ a^n) = (b^m \circ H \circ b^n) \). 
This implies that \( [b]_{m,0} \subseteq (a^m \circ H \circ a^n \circ H) = ((a^m \circ H \circ a^n) \circ H) = (b^m \circ H \circ b^n \circ H) \subseteq (b^m \circ H \circ b^n \circ H) \). 
Therefore, \( b \in (b^m \circ H \circ b^n \circ H) \) and thus, \( b \) is an \((m,n)\)-right weakly regular element of \( \mathcal{H}^n_m \)-class \( H \). Hence, \( H \) is \((m,n)\)-right weakly regular.

\[ \square \]

**Corollary 5.8.** An ordered semihypergroup \( H \) is \((\text{respectively, (m,0)-regular, (0,n)-regular, (m,n)-regular}) \) \((m,n)\)-right weakly regular if and only if each \( \mathcal{H}^n_m \)-class of \( H \) contains a \((\text{respectively, (m,0)-regular, (0,n)-regular, (m,n)-regular}) \) \((m,n)\)-right weakly regular element.

6 Conclusion

The main purpose of the present paper is to introduce the equivalence relations \( m\mathcal{I}, \mathcal{I}_n, \mathcal{B}^m_n \) and \( \mathcal{H}^n_m \) on an ordered semihypergroup and enhance the un-
understanding of different classes of ordered semihypergroups ((m, n)-regular, (m, 0)-regular, (0, n)-regular, (m, n)-right weakly regular) by considering the structural influence of the equivalence relations $mI_n, B^e_n, H^e_m$. In particular, if we take $m = 1 = n$, the equivalence relations $mI, I_n$ and $H^e_m$ are reduced to the equivalence relations $R, L$ and $H$ in ordered semihypergroup, respectively, which mimic the definition of the usual Green’s relations $R, L$ and $H$ in plain semihypergroups [11]. Also when we take $m = 1 = n$ in Theorems 1.9, 1.11, 4.1, 3.6, and 4.2, and Lemmas 4.1, 4.2, 4.3, 4.3, 5.1, and 5.2, then we obtain all the results for bi-hyperideals in an ordered semihypergroup and some characterizations of regular ordered semihypergroups, which is the main application of the results presented in this paper.

References


Ahsan Mahboob Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
Email: khanahsan56@gmail.com

Noor Mohammad Khan Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
Email: nm_khan123@yahoo.co.in

Bijan Davvaz Department of Mathematics, Yazd University, Yazd, Iran.
Email: dawaz@yazd.ac.ir