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The categories of lattice-valued maps, equalities, free objects, and $\mathcal{C}$-reticulation

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Dedicated to Professor George A. Grätzer

Abstract. In this paper, we study the concept of $\mathcal{C}$-reticulation for the category $\mathcal{C}$ whose objects are lattice-valued maps. The relation between the free objects in $\mathcal{C}$ and the $\mathcal{C}$-reticulation of rings and modules is discussed. Also, a method to construct $\mathcal{C}$-reticulation is presented, in the case where $\mathcal{C}$ is equational. Some relations between the concepts reticulation and satisfying equalities and inequalities are studied.

1 Introduction

In the theory of f-rings, K. Keimel (1968, 1971, [13], [14]) used $L_1(A)$ to get the Keimel's representation theory for f-rings, where A is an f-ring and $L_1(A)$ is the distributive lattice generated by the symbols $D_1(a)$, $a \in A$, subject to the relations

$$D_1(1) = 1, D_1(0) = 0$$

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$$D_1(a \wedge b) = D_1(a) \wedge D_1(b), D_1(a \vee b) = D_1(a) \vee D_1(b)$$

Subsequent to Keimel's work, J.F. Kennison (1976, [15]) constructed another representation by using $D_1 : A \to L_1(A), a \mapsto D_1(a)$.

A. Joyal (1976, [17]) used $L_2(A)$ to study generally the Zariski spectrum via the distributive lattice, where A is an arbitrary ring, and $L_2(A)$ is the distributive lattice generated by the symbols $D_2(a)$, $a \in A$, subject to the relations

$$D_2(1_A) = 1_{L_2(A)}, D_2(0_A) = 0_{L_2(A)}$$

$$D_2(ab) = D_2(a) \wedge D_2(b), D_2(a+b) \leq D_2(a) \vee D_2(b)$$

Also C.J. Mulvey (1979, [20], [21]), using $D_2 : A \to L_2(A)$, introduced the notion of the Gelfand ring and proved a representation theorem for Gelfand rings.

G.W. Brumfiel (1979, [6]) in the representation theory of partially ordered rings, used $L_3(A)$ with additional relation respect to the ring's order which is $D_3(a) \leq D_3(b)$ whenever $0 \leq a \leq b$.

Finally, Simmons (1980, [22]) extensively studied the notation $D : A \to L(A)$, and he called $L(A)$ the *reticulation* of A.

On the other hand, B. Banaschewski (1997, [2]) utilized the cozero part of the frame to study frames L and the f-rings $C(L)$, the pointfree version of $C(X)$. Also, in looking at pointfree version of the Gelfand duality, cozero elements play an important role [1]. Then, the other authors following Banaschewski, applied this tool in pointfree topology [3–5, 11, 12, 19]. The present author, A. Karimi (2006, [9]), generalized the concept of cozero elements to *cozero maps*, that is, maps $M \to L$ satisfying some relations (2.6 of this paper), where M is an ℓ-module and L is a frame. He used the cozero maps to introduce the concept of the cozero transformations, which is applied to obtain a general theorem containing both of Gelfand and Kakutani pointfree dualities as particular cases.

On the basis of these historical trends, in this paper, we introduce semi-cozero maps, and we extend it to the concept of *lattice-valued maps*. Also, we introduce $\mathcal{C}$-reticulation and discuss the relation between this concept and the free objects in $\mathcal{C}$, whose objects are lattice-valued maps.

The necessary background on lattices, ordered algebraic structures, universal algebra, and some category notations, are given in Section 2.

In Section 3, we introduce the notion of *semi-cozero maps* as a simple generalization of cozero maps, and we discuss some relations between semi-cozero maps and submodules and ideals.

In Section 4, we give the model of lattice-valued maps satisfying a set of equalities and inequalities, to generalize semi-cozero and cozero maps. Also, we define the categories involved.

Finally, in the Section 5, we look at the concept of *free lattice-valued maps* in the category $\mathcal{C}$ whose objects are lattice-valued maps. We introduce a $\mathcal{C}$-reticulation of B, and we show that $\mathcal{C}$-reticulation is closely related to the *free objects* in $\mathcal{C}$. We construct a $\mathcal{C}$-reticulation $c_B : B \to L(B)$ of B, in the cases where the objects of the category $\mathcal{C}$ satisfy $\sum$, where $\sum$ is a set of equalities and inequalities. Also, we deeply study the relations between the concepts of reticulation and satisfying equalities and inequalities. Finally, we introduce a concept which is a relation between equalities and inequalities, denoted by $\models^{\mathcal{B}}$, and the logical relation between this notion and satisfying equalities and inequalities is given (Corollary 5.16).

2 Background

Here we give the notions we need from the literature.

2.1 In this paper, all rings are commutative with identity and all modules are unitary.

Let A be a ring. An ideal I of A is called *prime* if $xy \in P$ implies $x \in I$ or $y \in I$. Also, I is called *radical* if $x^n \in I$ for some $n \in \mathbb{N}$ implies $x \in I$. It is clear that every prime ideal is a radical ideal.

2.2 A poset L is called a *lattice* if for every $a, b \in L$, both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist. We denote $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$. The top and the bottom elements are denoted by 1 and 0, respectively. We denote the two element lattice $\{0, 1\}$ by **2**.

A *prime element* of L, is an element $p \in L$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$.

A poset L is called a *complete lattice* if for every subset S of L, both $\sup S = \bigvee S$ and $\inf S = \bigwedge S$ exist. A complete lattice L is called a *frame* if for every subset S and element a of L, $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$.

2.3 [10] An abelian group G with a partial order $\leq$ is called an *abelian ℓ-group* if $(G, \leq)$ is a lattice, and $a \leq b$ implies $a+c \leq b+c$ for all $a, b, c \in G$.

For an abelian ℓ-group G, and $a, b \in G$, defining $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, $|a| = a \vee (-a)$, we have $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+ \wedge a^- = 0$, $|a + b| \leq |a| + |b|$.

A *partially ordered ring* (*po-ring*) is a ring A with a partial order $\leq$ such that $a \leq b$ and $r \geq 0$ imply $ra \leq rb$ and $a + c \leq b + c$ for all $c \in A$. A is called an ℓ-ring if its order is a lattice order.

Let A be a commutative po-ring with an identity 1. A *partially ordered module* M over A is an A-module with an order $\leq$ such that for every $a, b, c \in M$ and $r \in A$, $a \leq b$ and $r \geq 0$ imply $a + c \leq b + c$ and $ra \leq rb$. Then M is called an *ℓ-module* if it is also a lattice.

Suppose that A is an ordered ring and M is an ℓ-module over A. A submodule I of M is called an *ℓ-ideal* if $|x| \leq |a|$ and $a \in I$ imply $x \in I$.

2.4 [7] A *type* of algebras is a sequence τ of function symbols such that a non-negative integer n is assigned to each member λ of τ. This integer is called the arity (or rank) of λ and λ is said to be an n-ary function symbol. The set of all n-ary function symbols is denoted by τ_n.

If τ is a language of algebras, then an *algebra* A of type τ is an ordered pair (A, Λ), where A is a (nonempty) set and Λ is a family of n-ary operations on A indexed by the type τ such that corresponding to each n-ary function symbol λ in τ, there is an n-ary operation λ^A on A.

Let X be a set of (distinct) objects called variables. Let τ be a type of algebras. The set $T(X)$ of terms of type τ over X is the smallest set such that

(i) $X \cup \tau_0 \subset T(X)$.

(ii) If $p_1, ..., p_n \in T(X)$ and $\lambda \in \tau_n$, then the “string” $\lambda(p_1, ..., p_n) \in T(X)$.

For $p \in T(X)$ we often write p as $p(x_1, \cdots, x_n)$ to indicate that the variables occurring in p are among $x_1, \cdots, x_n \in A$. A term p is n-ary if the number of variables appearing explicitly in p is $\leq n$.

2.5 The category of all rings (commutative with identity) and ring homomorphisms between them is denoted by **Rng**. Let A be a fixed ring (commutative with identity). The category of all (unitary) modules over A with module homomorphisms is denoted by $\mathbf{Mod}(A)$. The category of all bounded lattices with lattice homomorphisms preserving $0, 1$ is denoted by $\mathbf{Latt_0^1}$. The category of all ℓ-rings and ℓ-ring homomorphisms between them is denoted by $\ell\mathbf{Rng}$. Let A be a fixed ordered ring. The category of all (unitary) ℓ-modules over A with ℓ-module homomorphisms is denoted by $\ell\mathbf{Mod}(A)$. Note that $\ell\mathbf{Rng}$ and $\ell\mathbf{Mod}(A)$ can be considered as subcategories of $\mathbf{Rng}$ and $\mathbf{Mod}(A)$, respectively.

2.6 [9] Suppose that M is an ℓ-module over a ring A, and L is a frame. A map $c : M \to L$ is said to be a *cozero map* if for every $x, y \in M$ and $a \in A$,

$$c(0) = 0, c(x + y) \leq c(x) \vee c(y), c(ax) \leq c(x),$$

$c(|x|) = c(x)$, and for every $x, y \geq 0$, $c(x \wedge y) = c(x) \wedge c(y)$, $c(x + y) = c(x) \vee c(y)$.

3 Semi-Cozero Maps

In this section, we introduce the semi-cozero maps, and discuss some relations between semi-cozero maps and submodules. Also we show, under this correspondence, the radical and strong semi-cozero maps are related to the radical and prime ideals, respectively. Finally, using of the notion of strong cozero maps, the Zariski topology is generalized to a weaker definition which is called the F-Zariski topology, and some connections between the Zariski topology and the F-Zariski topology are explained in Remark 3.5.

Definition 3.1. Let M be a module over a ring A, and L be a lattice. A *semi-cozero map* from M to L is a map $c : M \to L$ such that

(1) $c(0) = 0$,

(2) for every $x, y \in M$, $c(x + y) \leq c(x) \vee c(y)$,

(3) for every $a \in A$, $c(ax) \leq c(x)$.

In the case of $M = A$, $c : A \to L$ is called a *strong semi-cozero map* if $c(xy) = c(x) \wedge c(y)$, for all $x, y \in A$. And $c : A \to L$ is called a *radical semi-cozero map* if there exists $n \in \mathbb{N}$ such that $c(x^n) = c(x)$ for all $x \in A$.

There is a close correspondence between semi-cozero maps on a module M (ring A) and the submodules of M (ideals of A). Moreover, under this correspondence, radicals and strong semi-cozero maps are related to radicals and prime ideals, respectively. The next two propositions describe these correspondences.

Definition 3.2. Let M be an A-module. Let $c : M \to L$ be a cozero map, $a \in L$, and N be a submodule of M. Define $I(c, a) = \{x \in M : c(x) \leq a\}$

and $c(N,a) : A \to L$ given by $c(N,a)(x) = 0$ if $x \in N$ and $c(N,a)(x) = a$ if $x \notin N$. Define $\ker c = I(c,0)$ and $c_N = c(N,1)$.

Proposition 3.3. *With the above notations, $I(c,a)$ is a submodule of M and $c(N,a)$ is a semi-cozero map from M into L. In the particular case of $L = \mathbf{2}$, $\ker c_N = N$ and $c_{\ker c} = c$.*

Proof. First note that $0 \in I(c,a)$. Now, let $x, y \in I(c,a)$ and $r \in A$. We have $c(x+y) \leq c(x) \vee c(y) \leq a \vee a = a$, $c(rx) \leq c(x) \leq a$, and so $x+y, rx \in I(c,a)$. Hence $I(c,a)$ is a submodule. To check that $c(N,a)$ is a semi-cozero map, first note that $c(N,a)(0) = 0$ because $0 \in N$. Let $x, y \in M$. If $x \notin N$ or $y \notin N$, $c(N,a)(x) \vee c(N,a)(y) = a \geq c(N,a)(x+y)$, otherwise, $x, y \in N$, so $c(N,a)(x) \vee c(N,a)(y) = 0 = c(N,a)(x+y)$, and hence $c(N,a)(x+y) \leq c(N,a)(x) \vee c(N,a)(y)$. Finally, if $x \in N$, $c(N,a)(xy) = 0 = c(N,a)(x)$, and if $x \notin N$, $c(N,a)(xy) \leq a = c(N,a)(x)$. So $c(N,a)(xy) \leq c(N,a)(x)$. Therefore, $c(N,a)$ is a semi-cozero map. To check the second part in the case $L = \mathbf{2}$, we have $x \in \ker c_N \Leftrightarrow c_N(x) = 0 \Leftrightarrow x \in N$, so $\ker c_N = N$. Also, $c_{\ker c}(x) = 0 \Leftrightarrow x \in \ker c \Leftrightarrow c(x) = 0$, thus $c_{\ker c} = c$. □

Proposition 3.4. (1) *If $p \in L$ is a prime element and $c : A \to L$ is a strong semi-cozero map, then $I(c,p)$ is a prime ideal.*

(2) *If $c : A \to L$ is a radical semi-cozero map, then $I(c,p)$ is a radical ideal.*

(3) *The semi-cozero map $c(I,a)$ is strong if and only if I is a prime ideal of A.*

(4) *The semi-cozero map $c(I,a)$ is radical if and only if I is a radical ideal of A.*

(5) *In the case $L = \mathbf{2}$, c is strong if and only if $\ker c$ is a prime ideal.*

(6) *In the case $L = \mathbf{2}$, c is radical if and only if $\ker c$ is a radical ideal.*

Proof. (1) Suppose that $xy \in I(c,p)$. So $c(x) \wedge c(y) = c(xy) \leq p$. Since p is prime, $c(x) \leq p$ or $c(y) \leq p$, thus $x \in I(c,p)$ or $y \in I(c,p)$, and hence $I(c,p)$ is a prime ideal.

(2) Suppose that $x^n \in I(c,a)$, and so $c(x) = c(x^n) \leq a$, hence $x \in I(c,a)$. Therefore, $I(c,a)$ is a radical ideal.

(3) Assume that $c(I,a)$ is strong and $xy \in I$. So $c(I,a)(x) \wedge c(I,a)(y) = c(I,a)(xy) = 0$, and hence, by the definition of $c(I,a)$, $c(I,a)(x) = 0$ or $c(I,a)(y) = 0$. Therefore $x \in I$ or $y \in I$, that is, I is a prime ideal.

Conversely, suppose that I is a prime ideal. Let $x, y \in A$. If $x \notin I$ and $y \notin I$, then, since I is prime, $xy \notin I$, thus $c(I,a)(xy) = a = c(I,a)(x) \wedge c(I,a)(y)$. In other cases, $c(I,a)(xy) = 0 = c(I,a)(x) \wedge c(I,a)(y)$, and hence $c(I,a)$ is strong.

(4) Assume that $c(I,a)$ is radical, and $x^n \in I$. So $c(I,a)(x) = c(I,a)(x^n) = 0$, and hence $x \in I$. Therefore, I is a radical ideal. Conversely, suppose that I is a radical ideal. Then,

$$c(I,a)(x^n) = 0 \Leftrightarrow x^n \in I \Leftrightarrow x \in I \Leftrightarrow c(I,a) = 0$$

So $c(I,a)(x^n) = c(I,a)(x)$ for all $x \in A$.

(5) If c is strong, by (1) we have $\ker c$ is prime. Conversely, if $\ker c$ is prime, $c(xy) = 0 \Leftrightarrow xy \in \ker c \Leftrightarrow (x \in \ker c \;\text{ or }\; \mathrm{y} \in \ker \mathrm{c}) \Leftrightarrow (\mathrm{c(x)} = 0 \;\text{ or }\; \mathrm{c(y)} = 0) \Leftrightarrow \mathrm{c(x)} \wedge \mathrm{c(y)} = 0$. But $L = \mathbf{2}$, hence $c(xy) = c(x) \wedge c(y)$.

(6) Assume that c is a radical semi-cozero map. By (2), $\ker c$ is a radical ideal. Conversely, if $\ker c$ is a radical ideal, then

$$c(x^n) = 0 \Leftrightarrow x^n \in \ker c \Leftrightarrow x \in \ker c \Leftrightarrow c(x) = 0,$$

and since $L = \mathbf{2}$, we have $c(x^n) = c(x)$ for all $x \in A$. □

Remark 3.5. There is a one-one correspondence between prime ideals and strong semi-cozero maps. On the other hand, since prime ideals are used to construct the Zariski topology, strong semi-cozero maps can be used to make a more general Zariski topology which is called the F-*Zariski* topology for a nontrivial filter F of L, as follows:

Let A be a ring, L be a lattice and F be a filter of L such that $0 \notin F$. For every $a \in A$, define

$$U_a^F = \{c : A \to L | c \text{ is a strong semi} - \text{cozero map with } \mathrm{c(a)} \in \mathrm{F}\}.$$

Define $\mathcal{U}_A^F = \{U_a^F : a \in A\}$ and $\Upsilon_A^F = \{c : A \to L | c$ is a strong semi-cozero map$\}$. We have $U_a^F \cap U_b^F = U_{ab}^F$ and $U_0^F = \emptyset$. So, $\mathcal{U}_A^F$ is a basis for a topology on the set Υ_A^F, which is called the F-Zariski topology over the ring A. Note that the usual Zariski topology over A is equivalent to $\Upsilon_A^{\mathbf{1}}$, where $\mathbf{1} = \{1\}$ is the only filter of $\mathbf{2}$.

Now, suppose that $\phi : L \to M$ is a lattice morphism such that $\phi[F] = G$, where F and G are some fixed filters of L and M, respectively. Define

$\overline{\phi}: \Upsilon_A^F \to \Upsilon_A^G$ by $\overline{\phi}(c) = \phi \circ c$. For every $a \in A$ we have $\overline{\phi}^{-1}(U_a^G) = U_a^F$, so $\overline{\phi}$ is a continuous function. On the other hand, for any two filters F_1, F_2 of L such that $F_1 \subseteq F_2$, for all $a \in A$ we have $U_a^{F_1} \subseteq U_a^{F_2}$, and hence we can say that F_2-Zariski topology is weaker than F_1-Zariski topology over A, in other words, $id : \Upsilon_A^{F_1} \to \Upsilon_A^{F_2}$ is a continuous map.

Now, consider the inclusion lattice morphism $i : \mathbf{2} \to L$, given by $i(0) = 0, i(1) = 1_L$, where L is a bounded lattice. Since $i[\{1\}] = \{1_L\} \subseteq F$, $\bar{i} : \Upsilon_A^{\mathbf{1}} \to \Upsilon_A^{\{1_L\}}$ is an embedding of the Zariski topology into the $\{1_L\}$-Zariski topology over the ring A. Since the F-Zariski topology is weaker than the $\{1_L\}$-Zariski topology, it is weaker than the Zariski topology over A.

Moreover, for any lattice morphism $\mathbf{p} : L \to \mathbf{2}$ (any point of L) such that $\mathbf{p}[F] = \{1\}$, there is a continuous function $\overline{\mathbf{p}} : \Upsilon_A^F \to \Upsilon_A^{\mathbf{1}}$, from the F-Zariski topology to the Zariski topology over A.

4 Lattice-valued maps satisfying equalities and inequalities

In this section, suppose that A is a ring, $\mathcal{B}$ is a subcategory of **Rng** or **Mod**(A), $\mathcal{A}$ is a subcategory of **Rng**, and $\mathcal{L}$ is a subcategory of $\mathbf{Latt_0^1}$. Suppose that $\mathcal{B}$ consists of objects which are of the same type τ as algebraic structures. For example, if $\mathcal{B} = \mathbf{Rng}$, then every object B of $\mathcal{B}$ is of type $\tau =< +, \cdot, 0, 1 >$, and if $\mathcal{B} = \ell\mathbf{Rng}$ then every object B of $\mathcal{B}$ is of type $\tau =< +, \cdot, \vee, \wedge, 0, 1 >$. In this case, we say that $\mathcal{B}$ is a category of type τ. We assume a similar perspective for the category $\mathcal{L}$. Now, suppose that $\mathcal{B}$ is a subcategory of **Rng** or **Mod**(A) of type τ and let $\mathcal{L}$ be a subcategory of $\mathbf{Latt_0^1}$ of type τ'. Let X be a set. The set of all term functions of type τ is denoted by $T_\tau(X)$, and similarly define the notation $T_{\tau'}(X)$.

Definition 4.1. Let $p = p(x_1, \cdots, x_n)$ and $q = q(x_1, \cdots, x_n)$ be two n-ary term functions of type τ and τ', respectively. Let $\nu : T_\tau(X) \to T_{\tau'}(X)$ be a map. An *equality* is a pair $\nu(p(x_1, \cdots, x_n)) = q(\nu(x_1), \cdots, \nu(x_n))$ and is denoted by $\nu(p) = q(\nu)$. Also an *inequality* is a relation $\nu(p(x_1, \cdots, x_n)) \leq q(\nu(x_1), \cdots, \nu(x_n))$, and is denoted by $\nu(p) \leq q(\nu)$.

Definition 4.2. Let $c : B \to L$ be a map. We say that c *satisfies the equality* $\nu(p) = q(\nu)$, if $c(p(b_1, \cdots, b_n)) = q(c(b_1), \cdots, c(b_n))$ for all $b_i \in B$. And we say that c *satisfies the inequality* $\nu(p) \leq q(\nu)$, if $c(p(b_1, \cdots, b_n)) \leq$

$q(c(b_1), \cdots, c(b_n))$ for all $b_i \in B$. Let $\sum$ be a set of equalities and inequalities. We say that $c : B \to L$ satisfies $\sum$ if c satisfies all the elements of $\sum$.

Notation 4.3. Let $\mathcal{B}$ be a subcategory of **Rng** or $\mathbf{Mod}(A)$, and $\mathcal{L}$ be a subcategory of $\mathbf{Latt_0^1}$. Suppose that B_1, B_2 are two objects in $\mathcal{B}$ and L_1, L_2 are two objects in $\mathcal{L}$. Let $c_1 : B_1 \to L_1$ and $c_2 : B_2 \to L_2$ be two maps.

A *morphism* from c_1 to c_2 is a pair (α, f), where $\alpha : B_1 \to B_2$ is a morphism in $\mathcal{B}$ and $f : L_1 \to L_2$ is a morphism in $\mathcal{L}$ such that

$$\begin{array}{ccc} B_1 & \xrightarrow{\alpha} & B_2 \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_2} \\ L_1 & \xrightarrow{f} & L_2 \end{array}$$

commutes.

The resulting category is denoted by $\mathcal{B}\mathbf{map}\mathcal{L}$. Note that the composition is defined by $(\alpha_2, f_2)(\alpha_1, f_1) = (\alpha_2\alpha_1, f_2 f_1)$, where $(\alpha_1, f_1) : c \to c'$ and $(\alpha_2, f_2) : c' \to c''$ are two morphisms. For a given $\mathcal{B}$, the full subcategories of $\mathcal{B}\mathbf{map}\mathcal{L}$, consisting of all semi-cozero maps, all strong semi-cozero maps, all radical semi-cozero maps, and all cozaro maps, are denoted by $\mathcal{B}\mathbf{SemCoz}\mathcal{L}$, $\mathcal{B}\mathbf{StSemCoz}\mathcal{L}$, $\mathcal{B}\mathbf{RadSemCoz}\mathcal{L}$, and $\mathcal{B}\mathbf{Coz}\mathcal{L}$, respectively. Also, let $\sum$ be a set of equalities and inequalities. The full subcategory of all $c \in \mathcal{B}\mathbf{map}\mathcal{L}$ satisfying $\sum$ is denoted by $\mathcal{B}\sum\mathbf{map}\mathcal{L}$.

Let A be a ring, $\mathcal{B}$ be a subcategory of $\mathbf{Mod}(A)$, and $\mathcal{L}$ be a subcategory of $\mathbf{Latt_0^1}$. Consider the following equalities and inequalities of type $< +, 0, (a.-)_{a\in A} >$:

C1) $\nu(0) = 0$,

C2) $\nu(x + y) \le \nu(x) \vee \nu(y)$,

For every $a \in A$,

C3a) $\nu(ax) \le \nu(x)$.

And for the types $< +, 0, (a.-)_{a\in A}, \vee, \wedge >$ or $< +, ., 0, 1, \vee, \wedge >$ (ℓ-modules over an ordered ring A, in particular ℓ-rings)

C4) $\nu(|x|) = \nu(x)$,

C5) $\nu(|x| \wedge |y|) = \nu(|x|) \wedge \nu(|y|)$,

C6) $\nu(|x| + |y|) = \nu(|x|) \vee \nu(|y|)$.

Let $\mathcal{A}$ be a subcategory of **Rng**. Consider the following equalities and inequalities of type $< +, ., 0, 1 >$:

C7) $\nu(xy) \leq \nu(x)$,

C8) $\nu(xy) = \nu(x) \wedge \nu(y)$,

For every $n = 1, 2, \cdots$,

C9n) $\nu(x^n) = \nu(x)$.

Define

$\sum_1 = \{C1, C2\} \cup \{C3a : a \in A\}$,

$\sum_2 = \{C1, C2, C7\}$,

$\sum_3 = \{C1, C2, C8\}$,

$\sum_4 = \{C1, C2\} \cup \{C9n : n = 1, 2, \cdots\}$, and

$\sum_5 = \{C1, C2, C4, C5, C6\} \cup \{C3a : a \in A\}$.

By the above notations, we have

$\mathcal{B}\mathbf{SemCoz}\mathcal{L} = \mathcal{B}\sum_{\mathbf{1}} \mathbf{map}\mathcal{L}$,

$\mathcal{A}\mathbf{SemCoz}\mathcal{L} = \mathcal{A}\sum_{\mathbf{2}} \mathbf{map}\mathcal{L}$,

$\mathcal{A}\mathbf{StSemCoz}\mathcal{L} = \mathcal{A}\sum_{\mathbf{3}} \mathbf{map}\mathcal{L}$,

$\mathcal{A}\mathbf{RaSemCoz}\mathcal{L} = \mathcal{A}\sum_{\mathbf{4}} \mathbf{map}\mathcal{L}$, and

$\mathcal{B}\mathbf{Coz}\mathcal{L} = \mathcal{B}\sum_{\mathbf{5}} \mathbf{map}\mathcal{L}$

A subcategory $\mathcal{C}$ of $\mathcal{B}\mathbf{map}\mathcal{L}$ is called *equational* if there is a $\sum$ such that $\mathcal{C} = \mathcal{B}\sum \mathbf{map}\mathcal{L}$.

Definition 4.4. A subcategory $\mathcal{C}$ of $\mathcal{B}\mathbf{map}\mathcal{L}$ is called $\mathcal{B}$-*closed* if for every morphism $\alpha : B_1 \to B$ in $\mathcal{B}$ and an object $c : B \to L$ of $\mathcal{C}$, $c\alpha$ belongs to $\mathcal{C}$.

Also, it is $\mathcal{L}$-*closed* if for every morphism $f : L \to L_1$ in $\mathcal{L}$ and an object $c : B \to L$ of $\mathcal{C}$, fc belongs to $\mathcal{C}$.

Proposition 4.5. *Let $\sum$ be a set of equalities and inequalities. Then $\mathcal{B}\sum \mathbf{map}\mathcal{L}$ is both $\mathcal{B}$-closed and $\mathcal{L}$-closed.*

Proof. Suppose that $c : B \to L$ satisfies $\sum$. Let $\alpha : B_1 \to B$ and $f : L \to L_1$ be morphisms in $\mathcal{B}$ and $\mathcal{L}$, respectively. Let $\nu(p) = q(\nu)$ be an equality of $\sum$. Let $b_1, \cdots, b_n \in B$. Then

$$\begin{aligned} fc\alpha(p(b_1, \cdots, b_n)) &= fc(p(\alpha(b_1), \cdots, \alpha(b_n))) \\ &= f(q(c\alpha(b_1), \cdots, \alpha(b_n))) \\ &= q(f(c\alpha(b_1)), \cdots, f(\alpha(b_n))). \end{aligned}$$

So, $fc\alpha$ satisfies $\nu(p) = q(\nu)$. Let $\nu(p) \leq q(\nu)$ be an inequality in $\sum$. Then,

$$\begin{aligned}
&fc\alpha(p(b_1, \cdots, b_n)) \vee q(f(c\alpha(b_1)), \cdots, f(c\alpha(b_n))) \\
&= f(c(p(\alpha(b_1), \cdots, \alpha(b_n)))) \vee f(q(c\alpha(b_1), \cdots, c\alpha(b_n))) \\
&= f(c(p(\alpha(b_1), \cdots, \alpha(b_n))) \vee q((c\alpha(b_1)), \cdots, c\alpha(b_n))) \\
&= f(q(c\alpha(b_1), \cdots, c\alpha(b_n))) \\
&= q((fc\alpha(b_1)), \cdots, fc\alpha(b_n))).
\end{aligned}$$

Thus $fc\alpha$ satisfies $\nu(p) \leq q(\nu)$. Therefore $fc\alpha$ satisfies $\sum$. □

Theorem 4.6. *Let $c : A \to L$ and $c' : A' \to L'$ be two objects of $\mathcal{B}$**map**$\mathcal{L}$. Suppose that $(\alpha, f) : c \to c'$ is a morphism in the category $\mathcal{B}$**map**$\mathcal{L}$.*
(1) If α is onto and c satisfies $\sum$, then so does c'.
(2) If f is one-one and c' satisfies $\sum$, then so does c.

Proof. (1) Suppose that c satisfies $\sum$. Let $\nu(p) = q(\nu)$ be an equality of $\sum$. Let $y_1, \cdots, y_n \in B_1$. Since α is onto, there are $x_1, \cdots, x_n \in B$ such that $\alpha(x_i) = y_i$. We have

$$\begin{aligned}
c'(p(y_1, \cdots, y_n)) &= c'(p(\alpha(x_1), \cdots, \alpha(x_n))) \\
&= c'\alpha(p(x_1, \cdots, x_n)) \\
&= fc(p(x_1, \cdots, x_n)) \\
&= f(q(c(x_1), \cdots, c(x_n))) \\
&= q(fc(x_1), \cdots, fc(x_n)) \\
&= q(c'\alpha(x_1), \cdots, c'\alpha(x_n)) \\
&= q(c'(y_1), \cdots, c'(y_n)).
\end{aligned}$$

Let $\nu(p) \leq q(\nu)$ be an inequality in $\sum$. We have

$$\begin{aligned}
c'(p(y_1, \cdots, y_n)) &= c'(p(\alpha(x_1), \cdots, \alpha(x_n))) \\
&= c'\alpha(p(x_1, \cdots, x_n)) \\
&= fc(p(x_1, \cdots, x_n)) \\
&\leq f(q(c(x_1), \cdots, c(x_n))) \\
&= q(fc(x_1), \cdots, fc(x_n)) \\
&= q(c'\alpha(x_1), \cdots, c'\alpha(x_n)) \\
&= q(c'(y_1), \cdots, c'(y_n)),
\end{aligned}$$

and so c' satisfies $\sum$.

(2) Suppose that c' satisfies $\sum$. Let $\nu(p) = q(\nu)$ be an equality in $\sum$. Then

$$\begin{aligned} fc(p(x_1, \cdots, x_n)) &= c'\alpha(p(x_1, \cdots, x_n)) \\ &= c'(p(\alpha(x_1), \cdots, \alpha(x_n))) \\ &= q(c'\alpha(x_1), \cdots, c'\alpha(x_n))) \\ &= q(fc(x_1), \cdots, fc(x_n)) \\ &= f(q(c(x_1), \cdots, c(x_n))). \end{aligned}$$

Since f is one-one, $c(p(x_1, \cdots, x_n)) = q(c(x_1), \cdots, c(x_n))$.

Let $\nu(p) \leq q(\nu)$ be an inequality in $\sum$. Then

$$\begin{aligned} f(c(p(x_1, \cdots, x_n)) &= c'\alpha(p(x_1, \cdots, x_n))) \\ &= c'(p(\alpha(x_1), \cdots, \alpha(x_n)) \\ &\leq q(c'\alpha(x_1), \cdots, c'\alpha(x_n)) \\ &= q(fc(x_1), \cdots, fc(x_n)) \\ &= f(q(c(x_1), \cdots, c(x_n))). \end{aligned}$$

Since f is one-one, $c(p(x_1, \cdots, x_n)) \leq q(c(x_1), \cdots, c(x_n))$, and hence $c(p(x_1, \cdots, x_n)) \leq q(c(x_1), \cdots, c(x_n))$. Therefore, c satisfies $\sum$. $\square$

5 $\mathcal{C}$-reticulation and Free lattice-valued maps

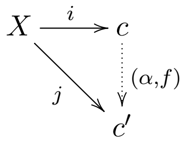
In this section, we discuss free objects in a subcategory $\mathcal{C}$ of the category $\mathcal{B}$**map**$\mathcal{L}$. To do this, we introduce a concept named *$\mathcal{C}$-reticulation* of an object B of $\mathcal{B}$. Then, we give some methods to construct the $\mathcal{C}$-reticulation for $\mathcal{C} = \mathcal{B}\sum$ **map**$\mathcal{L}$. Also, some relations between the concepts reticulation and satisfying equalities are studied.

Definition 5.1. Suppose that $\mathcal{B}$ is a subcategory of **Rng** or **Mod**(A), and suppose that $\mathcal{L}$ is a subcategory of $\mathbf{Latt_0^1}$. Let X be a set.

(1) Let $c : B \to L$ be a map. An *arrow* from X to c is a map $i : X \to B$, and is denoted by $i : X \to c$. For a morphism $(\alpha, f) : c \to c'$ in $\mathcal{B}$**map**$\mathcal{L}$ and an arrow $i : X \to c$, the composition of the morphism (α, f) and i is defined by αi, that is, $(\alpha, f) \circ i = \alpha i$.

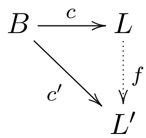
(2) Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$**map**$\mathcal{L}$. We say that $c : B \to L$ is *free* in $\mathcal{C}$, with respect to the arrow $i : X \to c$, if for every arrow $j : X \to c'$,

where c' in $\mathcal{C}$, there exists a unique morphism $(\alpha, f) : c \to c'$ in $\mathcal{C}$ such that $(\alpha, f) \circ i = j$ (that is $\alpha i = j$). In the other word,



commutes. Also, we say that c is a free object on X in $\mathcal{C}$.

Definition 5.2. Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$**map**$\mathcal{L}$. A map $c : B \to L$ is called a $\mathcal{C}$-*reticulation* of B if for every map $c' : B \to L'$ of $\mathcal{C}$, there exists a unique morphism $f : L \to L'$ in $\mathcal{L}$ such that



commutes.

Theorem 5.3. *Let $\mathcal{C}$ be a $\mathcal{B}$-closed subcategory of $\mathcal{B}$**map**$\mathcal{L}$. If $B(X)$ is a free object in the category $\mathcal{B}$ with respect to a map $i : X \to B$ and $c : B(X) \to L$ is a $\mathcal{C}$-reticulation of $B(X)$, then $c : B(X) \to L$ is a free object in $\mathcal{C}$, with respect to the arrow $i : X \to c$.*

Proof. Let $c' : B' \to L'$ be an object in $\mathcal{C}$. Assume that $j : X \to c'$ is an arrow. Since $B(X)$ is free, there is a unique morphism $\alpha : B(X) \to B'$ in $\mathcal{B}$ such that $\alpha i = j$. Consider the map $c'\alpha : B(X) \to L'$. Since $\mathcal{C}$ is $\mathcal{B}$-closed, $c'\alpha \in \mathcal{C}$ and, since the map $c : B(X) \to L$ is a $\mathcal{C}$-reticulation of $B(X)$, there is a unique morphism $f : L \to L'$ in $\mathcal{L}$ such that $fc = c'\alpha$. So $(\alpha, f) : c \to c'$ is a morphism in $\mathcal{C}$ such that $(\alpha, f) \circ i = \alpha i = j$. That is, $c : B \to L$ is a free object in $\mathcal{C}$ with respect to $i : X \to c$. □

Definition 5.4. Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$**map**$\mathcal{L}$. We say that $\mathcal{C}$ *has enough objects* if for every $B' \in \mathcal{B}$ there is a map $c' : B' \to L'$ such that $c' \in \mathcal{C}$.

Theorem 5.5. *Suppose that $\mathcal{C}$ is a subcategory of $\mathcal{B}$**map**$\mathcal{L}$ which has enough objects. If $c : B \to L$ is free in $\mathcal{C}$ with respect to $i : X \to c$, then B is free in $\mathcal{B}$ with respect to $i : X \to B$.*

Proof. Let $j : X \to B'$. By hypothesis, there is a map $c' : B' \to L'$ in $\mathcal{C}$. Since c is free in $\mathcal{C}$ with respect to $i : X \to c$, there is a unique morphism $(\alpha, f) : c \to c'$ such that $(\alpha, f)i = j$. Since $(\alpha, f)i = \alpha i$, there is a unique $\alpha : B \to B'$ such that $\alpha i = j$. Therefore, B is free in $\mathcal{B}$ with respect to $i : X \to B$. □

Let $\sum$ be a set of equalities and inequalities. Now, we construct a $\mathcal{C}$-reticulation of B, for $\mathcal{C} = \mathcal{B}\sum\mathbf{map}\mathcal{L}$ and a given $B \in \mathcal{B}$. Suppose that free objects exist in the category $\mathcal{L}$. The congruences of the objects in the category of $\mathcal{L}$ are called $\mathcal{L}$-congruence.

Let B be an object of $\mathcal{B}$. Consider the set of symbols indexed by B, $X = \{c_x : x \in B\}$. Suppose that $L(X)$ is the free object on X in $\mathcal{L}$. Let Θ be the $\mathcal{L}$-congruence generated by the following subset of $L(X) \times L(X)$: $\{(c_{p(b_1,\cdots,b_n)}, q(c_{b_1},\cdots,c_{b_n})) : \nu(p) = q(\nu)$ is an equality in $\sum, b_1,\cdots,b_n \in B\} \cup \{(c_{p(b_1,\cdots,b_n)} \vee q(c_{b_1},\cdots,c_{b_n}), q(c_{b_1},\cdots,c_{b_n})) : \nu(p) \leq q(\nu)$ is an inequality in $\sum, b_1,\cdots,b_n \in B\}$

Let $L(B) = \frac{L(X)}{\Theta}$ and $c_B : B \to L(B)$ be given by $c_B(x) = \overline{c_x} = c_x/\Theta$.

Theorem 5.6. *For $\mathcal{C} = \mathcal{B}\sum\mathbf{map}\mathcal{L}$, the map $c_B : B \to L(B)$ is a $\mathcal{C}$-reticulation of B.*

Proof. First we show that c_B is a $\sum$-map. Let $\nu(p) = q(\nu)$ be an equality in $\sum$. For every $b_1,\cdots,b_n \in B$,

$$\begin{aligned} c_B(p(b_1,\cdots,b_n)) &= \overline{c_{p(b_1,\cdots,b_n)}} \\ &= \overline{q(c_{b_1},\cdots,c_{b_n})} \\ &= q(\overline{c_{b_1}},\cdots,\overline{c_{b_n}}) \\ &= q(c_B(b_1),\cdots,c_B(b_n)). \end{aligned}$$

So, c_B satisfies the equality $\nu(p) = q(\nu)$. Let $\nu(p) \leq q(\nu)$ be an inequality in $\sum$. For every $b_1,\cdots,b_n \in B$,

$$\begin{aligned} c_B(p(b_1,\cdots,b_n)) \vee q(c_B(b_1),\cdots,c_B(b_n)) &= \overline{c_{p(b_1,\cdots,b_n)}} \vee q(\overline{c_{b_1}},\cdots,\overline{c_{b_n}}) \\ &= \overline{c_{p(b_1,\cdots,b_n)}} \vee \overline{q(c_{b_1},\cdots,c_{b_n})} \\ &= \overline{c_{p(b_1,\cdots,b_n)} \vee q(c_{b_1},\cdots,c_{b_n})} \\ &= \overline{q(c_{b_1},\cdots,c_{b_n})} \\ &= q(\overline{c_{b_1}},\cdots,\overline{c_{b_n}}) \\ &= q(c_B(b_1),\cdots,c_B(b_n)). \end{aligned}$$

Hence, $c_B(p(b_1, \cdots, b_n)) \leq q(c_B(b_1), \cdots, c_B(b_n))$, and thus c_B satisfies the inequality $\nu(p) \leq q(\nu)$. Therefore, c_B satisfies $\sum$.

Now, let $c' : B \to L'$ be a $\sum$-map. Consider the map $j : X \to L'$ given by $j(c_x) = c'(x)$ for all $x \in B$. Since $L(X)$ is an $\mathcal{L}$-free object on X, there exists a unique morphism $f : L(X) \to L'$ in $\mathcal{L}$ such that $fi = j$, where $i : X \to L(X)$ is the inclusion map. Now, we show that $\Theta \subseteq \ker f$. Let $\nu(p) = q(\nu)$ be an equality in $\sum$. For every $b_1, \cdots, b_n \in B$,

$$\begin{aligned} f(c_{p(b_1,\cdots,b_n)}) &= fi(c_{p(b_1,\cdots,b_n)}) \\ &= j(c_{p(b_1,\cdots,b_n)}) \\ &= c'(p(b_1, \cdots, b_n)) \\ &= q(c'(b_1), \cdots, c'(b_n)) \\ &= q(j(c_{b_1}), \cdots, j(c_{b_n})) \\ &= q(fi(c_{b_1}), \cdots, fi(c_{b_n})) \\ &= q(f(c_{b_1}), \cdots, f(c_{b_n})) \\ &= f(q(c_{b_1}, \cdots, c_{b_n})). \end{aligned}$$

So, we have $(c_{p(b_1,\cdots,b_n)}, q(c_{b_1}, \cdots, c_{b_n})) \in \ker f$.

Now, let $\nu(p) \leq q(\nu)$ be an inequality in $\sum$. For every $b_1, \cdots, b_n \in B$,

$$\begin{aligned} f(c_{p(b_1,\cdots,b_n)} \vee q(c_{b_1}, \cdots, c_{b_n})) &= f(c_{p(b_1,\cdots,b_n)}) \vee q(f(c_{b_1}), \cdots, f(c_{b_n})) \\ &= j(c_{p(b_1,\cdots,b_n)}) \vee q(j(c_{b_1}), \cdots, j(c_{b_n})) \\ &= c'(p(b_1, \cdots, b_n)) \vee q(c'(b_1), \cdots, c'(b_n)) \\ &= q(c'(b_1), \cdots, c'(b_n)) \\ &= q(f(c_{b_1}), \cdots, f(c_{b_n})) \\ &= f(q(c_{b_1}), \cdots, c_{b_n}). \end{aligned}$$

So we have $(c_{p(b_1,\cdots,b_n)} \vee q(c_{b_1}, \cdots, c_{b_n}), q(c_{b_1}, \cdots, c_{b_n})) \in \ker f$. Therefore $\Theta \subseteq \ker f$, by the definition of Θ. Define $\overline{f} : \frac{L}{\Theta} = L(B) \to L'$ by $\overline{f}(a/\Theta) = f(a)$. Then, $\overline{f}$ is a well-defined $\mathcal{L}$-morphism. But $\overline{f}(\overline{c}_x) = f(c_x)$ for all $x \in B$, so $\overline{f}c_B = fc_B = jc_B = c'$. To show the uniqueness of $\overline{f}$, let $g_1, g_2 : L(B) \to L'$ be such that $g_1c_B = c' = g_2c_B$. Consider the map $j : X \to L'$. For the bijection map $\delta : X \to B$ given by $\delta(c_x) = x$, we have $c'\delta = j$. Since $L(X)$ is $\mathcal{L}$-free, there is a unique $\mathcal{L}$-morphism $h : L(X) \to L'$ such that $hi = j$. Consider the natural quotient map $\gamma : L(X) \to L(B) = L(X)/\Theta$. Thus $g_1\gamma i = g_1c_B\delta = c'\delta = j$. Similarly, $g_2\gamma i = j$. So, by the uniqueness of h, $g_1\gamma = h = g_2\gamma$. Since γ is onto, $g_1 = g_2$. It proves that $c_B : B \to L(B)$ is a $\mathcal{C}$-reticulation of B. □

Definition 5.7. Let $\mathcal{C}$ be a subcategory of $\mathcal{B}\mathbf{map}\mathcal{L}$ and. $B \in \mathcal{B}$. The subcategory of $\mathcal{C}$ consisting of all maps $B \to L$ is denoted by $\mathcal{C}^B$, whose morphisms are of the form (id_B, f).

By the notation of $\mathcal{C}^B$, we have the following lemma the proof of which is straightforward.

Lemma 5.8. *A map $B \to L$ in $\mathcal{C}$ is a $\mathcal{C}$-reticulation of B if and only if it is an initial object of $\mathcal{C}^B$.*

Corollary 5.9. *Let $\mathcal{C}$ be a $\mathcal{L}$-closed subcategory of $\mathcal{B}\mathbf{map}\mathcal{L}$. If $c : B \to L$ is a $\mathcal{C}$-reticulation of B and $l : L \to L_1$ is an isomorphism in $\mathcal{L}$, then $lc : B \to L_1$ is a $\mathcal{C}$-reticulation of B. Conversely, if $c' : B \to L_1$ is another $\mathcal{C}$-reticulation of B, then there is a unique isomorphism $l : L \to L_1$ such that $lc = c'$.*

Proof. It is clear, using Lemma 5.8 and noting that in a category, isomorphisms preserves initial objects, and also two initial objects are isomorphic. □

Lemma 5.10. *Let $c : B \to L$ be a $\mathcal{C}$-reticulation of B. Suppose that $\mathcal{C}$ is $\mathcal{B}$-closed. If $\alpha : B_1 \to B$ is an isomorphism in $\mathcal{B}$, then $c\alpha : B_1 \to L$ is a $\mathcal{C}$-reticulation of B_1.*

Proof. Suppose that $\kappa : B_1 \to L_1$ is an object in $\mathcal{C}$. Consider the map $\kappa\alpha^{-1} : B \to L_1$. Since $\mathcal{C}$ is $\mathcal{B}$-closed, and $c : B \to L$ is a $\mathcal{C}$-reticulation of B, there exists a unique $f : L \to L_1$ in $\mathcal{L}$ such that $fc = \kappa\alpha^{-1}$, so $fc\alpha = \kappa$. Therefore $c\alpha : B_1 \to L$ is a $\mathcal{C}$-reticulation of B_1. □

Proposition 5.11. *Suppose that $\mathcal{C} = \mathcal{B}\sum\mathbf{map}\mathcal{L}$ has enough objects. If $c : B \to L$ is free in $\mathcal{C}$ then B is free and c is a $\mathcal{C}$-reticulation of B.*

Proof. By Theorem 5.5, B is free on a set X in $\mathcal{B}$. By Theorem 5.6, $c_B : B \to L(B)$ is a $\mathcal{C}$-reticulation of B. Also, by Proposition 4.5 and Theorem 5.3, $c_B : B \to L(B)$ is free on X in the category $\mathcal{C}$. Suppose that c and c_B are free over maps $i : X \to B$ and $j : X \to B$, respectively. Hence there is an isomorphism $(\alpha, f) : c \to c_B$ in $\mathcal{C}$, such that $\alpha j = i$. Thus $c_B\alpha = fc$, so $f^{-1}c_B\alpha = c$. By Corollary 5.9 and Lemma 5.10, $c = f^{-1}c_B\alpha : B \to L$ is a $\mathcal{C}$-reticulation of B. □

Theorem 5.12. *Let $\mathcal{B}$ be a subcategory of* **Rng** *or of* $\mathbf{Mod}(A)$*, $\mathcal{L}$ be a subcategory of* $\mathbf{Latt_0^1}$*, $\sum$ be a set of equalities and inequalities, and $\mathcal{C}$ be a subcategory of $\mathcal{B}\mathbf{map}\mathcal{L}$. If $c : B \to L$ is a $\mathcal{C}$-reticulation of B satisfying $\sum$, then every map $c' : B \to L' \in \mathcal{C}$ satisfies $\sum$.*

Proof. Let $c : B \to L$ be a $\mathcal{C}$-reticulation of B, and $c' : B \to L'$ be an arbitrary map in $\mathcal{C}$. Since c is a $\mathcal{C}$-reticulation of B, there is a unique morphism $f : L \to L'$ in $\mathcal{L}$ such that $fc = c'$, and so $(id_B, f) : c \to c'$ is a morphism in $\mathcal{C}$. By Theorem 4.6(1), since c satisfies $\sum$, c' satisfies $\sum$, too. $\square$

Remark 5.13. Theorem 5.12 has some beautiful consequences. For example, consider $\mathcal{C} = \mathcal{A}\mathbf{SemCoz}\mathcal{L}$ and $\sum = \{C8\}$. Let $c : A \to L$ be a $\mathcal{C}$-reticulation of A. Suppose that $\mathbf{2} \in \mathcal{L}$. If A in $\mathcal{A}$ has an ideal which is not prime, then no $\mathcal{C}$-reticulation of A is strong. Because, considering $c_I : A \to \mathbf{2}$ in $\mathcal{C}$, where I is an ideal of A which is not prime, by Proposition 3.4(2), c_I is not strong, so using Theorem 5.12, any $\mathcal{C}$-reticulation of A does not satisfies $\sum$, hence it can not to be strong.

The following theorem describes generally the reason of Remark 5.13 in the sense of reticulation.

Theorem 5.14. *Let $\mathcal{B}$ be a subcategory of* **Rng** *or of* $\mathbf{Mod}(A)$*, $\mathcal{L}$ be a subcategory of* $\mathbf{Latt_0^1}$*, $\sum$ be a set of equalities and inequalities. Suppose that $\sigma \notin \sum$. Let $\mathcal{C} = \mathcal{B}\sum\mathbf{map}\mathcal{L}$. If $c : B \to L$ is a $\mathcal{C}$-reticulation satisfying σ, then it is a $\mathcal{C}_1$-reticulation, where $\mathcal{C}_1 = \mathcal{B}\sum_{\mathbf{1}}\mathbf{map}\mathcal{L}$ and $\sum_1 = \sum\cup\{\sigma\}$.*

Proof. Using Theorem 5.6, let $c_1 : B \to L$ be a $\mathcal{C}_1$-reticulation. Since c is a $\mathcal{C}$-reticulation, there exists a unique lattice map $f : L \to L_1$ such that $fc = c_1$. Since c satisfies σ, so, using $\mathcal{C}_1$-reticulation of c_1, there exists a unique map $g : L_1 \to L$ such that $gc_1 = c$. Therefore, $fg = id_L$ and $gf = id_{L_1}$. Hence, $(id_B, f) : c \to c_1$ is an isomorphism, which completes the proof. $\square$

Theorem 5.14 is the main motivation of the following definition.

Definition 5.15. Let $\mathcal{B}$ be a subcategory of **Rng** or of $\mathbf{Mod}(A)$, $\mathcal{L}$ be a subcategory of $\mathbf{Latt_0^1}$, $\sum$ be a set of equalities and inequalities. Suppose that $\sigma \notin \sum$. Let $\mathcal{C} = \mathcal{B}\sum\mathbf{map}\mathcal{L}$. Let $\sum_1 = \sum\cup\{\sigma\}$, and $\mathcal{C}_1 = \mathcal{B}\sum_{\mathbf{1}}\mathbf{map}\mathcal{L}$.

We say that $\sum$ *generates* σ *over* $\mathcal{B}$, and we write $\sum \models^{\mathcal{B}} \sigma$, if any $\mathcal{C}$-reticulation of B is also a $\mathcal{C}_1$-reticulation of B, for all B in $\mathcal{B}$.

We finish the paper by the following corollary which is implied from Theorem 5.12

Corollary 5.16. *If c satisfies $\sum$ and $\sum \models^{\mathcal{B}} \sigma$, then c satisfies σ.*

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