Classification of monoids by Condition \((PWP_{ssc})\) of right acts

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Abstract. Condition \((PWP)\) which was introduced by Laan is related to the concept of flatness of acts over monoids. Golchin and Mohammadzadeh introduced Condition \((PWP_E)\) as a generalization of Condition \((PWP)\).

In this paper, we introduce Condition \((PWP_{ssc})\) which is much easier to check than Conditions \((PWP)\) and \((PWP_E)\) and does not imply them. Also principally weakly flat is a generalization of this condition. At first, general properties of Condition \((PWP_{ssc})\) will be given. Finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition \((PWP_{ssc})\) and also a classification of monoids \(S\) will be given for which all right \(S\)-acts satisfying some other flatness properties have Condition \((PWP_{ssc})\).

1 Introduction

For a monoid \(S\), with 1 as its identity, a set \(A\) (we consider nonempty) is called a right \(S\)-act, usually denoted by \(A_S\) (or simply \(A\)), if \(S\) acts on \(A\)

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unitarian from the right, that is, there exists a mapping \( A \times S \rightarrow A, (a, s) \mapsto as \), satisfying the conditions \((as)t = a(st)\) and \(a1 = a\), for all \(a \in A\) and \(s, t \in S\). Left acts are defined dually. The study of flatness properties of \(S\)-acts in general began in the early 1970s and a comprehensive survey of this research (up until the year 2000) is found in [14]. In [16], the principally weak form of Condition \((P)\) is defined, called Condition \((PWP)\), and in [8] the weak form of Condition \((PWP)\) is defined, called Condition \((PWP_E)\). In this paper we introduce Condition \((PWP_{ssc})\) and compare it with principally weakly flat, Condition \((PWP)\) and Condition \((PWP_E)\). At first general properties of this condition will be given and finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition \((PWP_{ssc})\) and also a classification of monoids \(S\) will be given for which all right \(S\)-acts satisfying some other flatness properties have Condition \((PWP_{ssc})\).

From now on by \(S\)-act we mean right \(S\)-act. Throughout this paper, \(S\) will always denote a monoid and \(A\) an \(S\)-act. For basic definitions and terminologies relating to semigroups and acts over monoids we refer the reader to [12] and [14].

2 General properties

In this section we introduce Condition \((PWP_{ssc})\) and give some results of it. We show that this condition can be transferred from the product of \(S\)-acts to their components and we give equivalences for which \(S^n\) satisfies Condition \((PWP_{ssc})\). We show that Condition \((PWP_{ssc})\) implies principal weak flatness but the converse is not true. For left PSF monoid, we show that the converse is also true. Also we see that Condition \((PWP_{ssc})\) does not imply Condition \((PWP)\) (Condition \((PWP_E)\)) and vise versa.

An element \(s\) of \(S\) is called right \(e\)-cancellable, for an idempotent \(e \in S\), if \(s = es\) and \(\ker\rho_s \leq \ker\rho_e\) (\(\rho_x\) is the right translation on \(S\), for every \(x \in S\), that is, \(\rho_x : S \rightarrow S, t \mapsto tx\), for every \(t \in S\)). \(S\) is called left \(PP\) if every principal left ideal of \(S\) is projective as a left \(S\)-act. This is equivalent to saying that every element \(s \in S\) is right \(e\)-cancellable for some idempotent \(e \in S\) (see [5]). \(S\) is called left PSF if every principal left ideal of \(S\) is strongly flat as a left \(S\)-act. This is equivalent to saying that \(S\) is right semi-cancellative, that is, whenever \(su = s'u\), for \(s, s', u \in S\), there exists \(r \in S\) such that \(u = ru\) and \(sr = s'r\) (see [17]).
We recall, from [14], that $A$ satisfies Condition (E) if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A) (\exists u \in S) (a = a'u \text{ and } us = us'),$$

and $A$ satisfies Condition (P) if for all $a, a' \in A$, $s, s' \in S$,

$$as = a's' \Rightarrow (\exists a'' \in A) (\exists u, u' \in S) (a = a''u, \ a' = a''u', \text{ and } us = u's'),$$

and $A$ is strongly flat if and only if it satisfies both Conditions (P) and (E).

Recall, from [16], that $A$ satisfies Condition (PWP) if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists u \in S) (au = a'u, \text{ and } us = vs).$$

**Definition 2.1.** We say that $A$ satisfies Condition strongly semi cancellative-(PWP) or Condition (PWPssc) if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists u \in S) (au = a'u, \text{ and } us = s).$$

**Theorem 2.2.** The following statements hold:

1. The one element act $\Theta_S$ satisfies Condition (PWPssc).
2. If $A$ satisfies Condition (PWPssc), then every subact of $A$ satisfies it.
3. If $A$ satisfies Condition (PWPssc), then every retract of $A$ satisfies it.
4. If $\prod_{i \in I} A_i$, where $A_i, i \in I$, are $S$-acts, satisfies Condition (PWPssc), then $A_i$ satisfies it, for every $i \in I$.
5. $\bigsqcup_{i \in I} A_i$, where each $A_i$ is an $S$-act, satisfies Condition (PWPssc) if and only if $A_i$ satisfies Condition (PWPssc), for every $i \in I$.
6. If $\{B_i | i \in I\}$ is a chain of subacts of $A$ and every $B_i, i \in I$, satisfies Condition (PWPssc), then $\bigcup_{i \in I} B_i$ satisfies it.
7. $S_S$ satisfies Condition (PWPssc) if and only if $S$ is right semi-cancellative.

**Proof.** Proofs are obvious. 

Recall that for any nonempty set $I$, $S^I_S$ is the product of a family of $S$ in $\text{Act-S}$. 
The converse of part (4) of Theorem 2.2 is not true. Consider the monoid $S$ of Example 1.6 of [23]. $S$ is left PSF and so $S_S$ satisfies Condition $(PWP_{ssc})$, by part (4) of Theorem 2.2, while $S^f_S$ is not principally weakly flat, by [23, Proposition 1.5], and so it does not satisfy Condition $(PWP_{ssc})$, by part (1) of Theorem 2.8.

An element $s \in S$ acts injectively on $A$ if $as = a's$, $a, a' \in A$, implies $a = a'$. If every $s \in S$ acts injectively on $A$, then we say that $S$ acts injectively on $A$ (see [14]).

The proof of Propositions 2.3 is clear.

**Proposition 2.3.** If $S$ acts injectively on $A_i$, for every $i \in I$, then $\prod_{i \in I}A_i$ satisfies Condition $(PWP_{ssc})$.

Obviously, $S$ acts injectively on $S_S$ if and only if $S$ is right cancellative.

**Corollary 2.4.** If $S$ is right cancellative, then $S^f_S$ satisfies Condition $(PWP_{ssc})$, for any nonempty set $I$.

Since $S$ acts injectively on $\Theta_i = \{\theta_i\}$, $\prod_{i \in I}\Theta_i$ satisfies Condition $(PWP_{ssc})$.

Recall, from [2], that for $S$, the cartesian product $S \times S$, equipped with the right $S$-action $(s, t)u = (su, tu)$, for $s, t, u \in S$, is called the diagonal act of $S$ and it is denoted by $D(S)$.

In the following theorem we obtain equivalent conditions for $S^n_S$ to satisfy Condition $(PWP_{ssc})$.

**Theorem 2.5.** For a natural number $n \geq 2$, the following statements are equivalent:

1. $S^n_S$ satisfies Condition $(PWP_{ssc})$;
2. $D(S)$ satisfies Condition $(PWP_{ssc})$;
3. $S$ is right semi-cancellative.

**Proof.** (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) This is clear, by parts (4) and (7) of Theorem 2.2.

(3) $\Rightarrow$ (1) Let $(x_1, x_2, \ldots, x_n)s = (y_1, y_2, \ldots, y_n)s$, for $x_1, y_1, \ldots, x_n, y_n, s \in S$. Then $x_is = y_is$, for $1 \leq i \leq n$. Since $S$ is right semi-cancellative, $x_is = y_is$ implies the existence of $v_i \in S$ such that $v_is = s$ and $x_1v_1 = y_1v_1$. Then $x_2v_1s = y_2v_1s$ implies the existence of $v_2 \in S$ such that $v_2s = s$ and $x_2v_1v_2 = y_2v_1v_2$. If $v = v_1v_2$, then

$$vs = v_1v_2s = s, \quad x_1v = x_1v_1v_2 = y_1v_1v_2 = y_1v, \quad x_2v = y_2v.$$
Continuing this procedure, there exists \( u \in S \) such that \( us = s \) and \( xiu = yi, \) for \( 1 \leq i \leq n. \) Thus we have \( (x_1, x_2, ..., x_n)u = (y_1, y_2, ..., y_n)u \) and so \( S^n_S \) satisfies Condition \( (PWP_{ssc}) \), as required.

Now we give an equivalence for cyclic \( S \)-act to satisfy Condition \( (PWP_{ssc}) \).

**Theorem 2.6.** Let \( \rho \) be a right congruence on \( S \). Then the \( S \)-act \( S/\rho \) satisfies Condition \( (PWP_{ssc}) \) if and only if

\[
(\forall x, y, s \in S) ((xs)\rho(ys) \implies (\exists r \in S)((xr)\rho(yr) \land rs = s)).
\]

**Proof.** Necessity. Suppose \((xs)\rho(ys)\), for \( x, y, s \in S \). Thus \([x]_\rho s = [y]_\rho s \). By assumption, there exists \( r \in S \) such that \([x]_\rho r = [y]_\rho r \) and \( rs = s \). Hence \((xr)\rho(yr)\), as required.

Sufficiency. Let \([x]_\rho s = [y]_\rho s \), for \( x, y, s \in S \). Thus \((xs)\rho(ys)\) and so, by assumption, there exists \( r \in S \) such that \((xr)\rho(yr)\) and \( rs = s \). Hence \([x]_\rho r = [y]_\rho r \). So \( S/\rho \) satisfies Condition \( (PWP_{ssc}) \), as required.

**Theorem 2.7.** Let \( w \in S \) and \( \rho = \rho(w, 1) \). The cyclic \( S \)-act \( S/\rho \) satisfies Condition \( (PWP_{ssc}) \) if and only if for every \( x, y, s \in S \) and \( m, n \in \mathbb{N}_0 \)

\[
w^m xs = w^n ys \implies (\exists p, q \in \mathbb{N}_0)(\exists r \in S)(w^p xr = w^q yr \land rs = s).
\]

**Proof.** Necessity. Let \( w^m xs = w^n ys \), for \( x, y, s \in S \) and \( m, n \in \mathbb{N}_0 \). Then we have \((xs)\rho(ys)\), by [14, III, Corollary 8.7]. By Theorem 2.6, there exists \( r \in S \) such that \( rs = s \) and \((xr)\rho(yr)\). Thus, by [14, III, Corollary 8.7], there exist \( p, q \in \mathbb{N}_0 \) such that \( w^p xr = w^q yr \).

Sufficiency. Let \((xs)\rho(ys)\), for \( x, y, s \in S \). Then, by [14, III, Corollary 8.7], there exist \( m, n \in \mathbb{N}_0 \) such that \( w^m xs = w^n ys \). By assumption, there exist \( r \in S \) and \( p, q \in \mathbb{N}_0 \) such that \( rs = s \) and \( w^p xr = w^q yr \). Hence \((xr)\rho(yr)\), by [14, III, Corollary 8.7], and so \( S/\rho \) satisfies Condition \( (PWP_{ssc}) \), by Theorem 2.6.

Notice that in the above theorem, if \( w = 1 \), then \( S/\rho = S/\rho(w, 1) = S/\rho(1, 1) = S/\Delta_S \cong S_S \), which by placing \( w \) in the above theorem, the part (7) of Theorem 2.2 is obtained.

**Theorem 2.8.** The following statements hold:

1. If \( A \) satisfies Condition \( (PWP_{ssc}) \), then \( A \) is principally weakly flat.
2. For left PSF monoid \( S \), \( A \) is principally weakly flat if and only if \( A \) satisfies Condition \( (PWP_{ssc}) \).
Proof. (1) Let \( as = a's \), for \( a \in A \) and \( s, s' \in S \). By assumption, there exists \( r \in S \) such that \( ar = a'r \) and \( rs = s \). Hence

\[
a \otimes s = a \otimes rs = ar \otimes s = a'r \otimes s = a' \otimes s
\]

in \( A \otimes Ss \). Thus \( A \) is principally weakly flat, as required.

(2) Necessity. Let \( as = a's \), for \( a \in A \) and \( s, s' \in S \). By assumption, there exist \( n \in \mathbb{N} \) and elements \( a_1, \ldots, a_n \in A \), \( s_1, t_1, \ldots, s_n, t_n \in S \) such that

\[
a = a_1 s_1 \]
\[
a_1 t_1 = a_2 s_2 \quad s_1 s = t_1 s \]
\[
a_2 t_2 = a_3 s_3 \quad s_2 s = t_2 s \]
\[
\ldots \]
\[
a_n t_n = a' \quad s_n s = t_n s.
\]

Since \( S \) is right semi-cancellative, \( s_1 s = t_1 s \) implies the existence of \( v_1 \in S \) such that \( v_1 s = s \) and \( s_1 v_1 = t_1 v_1 \). Then \( s_2 v_1 s = t_2 v_1 s \) implies the existence of \( v_2 \in S \) such that \( v_2 s = s \) and \( s_2 v_1 v_2 = t_2 v_1 v_2 \). If \( v = v_1 v_2 \), then

\[
us = v_1 v_2 s = s, \quad s_1 v = s_1 v_1 v_2 = t_1 v_1 v_2 = t_1 v, \quad s_2 v = t_2 v.
\]

Continuing this procedure, there exists \( u \in S \) such that \( us = s \) and \( s_i u = t_i u \), for \( 1 \leq i \leq n \). Thus we have

\[
a u = (a_1 s_1) u = a_1 (s_1 u) = a_1 (t_1 u) = (a_1 t_1) u = \ldots = (a_n t_n) u = a' u.
\]

So \( A \) satisfies Condition \( (PW \text{P}_{\text{ssc}}) \), as required.

Sufficiency. This is true, by (1). \( \square \)

The following example shows that the converse of part (1) of Theorem 2.8 is not true in general.

Example 2.9. Let \( J \) be a proper right ideal of \( S \). Let \( x, y, z \) be different symbols not belonging to \( S \). Define \( A(J) := (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J) \) and a right \( S \)-action on \( A(J) \) by

\[
(x, u)s = \begin{cases} (x, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}
\]
\[
(y, u)s = \begin{cases} (y, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}
\]

In this example, \( A(J) \) is not principally weakly flat.
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\[(z, u)s = (z, us).\]

Clearly $A(J)$ is an $S$-act. Let $S = \{0, 1, e, s\}$ with the table

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If $J = \{0, e\}$, then $J$ is a right ideal of $S$. Because $0 \in J$ and $e \in J$, $A(J) = \{(x, 1), (x, s), (y, 1), (y, s), (z, 0), (z, e)\}$, by [1, Proposition 2.2], is a flat $S$-act and so is principally weakly flat. But it does not satisfy Condition $(PWP_{ssc})$. Otherwise $(x, s)s = (y, s)s$ implies that there exists $u \in S$ such that $(x, s)u = (y, s)u$ and $us = s$. Because 1 is the only element such that $1s = s$, $(x, s) = (y, s)$, which is a contradiction.

Recall, from [8], that $A$ satisfies Condition $(PWP_E)$ if for all $a, a' \in A$, $s \in S$

\[as = a's \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(e, f \in E(S)) (ae = a''ue, a'f = a''vf, es = s = fs, \text{and} us = vs).\]

Clearly Condition $(PWP)$ implies Condition $(PWP_E)$.

Suppose $S$ is not semi-cancellative. Then $S_S$ does not satisfy Condition $(PWP_{ssc})$ while $S_S$ satisfies Condition $(PWP)$. Therefore Condition $(PWP)$ does not imply Condition $(PWP_{ssc})$.

Now, in the following example, we show that Condition $(PWP_{ssc})$ does not imply Condition $(PWP_E)$ and so does not imply Condition $(PWP)$. In this example we use the monoid $S$ of Example 1.6 of [23].

**Example 2.10.** Let $(I, \leq)$ be a totally ordered set with no successor for each element (as $\mathbb{R}$). Consider the commutative monoid $S = \{x_i^m| i \in I, m \in \mathbb{N}\} \cup \{1\}$ in which $x_i^m x_j^n$ equals to $x_j^n$ if $i < j$ and equals to $x_i^{m+n}$ if $i = j$. Then $S$ is left PSF and, since $S$ does not have any idempotent except 1, it is not left PP. Now let $A = \{a_1, a_2, a_3\}$ be a set such that $a_1 1 = a_i$ and $a_i s = a_3$, for $i = 1, 2, 3$ and any $s \in S$. Obviously $A$ is an $S$-act which does not satisfy Condition $(PWP_E)$. Otherwise $a_1 x_i^m = a_2 x_i^n$ implies that there exist $u, v, f^2 = f, e^2 = e \in S$, and $b \in A$ such that $a_1 e = bue, a_2 f = bv f,$
$e x_i^m = x_i = f x_i^m$, and $u x_i^m = v x_i^m$. Because 1 is the only idempotent such that $1 x_i^m = x_i$, $a_1 = bu$ implies $b = a_1$ and $u = 1$. Then $a_2 \neq a_1 v$ for every $v \in S$, which is a contradiction. Therefore $A$ does not satisfy Condition ($PWP_E$) and so does not satisfy Condition ($PWP$). On the other hand, $a_j x_i^m = a_3 = a_k x_i^m$, for $j, k = 1, 2, 3$ and any $x_i^m \in S$. Let $l < i$. Then $a_j x_i^m = a_3 = a_k x_i^m$ and $x_i^m x_i^m = x_i^m$, that is, $A$ satisfies Condition ($PWP_{ssc}$).

3 Classification by Condition ($PWP_{ssc}$) of acts

In this section we give a classification of monoids for which all (cyclic, monocy- 
cyclic) acts satisfy Condition ($PWP_{ssc}$) and give a classification of monoids 
for which all acts satisfying some other flatness properties have Condition 
($PWP_{ssc}$).

**Theorem 3.1.** The following statements are equivalent:

1. All $S$-acts satisfy Condition ($PWP_{ssc}$);
2. all $S$-acts generated by exactly two elements satisfy Condition ($PWP_{ssc}$);
3. all cyclic $S$-acts satisfy Condition ($PWP_{ssc}$);
4. all monocyclic $S$-acts of the form $S/\rho(s, s^2)$ ($s \in S$) satisfy Condition 
($PWP_{ssc}$);
5. all Rees factor $S$-acts satisfy Condition ($PWP_{ssc}$);
6. all divisible $S$-acts satisfy Condition ($PWP_{ssc}$);
7. all cofree $S$-acts satisfy Condition ($PWP_{ssc}$);
8. $S$ is regular.

**Proof.** Implications (1) ⇒ (2), (1) ⇒ (3) ⇒ (4), (3) ⇒ (5) are obvious.

Implications (1) ⇒ (6) ⇒ (7) are obvious because cofree ⇒ divisible.

(2) ⇒ (8) Let $s \in S$. If $s S = S$, then there exists $z \in S$ such that $sz = 1$. 
Therefore $sz s = s$ and $s$ is regular. Now let $s S \neq S$. Put

$$A = S \prod_{s \in S} S = \{ (l, x) | l \in S \setminus s S \} \cup s S \cup \{ (t, y) | t \in S \setminus s S \}.$$ 

Indeed $A =< (1, x), (1, y) >= (1, x) S \cup (1, y) S$. Since $A$ is generated by two 
different elements, by assumption, $A$ satisfies Condition ($PWP_{ssc}$). Now 
$s = (1, x) s = (1, y) s$ implies that there exists $r \in S$ such that $(1, x) r = 
(1, y) r$ and $r s = s$. Thus $r \in s S$ and so there exists $x \in S$ such that $r = s x$. 
Then $s = r s = s x s$, that is, $s$ is regular and so $S$ is regular.
(4) ⇒ (8) By part (1) of Theorem 2.8, all monocyclic \( S \)-acts of the form \( S/\rho(s, s^2) \) \( (s \in S) \) are principally weakly flat. Thus, by [14, IV, Theorem 6.6], \( S \) is regular.

(5) ⇒ (8) By part (1) of Theorem 2.8, all Rees factor \( S \)-acts are principally weakly flat. Thus, by [14, IV, Theorem 6.6], \( S \) is regular.

(7) ⇒ (8) Every \( S \)-act can be embedded in a cofree \( S \)-act. By assumption, every \( S \)-act is a subact of an \( S \)-act satisfying Condition \( \text{(PWPssc)} \). Thus every \( S \)-act satisfies Condition \( \text{(PWPssc)} \), by part (2) of Theorem 2.2, and so every \( S \)-act is principally weakly flat, by part (1) of Theorem 2.8. Therefore, by [14, IV, Theorem 6.6], \( S \) is regular.

(8) ⇒ (1) Every \( S \)-act is principally weakly flat, by [14, IV, Theorem 6.6]. Every regular monoid is left PP and so is left PSF. Thus every \( S \)-act satisfies Condition \( \text{(PWPssc)} \), by part (2) of Theorem 2.8. 

Recall, from [10], [9], and [15], that \( A \) satisfies Condition \( (E'P) \) if for all \( a \in A, s, s', z \in S, \)
\[
as = as', \; sz = s'z \Rightarrow (\exists a' \in A) \; (\exists u, u' \in S) \; (a = a'u = a'u' \; \text{and} \; us = u's').
\]
\( A \) satisfies Condition \( (EP) \) if for all \( a \in A, s, s' \in S \)
\[
as = as' \Rightarrow (\exists a' \in A) \; (\exists u, u' \in S) \; (a = a'u = a'u' \; \text{and} \; us = u's').
\]
\( A \) satisfies Condition \( (E') \) if for all \( a \in A, s, s', z \in S \)
\[
as = as', \; sz = s'z \Rightarrow (\exists a' \in A) \; (\exists u \in S) \; (a = a'u \; \text{and} \; us = us').
\]

**Theorem 3.2.** The following statements are equivalent:

(1) All \( S \)-acts satisfy Condition \( \text{(PWPssc)} \);

(2) all \( S \)-acts satisfying Condition \( (E) \) satisfy Condition \( \text{(PWPssc)} \);

(3) all \( S \)-acts satisfying Condition \( (E'P) \) satisfy Condition \( \text{(PWPssc)} \);

(4) \( S \) is regular.

**Proof.** Implications (1) ⇒ (3) ⇒ (2) are obvious, because \( (E) \Rightarrow (E'P) \).

(2) ⇒ (4) Let \( s \in S \). If \( sS = S \), then there exists \( x \in S \) such that \( xs = 1 \). Thus \( sxS = s \), that is, \( s \) is regular and so \( S \) is regular. Let \( sS \neq S \). Put

\[
A = S \bigsqcup_{S} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.
\]
Indeed
\[ B = \{(l,x) \mid l \in S \setminus sS\} \cup sS \cong S_s \cong \{(t,y) \mid t \in S \setminus sS\} \cup sS = C. \]

\( B \) and \( C \) are subacts of \( A \) which are generated by \((1,x)\) and \((1,y)\), respectively. \( A \) is generated by \((1,x)\) and \((1,y)\) because \( A = B \cup C \). By the above isomorphisms, \( B \) and \( C \) satisfy Condition \((E)\) and so \( A \) satisfies Condition \((E)\). Thus, by assumption, \( A \) satisfies Condition \((PWP_{ssc})\). Then \((1,x)s = (1,y)s\) implies that there exists \( r \in S \) such that \( rs = s \) and \((1,x)r = (1,y)r\). Therefore \( r \in sS \) and so there exists \( x \in S \) such that \( r = sx \). Hence \( s = rs = sx \), that is, \( s \) is regular. Thus \( S \) is regular, as required.

(4) \implies (1) This is true, by Theorem 3.1.

By the proof of Theorem 3.2, we conclude that the above theorem is true for finitely generated \( S \)-acts.

**Theorem 3.3.** Let \((*)\) be a property on \( S \)-acts such that every \( S \)-act satisfying Property \((*)\) is principally weakly flat and \( S_S \) satisfies Property \((*)\). Then the following statements are equivalent:

1. All \( S \)-acts satisfying Property \((*)\) satisfy Condition \((PWP_{ssc})\);
2. All finitely generated \( S \)-acts satisfying Property \((*)\) satisfy Condition \((PWP_{ssc})\);
3. All cyclic \( S \)-acts satisfying Property \((*)\) satisfy Condition \((PWP_{ssc})\);
4. \( S \) is left PSF.

**Proof.** Implications (1) \implies (2) \implies (3) are obvious.

(3) \implies (4) \( S \) is a cyclic \( S \)-act satisfying Property \((*)\) and so, by assumption, satisfies Condition \((PWP_{ssc})\). Therefore \( S \) is left PSF, by part (7) of Theorem 2.2.

(4) \implies (1) By assumption, every \( S \)-act satisfying Property \((*)\) is principally weakly flat. Since \( S \) is left PSF, principally weakly flat is equivalent to Condition \((PWP_{ssc})\), by Theorem 2.8. Therefore all \( S \)-acts satisfying Property \((*)\) satisfy Condition \((PWP_{ssc})\), as required.

**Theorem 3.4.** The following statements are equivalent:

1. All torsion free \( S \)-acts satisfy Condition \((PWP_{ssc})\);
2. All torsion free cyclic \( S \)-acts satisfy Condition \((PWP_{ssc})\);
(3) all torsion free Rees factor $S$-acts satisfy Condition $(PWP_{ssc})$;

(4) $S$ is left almost regular.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$ All torsion free Rees factor $S$-acts are principally weakly flat, by part (1) of Theorem 2.8. Thus $S$ is left almost regular, by [14, IV, Theorem 6.5].

$(4) \Rightarrow (1)$ All torsion free $S$-acts are principally weakly flat, by [14, IV, Theorem 6.5]. Also every left almost regular monoid is left PP, by the duality of [14, IV, Proposition 1.3], and so is left PSF. Thus all torsion free $S$-acts satisfy Condition $(PWP_{ssc})$, by part (2) of Theorem 2.8.

Recall, from [21], that $A$ is called $\mathcal{R}$-torsion free if for any $a, b \in A$ and $c \in S$, $c$ right cancellable, $ac = bc$ and $a \mathcal{R} b$ ($\mathcal{R}$ is Green’s equivalence) imply that $a = b$.

**Theorem 3.5.** The following statements are equivalent:

(1) All $\mathcal{R}$-torsion free $S$-acts satisfy Condition $(PWP_{ssc})$;

(2) all $\mathcal{R}$-torsion free $S$-acts generated by exactly two elements satisfy Condition $(PWP_{ssc})$;

(3) $S$ is regular.

Proof. Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so $s$ is regular. Let $sS \neq S$. Put

$$A = S \bigsqcup_{\sim} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$

By the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, $A$ is an $S$-act which is generated by two different elements $(1, x)$ and $(1, y)$ and also satisfy Condition $(E)$. Every $S$-act satisfying Condition $(E)$ is $\mathcal{R}$-torsion free, by [21, Proposition 1.2]. Thus $A$ is $\mathcal{R}$-torsion free and so, by assumption, satisfies Condition $(PWP_{ssc})$. Hence, by the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, $s$ is regular. Therefore $S$ is regular, as required.

$(3) \Rightarrow (1)$ Every $\mathcal{R}$-torsion free $S$-act is principally weakly flat, by [21, Theorem 4.5]. Since every regular monoid is left PP and so is left PSF, principally weakly flat is equivalent to Condition $(PWP_{ssc})$, by part (2) of Theorem 2.8. Therefore every $\mathcal{R}$-torsion free $S$-act satisfies Condition $(PWP_{ssc})$. \qed
We recall, from [14], that $A$ is (strongly) faithful if for $s, t \in S$ the equality $as = at$, for (some) all $a \in A$, implies that $s = t$.

**Theorem 3.6.** The following statements are equivalent:

1. All faithful $S$-acts satisfy Condition $(PWP_{ssc})$;
2. All faithful $S$-acts generated by exactly two elements satisfy Condition $(PWP_{ssc})$;
3. $S$ is regular.

**Proof.** Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sx s = s$ and so $s$ is regular. Let $sS \neq S$. Put $A = S \bigsqcup_{s \in S} S$. By the proof of part (2) $\Rightarrow$ (4) of Theorem 3.2, $A$ is an $S$-act which is generated by two different elements $(1, x)$ and $(1, y)$. Indeed $A$ is faithful and so, by assumption, satisfies Condition $(PWP_{ssc})$. Hence, by the proof of part (2) $\Rightarrow$ (4) of Theorem 3.2, $s$ is regular. Therefore $S$ is regular, as required.

(3) $\Rightarrow$ (1) All $S$-acts satisfy Condition $(PWP_{ssc})$, by Theorem 3.1. Thus all faithful $S$-acts satisfy Condition $(PWP_{ssc})$.

**Notation:** $C_l (C_r)$ is the set of all left (right) cancellable elements of $S$.

**Lemma 3.7.** The following statements are equivalent:

1. There exists at least one strongly faithful $S$-act;
2. $sS$ as an $S$-act is strongly faithful, for every $s \in S$;
3. $S$ as an $S$-act is strongly faithful;
4. $sS \subseteq C_l$, for every $s \in S$;
5. $S$ is left cancellative.

**Proof.** Implications (2) $\Rightarrow$ (1), (5) $\Rightarrow$ (4) and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (5) Let $A$ be a strongly faithful $S$-act and $sl = st$, for $s, l, t \in S$. Then $asl = ast$, for $a \in A$. Since $A$ is strongly faithful and $as \in A$, $l = t$. Therefore $S$ is left cancellative.

(5) $\Rightarrow$ (3) Let $S$ be left cancellative and $sl = st$, for $l, t, s \in S$. By assumption, $l = t$ and so $S$ is strongly faithful, as an $S$-act.

(4) $\Rightarrow$ (5) Let $sS \subseteq C_l$, for $s \in S$ and $rt = rl$, for $r, t, l \in S$. Then $(sr)t = (sr)l$. By assumption, $t = l$ and so $S$ is left cancellative.

(5) $\Rightarrow$ (2) Let $skt = skl$, for $sk \in sS$ and $t, l \in S$. By assumption, $t = l$ and so $sS$ is strongly faithful as an $S$-act.
By the above lemma, for $S$ there exists no strongly faithful $S$-act if and only if $S$ is not left cancellative.

**Theorem 3.8.** The following statements are equivalent:

1. All strongly faithful $S$-acts satisfy Condition $(PWP_{ssc})$;
2. All strongly faithful $S$-acts generated by exactly two elements satisfy Condition $(PWP_{ssc})$;
3. $S$ is not left cancellative or it is a group.

**Proof.** Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ If $S$ is not left cancellative, then $(3)$ is satisfied. Let $S$ be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sx = s$ and so $s$ is regular. Now let $sS \neq S$. Put

$$A = S \coprod_{s \in S} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$  

We have

$$B = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cong S \cong \{(t, y) \mid t \in S \setminus sS\} \cup sS = C$$

and

$$A = \langle (1, x), (1, y) \rangle = B \cup C.$$  

Since $S$ is left cancellative, $S_S$ is strongly faithful, by Lemma 3.7. By the above isomorphisms, $B$ and $C$ are strongly faithful as subacts of $A$. Thus $A$ is strongly faithful. Since $A$ is generated by two different elements $(1, x)$ and $(1, y)$, by assumption, $A$ satisfies Condition $(PWP_{ssc})$. By the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, $s$ is regular and so $S$ is regular. Thus for every $s \in S$ there exists $x \in S$ such that $sx = s$. Since $S$ is left cancellative, $xs = 1$. Thus every element in $S$ has left inverse and so $S$ is a group.

$(3) \Rightarrow (1)$ If $S$ is not left cancellative, then there exists no strongly faithful $S$-act, by Lemma 3.7. Thus $(1)$ is satisfied. If $S$ is left cancellative, then there exists at least a strongly faithful $S$-act, by Lemma 3.7. Since $S$ is a group, it is regular and so $(1)$ is satisfied, by Theorem 3.1.  

**Lemma 3.9.** Let $\rho$ be a right congruence on $S$. Then the following statements are equivalent:

1. $S/\rho$ is a strongly faithful cyclic $S$-act;
2. $\rho = \Delta_S$ and $S$ is left cancellative.

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By the above lemma, for $S$ there exists no strongly faithful $S$-act if and only if $S$ is not left cancellative.

**Theorem 3.8.** The following statements are equivalent:

1. All strongly faithful $S$-acts satisfy Condition $(PWP_{ssc})$;
2. All strongly faithful $S$-acts generated by exactly two elements satisfy Condition $(PWP_{ssc})$;
3. $S$ is not left cancellative or it is a group.

**Proof.** Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ If $S$ is not left cancellative, then $(3)$ is satisfied. Let $S$ be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sx = s$ and so $s$ is regular. Now let $sS \neq S$. Put

$$A = S \coprod_{s \in S} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$  

We have

$$B = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cong S \cong \{(t, y) \mid t \in S \setminus sS\} \cup sS = C$$

and

$$A = \langle (1, x), (1, y) \rangle = B \cup C.$$  

Since $S$ is left cancellative, $S_S$ is strongly faithful, by Lemma 3.7. By the above isomorphisms, $B$ and $C$ are strongly faithful as subacts of $A$. Thus $A$ is strongly faithful. Since $A$ is generated by two different elements $(1, x)$ and $(1, y)$, by assumption, $A$ satisfies Condition $(PWP_{ssc})$. By the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, $s$ is regular and so $S$ is regular. Thus for every $s \in S$ there exists $x \in S$ such that $sx = s$. Since $S$ is left cancellative, $xs = 1$. Thus every element in $S$ has left inverse and so $S$ is a group.

$(3) \Rightarrow (1)$ If $S$ is not left cancellative, then there exists no strongly faithful $S$-act, by Lemma 3.7. Thus $(1)$ is satisfied. If $S$ is left cancellative, then there exists at least a strongly faithful $S$-act, by Lemma 3.7. Since $S$ is a group, it is regular and so $(1)$ is satisfied, by Theorem 3.1.  

**Lemma 3.9.** Let $\rho$ be a right congruence on $S$. Then the following statements are equivalent:

1. $S/\rho$ is a strongly faithful cyclic $S$-act;
2. $\rho = \Delta_S$ and $S$ is left cancellative.
Proof. (1) ⇒ (2) Since $S/\rho$ is strongly faithful as an $S$-act, there exists at least one strongly faithful $S$-act. Hence $S$ is left cancellative, by Lemma 3.7. Now let $(s, t) \in \rho$, for $s, t \in S$. Then $[1]_\rho \cdot s = [s]_\rho = [t]_\rho = [1]_\rho \cdot t$. Thus $s = t$, since $S/\rho$ is strongly faithful, and so $\rho = \Delta_S$.

(2) ⇒ (1) $S/\rho = S/\Delta_S \cong S_S$. Since $S$ is left cancellative, $S_S \cong S/\rho$ is strongly faithful, by Lemma 3.7.

\begin{theorem}
The following statements are equivalent:
\begin{enumerate}
\item Every strongly faithful cyclic $S$-act satisfies Condition (\textnormal{PWP}_{ssc});
\item $S$ is not left cancellative or it is left PSF.
\end{enumerate}
\end{theorem}

\begin{proof}
(1) ⇒ (2) If $S$ is not left cancellative, then (2) is satisfied. Let $S$ be left cancellative. Then $S$, as a cyclic $S$-act, is strongly faithful, by Lemma 3.7. Thus, by assumption, $S_S$ satisfies Condition (\textnormal{PWP}_{ssc}) and so $S$ is left PSF, by part (7) of Theorem 2.2.

(2) ⇒ (1) If $S$ is not left cancellative, then there exists no strongly faithful $S$-act, by Lemma 3.7. Thus (1) is satisfied. If $S$ is left cancellative, then there exists at least one strongly faithful cyclic $S$-act, by Lemma 3.7. If $S/\rho$ is a strongly faithful cyclic $S$-act satisfying Condition (\textnormal{PWP}_{ssc}), then $\rho = \Delta_S$, by Lemma 3.9, and so $S/\rho \cong S_S$. By assumption, $S$ is left PSF and so $S/\rho \cong S_S$ satisfies Condition (\textnormal{PWP}_{ssc}), by part (7) of Theorem 2.2.
\end{proof}

\begin{theorem}
The following statements are equivalent:
\begin{enumerate}
\item There exists at least one strongly faithful cyclic $S$-act satisfying Condition (\textnormal{PWP}_{ssc});
\item $S$ is left cancellative and every strongly faithful cyclic $S$-act satisfies Condition (\textnormal{PWP}_{ssc});
\item $S$ is left cancellative and left PSF.
\end{enumerate}
\end{theorem}

\begin{proof}
(1) ⇒ (2) Since there exists at least one strongly faithful cyclic $S$-act, $S$ is left cancellative, by Lemma 3.7. If $S/\rho$ is a strongly faithful cyclic $S$-act satisfying Condition (\textnormal{PWP}_{ssc}), then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho \cong S_S$ satisfies Condition (\textnormal{PWP}_{ssc}) and so $S$ is left PSF, by part (7) of Theorem 2.2. Therefore every strongly faithful cyclic $S$-act satisfies Condition (\textnormal{PWP}_{ssc}), by Theorem 3.10.

(2) ⇒ (3) This is true, by Theorem 3.10.

(3) ⇒ (1) Since $S$ is left cancellative, there exists at least one strongly faithful cyclic $S$-act, by Lemma 3.7. If $S/\rho$ is a strongly faithful cyclic $S$-act,
then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho = S/\Delta_S \cong S_S$. Since $S$ is left PSF, $S/\rho \cong S_S$ satisfies Condition (PWP$_{ssc}$), by part (7) of Theorem 2.2.

We recall, from [22], that an act $A$ is called strongly torsion free if for $a, b \in A$ and any $s \in S$, the equality $as = bs$ implies $a = b$. It is obvious that every strongly torsion free $S$-act satisfies Condition (PWP$_{ssc}$).

Let $K_S$ be a right ideal of $S$. Then the Rees factor $S/K_S$ is simple if and only if $K_S = S$, by [14, I, Proposition 5.28]. Thus $S/K_S$ is simple if and only if $S/K_S = S/S_S \cong \Theta_S$. Since $\Theta_S$ satisfies Condition (PWP$_{ssc}$), by part (1) of Theorem 2.2, every simple Rees factor $S$-act satisfies Condition (PWP$_{ssc}$).

Recall, from [14], that a right ideal $K_S$ of $S$ satisfies Condition (LU) if for every $k \in K_S$ there exists $l \in K_S$ such that $lk = k$.

Lemma 3.12. Let $S \neq C_r$. Then the following statements hold:

1. $I = S \setminus C_r$ is a proper right ideal of $S$.
2. $S/I$ ($I = S \setminus C_r$) is a torsion free $S$-act.
3. If $S$ is left PSF, then the right ideal $I = S \setminus C_r$ satisfies Condition (LU).

Proof. (1) Since $S \neq C_r$ and $1 \in C_r$, $\emptyset \neq I \subset S$. Let $i \in I$ and $s \in S$. Then there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Thus $l_1 is = l_2 is$. If $is \in C_r$, then $l_1 = l_2$, which is a contradiction. Therefore $is \in I$ and so $I$ is a proper right ideal of $S$, as required.

(2) Let $sc \in I$, for $s \in S$ and $c \in C_r$. We claim that $s \in I$. Since $sc \in I$, there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 sc = l_2 sc$. Thus $l_1 s = l_2 s$. If $s \not\in I$, then $s \in C_r$ and so $l_1 = l_2$, which is a contradiction. Thus $s \in I$ and so the $S$-act $S/I$ is torsion free, by [14, III, Proposition 8.10].

(3) Let $i \in I$. Then $i$ is not right cancellable. Thus there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Since $S$ is left PSF, there exists $r \in S$ such that $l_1 r = l_2 r$ and $ri = i$, by part (7) of Theorem 2.2. If $r \not\in I$, then $l_1 = l_2$, which is a contradiction. Thus $r \in I$ and $ri = i$. Therefore $I$ satisfies Condition (LU).

We recall, from [20], that $A$ is called GP-flat if for every $s \in S$ and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes S$ implies the existence of a natural number $n$ such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes Ss^n$.

Lemma 3.13. Let $S$ be right cancellative. Then for every $S$-act we have
strongly torsion free ⇔ torsion free ⇔ GP-flat ⇔ principally weakly flat ⇔ Condition (PWP) ⇔ Condition (P′) ⇔ Condition (PWP_E) ⇔ Condition (PWPssc) ⇔ translation kernel flat ⇔ principally weakly kernel flat.

Proof. Since S is right cancellative, strongly torsion free is equivalent to torsion free, by definition. We always have strongly torsion free ⇒ Condition (PWP) (Condition (PWP_E), Condition (PWPssc), Condition (P′))⇒ principally weakly flat ⇒ GP-flat ⇒ torsion free. Thus strongly torsion free ⇔ torsion free ⇔ GP-flat ⇔ principally weakly flat ⇔ Condition (PWP) ⇔ Condition (PWP_E) ⇔ Condition (P′). Also we always have principally weakly kernel flat ⇒ translation kernel flat ⇒ Condition (PWP), by [3, Proposition 27]. Let a, a′ ∈ A and s, s′ ∈ S. as = a′s′ in A if and only if a ⊗ s = a′ ⊗ s′ in A ⊗ S (see [14, II, Proposition 5.13]). It is obvious that a ⊗ s = a′ ⊗ s′ in A ⊗ S if and only if a ⊗ (s, s) = a′ ⊗ (s′, s′) in A ⊗ Δ. Since S is right cancellative, ker ρ_z = sΔ for every z ∈ S. Now Condition (PWP) is equivalent to translation kernel flat, by [3, Proposition 5].

Theorem 3.14. Let (★) be a property on S-acts such that flat ⇒ Property (★) ⇒ torsion free. Then the following statements are equivalent:

1. S is left PSF and Property (★) in S-acts implies principally weakly kernel flat;
2. S is left PSF and Property (★) in S-acts implies translation kernel flat;
3. S is left PSF and Property (★) in S-acts implies Condition (PWP);
4. S is left PSF and Property (★) in S-acts implies Condition (P′);
5. S is right cancellative.

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious because principally weakly kernel flat ⇒ translation kernel flat ⇒ Condition (PWP), also implication (4) ⇒ (3) is obvious because Condition (P′) ⇒ Condition (PWP).

(3) ⇒ (5) Let S be not right cancellative and I = S \ C_r. Then I is a proper right ideal of S which satisfies Condition (LU), by Lemma 3.12. Put

\[ A = \biguplus_{I} S = \{(l, x) | l \in S \setminus I\} \cup I \cup \{(t, y) | t \in S \setminus I\}. \]
So $A$ is flat, by [14, III, Proposition 12.19]. By assumption, $A$ satisfies Condition (PWP). Let $i \in I$, then the equality $(1, x)i = (1, y)i$ implies there exist $a \in A$ and $u, v \in S$ such that $(1, x) = au$, $(1, y) = av$, and $ui = vi$. Hence there exist $l, t \in S \setminus I$ such that $(l, x) = a = (t, y)$, which is a contradiction. Thus $S$ is right cancellative, as required.

(5) $\Rightarrow$ (1) Since $S$ is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to principally weakly kernel flat, by Lemma 3.13. Thus, by assumption, every $S$-act satisfying (*) is principally weakly kernel flat.

(5) $\Rightarrow$ (4) Since $S$ is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to Condition ($P'$), by Lemma 3.13. Thus, by assumption, every $S$-act satisfying the Property (*)& satisfies Condition ($P'$).

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.14 is true for finitely generated $S$-acts. Furthermore the Property (*)& in Theorem 3.14 can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

Notice that in general [6, Theorem 2.8], [18, Theorems 2.6 and 2.8], [19, Lemma 2.12], and [20, Theorem 3.11] follow for every $S$-act, by putting the Property (*)& in Theorem 3.14 with any property as flat, weakly flat, principally weakly flat, and GP-flat.

**Corollary 3.15.** Let (*)& be a property on $S$-acts such that flat $\Rightarrow$ Property (*)& $\Rightarrow$ torsion free. Then the following statements are equivalent:

1. All $S$-acts satisfying Property (*)& are principally weakly kernel flat and satisfy Condition (PWP$_{ssc}$);
2. All $S$-acts satisfying Property (*)& are translation kernel flat and satisfy Condition (PWP$_{ssc}$);
3. All $S$-acts satisfying Property (*)& satisfy Conditions (PWP) and (PWP$_{ssc}$);
4. All $S$-acts satisfying Property (*)& satisfy Conditions ($P'$) and (PWP$_{ssc}$);
5. $S$ is right cancellative.

**Proof.** Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, because principally weakly kernel flat $\Rightarrow$ translation kernel flat $\Rightarrow$ Condition (PWP), also implication (4) $\Rightarrow$ (3) is obvious because Condition ($P'$) $\Rightarrow$ Condition (PWP).
Since $S$ is flat, by assumption, it satisfies Property $(\ast)$ and so it satisfies Condition $(PWP_{ssc})$. Therefore $S$ is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every $S$-act satisfying Property $(\ast)$ satisfies Condition $(PWP)$ and thus $S$ is right cancellative, by Theorem 3.14.

Since $S$ is right cancellative, every $S$-act satisfying Property $(\ast)$ is principally weakly kernel flat, by Theorem 3.14. Also, since $S$ is right cancellative, $S$ is left almost regular and so every torsion free $S$-act satisfies Condition $(PWP_{ssc})$, by Theorem 3.4. Therefore every $S$-act satisfying Property $(\ast)$ satisfies Condition $(PWP_{ssc})$.

Since $S$ is right cancellative, torsion free is equivalent to Condition $(P')$, by Lemma 3.13. Therefore, by assumption, every $S$-act satisfying Property $(\ast)$ satisfies Condition $(P')$. Also, since $S$ is right cancellative, $S$ is left almost regular and so every torsion free $S$-act satisfies Condition $(PWP_{ssc})$, by Theorem 3.4. Therefore every $S$-act satisfying Property $(\ast)$ satisfies Condition $(PWP_{ssc})$, as required.

Note that, since Theorem 3.14 holds for finitely generated $S$-acts, Corollary 3.15 is also true for finitely generated $S$-acts. Furthermore the Property $(\ast)$ in the above corollary can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

**Lemma 3.16.** Let $S$ be right cancellative. Then for every $S$-act we have

weakly pullback flat $\iff$ weakly kernel flat $\iff$ $(WP) \iff$ Condition $(P) \iff$ Condition $(P_E) \iff$ flat $\iff$ weakly flat.

**Proof.** Since $S$ is right cancellative, by [4, Proposition 1], Condition $(P) \iff$ flat $\iff$ weakly flat. Also, since Condition $(P) \Rightarrow (WP) \Rightarrow$ weakly flat, Condition $(WP) \iff$ weakly flat. We always have weakly pullback flat $\Rightarrow$ weakly kernel flat $\Rightarrow$ weakly flat. Thus weakly pullback flat $\iff$ weakly kernel flat $\iff$ weakly flat, by [19, Theorem 2.14]. Since $S$ is right cancellative, it is left PP and so Condition $(P_E) \iff$ weakly flat, by [11, Theorem 2.5].

**Theorem 3.17.** Let $(\ast)$ be a property on $S$-acts such that flat $\Rightarrow$ Property $(\ast)$ $\Rightarrow$ weakly flat. Then the following statements are equivalent:

1. $S$ is left PSF and Property $(\ast)$ in $S$-acts implies weakly pullback flat;
2. $S$ is left PSF and Property $(\ast)$ in $S$-acts implies Condition $(P)$;
3. $S$ is right cancellative.
Proof. The implication (1) ⇒ (2) is obvious because weakly pullback flat ⇒ Condition (P).

(2) ⇒ (3) All flat S-acts satisfy Condition (WP), since flat implies Property (⋆), by assumption, Property (⋆) implies Condition (P) and Condition (P) implies Condition (WP). Thus S is right cancellative, by [19, Theorem 2.14].

(3) ⇒ (1) Since S is right cancellative, S is left PSF and every weakly flat S-act is weakly pullback flat, by [19, Theorem 2.14]. Since Property (⋆) implies weakly flat, Property (⋆) for S-acts implies weakly pullback flat, as required.

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.17 is true for finitely generated S-acts. Furthermore Property (⋆) in Theorem 3.17 can be properties as flat and weakly flat.

Notice that we have Theorem 2 of [4] and Theorem 5.4 of [13] as a corollary of Theorem 3.17. Furthermore these theorems are true for finitely generated S-acts.

**Corollary 3.18.** Let (⋆) be a property on S-acts such that flat ⇒ Property (⋆) ⇒ weakly flat. Then the following statements are equivalent:

1. All S-acts satisfying Property (⋆) are weakly pullback flat and satisfy Condition ($PWP_{ssc}$);
2. all S-acts satisfying Property (⋆) are weakly kernel flat and satisfy Condition ($PWP_{ssc}$);
3. all S-acts satisfying Property (⋆) satisfy Conditions ($WP$) and ($PWP_{ssc}$);
4. all S-acts satisfying Property (⋆) satisfy Conditions ($P$) and ($PWP_{ssc}$);
5. S is right cancellative.

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious, because weakly pullback flat ⇒ weakly kernel flat ⇒ Condition (WP), also implications (1) ⇒ (4) ⇒ (3) are obvious, because weakly pullback flat ⇒ Condition (P) ⇒ Condition (WP).

(3) ⇒ (5) Since $S_S$ is flat, by assumption, satisfies Property (⋆) and so satisfies Condition ($PWP_{ssc}$). Therefore S is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every S-act satisfying Property (⋆) satisfies Conditions (WP) and thus S is right cancellative, by Theorem 3.17.

(5) ⇒ (1) Since S is right cancellative, every S-act satisfying Property (⋆) is weakly pullback flat, by Theorem 3.17. Also S is left PSF, by Theorem
3.17. Since Property $(\ast)$ implies principally weakly flat, every $S$-act satisfying Property $(\ast)$ satisfies Condition $(PWP_{ssc})$, by Theorem 2.8(2).

Using a similar argument as in the proof of the above corollary, we conclude that Corollary 3.18 is true for finitely generated $S$-acts. Furthermore Property $(\ast)$ in Corollary 3.18 can be properties as flat and weakly flat.

**Theorem 3.19.** The following statements are equivalent:

1. Every Rees factor $S$-act satisfying Condition $(P)$ satisfies Condition $(PWP_{ssc})$;
2. Every free Rees factor $S$-act satisfies Condition $(PWP_{ssc})$;
3. $S$ does not contain a left zero or it is left PSF.

**Proof.** Implications $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ If $S$ does not contain a left zero, then $(3)$ is true. Let $z$ be a left zero of $S$. Then $K_S = zS = \{z\}$ is a right ideal of $S$. Since $|K_S| = 1$, $S/K_S \cong S_S$ is free, by [14, I, Proposition 5.22], and so, by assumption, $S/K_S \cong S_S$ satisfies Condition $(PWP_{ssc})$. Thus $S$ is left PSF, by part (7) of Theorem 2.2.

$(3) \Rightarrow (1)$ Let $K_S$ be a right ideal of $S$ such that $S/K_S$ satisfies Condition $(P)$. If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ and so $S/K_S \cong \Theta_S$ satisfies Condition $(PWP_{ssc})$, by part (1) of Theorem 2.2. If $K_S \neq S$, then $|K_S| = 1$, by [14, III, Proposition 13.9]. If $z \in K_S$, then $K_S = zS = \{z\}$, that is, $z$ is a left zero of $S$. Since $S$ contains a left zero, $S$ is left PSF, by assumption, and so $S/K_S \cong S_S$ satisfies Condition $(PWP_{ssc})$, by part (7) of Theorem 2.2.

**Remark**

The referee had told the new property lies between strong torsion freeness and principal flatness, and it looks very similar to the right semi-cancellativity of a monoid. So why not, for example, to say “$A$ is semi-cancellative” instead of “$A$ satisfies Condition $(PWP_{ssc})$”? We did not change the naming, since in our definition the first part $(as = a's)$ is similar to Condition $(PWP)$ and the last part $(au = a'u$ and $us = s)$ is similar to semi-cancellative. Also semi-cancellativity of monoids is equivalent to PSF monoids, while the property PSF is not defined for $S$-acts. So we thought that the assertion “$A$ is semi-cancellative” is ambiguous.
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