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Completeness results for metrized rings and lattices

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Dedicated to George Grätzer, and to the memory of Jonathan Gleason

Abstract. The Boolean ring B of measurable subsets of the unit interval, modulo sets of measure zero, has proper *radical* ideals (for example, $\{0\}$) that are closed under the natural metric, but has no *prime* ideal closed under that metric; hence closed radical ideals are not, in general, intersections of closed prime ideals. Moreover, B is known to be complete in its metric. Together, these facts answer a question posed by J. Gleason. From this example, rings of arbitrary characteristic with the same properties are obtained.

The result that B is complete in its metric is generalized to show that if L is a lattice given with a metric satisfying identically *either* the inequality $d(x \vee y,\, x \vee z) \leq d(y, z)$ *or* the inequality $d(x \wedge y,\, x \wedge z) \leq d(y, z)$, and if in L every increasing Cauchy sequence converges and every decreasing Cauchy sequence converges, then every Cauchy sequence in L converges; that is, L is complete as a metric space.

We show by example that if the above inequalities are replaced by the

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weaker conditions $d(x,\, x\vee y) \leq d(x,y)$, respectively $d(x,\, x\wedge y) \leq d(x,y)$, the completeness conclusion can fail.

We end with two open questions.

1 Overview: a ring-theoretic question, culminating in a lattice-theoretic result

A standard result of ring theory says that if I is an ideal of a commutative ring R, then the nil radical of I (the ideal of elements having some power in I) is the intersection of the prime ideals of R containing I [1, Proposition 10.2.9, p. 352].

Jonathan Gleason[1] (personal communication) asked the present author about a possible generalization of that result. Namely, suppose R is a topological commutative ring. For any ideal I of R, let $\sqrt{I}$ denote the least *closed* ideal $J \supseteq I$ such that J contains every element x such that $x^n \in J$ for some $n \geq 1$. Must $\sqrt{I}$ be the intersection of all closed *prime* ideals containing I? If not in general, does this become true if R is complete with respect to the given topology?

We shall see that the answer is negative: If R is the Boolean ring of measurable subsets of the unit interval modulo sets of measure zero, topologized using the metric given by the measure of the symmetric difference of such sets, then R is complete in that metric, and $\{0\} = \sqrt{\{0\}}$ (defined as above); but R has no closed prime ideals, so $\{0\}$ is not an intersection of such ideals. We give the details in §2, and note in §3 how to get, from this characteristic-2 example, examples of arbitrary characteristic.

The one not-so-obvious property of our example is the completeness of B as a metric space. In §4 (which is independent of §§2-3) we note that this can be deduced from a standard result of measure theory, and then prove a general result on when a metrized *lattice* is complete, which yields an alternate proof.

In §5 we give a curious counterexample to a version of that general completeness result with a weakened hypothesis.

[1] Jonathan Gleason, then a graduate student in mathematics at the University of California, Berkeley, raised this question shortly before his unexpected tragic death in January, 2018.

2 The Boolean example

Most of the desired properties of the example sketched above are straightforward to verify.

Recall that for any set X, the subsets of X form a Boolean ring under the operations

$$0 = \emptyset, \quad 1 = X, \quad S + T = S \cup T \setminus (S \cap T), \quad ST = S \cap T. \tag{2.1}$$

Now let B_0 be the set of measurable subsets of the unit interval $[0, 1]$, and for $S \in B_0$, let $\mu(S) \in [0, 1]$ be its measure. B_0 clearly forms a subring of the Boolean ring of subsets of $[0, 1]$, and for any $S,\, T \in B_0$, we see from the above definition of $S + T$ that

$$\mu(S \cup T) \;=\; \mu(S + T) + \mu(S \cap T) \;=\; \mu(S) + \mu(T) - \mu(S \cap T). \tag{2.2}$$

For $S,\, T \in B_0$, let

$$d_0(S,\, T) \;=\; \mu(S + T). \tag{2.3}$$

Then d_0 is a pseudometric, that is, for all $S,\, T,\, U \in B_0$,

$$d_0(S,\, T) \;\geq\; 0, \tag{2.4}$$

$$S = T \;\;\Longrightarrow\;\; d_0(S,\, T) = 0, \tag{2.5}$$

$$d_0(T,\, S) \;=\; d_0(S,\, T), \tag{2.6}$$

$$d_0(S,\, T) + d_0(T,\, U) \;\geq\; d_0(S, U). \tag{2.7}$$

Here (2.4)-(2.6) are immediate. To get (2.7), note that the second equality of (2.2) gives the inequality $\mu(S) + \mu(T) \geq \mu(S + T)$. Putting $S + T$ and $T + U$ in place of S and T in that relation gives (2.7).

The definition (2.3), and the identities of Boolean rings, show that the Boolean operations behave nicely under d_0:

$$d_0(S{+}U,\, T{+}U) \;=\; d_0(S,\, T), \tag{2.8}$$

$$d_0(SU,\, TU) \;\leq\; d_0(S,\, T). \tag{2.9}$$

These relations show in particular that if S and T are close to one another under d_0, then the results of adding U to S and T are also close to one another, as are the results of multiplying S and T by U, in each case in a uniform way, whence addition and multiplication are continuous with respect to d_0.

Now let B be the quotient ring

$$B \;=\; B_0 \,/\, \{S \mid \mu(S) = 0\}. \tag{2.10}$$

Let us write $[S] \in B$ for the residue class of an element $S \in B_0$, and define

$$d([S],[T]) \;=\; d_0(S,T) \;=\; \mu(S+T) \quad \text{for} \quad [S],\, [T] \in B. \tag{2.11}$$

Then (2.4)-(2.9) clearly carry over to d, with the "$\Longrightarrow$" of (2.5) strengthened to "$\Longleftrightarrow$". Thus, d is a metric, and

Lemma 2.1. *The operations of the Boolean ring B of measurable subsets of $[0,1]$ modulo sets of measure zero are continuous in the metric d of* (2.11). □

Being a Boolean ring, B has no nonzero nilpotent elements, hence the ideal $\{0\}$ of B trivially contains every $x \in B$ such that $x^n \in \{0\}$ for some $n \geq 1$; and being a singleton, $\{0\}$ is closed in the metric topology. So $\sqrt{\{0\}} = \{0\}$ under the definition of $\sqrt{I}$ suggested by Gleason.

Is $\{0\}$ an intersection of closed prime ideals? A negative answer follows from

Lemma 2.2. *Every prime ideal P of the ring B has for topological closure the whole ring B.*

Hence B has no closed prime ideals.

Proof. Given a prime ideal P, let us first show that P has elements arbitrarily close to 1, that is, elements $[1+U]$ such that U has arbitrarily small measure. To do this, we shall show that whenever $[1+U] \in P$, there exists $[1+S] \in P$ with $\mu(S) = \mu(U)/2$.

Indeed, let us write U as the union of two disjoint measurable subsets S and T, each of measure $\mu(U)/2$. (For example, one can take $S = U \cap [0,t]$ and $T = U \cap (t,1]$ for appropriate $t \in [0,1]$. Such a t exists by continuity of $\mu(U \cap [0,t])$ in t.) Since $[1+S][1+T] = [1+U] \in P$, one of $[1+S]$, $[1+T]$ belongs to P.

So P indeed has elements arbitrarily close to 1. Multiplying arbitrary $[V] \in B$ by such elements, we see that P has elements arbitrarily close to $[V]$; so the closure of P contains every $[V] \in B$, as claimed.

Hence no prime ideal P is itself closed, giving the final assertion. $\square$

So $\{0\} = \sqrt{\{0\}}$ is not an intersection of closed prime ideals. Since B is complete in the metric μ (to be proved in two ways in §4), B answers the strongest form of Gleason's question.

(To be precise, Gleason's question concerned completeness of a topological ring R in the *uniform structure* arising from additive translates of neighborhoods of 0. Our metrized ring has additive-translation-invariant metric by (2.8), hence completeness in the uniform structure so arising from the metric topology is equivalent to completeness in the metric.)

3 Non-Boolean algebras

Is the behavior of above example limited to Boolean rings, or perhaps to rings of nonzero characteristic? No. We note below how to generalize the construction of the preceding section to algebras in the sense of General Algebra (a.k.a. Universal Algebra), and observe that when the algebras in question are rings (of arbitrary characteristic), these give more varied examples of the properties proved for B.

We start with the analog of B_0.

Definition 3.1. For X a set, let $X^{[0,1]}$ denote the set of all X-valued functions on the unit interval, and for each $f \in X^{[0,1]}$ and $x \in X$, let

$$f_x \;=\; \{t \in [0,1] \mid f(t) = x\}. \tag{3.1}$$

Let X' denote the subset of $X^{[0,1]}$ consisting of those f such that

$$\text{for all } x \in X, \text{ the set } f_x \subseteq [0,1] \text{ is measurable,} \tag{3.2}$$

and

$$\text{the image of } f, \text{ that is, } \{x \in X \mid f_x \neq \emptyset\}, \text{ is countable (that is, finite or countably infinite).} \tag{3.3}$$

For $f, g \in X'$, let

$$d'(f,\, g) = \mu(\{t \in [0,1] \mid f(t) \neq g(t)\}) = \left(\textstyle\sum_{x \in X} d_0(f_x,\, g_x)\right) / \, 2, \tag{3.4}$$

where d_0 is the pseudometric on measurable subsets of $[0,1]$ defined in (2.3).

The final equality of (3.4) is, intuitively, a consequence of the fact that every $t \in [0,1]$ such that $f(t) \neq g(t)$ contributes twice to the summation in the final term: via the summand with $x = f(t)$ and the summand with $x = g(t)$. That idea is easily formalized to show that that sum is indeed twice the middle term of (3.4).

Lemma 3.2. *For any set* X, *the function* d' *defined by* (3.4) *is a pseudometric on* X'.

For any finitary operation $u : X^n \to X$, *the induced pointwise operation* $u^{[0,1]} : (X^{[0,1]})^n \to X^{[0,1]}$ *carries* $(X')^n$ *to* X', *and for* $f^{(0)}, \dots, f^{(n-1)}$, $g^{(0)}, \dots, g^{(n-1)} \in X'$, *we have*

$$\begin{aligned} d'(u^{[0,1]}(f^{(0)}, \dots, f^{(n-1)}),\ u^{[0,1]}(g^{(0)}, \dots, g^{(n-1)})) \\ \leq\ d'(f^{(0)},\, g^{(0)}) + \dots + d'(f^{(n-1)},\, g^{(n-1)}). \end{aligned} \tag{3.5}$$

Proof. That d' is a pseudometric is straightforward. (The triangle inequality is verified using the rightmost expression of (3.4) and the fact that d_0 is a pseudometric.)

We need to know next that given $u : X^n \to X$, and $f^{(0)}, \dots, f^{(n-1)} \in X'$, we have $u^{[0,1]}(f^{(0)}, \dots, f^{(n-1)}) \in X'$.

Note that for each $x \in X$, $u^{[0,1]}(f^{(0)}, \dots, f^{(n-1)})_x$ will be the union, over all n-tuples $(x_0, \dots, x_{n-1})$ satisfying $u(x_0, \dots, x_{n-1}) = x$, of the sets

$$f^{(0)}_{x_0} \cap \dots \cap f^{(n-1)}_{x_{n-1}}. \tag{3.6}$$

Now for each $i \in n$, only countably many values of x_i make $f^{(i)}_{x_i}$ nonempty, so only countably many n-tuples $(x_0, \dots, x_{n-1})$ make the intersection (3.6) nonempty; and by (3.2), each of those n-fold intersections is measurable; so for each x, $u^{[0,1]}(f^{(0)}, \dots, f^{(n-1)})_x$ is a countable union of measurable sets, hence measurable; i.e., $u^{[0,1]}(f^{(0)}, \dots, f^{(n-1)})$ satisfies the condition of (3.2). It also satisfies the condition of (3.3), since the countably many cases where (3.6) is nonempty lead to only countably many possibilities for the element $u(x_0, \dots, x_{(n-1)})$. So $u^{[0,1]}$ carries $(X')^n$ to X'.

The inequality (3.5) follows from the fact that $u(f^{(0)}, \dots, f^{(n-1)})$ and $u(g^{(0)}, \dots, g^{(n-1)})$ can differ only at points $t \in [0,1]$ where $f^{(i)}$ and $g^{(i)}$ differ for at least one i. □

We now want to deduce the corresponding results with the set of functions X' replaced by the set of equivalence classes of such functions under the relation of differing on a set of measure zero. We will need the following observation.

Lemma 3.3. *As in §2, let B_0 denote the Boolean ring of measurable subsets of* $[0,1]$.

Let S_0, S_1, … be a countable (that is, finite or countably infinite) family of elements of B_0 such that

$$\mu(S_i \cap S_j) \;=\; 0 \quad \textit{whenever } i \neq j, \tag{3.7}$$

and

$$\textstyle\sum_i \mu(S_i) \;=\; 1. \tag{3.8}$$

Then there exist $T_0, T_1, \ldots \in B_0$ such that

$$d_0(S_i, T_i) \;=\; 0 \quad (i = 0, 1, \ldots), \tag{3.9}$$

and the T_i partition $[0,1]$, *that is, satisfy the two conditions*

$$T_i \cap T_j \;=\; \emptyset \textit{ whenever } i \neq j \ (\textit{cf.}\ (3.7)) \tag{3.10}$$

and

$$\textstyle\bigcup_i T_i \;=\; [0,1] \ (\textit{cf.}\ (3.8)). \tag{3.11}$$

Proof. Let

$$T_i \;=\; S_i \setminus \textstyle\bigcup_{0 \leq j < i} S_j \;\text{ for }\; i > 0, \tag{3.12}$$

and

$$T_0 \;=\; [0,1] \setminus \textstyle\bigcup_{i>0} T_i. \tag{3.13}$$

These sets are clearly measurable and partition $[0,1]$.

Since the sets S_j whose members are removed from S_i in (3.12) have, by (3.7), only a set of measure zero in common with S_i, we see that for $i > 0$, T_i differs from S_i in a set of measure zero, giving (3.9) for such i.

Also by (3.12), no T_i with $i > 0$ contains elements of S_0, so by (3.13), $T_0 \supseteq S_0$; hence to prove the $i = 0$ case of (3.9), it suffices to show that $\mu(T_0) = \mu(S_0)$. To do this, note that since the T_i partition $[0,1]$, we have $\sum_{i\geq 0} \mu(T_i) = \mu([0,1]) = 1 = \sum_{i\geq 0} \mu(S_i)$ by (3.8). If we subtract from that relation the equations $\mu(T_i) = \mu(S_i)$ for all $i > 0$, which follow from the cases of (3.9) already obtained, we get the desired $i = 0$ case. □

Now – still assuming the completeness of B, to be obtained in the next section – we can get

Proposition 3.4. *For X a set, let X^* denote the quotient of X' (defined in* Definition 3.1) *by the equivalence relation $d'(f, g) = 0$, and let d^* be the metric on X^* induced by d'.*

Then X^*, under the metric d^*, is a complete metric space.

Proof. Consider any Cauchy sequence $[f^{(0)}], [f^{(1)}], \cdots \in X^*$, where $f^{(0)}, f^{(1)}, \cdots \in X'$. For each n, the set of elements $x \in X$ such that $f_x^{(n)}$ is nonempty is countable by (3.3); hence there exists a countable (possibly finite) list of distinct such elements:

$$\{x_0, x_1, \dots\} \ = \ \{x \in X \mid (\exists\, n)\, f_x^{(n)} \neq \emptyset\}. \tag{3.14}$$

Now (writing d_0 and d, as in the preceding section, for our pseudometric on B_0 and metric on B), we have for all i, m, n,

$$\begin{aligned} d([f_{x_i}^{(m)}], [f_{x_i}^{(n)}]) \ &= \ d_0(f_{x_i}^{(m)}, f_{x_i}^{(n)}) \\ &\leq \ d'(f^{(m)}, f^{(n)}) \ = \ d^*([f^{(m)}], [f^{(n)}]); \end{aligned} \tag{3.15}$$

hence the Cauchyness of the sequence of elements $[f^{(n)}] \in X^*$ implies, for each x_i, the Cauchyness of the sequence of $[f_{x_i}^{(n)}] \in B$. Hence by the completeness of B, for each i the sequence $[f_{x_i}^{(0)}], [f_{x_i}^{(1)}], \dots$ converges to an element which we shall write $[S_i]$, choosing an arbitrary representative $S_i \in B_0$ of the limit of that sequence in B.

I claim that these sets S_i satisfy (3.7) and (3.8). To get the first of these equations for given $i \neq j$, note that for any $\varepsilon > 0$, one can choose n such that $d_0(f_{x_i}^{(n)}, S_i) < \varepsilon/2$ and $d_0(f_{x_j}^{(n)}, S_j) < \varepsilon/2$. Since $f_{x_i}^{(n)}$ and $f_{x_j}^{(n)}$ are disjoint, we see that S_i and S_j intersect in a set of measure at most ε. Since this holds for all $\varepsilon > 0$, they intersect in a set of measure 0.

To get (3.8), note that for any $\varepsilon > 0$ we may choose m such that for all $n \geq m$ we have $d^*([f^{(m)}],\, [f^{(n)}]) \leq \varepsilon/3$, equivalently,

$$d'(f^{(m)},\, f^{(n)}) \ \leq\ \varepsilon/3. \tag{3.16}$$

Since $f_{x_0}^{(m)}, f_{x_1}^{(m)}, \ldots$ partition $[0,1]$, we can also choose j such that

$$\mu(f_{x_0}^{(m)}) + \cdots + \mu(f_{x_j}^{(m)}) \ \geq\ 1 - \varepsilon/3. \tag{3.17}$$

For $n \geq m$, (3.16) guarantees that $d_0(f_{x_0}^{(m)},\, f_{x_0}^{(n)}) + \cdots + d_0(f_{x_j}^{(m)},\, f_{x_j}^{(n)}) \leq 2\varepsilon/3$ (see (3.4)), equivalently, $d([f_{x_0}^{(m)}],\, [f_{x_0}^{(n)}]) + \cdots + d([f_{x_j}^{(m)}],\, [f_{x_j}^{(n)}]) \leq 2\varepsilon/3$, hence passing to the limit as $n \to \infty$, $d([f_{x_0}^{(m)}],\, [S_0]) + \cdots + d([f_{x_j}^{(m)}],\, [S_j]) \leq 2\varepsilon/3$; and combining with (3.17) we get $\mu(S_0) + \cdots + \mu(S_j) \geq 1 - \varepsilon$, equivalently, $\mu(S_0 \cup \cdots \cup S_j) \geq 1 - \varepsilon$. Since this holds for all ε, we have $\sum_i \mu(S_i) = \mu(\bigcup_i S_i) \geq 1$; and since a subset of $[0,1]$ cannot have measure larger than 1, we get (3.8).

Lemma 3.3 now gives us a partition of $[0,1]$ into sets T_i which differ from the S_i by sets of measure zero. If we define $f \in X'$ by

$$f_{x_i} \ =\ T_i \ \text{ for all } i \text{ (whence, by (3.11), } f_x = \emptyset \text{ for all } x \text{ not of the form } x_i), \tag{3.18}$$

then $[f] \in X^*$ is a limit of the given Cauchy sequence $[f^{(0)}],\, [f^{(1)}],\, \ldots$, proving completeness. □

Remark: If in (3.3) we had allowed uncountable cardinalities, we would not have been able to use basic properties of measure, for example, in concluding that the set in the middle term of (3.4) was measurable, and in proving in Lemma 3.2, that $u^{[0,1]}$ carries $(X')^n$ to X'. On the other hand, if we had required the set of x making f_x nonempty to be finite, our X^* would not have been complete, except in the case where X was finite. So countability is the only choice that gives our construction X^* the desired properties.

We have not yet called on (3.5). It implies that our construction behaves nicely on algebras:

Proposition 3.5. *Suppose A is an algebra in the sense of General Algebra, that is, a set given with a (finite or infinite) family of operations, each of finite arity, and let the set A^* be defined as in Proposition 3.4.*

Then for each operation $u : A^n \to A$ of A, the operation u^* of A^* described by

$$u^*([f_0], \dots, [f_{n-1}]) \;=\; [u^{[0,1]}(f_0, \dots, f_{n-1})] \tag{3.19}$$

is well-defined, and uniformly continuous in the metric d^*; indeed, it satisfies Lipschitz condition

$$\begin{aligned} &d^*(u^*([f^{(0)}], \dots, [f^{(n-1)}]),\, u^*([g^{(0)}], \dots, [g^{(n-1)}])) \\ &\qquad \le\; d^*([f^{(0)}],\, [g^{(0)}]) + \dots + d^*([f^{(n-1)}],\, [g^{(n-1)}]). \end{aligned} \tag{3.20}$$

The resulting algebra A^* satisfies all identities satisfied by A. In fact, every finite set of elements of A^* is contained in a subalgebra of A^* isomorphic to a countable direct product of copies of A.

Sketch of proof. By the case of (3.5) where the $d'(f^{(i)}, g^{(i)})$ are all zero, the operations $u^{[0,1]}$ of A' respect the equivalence relation used in defining the set A^*, so (3.19) gives well-defined operations. The general case of (3.5) then gives the Lipschitz inequality (3.20), and in particular, continuity. Finally, given any finite family of elements $[f^{(0)}], \dots, [f^{(N-1)}]$ of A^*, the countably many nonempty sets $f_a^{(i)}$ $(0 \le i < N,\ a \in A)$ yield a decomposition of $[0,1]$ into countably many intersections as in (3.6), on each of which all of $f^{(0)}, \dots, f^{(N-1)}$ are constant. Dropping those intersections that have measure zero, and looking at the algebra of members of A' that are constant on the remaining countably many subsets, we see that this contains $f^{(0)}, \dots, f^{(N-1)}$, and has as its image in A^* a subalgebra isomorphic to a countable direct product of copies of A. □

Finally, some observations specific to rings.

Proposition 3.6. *Let A be an associative unital ring. Then in the complete metrized ring A^* arising by the construction of Proposition 3.5, the closure of every prime ideal is all of A^*; hence A^* has no closed prime ideals.*

On the other hand, if A is commutative and has no nonzero nilpotent elements, then the topological radical $\sqrt{\{0\}}$, defined as in §1, *is* $\{0\}$.

Sketch of proof. Let P be a prime ideal of A^*. Writing 1_S for the characteristic function with values in $\{0,1\} \subseteq A$ of a subset $S \subseteq [0,1]$, let us show that P contains elements $[1_{[0,1]\setminus U}]$ for sets U of arbitrarily small positive measure. Clearly, it contains $[1_{[0,1]\setminus U}]$ for $U = [0,1]$. Given any U such that $[1_{[0,1]\setminus U}] \in P$, let us partition U into two disjoint measurable subsets S and T of equal measure. Since $\{0,1\}$-valued functions are central in A', so are their images in A^*, so we have

$$\begin{aligned} [1_{[0,1]\setminus S}]\ A^*\ [1_{[0,1]\setminus T}] \ &= \ [1_{[0,1]\setminus S}]\ [1_{[0,1]\setminus T}]\ A^* \\ &= \ [1_{[0,1]\setminus (S\cup T)}]\ A^* \ = \ [1_{[0,1]\setminus U}]\ A^* \ \subseteq P, \end{aligned} \tag{3.21}$$

so as P is prime, one of $[1_{[0,1]\setminus S}]$, $[1_{[0,1]\setminus T}]$ belongs to P; so we have cut in half the measure of our set U with $[1_{[0,1]\setminus U}] \in P$. Since $d([1_{[0,1]\setminus U}],\, 1) = \mu(U)$, we have, as in the proof of Lemma 2.2, found elements of P arbitrarily close to 1, and can deduce that the closure of P is all of A^*.

The final assertion is straightforward. □

Thus, for A a commutative ring without nilpotents, the rings A^* generalize the properties of the example B of the preceding section. That B is, of course, the case of this construction with $A = \mathbb{Z}/2\mathbb{Z}$.

If we take $A = \mathbb{Z}/n\mathbb{Z}$ for an arbitrary positive integer n, then A may have nilpotents, so the last sentence of Proposition 3.6 does not apply. Nevertheless, from the fact that the nil ideal of A is finite, and so in particular, has a bound on the order of nilpotence of its elements, it is easy to deduce that the nilpotent elements of A^* form a closed ideal N. Thus, $N = \sqrt{N}$ is a proper ideal of A^*, hence again, not an intersection of closed prime ideals; so as stated in the Abstract, we get counterexamples of all characteristics to the statement J. Gleason asked about.

Remark: The development of the above results in terms of measurable X-valued functions on $[0,1]$, modulo disagreement on sets of measure zero, feels artificial. Surely one should be able to perform our constructions abstractly in terms of the set X, the Boolean ring B, and the real-valued function on B induced by the measure on $[0,1]$, and then generalize it to get such results with B replaced by any Boolean ring with an appropriate real-valued function.

If we were interested in maps $[0,1] \to X$ assuming only *finitely many* values, then the analog of X^* could be described as the set of continuous functions from the Stone space of B to the discrete space X. But for maps allowed to assume countably many values, the function corresponding to the metric seems to be needed in defining X^*. I leave the proper formulation and generalization of that construction to experts in the subject. Cf. [2, Chapter 31].

In contrast, the results of the next section will be obtained in a satisfyingly general context.

4 Completeness

When I first looked at the Boolean ring B of measurable subsets of $[0,1]$, modulo sets of measure zero, as a possible answer to J. Gleason's question, the one property that was not clear was completeness in the natural metric, though it seemed likely.

One might naively hope to prove completeness by showing that every sequence of measurable subsets of $[0,1]$ whose images in B form a Cauchy sequence "converges almost everywhere" on $[0,1]$; i.e., that almost every $t \in [0,1]$ belongs either to all but finitely many members of the sequence, or to only finitely many. But this is not so; a counterexample [3, Exercise 22(6), p.94] is the sequence whose first term is $[0,1]$, whose next two are $[0,\ 1/2]$ and $[1/2,\ 1]$, whose next three are $[0,\ 1/3]$, $[1/3,\ 2/3]$, $[2/3,\ 1]$, and, generally, whose $1{+}2{+}\ldots{+}(n{-}1){+}\,i$-th term for $1 \leq i \leq n$ is $[(i-1)/n,\ i/n]$. The measures of these sets approach zero, so the sequence approaches $\emptyset$ in our metric; but clearly every $t \in [0,1]$ occurs in infinitely many of these sets. Looking at this example, one might still hope that given a Cauchy sequence in B, almost every $t \in [0,1]$ has the property that the terms S_i which contain t are either "eventually scarce", or have eventually scare complement. But this, too, fails; to see this, take the above example, and "stretch it out" by repeating the m-th term 2^m times successively, for each m.

However, an online search turned up a proof of the desired completeness statement in a set of exercises [5] (in particular, point 6 on p. 2). I cited that, in the first draft of this note, as the only reference for the result that I could find. David Handelman then pointed out that the desired statement follows immediately from the standard fact that L^1 of the unit interval is complete in its natural metric ([4, Theorem VI.3.4, p.133], [3, Theorem 22.E, p.93]), on identifying measurable sets with their characteristic functions. (And indeed, in [3, Exercise 40(1), p.169], the reader is asked to deduce the result we want from that result about L^1.) Subsequently, Hannes Thiel pointed me to a result of the desired sort proved for a large class of Boolean rings with measure-like $[0,1]$-valued functions [2, Theorem 323G(c)]. (The condition there called *localizability* means, roughly, that the Boolean ring has "enough" elements of finite measure, and has joins of arbitrary subsets.)

In all these sources, the key to the proof of completeness is to pass from an arbitrary Cauchy sequence to a subsequence with the property that the distance between the i-th and $i{+}1$-st terms is $\leq 2^{-i}$. Rather magically, a sequence with this property does indeed converge almost everywhere, giving a limit of the original Cauchy sequence.

In fact, this trick can be abstracted from the context of measure theory to that of lattices (or even semilattices) as in the next theorem, from which we will recover, as a corollary, the result on measurable sets modulo null sets.

Since we no longer need the notation "f_x" of the preceding section for the point-set at which a function takes on the value x, we will henceforth use subscripts in the conventional way to index terms of sequences.

We remark that the condition that a metrized lattice be complete as a metric space, obtained in the theorem, is independent of its completeness as a lattice, i.e., the existence of least upper bounds and greatest lower bounds of not necessarily finite subsets (though the condition that *certain* infinite least upper bounds and greatest lower bounds exist will be key to the argument). For instance, any lattice, under the metric that makes $d(x,y)=1$ whenever $x\neq y$, is complete as a metric space, and, indeed, satisfies the hypotheses of the next theorem, but need not be complete as a lattice. Inversely, the totally ordered subset of the real numbers $\{-2\}\cup(-1,1)\cup\{2\}$ is complete as a lattice, but not as a metric space.

Theorem 4.1. *Let L be a lattice, whose underlying set is given with a metric d which satisfies identically at least one of the inequalities*

$$d(x\vee y,\, x\vee z)\ \leq\ d(y,\, z)\quad (x,y,z\in L), \tag{4.1}$$

$$d(x\wedge y,\, x\wedge z)\ \leq\ d(y,\, z)\quad (x,y,z\in L) \tag{4.2}$$

(*or, more generally, let L be an upper semilattice satisfying* (4.1)*, or a lower semilattice satisfying* (4.2)).

Suppose moreover that in L

every increasing Cauchy sequence $x_0\leq x_1\leq\ldots\leq x_n\leq\ldots$ converges, (4.3) *and likewise*

every decreasing Cauchy sequence $x_0 \geq x_1 \geq \ldots \geq x_n \geq \ldots$ *converges.* (4.4)

Then every Cauchy sequence in L *converges; i.e.,* L *is complete as a metric space.*

Proof. It suffices to prove the case where L is an upper semilattice satisfying (4.1), since this includes the case where L is a lattice satisfying (4.1), while the cases where L is a lower semilattice or lattice satisfying (4.2) follow by duality. So assume L such an upper semilattice.

In proving L complete, it suffices to show convergence of sequences $x_0, x_1, \ldots$ such that

$$\textstyle\sum_{i\geq 0} d(x_i,\, x_{i+1}) \;<\; \infty, \tag{4.5}$$

since every Cauchy sequence has such a subsequence (for example, one chosen so that $d(x_i,\, x_{i+1}) \leq 2^{-i}$, as in [3], [4] and [5]), and if a subsequence of a Cauchy sequence converges, so does the whole sequence.

So let the sequence $x_0, x_1, \ldots$ satisfy (4.5), and let us define

$$x_{h,j} \;=\; x_h \vee x_{h+1} \vee \cdots \vee x_j \quad \text{for } h \leq j. \tag{4.6}$$

Note that if in (4.1) we put $x = x_{h,j}$, $y = x_j$, $z = x_{j+1}$, we get

$$d(x_{h,j},\, x_{h,j+1}) \;\leq\; d(x_j,\, x_{j+1}). \tag{4.7}$$

Also, for $h \leq j \leq k$, the triangle inequality (applied $k-j-1$ times) gives $d(x_{h,j},\, x_{h,k}) \leq \sum_{j\leq \ell<k} d(x_{h,\ell},\, x_{h,\ell+1})$. Applying (4.7) to each term of this summation, we get

$$d(x_{h,j},\, x_{h,k}) \;\leq\; \textstyle\sum_{j\leq \ell<k} d(x_\ell,\, x_{\ell+1}). \tag{4.8}$$

In particular, for each j, the distance from $x_{h,j}$ to any of the later terms $x_{h,k}$ is $\leq \sum_{j\leq \ell<\infty} d(x_\ell,\, x_{\ell+1})$. As $j \to \infty$, this sum approaches 0, so (still for fixed h) the elements $x_{h,j}$ ($j = h,\, h{+}1, \ldots$) form an increasing Cauchy sequence. By (4.3) this sequence will converge; let

$$x_{h,\infty} \;=\; \lim_{j\to\infty} x_{h,j}. \tag{4.9}$$

Note that, by (4.8), (4.9), and the continuity of d in the topology it defines, we have

$$d(x_{h,j},\, x_{h,\infty}) \;\leq\; \textstyle\sum_{j\leq \ell<\infty} d(x_\ell,\, x_{\ell+1}) \;\text{ for } h\leq j. \tag{4.10}$$

Note next that

$$x_{h,j} \;\geq\; x_{i,j} \;\text{ for } h\leq i\leq j. \tag{4.11}$$

I claim that this implies that

$$x_{h,\infty} \;\geq\; x_{i,\infty} \;\text{ for } h\leq i. \tag{4.12}$$

Indeed, (4.11) and (4.12) are respectively equivalent to the conditions $d(x_{h,j},\, x_{h,j}\vee x_{i,j}) = 0$ and $d(x_{h,\infty},\, x_{h,\infty}\vee x_{i,\infty}) = 0$, and the latter can be obtained from the former using (4.9).

So the elements $x_{h,\infty}$ $(h = 0, 1, \dots)$ form a decreasing sequence. I claim that this sequence, too, is Cauchy; in fact, that

$$d(x_{h,\infty},\, x_{i,\infty}) \;\leq\; \textstyle\sum_{h\leq \ell<i} d(x_\ell,\, x_{\ell+1}) \;\text{ for } h\leq i. \tag{4.13}$$

Namely, by essentially the same argument used to prove (4.8), one sees that for every $j \geq i$, $d(x_{h,j},\, x_{i,j}) \leq \sum_{h\leq \ell<i} d(x_\ell,\, x_{\ell+1})$; and by continuity of d, this again carries over to the limit as $j \to \infty$. Hence by (4.4), the terms of (4.12) converge, and we can define

$$x_{\infty,\infty} \;=\; \lim_{h\to\infty} x_{h,\infty}. \tag{4.14}$$

Finally, note that for every $h \geq 0$,

$$\begin{aligned} d(x_h,\, x_{\infty,\infty}) \;&=\; d(x_{h,h},\, x_{\infty,\infty}) \\ &\leq\; d(x_{h,h},\, x_{h,\infty}) + d(x_{h,\infty},\, x_{\infty,\infty}) \\ &\leq\; \textstyle\sum_{h\leq \ell<\infty} d(x_\ell,\, x_{\ell+1}) + \sum_{h\leq \ell<\infty} d(x_\ell,\, x_{\ell+1}) \\ &=\; 2\textstyle\sum_{h\leq \ell<\infty} d(x_\ell,\, x_{\ell+1}), \end{aligned} \tag{4.15}$$

and that this sum approaches 0 as $h \to \infty$. Hence the x_h converge,

$$\lim_{h\to\infty} x_h \;=\; x_{\infty,\infty}, \tag{4.16}$$

completing the proof of the theorem. □

The first assertion of the following corollary clearly includes the completeness result called on in §§2-3. The remaining two assertions are further generalizations.

Corollary 4.2. *Let M be a measure space of finite total measure, and B the Boolean ring of measurable subsets of M modulo sets of measure zero. For S a measurable subset of M, let $[S]$ denote its image in B. Then B is complete with respect to the metric*

$$d([S],\,[T]) \;=\; \mu(S+T), \tag{4.17}$$

More generally, if M is a measure space not necessarily of finite total measure, and C a positive real constant, and we define

$$d_C([S],\,[T]) \;=\; \min(\mu(S+T),\,C), \tag{4.18}$$

then the Boolean ring B is complete with respect to d_C.

Alternatively, if, in the latter situation, we define B_{fin} to be the nonunital *Boolean ring of measurable sets of* finite *measure modulo sets of measure zero, then B_{fin} is again complete with respect to the metric d of* (4.17).

In all these cases, the operations of our Boolean ring are continuous in the metric named.

Proof. In each of these cases, the lattice operations on our structure are easily shown to satisfy (4.1)-(4.4) with respect to the indicated metric. (In the case of the metric d_C of (4.18), note that in any Cauchy sequence, all but finitely many terms must have the property that their distances from all later terms are $< C$, so that the definition (4.18), applied to those distances, reduces to (4.17).) Hence Theorem 4.1 gives completeness. □

The standard result mentioned earlier, that L^1 of the unit interval (indeed, of any measure space) is complete in its natural metric, follows similarly, on regarding L^1 as a lattice under pointwise max and min.

We remark that in any partially ordered set with a metric, the conjunction of conditions (4.3) and (4.4) above is easily shown equivalent to the single condition that every Cauchy sequence whose members form a chain under the partial ordering converges. But the pair of conditions as stated seems easier to work with. In particular, it is easy to see that it holds for measurable sets modulo sets of measure zero in a measure space.

Concerning conditions (4.1) and (4.2), note that these are equivalent to Lipschitz continuity of $\vee$, respectively $\wedge$, with Lipschitz constant 1. One could generalize the proof of Theorem 4.1 to allow any Lipschitz constant; in fact, I suspect that a version of the theorem could be proved – at the cost of more complicated arguments – with these conditions weakened to say that $\vee$, respectively, $\wedge$, is uniformly continuous; equivalently, that there exists a function u from the positive reals to the positive reals satisfying

$$\lim_{t\to 0} u(t) \ = \ 0, \tag{4.19}$$

such that

$$d(x\vee y,\, x\vee z) \ \leq \ u(d(y,z)) \quad (x,y,z\in L), \tag{4.20}$$

respectively,

$$d(x\wedge y,\, x\wedge z) \ \leq \ u(d(y,z)) \quad (x,y,z\in L). \tag{4.21}$$

The idea would be to choose from a general Cauchy sequence a subsequence for which the distance between i-th and $i{+}1$-st terms decreases rapidly enough not only to make these distances have a convergent sum, but to have the corresponding property after taking into account the effect of the u in (4.20) or (4.21) under the iterated application of that inequality in the proof. But I don't know whether there are situations where metrics arise that would make it worth trying to prove such a result.

5 Counterexample: a natural lattice under a strange metric

Another pair of conditions weaker than (4.1) and (4.2), which I at one point thought might be able to replace those two hypotheses in Theorem 4.1, are

$$d(x,\, x\vee y) \ \leq \ d(x,y) \quad (x,y\in L), \tag{5.1}$$

and

$$d(x,\, x\wedge y) \ \leq \ d(x,y) \quad (x,y\in L). \tag{5.2}$$

One can in fact prove from (5.1) that

$$d(x_0,\, x_0\vee\dots\vee x_i) \ \leq \ \textstyle\sum_{0\leq \ell<i} d(x_\ell,\, x_{\ell+1}) \quad (x_0,\dots,x_i\in L). \tag{5.3}$$

Indeed, (5.1) gives (if $i > 0$) $d(x_0,\, x_0 \vee x_1 \vee \cdots \vee x_i) \leq d(x_0,\, x_1 \vee \cdots \vee x_i)$; by the triangle inequality, the right-hand side is $\leq d(x_0,\, x_1) + d(x_1,\, x_1 \vee \cdots \vee x_i)$. The second of these terms is (if $i > 1$) similarly bounded by $d(x_1,\, x_2) + d(x_2,\, x_2 \vee \cdots \vee x_i)$, and this procedure, iterated, gives (5.3). But one cannot similarly get

$$d(x_0 \vee \cdots \vee x_i,\ x_0 \vee \cdots \vee x_j) \ \leq\ \textstyle\sum_{i \leq \ell < j} d(x_\ell,\, x_{\ell+1}) \qquad (0 \leq i \leq j,\ x_0, \ldots, x_j \in L) \tag{5.4}$$

as would be needed to carry out the argument used in the proof of Theorem 4.1.

I give below examples showing that that theorem in fact does not hold with (5.1) and (5.2) in place of (4.1) and (4.2). We will first get an example for upper semilattices and (5.1), then note how to modify it to make the semilattice into a lattice. Applying duality, one gets examples for the remaining two cases of the theorem.

To start the construction, let M be any metric space, with metric d_M, and for any finite nonempty subset S of M, define its diameter,

$$\mathrm{diam}(S) \ =\ \max_{x,y \in S}\, (d_M(x, y)). \tag{5.5}$$

Now let L be the upper semilattice of all finite nonempty subsets of M, under the operation of union; and for $S, T \in L$ define

$$d_L(S,\, T) \ =\ \begin{cases} 0 & \text{if } S = T, \\ \mathrm{diam}(S \cup T) & \text{if } S \neq T. \end{cases} \tag{5.6}$$

It is straightforward that d_L is a metric on L; the only step requiring thought is the triangle inequality $d_L(S, U) \leq d_L(S, T) + d_L(T, U)$ in the case where the three sets S, T and U are distinct and the maximum defining the left-hand side of the desired inequality is given by the distance between some $x \in S$ and some $y \in U$. In that case, taking *any* $z \in T$, one sees that $d_L(S, U) = d_M(x, y) \leq d_M(x, z) + d_M(z, y) \leq d_L(S, T) + d_L(T, U)$, as required.

Under this metric, (5.1) is also immediate: writing that relation as $d(S,\, S \cup T) \leq d(S, T)$, we see that unless $T \subseteq S$, the two sides both equal $\mathrm{diam}(S \cup T)$, while if $T \subseteq S$, the left-hand side is zero.

I claim next that L has no infinite strictly increasing Cauchy sequences. Indeed, given $S_0 \subsetneqq S_1 \subsetneqq \cdots \in L$, the set S_1 must have more than one element, hence have nonzero diameter; and from (5.6) we see that for every $i \geq 1$, $d_L(S_i, S_{i+1}) \geq \mathrm{diam}(S_1)$, so the distances between successive terms of the sequence do not approach 0. L also has no infinite strictly decreasing Cauchy sequences, since any strictly decreasing sequence of sets starting with a finite set is finite. So, trivially, (4.3) and (4.4) hold.

Note also that every non-singleton $S \in L$ has distance at least $\mathrm{diam}(S)$ from every other element of L, so it is an isolated point. It follows that the set of non-singleton elements of L is open in L, so the set of singleton elements is a closed set, which is easily seen to be isometric to $M : d_L(\{x\}, \{y\}) = d_M(x, y)$.

Hence if we take for M a non-complete metric space, then a non-convergent Cauchy sequence in M yields a non-convergent Cauchy sequence in L. So the semilattice L is non-complete, despite satisfying (5.1), (4.3) and (4.4).

To get an example which is a lattice, we pass from L as above to $L' = L \cup \{\emptyset\}$, which is clearly a lattice under union and intersection. The only problem is how to extend the metric d_L to L'. We may in fact use *any* extension of d_L to a metric on that overset which does not sabotage the non-completeness of L; for (5.1) holds automatically when x and y are comparable, and $\emptyset$ is comparable to every element of L'. So, for instance, we might fix some $P \in L$, and define

$$d_{L'}(\emptyset, S) \;=\; 1 + d_L(P, S) \quad \text{for all } S \neq \emptyset. \tag{5.7}$$

Since this makes $\emptyset$ an isolated point, it leaves the image of M in L' closed, so L' remains non-complete.

We remark that the join operation of L, and hence of L', is in general discontinuous; for given any non-eventually-constant sequence $x_0, x_1, \ldots$ in M that approaches a limit $y \in M$, we know that $\{x_0\}, \{x_1\}, \ldots$ approach $\{y\}$ in L; but for any $z \neq y$, if we apply $- \vee \{z\}$ we get the sequence $\{x_0, z\}, \{x_1, z\}, \ldots$, which cannot approach the isolated point $\{y, z\}$.

So is it plausible that continuity of the meet and join operations, combined with (5.1), (4.3) and (4.4), would imply completeness of a metric lattice? Still no: if we apply the above construction of L with M taken to be a discrete non-complete metric space (for example, $\{n^{-1} \mid n \geq 1\} \subseteq [0, 1]$), then L is also discrete. (The only elements we don't already know are isolated are the singletons $\{x\}$ for $x \in M$; but taking ε such that the ball of radius ε about x in M contains no other points, we find that the ball of radius ε about $\{x\}$ in L also contains no other points.) Hence L', metrized as in (5.7), is also discrete; and any operation on a discrete space is continuous, though L' is, we have shown, non-complete.

6 Open questions

I have not examined the question of whether the *conjunction* of (5.1) and (5.2) might somehow force a metrized lattice satisfying (4.3) and (4.4) to be complete as a metric space. (In the discrete L' constructed above satisfying (5.1), the inequality (5.2), that is, $d_{L'}(S,\, S{\cap}T) \leq d_{L'}(S,\, T)$, holds when $S{\cap}T \neq \emptyset$, but not, in general, when $S \cap T = \emptyset$.)

The referee has asked whether Gleason's original question has the same answer if restricted to the case where R an integral domain. I do not know whether this is so, with or without the assumption that R is complete in the given topology. It seems an interesting question.

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References

[1] Cohn, P. M., "Basic Algebra. Groups, Rings and Fields", Springer, 2003.

[2] Fremlin, D. H., "Measure Theory. Vol. 3. Measure Algebras", corrected second printing of the 2002 original. Torres Fremlin, 2004.

[3] Halmos, P. R., "Measure Theory", D. Van Nostrand Company, 1950.

[4] Lang, S., "Real and Functional Analysis. Third edition", Graduate Texts in Mathematics 142, Springer, 1993.

[5] Mennucci, A., *The metric space of (measurable) sets, and Carathéodory's theorem*, (2013), 3 Pages, readable at `http://dida.sns.it/dida2/cl/13-14/folde2/pdf1` .

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