On \(GPW\)-flat acts

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Abstract. In this article, we present \(GPW\)-flatness property of acts over monoids, which is a generalization of principal weak flatness. We say that a right \(S\)-act \(A_S\) is \(GPW\)-flat if for every \(s \in S\), there exists a natural number \(n = n(s,A_S) \in \mathbb{N}\) such that the functor \(A_S \otimes S -\) preserves the embedding of the principal left ideal \(S(S^n)\) into \(S\). We show that a right \(S\)-act \(A_S\) is \(GPW\)-flat if and only if for every \(s \in S\) there exists a natural number \(n = n(s,A_S) \in \mathbb{N}\) such that the corresponding \(\phi\) is surjective for the pullback diagram \(P(S^n,S^n,t,t,S)\), where \(t : S(S^n) \rightarrow SS\) is a monomorphism of left \(S\)-acts. Also we give some general properties and a characterization of monoids for which this condition of their acts implies some other properties and vice versa.

1 Introduction


In this article, in Section 2, we introduce a generalization of principal weak flatness, called \(GPW\)-flatness and will give some general properties.

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In Section 3, we give conditions for a (Rees factor) cyclic act to be \( GPW \)-flat. In Section 4, we give a characterization of monoids over which all right \( S \)-acts are \( GPW \)-flat and also a characterization of monoids \( S \) for which this condition of their right \( S \)-acts implies some other properties and vice versa.

In this paper \( S \) will stand for a monoid and \( N \) the set of natural numbers. A nonempty set \( A \) is called a \textit{right} \( S \)-\textit{act}, denoted \( A_S \), if there exists a mapping \( A \times S \rightarrow A, (a, s) \mapsto as \), such that \((as)t = a(st)\) and \(a1 = a\), for all \(a \in A\) and all \(s, t \in S\). An act \( A_S \) is called \textit{weakly flat} if the functor \( A_S \otimes S- \) preserves all embeddings of left ideals into \( S \). An act \( A_S \) is called \textit{principally weakly flat} if the functor \( A_S \otimes S- \) preserves all embeddings of principal left ideals into \( S \). A right \( S \)-act \( A_S \) is called \textit{torsion free} if \( ac = a'c \) for any \( a, a' \in A_S \) and right cancellable element \( c \in S \) implies \( a = a' \). A right \( S \)-act \( A_S \) satisfies \textit{Condition (P)} if for every \(a, a' \in A_S, s, t \in S, as = a't \) implies that \(a = a''u, a' = a''v \) and \(us = vt \) for some \(a'' \in A_S, u, v \in S\). A right \( S \)-act \( A_S \) satisfies \textit{Condition (E)} if for every \(a \in A_S, s, t \in S, as = at \) implies that \(a = a'u \) and \(us = ut \) for some \(a' \in A_S, u \in S\). A right \( S \)-act \( A_S \) satisfies \textit{Condition (P')} if for every \(a, a' \in A_S, s, t, z \in S, as = a't \) and \(sz = tz \) imply that \(a = a''u, a' = a''v \) and \(us = vt \) for some \(a'' \in A_S, u, v \in S\).

Let \( K \) be a proper right ideal of \( S \). If \(x, y,\) and \(z\) denote elements not belonging to \( S \), define \( A(K) = \left( \{x, y\} \times (S \setminus K) \right) \cup \{z\} \times K \), and define a right \( S \)-action on \( A(K) \) by

\[
(x, v)s = \begin{cases} 
(x, vs) & vs \notin K \\
(z, vs) & vs \in K 
\end{cases}
\]

\[
(y, v)s = \begin{cases} 
(y, vs) & vs \notin K \\
(z, vs) & vs \in K 
\end{cases}
\]

\[
(z, v)s = (z, vs).
\]

Then clearly \( A(K) \) is a right \( S \)-act.
2 General properties

In this section, we introduce GPW-flatness property of acts and will give some general properties.

Definition 2.1. A right $S$-act $A_S$ is called GPW-flat if for every $s \in S$, there exists $n = n(s,A_S) \in \mathbb{N}$, such that the functor $A_S \otimes_S -$ preserves the embedding of the principal left ideal $s(Ss^n)$ into $SS$.

Clearly every principally weakly flat right $S$-act is GPW-flat, but, by the following example, we see that the converse is not true.

Example 2.2. Suppose $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. Clearly the right Rees factor $S$-act $S/K$ is GPW-flat, but it is not principally weakly flat.

Proposition 2.3. For any right $S$-act $A_S$, the following statements are equivalent:

1. $A_S$ is GPW-flat.
2. For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$.
3. For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$.
4. For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies that

\[
a = a_1s_1 \\
a_1t_1 = a_2s_2 & s_1s^n = t_1s^n \\
a_2t_2 = a_3s_3 & s_2s^n = t_2s^n \\
& \vdots \\
a_kt_k = a' & s_ks^n = t_ks^n,
\]

for some $k \in \mathbb{N}$ and elements $a_1, \ldots, a_k \in A_S$, $s_1, t_1, \ldots, s_k, t_k \in S$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious by Definition 2.1.

(2) $\Leftrightarrow$ (4) This is obvious by [9, II, Lemma 5.5].
Corollary 2.4. Suppose that $S$ is an idempotent or right cancellative monoid. Then for a right $S$-act $A_S$ the following statements are equivalent:

(i) $A_s$ is principally weakly flat.

(ii) $A_s$ is $GPW$-flat.

Proof. This is obvious by Proposition 2.3.

Proposition 2.5. Every $GPW$-flat right $S$-act is torsion free.

Proof. This is obvious by Proposition 2.3.

Example 2.6. Let $S = \mathbb{N} \cup G$, where $\mathbb{N}$ is the set of natural numbers and $G$ is a nontrivial group with unit element $e$ and define the multiplication on $S$ as $gn = ng = n$ for every $g \in G$ and $n \in \mathbb{N}$. Clearly all right $S$-acts are torsion free by [9, IV, Theorem 6.1], but not all right $S$-acts are $GPW$-flat, see Theorem 4.5.

The following strict implications exist for different flatness properties of acts:

Weakly flat $\Rightarrow$ Principally weakly flat $\Rightarrow$ $GPW$-flat $\Rightarrow$ Torsion free.

In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of $\varphi$ corresponding to some pullback diagram $P(M,N,f,g,Q)$ where $f : sM \rightarrow sQ$ and $g : sN \rightarrow sQ$ are homomorphisms of left $S$-acts.

Similar to [10, Proposition 2], we have the following proposition.

Proposition 2.7. A right $S$-act $A_S$ is $GPW$-flat if and only if for every $s \in S$ there exists $n = n(s,A_S) \in \mathbb{N}$ such that the corresponding $\varphi$ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : S(Ss^n) \rightarrow SS$ is a monomorphism of left $S$-acts.

Proposition 2.8. The following statements hold:

1. Any retract of a $GPW$-flat right $S$-act is $GPW$-flat.
2. If $A = \coprod_{i \in I} A_i$ is $GPW$-flat, then $A_i$ is $GPW$-flat for every $i \in I$.
3. $S_S$ is $GPW$-flat.
4. $\Theta_S$ is $GPW$-flat.
Proof. (3) and (4) are obvious.

(1) Suppose that $B_S$ is a $GPW$-flat right $S$-act and $A_S$ is a retract of $B_S$. Then there exist homomorphisms $f : B_S \to A_S$ and $f' : A_S \to B_S$, such that $ff' = id_{A_S}$. Let $s \in S$. Since $B_S$ is $GPW$-flat, there exists $n \in \mathbb{N}$ such that the equality $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes_S S^n$ implies that $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes_S (Ss^n)$, for any $b, b' \in B_S$ by (2) of Proposition 2.3. Let $as^n = a's^n$ for $a, a' \in A_S$. Then $f'(as^n) = f'(a's^n)$ and so $f'(a)s^n = f'(a')s^n$. Since $f'(a), f'(a') \in B_S$, $B_S$ is $GPW$-flat and $f'(a) \otimes s^n = f'(a') \otimes s^n$ in $B_S \otimes_S S^n$, we have

$$f'(a) = b_1s_1$$
$$b_1t_1 = b_2s_2 \quad s_1s^n = t_1s^n$$
$$b_2t_2 = b_3s_3 \quad s_2s^n = t_2s^n$$
$$\vdots$$
$$b_kt_k = f'(a') \quad s_k s^n = t_k s^n,$$

where $b_1, \ldots, b_k \in B_S$, $s_1, t_1, \ldots, s_k, t_k \in S$, by (4) of Proposition 2.3. Thus $f(f'(a)) = f(b_1s_1)$ and so $a = f(b_1)s_1$. Similarly, $f(b_{i-1})s_{i-1} = f(b_i)s_i$, $2 \leq i \leq k$, and $a' = f(b_k)t_k$. Hence

$$a \otimes s^n = f(b_1)s_1 \otimes s^n = f(b_1) \otimes s_1s^n = f(b_1) \otimes t_1s^n = f(b_1)t_1 \otimes s^n$$
$$f(b_2)s_2 \otimes s^n = f(b_2) \otimes s_2s^n = \ldots = f(b_k)t_k \otimes s^n = a' \otimes s^n$$

in $A_S \otimes_S (Ss^n)$.

(2) Suppose that $A = \coprod_{i \in I} A_i$ is $GPW$-flat right $S$-act and let $s \in S$. Let $j \in I$. By assumption, there exists $n \in \mathbb{N}$ such that $as^n = a's^n$ for $a, a' \in A_S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$. Let $as^n = a's^n$ for $a, a' \in A_j$. Thus $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$, by assumption. Hence $a \otimes s^n = a' \otimes s^n$ in $A_j \otimes_S (Ss^n)$, by [2, Corollary 2.3].

A monoid $S$ is called left almost regular if for every $s \in S$

$$s_1c_1 = sr_1$$
$$s_2c_2 = s_1r_2$$
$$\vdots$$
$$s_mc_m = s_{m-1}r_m$$
$$s = s_mr_s,$$
for some $r, r_1, \ldots, r_m, s_1, s_2, \ldots, s_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Clearly every regular and right cancellative monoid is left almost regular.

As, by [9, IV, Theorem 6.5], over a left almost regular monoid every torsion free right $S$-act is principally weakly flat, the following proposition is easily checked.

**Proposition 2.9.** Let $S$ be a left almost regular monoid, and $A_S$ be a right $S$-act. Then the following statements are equivalent:

1. $A_S$ is principally weakly flat.
2. $A_S$ is GPW-flat.
3. $A_S$ is torsion free.

**Theorem 2.10.** For a proper right ideal $K$ of monoid $S$ the following statements are equivalent:

1. $(\forall s \in S, \exists n \in \mathbb{N}) (\forall l \in S \setminus K) (ls^n \in K \Rightarrow (\exists k \in K, ls^n = ks^n))$.
2. $A(K)$ is GPW-flat.

**Proof.** (1) $\Rightarrow$ (2) Let $s \in S$. Then, by assumption, there exists $n \in \mathbb{N}$ such that (1) is established. Let $as^n = a's^n$ for $a, a' \in A(K)$. Since $(x, 1)S \cong S_S \cong (y, 1)S$ (every free right $S$-act is GPW-flat), without loss of generality, we can take $a = (x, r_1), a' = (y, r_2)$, where $r_1, r_2 \in S \setminus K$. Since $(x, r_1)s^n = (y, r_2)s^n$, we have $r_1s^n = r_2s^n \in K$, and so there exists $k \in K$ such that $r_1s^n = ks^n = r_2s^n$. Hence

$$(x, r_1) \otimes s^n = (x, 1) \otimes r_1s^n = (x, 1) \otimes ks^n = (y, 1) \otimes ks^n = (y, r_2) \otimes s^n$$

in $A(K) \otimes S(Ss^n)$.

(2) $\Rightarrow$ (1) Let $A(K)$ be GPW-flat and suppose $s \in S$. Then there exists $n \in \mathbb{N}$ such that $A(K) \otimes S$ preserves the embedding $\iota : S(Ss^n) \rightarrow SS$. Now let $l \in S \setminus K$ such that $ls^n \in K$. Then clearly $(x, l)s^n = (y, l)s^n$. By Proposition 2.3, we have

$$(x, l) = (w_1, u_1)s_1$$
$$(w_1, u_1)t_1 = (w_2, u_2)s_2$$
\vdots
$$(w_{m-1}, u_{m-1})t_{m-1} = (w_m, u_m)s_m$$
$$(w_m, u_m)t_m = (y, l)$$
$$s_1s^n = t_1s^n$$
\vdots
$$s_{m-1}s^n = t_{m-1}s^n$$
$$s_ms^n = t_ms^n.$$
for some $m \in \mathbb{N}$, $u_1, \ldots, u_m \in S$, $s_1, t_1, \ldots, s_m, t_m \in S$, and $w_1, \ldots, w_m \in \{x, y, z\}$. By definition of $A(K)$, there exists $i \in \{1, \ldots, m-1\}$ such that $w_i \neq w_{i+1}$, and so there exists $k \in K$ such that $u_i t_i = u_{i+1} s_{i+1} = k$. Hence we have

$$l s^n = u_1 s_1 s^n = u_1 t_1 s^n = u_2 s_2 s^n = \ldots = u_i t_i s^n = k s^n,$$

as required.

Recall from [9, III, Definition 10.14] that an element $s$ of a monoid $S$ is called right $e$-cancellable for an idempotent $e \in S$ if $s = es$ and $\ker \rho_s \leq \ker \rho_e$. A monoid $S$ is called left PP if every element $s \in S$ is right $e$-cancellable for some idempotent $e \in S$. It is obvious that every regular and every right cancellative monoid is left PP. An element $s \in S$ is right semi-cancellable if the equality $xs = ys$ for any $x, y \in S$ implies that there exists $r \in S$ such that $rs = s$ and $xr = yr$. A monoid $S$ is called left PSF if every element $s \in S$ is right semi-cancellable. Clearly every left PP monoid is left PSF.

**Proposition 2.11.** Suppose that $S$ is a left PP monoid. An act $A_S$ is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $es^n = s^n$ and $ae = a'e$ for some $e^2 = e \in S$.

**Proof.** This is obvious by [9, III, Theorem 10.16].

For a left PSF monoid, similar to argument used in [11, Proposition 2.5], we can show the following proposition.

**Proposition 2.12.** Suppose that $S$ is a left PSF monoid. An act $A_S$ is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $rs^n = s^n$ and $ar = a'r$ for some $r \in S$.

**Corollary 2.13.** For a left PSF monoid $S$, the following statements are equivalent:

1. $\prod_{i=1}^k A_i$ is GPW-flat.
2. For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $\alpha_i, \alpha'_i \in A_i, 1 \leq i \leq k$, if $(\alpha_1, \alpha_2, \ldots, \alpha_k)s^n = (\alpha'_1, \alpha'_2, \ldots, \alpha'_k)s^n$, then $us^n = s^n$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)u = (\alpha'_1, \alpha'_2, \ldots, \alpha'_k)u$ for some $u \in S$.

**Proof.** This is obvious by Proposition 2.12.
Proposition 2.14. For any family \( \{ A_i \}_{i \in I} \) of right \( S \)-acts, if \( \prod_{i \in I} A_i \) is \( GPW \)-flat, then \( A_i \) is \( GPW \)-flat, for every \( i \in I \).

Proof. Let \( \prod_{i \in I} A_i \) be \( GPW \)-flat and let \( s \in S \) and \( i \in I \). By assumption there exists \( n \in \mathbb{N} \) such that the functor \( \prod_{i \in I} A_i \otimes S \) preserves the embedding \( \iota : S(Ss^n) \to S \). Let \( a_is^n = a'_is^n \) for any \( a_i, a'_i \in A_i \) and suppose \( a_j \in (A_j)_S \) for \( j \neq i \). If

\[
\begin{align*}
  a_k &= \begin{cases} 
    a_i & \text{if } k = i \\
    a_j & \text{if } k \neq i
  \end{cases} \\
  a'_k &= \begin{cases} 
    a'_i & \text{if } k = i \\
    a_j & \text{if } k \neq i
  \end{cases}
\end{align*}
\]

Then \( (a_k)Is^n = (a'_k)Is^n \) and so \( (a_k)I \otimes s^n = (a'_k)I \otimes s^n \) in \( \prod_{i \in I} A_i \otimes_S (Ss^n) \), by Proposition 2.3. Now we have \( a_i \otimes s^n = a'_i \otimes s^n \) in \( A_i \otimes_S (Ss^n) \), by [14, Remark 3.1], and so \( A_i \) is \( GPW \)-flat.

Golchin in [3] showed that if \( S = G \cup I \) where \( G \) is a group and \( I \) is an ideal of \( S \) and \( A \) is a right \( S \)-act that is ((principally) weakly) flat, torsion free, satisfies Condition \((P)\) or \((PE)\) as a right \( I^1 \)-act, then it has these properties as a right \( S \)-act. Similarly, we can show the following theorem for \( GPW \)-flatness.

Theorem 2.15. Let \( S = G \cup I \) and let \( A \) be a right \( S \)-act. If \( A \) is \( GPW \)-flat as a right \( I^1 \)-act, then it is \( GPW \)-flat as a right \( S \)-act.

Proof. This is obvious by Proposition 2.3.

\[ \square \]

3 \( GPW \)-flatness of (Rees factor) cyclic acts

In this section, we give conditions for a (Rees factor) cyclic act to be \( GPW \)-flat.

Proposition 3.1. Suppose that \( \rho \) is a right congruence on a monoid \( S \). Then the following statements are equivalent:

(i) \( S/\rho \) is \( GPW \)-flat.

(ii) \( (\forall s \in S)(\exists n \in \mathbb{N})(\forall u, v \in S)((us^n)\rho(vs^n) \Rightarrow u(\rho \lor \ker \rho_{s^n})v) \).

\[ \square \]
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Proof. (i) ⇒ (ii) Let \( s \in S \). Since the right \( S \)-act \( S/\rho \) is GPW-flat, there exists \( n \in \mathbb{N} \) such that the functor \( (S/\rho)_S \otimes S - \) preserves the embedding \( \iota : S(Ss^n) \to SS \). Now suppose that \((us^n)(vs^n)\) for \( u, v \in S \). Thus \([v]_\rho \otimes s^n = [v]_\rho \otimes s^n\) in \((S/\rho)_S \otimes SS\) and so \([v]_\rho \otimes s^n = [v]_\rho \otimes s^n\) in \((S/\rho)_S \otimes S(Ss^n)\). Hence \( u(\rho \circ \ker \rho_{s^n})v \), by [9, III, Lemma 10.6].

(ii) ⇒ (i) Let \( s \in S \). By assumption, there exists \( n \in \mathbb{N} \) such that \((us^n)(vs^n)\), for every \( u, v \in S \), implies that \( u(\rho \circ \ker \rho_{s^n})v \). Suppose \([v]_\rho \otimes s^n = [v]_\rho \otimes s^n\) in \((S/\rho)_S \otimes SS\), thus \((us^n)(vs^n)\). Now, by assumption, \( u(\rho \circ \ker \rho_{s^n})v \) and so, by [9, III, Lemma 10.6], \([v]_\rho \otimes s^n = [v]_\rho \otimes s^n\) in \((S/\rho)_S \otimes S(Ss^n)\). Hence \( S/\rho \) is GPW-flat, by Proposition 2.3.

Corollary 3.2. The principal right ideal \( zS \) is GPW-flat if and only if for every \( s \in S \), there exists \( n \in \mathbb{N} \) such that for any \( x, y \in S \), \( zxs^n = zys^n \) implies that \( x(\ker \lambda_z \circ \ker \rho_{s^n})y \).

Proof. Since \( zS \cong S/\ker \lambda_z \), by Proposition 3.1, it suffices to take \( \rho = \ker \lambda_z \).

Theorem 3.3. Suppose that \( K \) is a right ideal of \( S \). Then \( S/K \) is GPW-flat if and only if for every \( s \in S \) there exists a natural number \( n \in \mathbb{N} \) such that \( ls^n \in K \), for \( l \in S \setminus K \) implies that \( ls^n = ks^n \), for some \( k \in K \).

Proof. If \( K = S \), then \( S/K \cong \Theta_S \) is GPW-flat by (3) of Proposition 2.8. Thus suppose that \( K \) is a proper right ideal of \( S \).

Necessity. Suppose that \( S/K \) is GPW-flat for the proper right ideal \( K \) of \( S \) and let \( s \in S \). Then there exists \( n \in \mathbb{N} \) such that the functor \( A_S \otimes S - \) preserves the embedding \( \iota : S(Ss^n) \to SS \). Now suppose \( ls^n \in K \) for \( l \in S \setminus K \). Then \([l] \otimes s^n = [j] \otimes s^n\) in \( S/K \otimes SS \), for any \( j \in K \) and so, by Proposition 2.3, there exist \( m \in \mathbb{N}, p_1, \ldots, p_m, s_1, t_1, \ldots, s_m, t_m \in S \) such that

\[
[l] = [p_1]s_1
\]
\[
[p_1]t_1 = [p_2]s_2 \quad s_1s^n = t_1s^n
\]
\[
[p_2]t_2 = [p_3]s_3 \quad s_2s^n = t_2s^n
\]
\[\vdots \]
\[
[p_m]t_m = [j] \quad s_ms^n = t_ms^n.
\]
Since \( j \in K \), we have \( p_{m}t_{m} \in K \). Let \( q \) be the least number such that \( q \in \{1, \ldots, m\} \) and \( q_{t} \in K \). Let \( k = q_{t} \), then \( p_{q_{t}} = p_{q_{t}t_{q}} = p_{q}s_{q} \), and so

\[
ls^{n} = p_{1}s_{1}s^{n} = p_{1}t_{1}s^{n} = p_{2}s_{2}s^{n} = \ldots = \]
\[
p_{q_{t}t_{q}t_{q} - 1}s^{n} = p_{q}s_{q}s^{n} = p_{q_{t}q}s^{n} = ks^{n}.
\]

**Sufficiency.** Let \( K \) be a right ideal of \( S \) and let \( s \in S \). Thus there exists \( n \in \mathbb{N} \) such that \( ls^{n} \in K \), for \( l \in S \setminus K \) implies that \( ls^{n} = ks^{n} \) for some \( k \in K \), by assumption. Let for any \( u, v \in S \), \([u] \otimes s^{n} = [v] \otimes s^{n} \) in \( S/K \otimes_{S} S \). Thus there are four cases as follows:

**Case 1.** \( u, v \in K \). Then it is clear that \([u] = [v]\) in \( S/K \) and so \([u] \otimes s^{n} = [v] \otimes s^{n} \) in \( S/K \otimes_{S}(Ss^{n}) \).

**Case 2.** \( u \in K, v \in S \setminus K \). Then there exists \( k \in K \) such that \( vs^{n} = ks^{n} \), by assumption. Then

\[
[u] \otimes s^{n} = [k] \otimes s^{n} = [1] \otimes ks^{n} = [1] \otimes vs^{n} = [v] \otimes s^{n}
\]
in \( S/K \otimes_{S}(Ss^{n}) \).

**Case 3.** \( u \in S \setminus K, v \in K \). It is similar to the Case 2.

**Case 4.** \( u, v \in S \setminus K \). Then from \([u] \otimes s^{n} = [v] \otimes s^{n} \) in \( S/K \otimes_{S} S \), we have either \( us^{n} = vs^{n} \) or \( us^{n}, vs^{n} \in K \). If \( us^{n} = vs^{n} \), the result follows. Otherwise, \( us^{n} = ks^{n} \) and \( vs^{n} = ls^{n} \) for some \( k, l \in K \), by assumption. So

\[
[u] \otimes s^{n} = [1] \otimes us^{n} = [1] \otimes ks^{n} = [k] \otimes s^{n} =
\]
\[
[l] \otimes s^{n} = [1] \otimes ls^{n} = [1] \otimes vs^{n} = [v] \otimes s^{n}
\]
in \( S/K \otimes_{S}(Ss^{n}) \).

\[ \square \]

### 4 Characterization of monoids by GPW-flatness of acts

Now we classify monoids over which all right \( S \)-acts are GPW-flat and also monoids over which some other properties imply GPW-flatness and vice versa.

A monoid \( S \) is called **regular** if for every \( s \in S \) there exists \( x \in S \) such that \( s = sx \).

**Definition 4.1.** An element \( s \in S \) is called **eventually regular** if \( s^{n} \) is regular for some \( n \in \mathbb{N} \). That is, \( s^{n} = s^{n}xs^{n} \) for some \( n \in \mathbb{N} \) and \( x \in S \). A monoid \( S \) is called **eventually regular** if every \( s \in S \) is eventually regular.
Obviously every regular monoid is eventually regular.

**Definition 4.2.** An element $s \in S$ is called *eventually left almost regular* if

$$
\begin{align*}
    s_1 c_1 &= s^n r_1 \\
    s_2 c_2 &= s_1 r_2 \\
    &\vdots \\
    s_m c_m &= s_{m-1} r_m \\
    s^n &= s_m r s^n,
\end{align*}
$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. In other words $s \in S$ is called *eventually left almost regular* if $s^n$ is left almost regular for some $n \in \mathbb{N}$.

If every element of a monoid $S$ is eventually left almost regular, then $S$ is called *eventually left almost regular*.

It is clear that every left almost regular monoid is eventually left almost regular, and also every eventually regular monoid is eventually left almost regular.

**Example 4.3.** Let $S = \{1, 0, e, f, a\}$ be the monoid with the following table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>e</th>
<th>f</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>e</td>
<td>f</td>
<td>a</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>0</td>
<td>e</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>f</td>
<td>f</td>
<td>0</td>
<td>0</td>
<td>f</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Clearly $S$ is eventually regular and so it is eventually left almost regular. But $S$ is not regular, because $a \in S$ is not regular. Also $S$ is not left almost regular, since $a \in S$ is not left almost regular.

**Theorem 4.4.** The following statements are equivalent:

1. $S$ is an eventually left almost regular monoid.
2. All torsion free right Rees factor acts over $S$ are GPW-flat.
3. All torsion free cyclic right $S$-acts are GPW-flat.
4. All torsion free finitely generated right $S$-acts are GPW-flat.
5. All torsion free right $S$-acts are GPW-flat.
Proof. (5) ⇒ (4) ⇒ (3) ⇒ (2) are clear.

(2) ⇒ (1) Suppose that all torsion free right Rees factor $S$-acts are GPW-flat and let $s \in S$. Let $K(s)$ be the subset of $S$ consisting of all elements $t \in S$ such that

$$s_1 c_1 = s^n r_1$$
$$s_2 c_2 = s_1 r_2$$
$$\vdots$$
$$s_{m-1} c_{m-1} = s_{m-2} r_{m-1}$$
$$t c_m = s_{m-1} r_m,$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. We see that $s^n \in K(s)$ for some $n \in \mathbb{N}$, and so $K(s)$ is non-empty, because, if $m = 1$ and $c_1 = r_1 = 1$, then $t = s^n$ has the required property mentioned. Now let $J = \bigcup_{t \in K(s)} tS$. Let $s'c \in J$, for $s' \in S$ and $c$ right cancellable. Then $s'c \in tS$ for some $t \in K(s)$, and so we have

$$s_1 c_1 = s^n r_1$$
$$s_2 c_2 = s_1 r_2$$
$$\vdots$$
$$s_{m-1} c_{m-1} = s_{m-2} r_{m-1}$$
$$t c_m = s_{m-1} r_m$$
$$s'c = tr_{m+1},$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m, r_{m+1} \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Thus $s' \in J$, and so $S/J$ is torsion free by [9, III, Proposition 8.10]. Hence $S/J$ is GPW-flat by assumption and so by Theorem 3.3, for $s^n \in J$, there exists $t r \in J$ such that $s^n = t r s^n$, where $t \in K(s)$, and $r \in S$. Now $s^n = t r s^n$ and $t \in K(s)$ implies that $s$ is eventually left almost regular.

(1) ⇒ (5) Let $S$ be an eventually left almost regular monoid and suppose $A_S$ is a torsion free right $S$-act and let $s \in S$. Since $s$ is eventually left almost
regular, we have
\[ s_1c_1 = s^n r_1 \]
\[ s_2c_2 = s_1 r_2 \]
\[ \vdots \]
\[ s_m c_m = s_{m-1} r_m \]
\[ s^n = s_m r s^n, \]
for some \( n \in \mathbb{N} \), the elements \( s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S \) and the right cancellable elements \( c_1, c_2, \ldots, c_m \in S \). Let \( as^n = a's^n \) for \( a, a' \in A_S \). Using torsion freeness, it can easily be seen that \( as_m r = a's_m r \). Hence we have
\[
    a \otimes s^n = a \otimes s_m r s^n = as_m r \otimes s^n = a's_m r \otimes s^n = a' \otimes s_m r s^n = a' \otimes s^n
\]
in \( A_S \otimes S(Ss^n) \) and so \( A_S \) is GPW-flat, as required. \( \square \)

A right \( S \)-act \( A_S \) is a generator if for any distinct homomorphisms \( \alpha, \beta : X_S \to Y_S \), there exists a homomorphism \( f : A_S \to X_S \) such that \( \alpha f \neq \beta f \). Equivalently, a right \( S \)-act \( A_S \) is a generator if and only if there exists an epimorphism \( \pi : A_S \to S_S \) ([9, II, Theorem 3.16]).

As we know, \( S \times A_S \) is a generator for each right \( S \)-act \( A_S \).

**Theorem 4.5.** The following statements are equivalent:
1. \( S \) is an eventually regular monoid.
2. A right \( S \)-act \( A_S \) is GPW-flat if \( \text{Hom}(A_S, S_S) \neq \emptyset \).
3. \( S \times A_S \) is GPW-flat for every generator right \( S \)-act \( A_S \).
4. \( S \times A_S \) is GPW-flat for every right \( S \)-act \( A_S \).
5. All generator right \( S \)-acts are GPW-flat.
6. All right Rees factor \( S \)-acts are GPW-flat.
7. All cyclic right \( S \)-acts are GPW-flat.
8. All right \( S \)-acts are GPW-flat.

**Proof.** (8) \( \Rightarrow \) (7) \( \Rightarrow \) (6), (8) \( \Rightarrow \) (5), (8) \( \Rightarrow \) (4) \( \Rightarrow \) (3), (2) \( \Rightarrow \) (4), (5) \( \Rightarrow \) (4) and (8) \( \Rightarrow \) (2) are obvious.

(4) \( \Rightarrow \) (8) This is valid by Proposition 2.14.

(6) \( \Rightarrow \) (1) If all right Rees factor acts over \( S \) are GPW-flat, then all right Rees factor acts over \( S \) are torsion free. So every right cancellable element
of $S$ is right invertible, by [9, IV, Theorem 6.1], but by Theorem 4.4, $S$ is
eventually left almost regular. Now let $s \in S$. Then

$$s_1c_1 = s_1^nr_1$$
$$s_2c_2 = s_1r_2$$
$$\vdots$$
$$s_mc_m = s_{m-1}r_m$$
$$s^n = s_mr^ns^n,$$

Multiplying both sides of the equalities in the above scheme by $c_i^{-1}$ for $i \in \{1, \ldots, m\}$, respectively, we get

$$s_1 = s^n r_1 c_1^{-1}$$
$$s_2 = s_1 r_2 c_1^{-1}$$
$$\vdots$$
$$s_m = s_{m-1} r_m c_m^{-1}.$$

Thus

$$s^n = s_mr^ns^n = s_{m-1} r_m c_m^{-1} r^ns^n = s_{m-2} r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r^ns^n = \ldots$$
$$= s^n r_1 c_1^{-1} \ldots r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r^ns^n,$$

and so $s$ is eventually regular, as required.

(1) $\Rightarrow$ (8) Suppose that $A_S$ is a right $S$-act and let $s \in S$. By Proposition
2.3, we have to show that there exists $m \in \mathbb{N}$ such that for any $a, a' \in A_S$, if
$a \otimes s^m = a' \otimes s^m$ in $A_S \otimes S$, then $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes s(St^n)$. Since $s$ is eventually regular, there exist $n \in \mathbb{N}$ and $t \in S$, such that $s^n = s^nts^n$. If $m = n$. Let $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes S$ for any $a, a' \in A_S$, then

$$a \otimes s^n = a \otimes s^nts^n = a s^n \otimes ts^n = a' s^n \otimes ts^n = a' \otimes s^nts^n = a' \otimes s^n$$
in $A_S \otimes s(St^n)$ and so $A_S$ is GPW-flat.

(3) $\Rightarrow$ (4) Suppose that $A_S$ is a right act over $S$. As we show in the proof
of (5) $\Rightarrow$ (4), $S \times A_S$ is a generator and so, by assumption, $S \times (S \times A_S)$ is
GPW-flat, which means that $S \times A_S$ is GPW-flat, by Proposition 2.14. □
It is obvious that Condition \((P)\) implies \(GPW\)-flatness, but the following example shows that this is not the case for Condition \((E)\).

**Example 4.6.** Let \(S = (\mathbb{N}, \cdot)\) be the monoid of natural numbers with multiplication and let \(A_\mathbb{N} = \mathbb{N} \coprod 2\mathbb{N} \mathbb{N}\). Then \(A_\mathbb{N}\) satisfies Condition \((E)\), but it is not \(GPW\)-flat.

Now, the question is that: What is the structure of monoids over which Condition \((E)\) of their acts implies \(GPW\)-flatness?

**Theorem 4.7.** For a right cancellative monoid \(S\) the following statements are equivalent:

1. \(\prod_{i \in I} A_i\) is principally weakly flat, for any family \(\{A_i\}_{i \in I}\) of right \(S\)-acts.
2. \(\prod_{i \in I} A_i\) is \(GPW\)-flat, for any family \(\{A_i\}_{i \in I}\) of right \(S\)-acts.
3. \(\prod_{i \in I} A_i\) is torsion free, for any family \(\{A_i\}_{i \in I}\) of right \(S\)-acts.
4. \(S\) is a group.

**Proof.** (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) and (4) \(\Rightarrow\) (1) are obvious.

(3) \(\Rightarrow\) (4) This is obvious, by [14, Remark 3.1] and [9, IV. Theorem 6.1].

An element \(a \in A_S\) is called divisible by \(s \in S\) if there exists \(b \in A_S\), such that \(bs = a\). An act \(A_S\) is said to be divisible if \(Ac = A\), for any left cancellable element \(c \in S\). It is clear that \(A_S\) is divisible if and only if every element of \(A_S\) is divisible by any left cancellable element of \(S\).

**Theorem 4.8.** The following statements are equivalent:

1. All right \(S\)-acts are divisible.
2. All \(GPW\)-flat right \(S\)-acts are divisible.
3. All \(GPW\)-flat finitely generated right \(S\)-acts are divisible.
4. All \(GPW\)-flat cyclic right \(S\)-acts are divisible.
5. All \(GPW\)-flat monocular right \(S\)-acts are divisible.
6. All left cancellable elements of \(S\) are left invertible.

**Proof.** (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4) \(\Rightarrow\) (5) are obvious.

(5) \(\Rightarrow\) (6) For every \(s \in S\) we have \(S/\rho(s, s) = S_S/\Delta_S \cong S_S\), by (3) of Proposition 2.8, \(S_S\) is \(GPW\)-flat, and so it is divisible by assumption. Thus \(Sc = S\), for any left cancellable element \(c \in S\). Thus, there exists \(s \in S\) such that \(sc = 1\), and so \(c\) is left invertible, as required.

(6) \(\Rightarrow\) (1) It is clear from [9, III, Proposition 2.2].
Recall, from [15], that a right $S$-act $A_S$ is called strongly torsion free if the equality $as = a's$, for $a, a' \in A_S$ and $s \in S$ implies $a = a'$. It is clear that every strongly torsion free right $S$-act is GPW-flat, but not the converse.

**Theorem 4.9.** The following statements are equivalent:

1. All GPW-flat right $S$-acts are strongly torsion free.
2. All GPW-flat finitely generated right $S$-acts are strongly torsion free.
3. All GPW-flat cyclic right $S$-acts are strongly torsion free.
4. $S$ is right cancellative monoid.

**Proof.** This follows from [15, Theorem 3.1]. \hfill \square

Recall from [6] that a right $S$-act $A_S$ is $E$-torsion free if for any $a, a' \in A_S$ and $e \in E(S)$, $ae = a' e$ implies $a = a'$.

**Theorem 4.10.** The following statements are equivalent:

1. All GPW-flat right $S$-acts are $E$-torsion free.
2. All GPW-flat finitely generated right $S$-acts are $E$-torsion free.
3. All GPW-flat cyclic right $S$-acts are $E$-torsion free.
4. $E(S) = \{1\}$.

**Proof.** This is obvious by [6, Theorem 3.1]. \hfill \square

Recall from [1, Definition 1] that a right $S$-act $A_S$ is called principally weakly kernel flat (PWKF) if the corresponding $\phi$ is bijective for the pullback diagram $P(Ss, Ss, f, f, S) (s \in S)$, and $A_S$ is translation kernel flat (TKF) if the corresponding $\phi$ is bijective for the pullback diagram $P(S, S, f, f, S)$.

**Theorem 4.11.** The following statements on a monoid $S$ are equivalent:

1. All GPW-flat right $S$-acts are PWKF and $S$ is left PSF.
2. All GPW-flat right $S$-acts are TKF and $S$ is left PSF.
3. All GPW-flat right $S$-acts satisfy Condition (PWP) and $S$ is left PSF.
4. All GPW-flat right $S$-acts satisfy Condition ($P'$) and $S$ is left PSF.
5. $S$ is right cancellative.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear.

5) $\Rightarrow$ (1) By [13, Theorem 2.12] and Corollary 2.4, it is clear.

3) $\Rightarrow$ (5) This is obvious, by [12, Theorem 2.8].

(5) $\Rightarrow$ (4) Since $S$ is right cancellative, $S$ is left PSF. By [4, Theorem 2.8] and Corollary 2.4, it is clear. \hfill \square
Theorem 4.12. The following statements on a monoid $S$ are equivalent:

(1) All GPW-flat right $S$-acts are PWKF and there exists a regular left $S$-act.
(2) All GPW-flat right $S$-acts are TKF and there exists a regular left $S$-act.
(3) All GPW-flat right $S$-acts satisfy Condition (PWP) and there exists a regular left $S$-act.
(4) All GPW-flat right $S$-acts satisfy Condition ($P'$) and there exists a regular left $S$-act.
(5) $|E(S)| = 1$ and there exists a regular left $S$-act.
(6) $S$ is right cancellative.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear.
(6) $\Rightarrow$ (4) This is clear by [4, Theorem 2.9] and Corollary 2.4.
(5) $\Leftrightarrow$ (6) This is clear by [12, Theorem 2.9].
(3) $\Rightarrow$ (6) This is clear by [12, Theorem 2.9].
(6) $\Rightarrow$ (1) This is clear by [13, Theorem 2.18] and Corollary 2.4. \hfill \Box

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