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On GPW-Flat Acts

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Abstract. In this article, we present GPW-flatness property of acts over monoids, which is a generalization of principal weak flatness. We say that a right S-act A_S is GPW-flat if for every $s \in S$, there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. We show that a right S-act A_S is GPW-flat if and only if for every $s \in S$ there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that the corresponding φ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \to {}_SS$ is a monomorphism of left S-acts. Also we give some general properties and a characterization of monoids for which this condition of their acts implies some other properties and vice versa.

1 Introduction

In 1970, Kilp [7] initiated a study of flatness of acts. In 1983, further investigation of (principal) weak version of flatness was done by Kilp [8]. In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of φ corresponding to some pullback diagram.

In this article, in Section 2, we introduce a generalization of principal weak flatness, called GPW-flatness and will give some general properties.

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In Section 3, we give conditions for a (Rees factor) cyclic act to be GPW-flat. In Section 4, we give a characterization of monoids over which all right S-acts are GPW-flat and also a characterization of monoids S for which this condition of their right S-acts implies some other properties and vice versa.

In this paper S will stand for a monoid and $\mathbb{N}$ the set of natural numbers. A nonempty set A is called a *right S-act*, denoted A_S, if there exists a mapping $A \times S \to A, (a, s) \mapsto as$, such that $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. An act A_S is called *weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of left ideals into S. An act A_S is called *principally weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of principal left ideals into S. A right S-act A_S is called *torsion free* if $ac = a'c$ for any $a, a' \in A_S$ and right cancellable element $c \in S$ implies $a = a'$. A right S-act A_S satisfies *Condition* (P) if for every $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$. A right S-act A_S satisfies *Condition* (E) if for every $a \in A_S$, $s, t \in S$, $as = at$ implies that $a = a'u$ and $us = ut$ for some $a' \in A_S, u \in S$. A right S-act A_S satisfies *Condition* (PWP) if for every $a, a' \in A_S$, $s \in S$, $as = a's$ implies that $a = a''u, a' = a''v$ and $us = vs$ for some $a'' \in A_S$, $u, v \in S$. A right S-act A_S satisfies *Condition* (P') if for every $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$.

Let K be a proper right ideal of S. If x, y, and z denote elements not belonging to S, define $A(K) = (\{x, y\} \times (S \setminus K)) \bigcup (\{z\} \times K)$, and define a right S-action on $A(K)$ by

$$(x, v)s = \begin{cases} (x, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(y, v)s = \begin{cases} (y, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(z, v)s = (z, vs).$$

Then clearly $A(K)$ is a right S-act.

2 General properties

In this section, we introduce GPW-flatness property of acts and will give some general properties.

Definition 2.1. A right S-act A_S is called *GPW-flat* if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$, such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$.

Clearly every principally weakly flat right S-act is GPW-flat, but, by the following example, we see that the converse is not true.

Example 2.2. Suppose $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. Clearly the right Rees factor S-act S/K is GPW-flat, but it is not principally weakly flat.

Proposition 2.3. *For any right S-act A_S, the following statements are equivalent*:

(1) *A_S is GPW-flat.*

(2) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SS$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$.*

(3) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$.*

(4) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies that*

$$\begin{array}{rlrl} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 s^n &= t_1 s^n \\ a_2 t_2 &= a_3 s_3 & s_2 s^n &= t_2 s^n \\ &\vdots & &\vdots \\ a_k t_k &= a' & s_k s^n &= t_k s^n, \end{array}$$

for some $k \in \mathbb{N}$ and elements $a_1, \ldots, a_k \in A_S$, $s_1, t_1, \ldots, s_k, t_k \in S$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious by Definition 2.1.

(2) $\Leftrightarrow$ (4) This is obvious by [9, II, Lemma 5.5]. □

Corollary 2.4. *Suppose that S is an idempotent or right cancellative monoid. Then for a right S-act A_S the following statements are equivalent*:

(i) *A_S is principally weakly flat.*

(ii) *A_S is GPW-flat.*

Proof. This is obvious by Proposition 2.3. □

Proposition 2.5. *Every GPW-flat right S-act is torsion free.*

Proof. This is obvious by Proposition 2.3. □

Example 2.6. Let $S = \mathbb{N} \cup G$, where $\mathbb{N}$ is the set of natural numbers and G is a nontrivial group with unit element e and define the multiplication on S as $gn = ng = n$ for every $g \in G$ and $n \in \mathbb{N}$. Clearly all right S-acts are torsion free by [9, IV, Theorem 6.1], but not all right S-acts are GPW-flat, see Theorem 4.5.

The following strict implications exist for different flatness properties of acts:

Weakly flat $\Rightarrow$ *Principally weakly flat* $\Rightarrow$ *GPW-flat* $\Rightarrow$ *Torsion free.*

In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of φ corresponding to some pullback diagram $P(M, N, f, g, Q)$ where $f : {}_SM \to {}_SQ$ and $g : {}_SN \to {}_SQ$ are homomorphisms of left S-acts.

Similar to [10, Proposition 2], we have the following proposition.

Proposition 2.7. *A right S-act A_S is GPW-flat if and only if for every $s \in S$ there exists $n = n_{(s,A_S)} \in \mathbb{N}$ such that the corresponding φ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \to {}_SS$ is a monomorphism of left S-acts.*

Proposition 2.8. *The following statements hold:*

(1) *Any retract of a GPW-flat right S-act is GPW-flat.*

(2) *If $A = \coprod_{i \in I} A_i$ is GPW-flat, then A_i is GPW-flat for every $i \in I$.*

(3) *S_S is GPW-flat.*

(4) Θ_S *is GPW-flat.*

Proof. (3) and (4) are obvious.

(1) Suppose that B_S is a GPW-flat right S-act and A_S is a retract of B_S. Then there exist homomorphisms $f : B_S \to A_S$ and $f' : A_S \to B_S$, such that $ff' = id_{A_S}$. Let $s \in S$. Since B_S is GPW-flat, there exists $n \in \mathbb{N}$ such that the equality $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes {}_S S$ implies that $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes {}_S (Ss^n)$, for any $b, b' \in B_S$ by (2) of Proposition 2.3. Let $as^n = a's^n$ for $a, a' \in A_S$. Then $f'(as^n) = f'(a's^n)$ and so $f'(a)s^n = f'(a')s^n$. Since $f'(a), f'(a') \in B_S$, B_S is GPW-flat and $f'(a) \otimes s^n = f'(a') \otimes s^n$ in $B_S \otimes {}_S S$, we have

$$\begin{aligned} f'(a) &= b_1 s_1 & & \\ b_1 t_1 &= b_2 s_2 & s_1 s^n &= t_1 s^n \\ b_2 t_2 &= b_3 s_3 & s_2 s^n &= t_2 s^n \\ &\vdots & &\vdots \\ b_k t_k &= f'(a') & s_k s^n &= t_k s^n, \end{aligned}$$

where $b_1, \ldots, b_k \in B_S$, $s_1, t_1, \ldots, s_k, t_k \in S$, by (4) of Proposition 2.3. Thus $f(f'(a)) = f(b_1 s_1)$ and so $a = f(b_1)s_1$. Similarly, $f(b_{i-1})t_{i-1} = f(b_i)s_i$, $2 \leq i \leq k$, and $a' = f(b_k)t_k$. Hence

$$\begin{aligned} a \otimes s^n &= f(b_1)s_1 \otimes s^n = f(b_1) \otimes s_1 s^n = f(b_1) \otimes t_1 s^n = f(b_1)t_1 \otimes s^n \\ &= f(b_2)s_2 \otimes s^n = f(b_2) \otimes s_2 s^n = \ldots = f(b_k)t_k \otimes s^n = a' \otimes s^n \end{aligned}$$

in $A_S \otimes {}_S (Ss^n)$.

(2) Suppose that $A = \coprod_{i \in I} A_i$ is GPW-flat right S-act and let $s \in S$. Let $j \in I$. By assumption, there exists $n \in \mathbb{N}$ such that $as^n = a's^n$ for $a, a' \in A_S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S (Ss^n)$. Let $as^n = a's^n$ for $a, a' \in A_j$. Thus $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S (Ss^n)$, by assumption. Hence $a \otimes s^n = a' \otimes s^n$ in $A_j \otimes {}_S (Ss^n)$, by [2, Corollary 2.3]. □

A monoid S is called *left almost regular* if for every $s \in S$

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s &= s_m r s, \end{aligned}$$

for some $r, r_1, \ldots, r_m$, s_1, s_2, $\ldots$, $s_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Clearly every regular and right cancellative monoid is left almost regular.

As, by [9, IV, Theorem 6.5], over a left almost regular monoid every torsion free right S-act is principally weakly flat, the following proposition is easily checked.

Proposition 2.9. *Let S be a left almost regular monoid, and A_S be a right S-act. Then the following statements are equivalent*:

(1) *A_S is principally weakly flat.*

(2) *A_S is GPW-flat.*

(3) *A_S is torsion free.*

Theorem 2.10. *For a proper right ideal K of monoid S the following statements are equivalent*:

(1) $\big(\forall s \in S, \exists n \in \mathbb{N}\big)\big(\forall l \in S \setminus K\big)\Big(ls^n \in K \Rightarrow \big(\exists k \in K, ls^n = ks^n\big)\Big)$.

(2) *$A(K)$ is GPW-flat.*

Proof. (1) ⇒ (2) Let $s \in S$. Then, by assumption, there exists $n \in \mathbb{N}$ such that (1) is established. Let $as^n = a's^n$ for $a, a' \in A(K)$. Since $(x,1)S \cong S_S \cong (y,1)S$ (every free right S-act is GPW-flat), without loss of generality, we can take $a = (x, r_1), a' = (y, r_2)$, where $r_1, r_2 \in S \setminus K$. Since $(x, r_1)s^n = (y, r_2)s^n$, we have $r_1 s^n = r_2 s^n \in K$, and so there exists $k \in K$ such that $r_1 s^n = ks^n = r_2 s^n$. Hence

$$(x, r_1) \otimes s^n = (x, 1) \otimes r_1 s^n = (x, 1) \otimes ks^n = (y, 1) \otimes ks^n = (y, r_2) \otimes s^n$$

in $A(K) \otimes {}_S(Ss^n)$.

(2) ⇒ (1) Let $A(K)$ be GPW-flat and suppose $s \in S$. Then there exists $n \in \mathbb{N}$ such that $A(K) \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now let $l \in S \setminus K$ such that $ls^n \in K$. Then clearly $(x, l)s^n = (y, l)s^n$. By Proposition 2.3, we have

$$\begin{array}{rlr} (x,l) &= (w_1,u_1)s_1 & \\ (w_1,u_1)t_1 &= (w_2,u_2)s_2 & s_1s^n = t_1s^n \\ &\vdots & \vdots \\ (w_{m-1},u_{m-1})t_{m-1} &= (w_m,u_m)s_m & s_{m-1}s^n = t_{m-1}s^n \\ (w_m,u_m)t_m &= (y,l) & s_ms^n = t_ms^n, \end{array}$$

for some $m \in \mathbb{N}$, $u_1, \ldots, u_m \in S$, $s_1, t_1, \ldots, s_m, t_m \in S$, and $w_1, \ldots, w_m \in \{x, y, z\}$. By definition of $A(K)$, there exists $i \in \{1, \ldots, m-1\}$ such that $w_i \neq w_{i+1}$, and so there exists $k \in K$ such that $u_i t_i = u_{i+1} s_{i+1} = k$. Hence we have

$$ls^n = u_1 s_1 s^n = u_1 t_1 s^n = u_2 s_2 s^n = \ldots = u_i t_i s^n = ks^n,$$

as required. □

Recall from [9, III, Definition 10.14] that an element s of a monoid S is called *right e-cancellable* for an idempotent $e \in S$ if $s = es$ and $\ker \rho_s \leq \ker \rho_e$. A monoid S is called *left PP* if every element $s \in S$ is right e-cancellable for some idempotent $e \in S$. It is obvious that every regular and every right cancellative monoid is left PP. An element $s \in S$ is *right semi-cancellable* if the equality $xs = ys$ for any $x, y \in S$ implies that there exists $r \in S$ such that $rs = s$ and $xr = yr$. A monoid S is called *left PSF* if every element $s \in S$ is right semi-cancellable. Clearly every left PP monoid is left PSF.

Proposition 2.11. *Suppose that S is a left PP monoid. An act A_S is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $es^n = s^n$ and $ae = a'e$ for some $e^2 = e \in S$.*

Proof. This is obvious by [9, III, Theorem 10.16]. □

For a left PSF monoid, similar to argument used in [11, Proposition 2.5], we can show the following proposition.

Proposition 2.12. *Suppose that S is a left PSF monoid. An act A_S is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $rs^n = s^n$ and $ar = a'r$ for some $r \in S$.*

Corollary 2.13. *For a left PSF monoid S, the following statements are equivalent:*

(1) *$\prod_{i=1}^{k} A_i$ is GPW-flat.*

(2) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $\alpha_i, \alpha'_i \in A_i, 1 \leq i \leq k$, if $(\alpha_1, \alpha_2, \ldots, \alpha_k)s^n = (\alpha'_1, \alpha'_2, \ldots, \alpha'_k)s^n$, then $us^n = s^n$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)u = (\alpha'_1, \alpha'_2, \ldots, \alpha'_k)u$ for some $u \in S$.*

Proof. This is obvious by Proposition 2.12. □

Proposition 2.14. *For any family $\{A_i\}_{i\in I}$ of right S-acts, if $\prod_{i\in I} A_i$ is GPW-flat, then A_i is GPW-flat, for every $i \in I$.*

Proof. Let $\prod_{i\in I} A_i$ be GPW-flat and let $s \in S$ and $i \in I$. By assumption there exists $n \in \mathbb{N}$ such that the functor $\prod_{i\in I} A_i \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Let $a_is^n = a'_is^n$ for any $a_i, a'_i \in A_i$ and suppose $a_j \in (A_j)_S$ for $j \neq i$. If

$$a_k = \begin{cases} a_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

$$a'_k = \begin{cases} a'_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

Then $(a_k)_Is^n = (a'_k)_Is^n$ and so $(a_k)_I \otimes s^n = (a'_k)_I \otimes s^n$ in $\prod_{i\in I} A_i \otimes_S (Ss^n)$, by Proposition 2.3. Now we have $a_i \otimes s^n = a'_i \otimes s^n$ in $A_i \otimes_S (Ss^n)$, by [14, Remrak 3.1], and so A_i is GPW-flat. □

Golchin in [3] showed that if $S = G\dot{\cup}I$ where G is a group and I is an ideal of S and A is a right S-act that is ((principally) weakly) flat, torsion free, satisfies Condition (P) or (P_E) as a right I^1 -act, then it has these properties as a right S-act. Similarly, we can show the following theorem for GPW-flatness.

Theorem 2.15. *Let $S = G\dot{\cup}I$ and let A be a right S -act. If A is GPW-flat as a right I^1 -act, then it is GPW-flat as a right S -act.*

Proof. This is obvious by Proposition 2.3. □

3 GPW-flatness of (Rees factor) cyclic acts

In this section, we give conditions for a (Rees factor) cyclic act to be GPW-flat.

Proposition 3.1. *Suppose that ρ is a right congruence on a monoid S. Then the following statements are equivalent:*

(i) *S/ρ is GPW-flat.*

(ii) $\big(\forall s \in S\big)\big(\exists n \in \mathbb{N}\big)\big(\forall u, v \in S\big)\Big((us^n)\rho(vs^n) \Rightarrow u(\rho \vee \ker \rho_{s^n})v\Big)$.

Proof. (i) $\Rightarrow$ (ii) Let $s \in S$. Since the right S-act S/ρ is GPW-flat, there exists $n \in \mathbb{N}$ such that the functor $(S/\rho)_S \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now suppose that $(us^n)\rho(vs^n)$ for $u, v \in S$. Thus $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_SS$ and so $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S(Ss^n)$. Hence $u(\rho \vee \ker \rho_{s^n})v$, by [9, III, Lemma 10.6].

(ii) $\Rightarrow$ (i) Let $s \in S$. By assumption, there exists $n \in \mathbb{N}$ such that $(us^n)\rho(vs^n)$, for every $u, v \in S$, implies that $u(\rho \vee \ker \rho_{s^n})v$. Suppose $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_SS$, thus $(us^n)\rho(vs^n)$. Now, by assumption, $u(\rho \vee \ker \rho_{s^n})v$ and so, by [9, III, Lemma 10.6], $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S(Ss^n)$. Hence S/ρ is GPW-flat, by Proposition 2.3. □

Corollary 3.2. *The principal right ideal zS is GPW-flat if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for any $x, y \in S$, $zxs^n = zys^n$ implies that $x(\ker \lambda_z \vee \ker \rho_{s^n})y$.*

Proof. Since $zS \cong S/\ker \lambda_z$, by Proposition 3.1, it suffices to take $\rho = \ker \lambda_z$. □

Theorem 3.3. *Suppose that K is a right ideal of S. Then S/K is GPW-flat if and only if for every $s \in S$ there exists a natural number $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$, for some $k \in K$.*

Proof. If $K = S$, then $S/K \cong \Theta_S$ is GPW-flat by (3) of Proposition 2.8. Thus suppose that K is a proper right ideal of S.

Necessity. Suppose that S/K is GPW-flat for the proper right ideal K of S and let $s \in S$. Then there exists $n \in \mathbb{N}$ such that the functor $A_S \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now suppose $ls^n \in K$ for $l \in S \setminus K$. Then $[l] \otimes s^n = [j] \otimes s^n$ in $S/K \otimes {}_SS$, for any $j \in K$ and so, by Proposition 2.3, there exist $m \in \mathbb{N}$, $p_1, \ldots, p_m, s_1, t_1, \ldots, s_m, t_m \in S$ such that

$$\begin{aligned}
[l] &= [p_1]s_1 & & \\
[p_1]t_1 &= [p_2]s_2 & s_1s^n &= t_1s^n \\
[p_2]t_2 &= [p_3]s_3 & s_2s^n &= t_2s^n \\
&\vdots & &\vdots \\
[p_m]t_m &= [j] & s_ms^n &= t_ms^n.
\end{aligned}$$

Since $j \in K$, we have $p_m t_m \in K$. Let q be the least number such that $q \in \{1, \dots, m\}$ and $p_q t_q \in K$. Let $k = p_q t_q$, then $p_{q-1} t_{q-1} = p_q s_q$, and so

$$ls^n = p_1 s_1 s^n = p_1 t_1 s^n = p_2 s_2 s^n = \dots = \\ p_{q-1} t_{q-1} s^n = p_q s_q s^n = p_q t_q s^n = ks^n.$$

Sufficiency. Let K be a right ideal of S and let $s \in S$. Thus there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$ for some $k \in K$, by assumption. Let for any $u, v \in S$, $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$. Thus there are four cases as follows:

Case 1. $u, v \in K$. Then it is clear that $[u] = [v]$ in S/K and so $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S (Ss^n)$.

Case 2. $u \in K, v \in S \setminus K$. Then there exists $k \in K$ such that $vs^n = ks^n$, by assumption. Then

$$[u] \otimes s^n = [k] \otimes s^n = [1] \otimes ks^n = [1] \otimes vs^n = [v] \otimes s^n$$

in $S/K \otimes_S (Ss^n)$.

Case 3. $u \in S \setminus K, v \in K$. It is similar to the Case 2.

Case 4. $u, v \in S \setminus K$. Then from $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$, we have either $us^n = vs^n$ or $us^n, vs^n \in K$. If $us^n = vs^n$, the result follows. Otherwise, $us^n = ks^n$ and $vs^n = ls^n$ for some $k, l \in K$, by assumption. So

$$[u] \otimes s^n = [1] \otimes us^n = [1] \otimes ks^n = [k] \otimes s^n = \\ [l] \otimes s^n = [1] \otimes ls^n = [1] \otimes vs^n = [v] \otimes s^n$$

in $S/K \otimes_S (Ss^n)$. □

4 Characterization of monoids by GPW-flatness of acts

Now we classify monoids over which all right S-acts are GPW-flat and also monoids over which some other properties imply GPW-flatness and vice versa.

A monoid S is called *regular* if for every $s \in S$ there exists $x \in S$ such that $s = sxs$.

Definition 4.1. An element $s \in S$ is called *eventually regular* if s^n is regular for some $n \in \mathbb{N}$. That is, $s^n = s^n x s^n$ for some $n \in \mathbb{N}$ and $x \in S$. A monoid S is called *eventually regular* if every $s \in S$ is eventually regular.

Obviously every regular monoid is eventually regular.

Definition 4.2. An element $s \in S$ is called *eventually left almost regular* if

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_mc_m &= s_{m-1}r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. In other words $s \in S$ is called *eventually left almost regular* if s^n is left almost regular for some $n \in \mathbb{N}$.

If every element of a monoid S is eventually left almost regular, then S is called *eventually left almost regular.*

It is clear that every left almost regular monoid is eventually left almost regular, and also every eventually regular monoid is eventually left almost regular.

Example 4.3. Let $S = \{1, 0, e, f, a\}$ be the monoid with the following table

	1	0	e	f	a
1	1	0	e	f	a
0	0	0	0	0	0
e	e	0	e	a	a
f	f	0	0	f	0
a	a	0	0	a	0

Clearly S is eventually regular and so it is eventually left almost regular. But S is not regular, because $a \in S$ is not regular. Also S is not left almost regular, since $a \in S$ is not left almost regular.

Theorem 4.4. *The following statements are equivalent*:

(1) *S is an eventually left almost regular monoid.*

(2) *All torsion free right Rees factor acts over S are GPW-flat.*

(3) *All torsion free cyclic right S-acts are GPW-flat.*

(4) *All torsion free finitely generated right S-acts are GPW-flat.*

(5) *All torsion free right S-acts are GPW-flat.*

Proof. (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) Suppose that all torsion free right Rees factor S-acts are GPW-flat and let $s \in S$. Let $K(s)$ be the subset of S consisting of all elements $t \in S$ such that

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. We see that $s^n \in K(s)$ for some $n \in \mathbb{N}$, and so $K(s)$ is non-empty, because, if $m = 1$ and $c_1 = r_1 = 1$, then $t = s^n$ has the required property mentioned. Now let $J = \bigcup_{t \in K(s)} tS$. Let $s'c \in J$, for $s' \in S$ and c right cancellable. Then $s'c \in tS$ for some $t \in K(s)$, and so we have

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m \\ s'c &= tr_{m+1}, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m, r_{m+1} \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Thus $s' \in J$, and so S/J is torsion free by [9, III, Proposition 8.10]. Hence S/J is GPW-flat by assumption and so by Theorem 3.3, for $s^n \in J$, there exists $tr \in J$ such that

$s^n = trs^n$, where $t \in K(s)$, and $r \in S$. Now $s^n = trs^n$ and $t \in K(s)$ implies that s is eventually left almost regular.

(1) ⇒ (5) Let S be an eventually left almost regular monoid and suppose A_S is a torsion free right S-act and let $s \in S$. Since s is eventually left almost regular, we have

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Let $as^n = a's^n$ for $a, a' \in A_S$. Using torsion freeness, it can easily be seen that $as_m r = a's_m r$. Hence we have

$$a \otimes s^n = a \otimes s_m rs^n = as_m r \otimes s^n = a's_m r \otimes s^n = a' \otimes s_m rs^n = a' \otimes s^n$$

in $A_S \otimes {}_S(Ss^n)$ and so A_S is GPW-flat, as required. □

A right S-act A_S is a generator if for any distinct homomorphisms $\alpha, \beta : X_S \to Y_S$, there exists a homomorphism $f : A_S \to X_S$ such that $\alpha f \neq \beta f$. Equivalently, a right S-act A_S is a generator if and only if there exists an epimorphism $\pi : A_S \to S_S$ ([9, II, Theorem 3.16]).

As we know, $S \times A_S$ is a generator for each right S-act A_S.

Theorem 4.5. *The following statements are equivalent*:

(1) *S is an eventually regular monoid.*

(2) *A right S-act A_S is GPW-flat if $Hom(A_S, S_S) \neq \emptyset$.*

(3) *$S \times A_S$ is GPW-flat for every generator right S-act A_S.*

(4) *$S \times A_S$ is GPW-flat for every right S-act A_S.*

(5) *All generator right S-acts are GPW-flat.*

(6) *All right Rees factor S-acts are GPW-flat.*

(7) *All cyclic right S-acts are GPW-flat.*

(8) *All right S-acts are GPW-flat.*

Proof. (8) $\Rightarrow$ (7) $\Rightarrow$ (6), (8) $\Rightarrow$ (5), (8) $\Rightarrow$ (4) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), (5) $\Rightarrow$ (4) and (8) $\Rightarrow$ (2) are obvious.

(4) $\Rightarrow$ (8) This is valid by Proposition 2.14.

(6) $\Rightarrow$ (1) If all right Rees factor acts over S are GPW-flat, then all right Rees factor acts over S are torsion free. So every right cancellable element of S is right invertible, by [9, IV, Theorem 6.1], but by Theorem 4.4, S is eventually left almost regular. Now let $s \in S$. Then

$$\begin{aligned} s_1 c_1 &= s^n r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

Multiplying both sides of the equalities in the above scheme by c_i^{-1} for $i \in \{1, \ldots, m\}$, respectively, we get

$$\begin{aligned} s_1 &= s^n r_1 c_1^{-1} \\ s_2 &= s_1 r_2 c_i^{-1} \\ &\vdots \\ s_m &= s_{m-1} r_m c_m^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} s^n = s_m r s^n = s_{m-1} r_m c_m^{-1} r s^n &= s_{m-2} r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n = \ldots \\ &= s^n r_1 c_1^{-1} \ldots r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n, \end{aligned}$$

and so s is eventually regular, as required.

(1) $\Rightarrow$ (8) Suppose that A_S is a right S-act and let $s \in S$. By Proposition 2.3, we have to show that there exists $m \in \mathbb{N}$ such that for any $a, a' \in A_S$, if $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes {}_S S$, then $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes {}_S (Ss^m)$. Since

s is eventually regular, there exist $n \in \mathbb{N}$ and $t \in S$, such that $s^n = s^n t s^n$. If $m = n$. Let $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S S$ for any $a, a' \in A_S$, then

$$a \otimes s^n = a \otimes s^n t s^n = a s^n \otimes t s^n = a' s^n \otimes t s^n = a' \otimes s^n t s^n = a' \otimes s^n$$

in $A_S \otimes {}_S(Ss^n)$ and so A_S is GPW-flat.

(3) $\Rightarrow$ (4) Suppose that A_S is a right act over S. As we show in the proof of (5) $\Rightarrow$ (4), $S \times A_S$ is a generator and so, by assumption, $S \times (S \times A_S)$ is GPW-flat, which means that $S \times A_S$ is GPW-flat, by Proposition 2.14. □

It is obvious that Condition (P) implies GPW-flatness, but the following example shows that this is not the case for Condition (E).

Example 4.6. Let $S = (\mathbb{N}, .)$ be the monoid of natural numbers with multiplication and let $A_{\mathbb{N}} = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N}$. Then $A_{\mathbb{N}}$ satisfies Condition (E), but it is not GPW-flat.

Now, the question is that: What is the structure of monoids over which Condition (E) of their acts implies GPW-flatness?

Theorem 4.7. *For a right cancellative monoid S the following statements are equivalent*:

(1) $\prod_{i \in I} A_i$ *is principally weakly flat, for any family* $\{A_i\}_{i \in I}$ *of right S-acts.*

(2) $\prod_{i \in I} A_i$ *is GPW-flat, for any family* $\{A_i\}_{i \in I}$ *of right S-acts.*

(3) $\prod_{i \in I} A_i$ *is torsion free, for any family* $\{A_i\}_{i \in I}$ *of right S-acts.*

(4) *S is a group.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (4) This is obvious, by [14, Remark 3.1] and [9, IV. Theorem 6.1]. □

An element $a \in A_S$ is called divisible by $s \in S$ if there exists $b \in A_S$, such that $bs = a$. An act A_S is said to be divisible if $Ac = A$, for any left cancellable element $c \in S$. It is clear that A_S is divisible if and only if every element of A_S is divisible by any left cancellable element of S.

Theorem 4.8. *The following statements are equivalent:*

1. *All right S-acts are divisible.*
2. *All GPW-flat right S-acts are divisible.*
3. *All GPW-flat finitely generated right S-acts are divisible.*
4. *All GPW-flat cyclic right S-acts are divisible.*
5. *All GPW-flat monocyclic right S-acts are divisible.*
6. *All left cancellable elements of S are left invertible.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6) For every $s \in S$ we have $S/\rho(s,s) = S_S/\Delta_S \cong S_S$, by (3) of Proposition 2.8, S_S is GPW-flat, and so it is divisible by assumption. Thus $Sc = S$, for any left cancellable element $c \in S$. Thus, there exists $s \in S$ such that $sc = 1$, and so c is left invertible, as required.

(6) $\Rightarrow$ (1) It is clear from [9, III, Proposition 2.2]. □

Recall, from [15], that a right S-act A_S is called *strongly torsion free* if the equality $as = a's$, for $a, a' \in A_S$ and $s \in S$ implies $a = a'$. It is clear that every strongly torsion free right S-act is GPW-flat, but not the converse.

Theorem 4.9. *The following statements are equivalent:*

1. *All GPW-flat right S-acts are strongly torsion free.*
2. *All GPW-flat finitely generated right S-acts are strongly torsion free.*
3. *All GPW-flat cyclic right S-acts are strongly torsion free.*
4. *S is right cancellative monoid.*

Proof. This follows from [15, Theorem 3.1]. □

Recall from [6] that a right S-act A_S is *E-torsion free* if for any $a, a' \in A_S$ and $e \in E(S)$, $ae = a'e$ implies $a = a'$.

Theorem 4.10. *The following statements are equivalent:*

1. *All GPW-flat right S-acts are E-torsion free.*

(2) *All GPW-flat finitely generated right S-acts are E-torsion free.*

(3) *All GPW-flat cyclic right S-acts are E-torsion free.*

(4) $E(S) = \{1\}$.

Proof. This is obvious by [6, Theorem 3.1]. □

Recall from [1, Definition 1] that a right S-act A_S is called *principally weakly kernel flat* (PWKF) if the corresponding φ is bijective for the pullback diagram $P(Ss, Ss, f, f, S)$ $(s \in S)$, and A_S is *translation kernel flat (TKF)* if the corresponding φ is bijective for the pullback diagram $P(S, S, f, f, S)$.

Theorem 4.11. *The following statements on a monoid S are equivalent*:

(1) *All GPW-flat right S-acts are $PWKF$ and S is left PSF.*

(2) *All GPW-flat right S-acts are TKF and S is left PSF.*

(3) *All GPW-flat right S-acts satisfy Condition (PWP) and S is left PSF.*

(4) *All GPW-flat right S-acts satisfy Condition (P') and S is left PSF.*

(5) *S is right cancellative.*

Proof. (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are clear.

(5) ⇒ (1) By [13, Theorem 2.12] and Corollary 2.4, it is clear.

(3) ⇒ (5) This is obvious, by [12, Theorem 2.8].

(5) ⇒ (4) Since S is right cancellative, S is left PSF. By [4, Theorem 2.8] and Corollary 2.4, it is clear. □

Theorem 4.12. *The following statements on a monoid S are equivalent*:

(1) *All GPW-flat right S-acts are $PWKF$ and there exists a regular left S-act.*

(2) *All GPW-flat right S-acts are TKF and there exists a regular left S-act.*

(3) *All GPW-flat right S-acts satisfy Condition (PWP) and there exists a regular left S-act.*

(4) *All GPW-flat right S-acts satisfy Condition (P') and there exists a regular left S-act.*

(5) $|E(S)| = 1$ *and there exists a regular left S-act.*

(6) *S is right cancellative.*

Proof. (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are clear.

(6) ⇒ (4) This is clear by [4, Theorem 2.9] and Corollary 2.4.

(5) ⇔ (6) This is clear by [12, Theorem 2.9].

(3) ⇒ (6) This is clear by [12, Theorem 2.9].

(6) ⇒ (1) This is clear by [13, Theorem 2.18] and Corollary 2.4. □

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References

[1] Bulman-Fleming, S., Kilp, M., and Laan, V., *Pullbacks and flatness properties of acts* II, Comm. Algebra 29(2) (2001), 851-878.

[2] Golchin, A., *Flatness and coproducts*, Semigroup Forum 72(3) (2006), 433-440.

[3] Golchin, A., *On flatness of acts*, Semigroup Forum 67(2) (2003), 262-270.

[4] Golchin, A. and Mohammadzadeh, H., *On Condition (P')*, Semigroup Forum 86(2) (2013), 413-430.

[5] Golchin, A. and Mohammadzadeh, H., *On regularity of Acts*, J. Sci. Islam. Repub. Iran 19(4) (2008), 339-345.

[6] Golchin, A., Zare, A., and Mohammadzadeh, H., *E-torsion free acts over monoids*, Thai J. Math. 14(1) (2015), 93-114.

[7] Kilp, M., *On flat acts* (Russian), Tatru UL. Toimetisted, 253 (1970), 66-72.

[8] Kilp, M., *Characterization of monoids by properties of their left Rees factors*, Tatru UL. Toimetisted, 640 (1983), 29-37.

[9] Kilp, M., Knauer, U., and Mikhalev, A., "Monoids, Acts and Categories", De Gruyter, 2000.

[10] Laan, V., *Pullbacks and flatness properties of acts* I., Comm. Algebra 29(2) (2001), 829-850.

[11] Nouri, L., Golchin, A., and Mohammadzadeh, H., *On properties of product acts over monoids*, Comm. Algebra 43(5) (2015), 1854-1876.

[12] Qiao, H., *Some new characterizations of right cancellative monoids by Condition* (PWP), Semigroup Forum 71(1) (2005), 134-139.

[13] Qiao, H., Limin, W., and Zhongkui, L., *On some new characterizations of right cancellative monoids by flatness properties*, Arab. J. Sci. Eng. 32(1) (2007), 75-82.

[14] Sedaghatjoo, M., Khosravi, R., and Ershad, M., *Principally weakly and weakly coherent monoids*, Comm. Algebra 37(12) (2009), 4281-4295.

[15] Zare, A., Golchin, A., and Mohammadzadeh, H., *Strongly torsion free acts over monoids*, Asian-Eur. J. Math. 6(3) (2013), 1350049.

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